Total light deflection in the field of an axisymmetric body at rest with full mass and spin multipole structure

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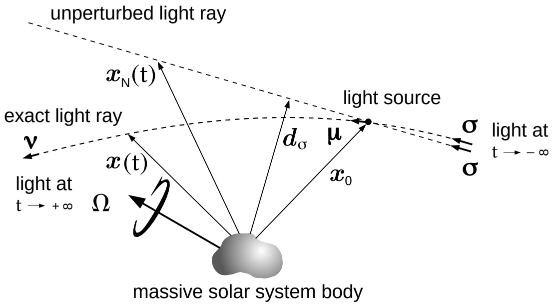
1.1 Light trajectory in curved space-time



- light trajectory in harmonic coordinates

$$\boxed{\boldsymbol{x}(t) = \boldsymbol{x}_0 + c\,(t - t_0)\,\boldsymbol{\sigma} + \Delta\boldsymbol{x}(t)} \qquad (1)$$

- tangent vectors of light ray at past and future infinity

$$\boldsymbol{\sigma} = \left.\frac{\dot{\boldsymbol{x}}(t)}{c}\right|_{t\to-\infty} \tag{2}$$

$$\boldsymbol{\nu} = \left.\frac{\dot{\boldsymbol{x}}(t)}{c}\right|_{t\to+\infty} \tag{3}$$

- total light deflection is angle between $\boldsymbol{\sigma}$ and $\boldsymbol{\nu}$

$$\boxed{\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}) = \arcsin|\boldsymbol{\sigma} \times \boldsymbol{\nu}|}$$

1.2 Light deflection in field of monopole

- tangent vector at plus infinity

$$\boxed{\boldsymbol{\nu} = \boldsymbol{\sigma} + \boldsymbol{\nu}_{1PN}^{M_0}}$$

with

$$\boxed{\boldsymbol{\nu}_{\rm 1PN}^{M_0} = -\frac{4GM}{c^2\, d_\sigma}\, \frac{\boldsymbol{d}_\sigma}{d_\sigma}}$$

- angle of total light deflection

$$\boxed{\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\rm 1PN}^{M_0}\right) = \frac{4GM}{c^2}\, \frac{1}{d_\sigma}}$$

- $\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\rm 1PN}^{M_0}\right) = 1.75''$ for grazing ray at Sun

1.3 Light deflection in field of monopole in uniform rotation

- tangent vector at plus infinity

$$\boldsymbol{\nu} = \boldsymbol{\sigma} + \boldsymbol{\nu}_{\rm 1PN}^{M_0} + \boldsymbol{\nu}_{\rm 1.5PN}^{S_1}$$

with angular momentum $\boldsymbol{S} = \kappa^2\, M\, \Omega\, P^2\, \boldsymbol{e}_3$ and

$$\boldsymbol{\nu}_{\rm 1.5PN}^{S_1} = -\frac{4G}{c^3\,(d_\sigma)^2}\left[2\,\frac{(\boldsymbol{\sigma}\times\boldsymbol{d}_\sigma)\cdot\boldsymbol{S}}{d_\sigma}\,\frac{\boldsymbol{d}_\sigma}{d_\sigma} + (\boldsymbol{\sigma}\times\boldsymbol{S})\right]$$

- angle of total light deflection: $\delta\left(\boldsymbol{\sigma},\boldsymbol{\nu}_{\rm 1PN}^{M_0}\right) + \delta\left(\boldsymbol{\sigma},\boldsymbol{\nu}_{\rm 1.5PN}^{S_1}\right)$

$$\delta\left(\boldsymbol{\sigma},\boldsymbol{\nu}_{\rm 1.5PN}^{S_1}\right) = \frac{4GM}{c^3\,(d_\sigma)^2}\,\frac{(\boldsymbol{\sigma}\times\boldsymbol{d}_\sigma)\cdot\boldsymbol{S}}{d_\sigma}$$

- $\delta\left(\boldsymbol{\sigma},\boldsymbol{\nu}_{\rm 1.5PN}^{S_1}\right) = 0.7\,\mu{\rm as}$ for grazing ray at Sun

2. Tangent vector and light deflection for arbitrary body

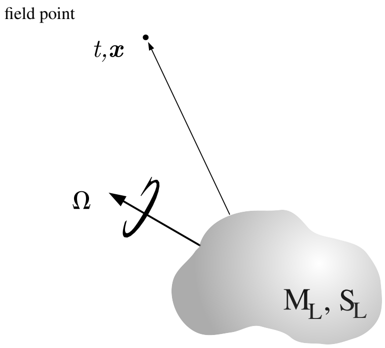
- multipole decomposition of tangent vector at future infinity

$$\boldsymbol{\nu} = \boldsymbol{\sigma} + \sum_{l=0}^{\infty} \boldsymbol{\nu}_{\rm 1PN}^{M_L} + \sum_{l=1}^{\infty} \boldsymbol{\nu}_{\rm 1.5PN}^{S_L} + \mathcal{O}\left(c^{-4}\right)$$

- multipole decomposition of total light deflection

$$\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}\right) = \sum_{l=0}^{\infty} \delta\Big(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\rm 1PN}^{M_L}\Big) + \sum_{l=1}^{\infty} \delta\Big(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\rm 1.5PN}^{S_L}\Big) + \mathcal{O}\left(c^{-4}\right)$$

2.1 Metric of arbitrary body at rest



- post-Newtonian expansion of metric tensor

$$g_{\mu\nu}(t, \boldsymbol{x}) = \eta_{\mu\nu} + h^{(2)}_{\mu\nu}(t, \boldsymbol{x}) + h^{(3)}_{\mu\nu}(t, \boldsymbol{x}) + \mathcal{O}\left(c^{-4}\right)$$

- metric perturbations in time-independent case [1, 2, 3] in canonical gauge:

$$
\begin{aligned}
h_{00}^{(2)}(\boldsymbol{x}) &= \frac{2}{c^2} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{M}_L \, \hat{\partial}_L \frac{1}{r} \\
h_{0i}^{(3)}(\boldsymbol{x}) &= \frac{4}{c^3} \sum_{l=1}^{\infty} \frac{(-1)^l \; l}{(l+1)!} \epsilon_{iab} \, \hat{S}_{bL-1} \, \hat{\partial}_{aL-1} \frac{1}{r} \\
h_{ij}^{(2)}(\boldsymbol{x}) &= \frac{2}{c^2} \, \delta_{ij} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \hat{M}_L \, \hat{\partial}_L \frac{1}{r}
\end{aligned}
$$

- with the STF differential operator

$$
\hat{\partial}_L = \mathrm{STF}_{i_1 \dots i_l} \, \frac{\partial}{\partial x^{i_1}} \, \dots \frac{\partial}{\partial x^{i_l}}
$$

2.2 The mass and spin multipoles for arbitrary body

- mass multipoles [3]

$$\hat{M}_L = \int d^3x \; \hat{x}_L \; \Sigma$$

- spin multipoles [3]

$$\hat{S}_L = \int d^3x \; \epsilon_{jk<i_l} \, \hat{x}_{L-1>} \; x^j \; \Sigma^k$$

- where $\Sigma = \left(T^{00} + T^{kk}\right)/c^2$ and $\Sigma^k = T^{0k}/c$ with $T^{\alpha\beta}$ is the stress-energy tensor of body

2.3 Geodesic equation

- geodesic equation is given by

$$\frac{\ddot{x}^i(t)}{c^2} = +\frac{\partial h_{00}^{(2)}}{\partial x^i} - 2\,\frac{\partial h_{00}^{(2)}}{\partial x^j}\,\sigma^i\sigma^j + \frac{\partial h_{0j}^{(3)}}{\partial x^i}\,\sigma^j - \frac{\partial h_{0i}^{(3)}}{\partial x^j}\,\sigma^j - \frac{\partial h_{0j}^{(3)}}{\partial x^k}\,\sigma^i\sigma^j\sigma^k$$

- It is **difficult to integrate the geodesic equation**, because one has first to perform the differentiations in metric and geodesic equation and afterwards to insert the unperturbed light ray and then to perform the integrations

2.4 Advanced integration method

- advanced integration methods by Kopeikin (1997) [4]
  based on new parameters

$$c\tau = \boldsymbol{\sigma} \cdot \boldsymbol{x}_{\rm N}$$

  and

$$\xi^i = P^i_j\, x^j_{\rm N}$$

- where the operator

$$P^{ij} = \delta^{ij} - \sigma^i \sigma^j$$

  projects onto the plane perpendicular to vector $\boldsymbol{\sigma}$

2.5 Metric in terms of new parameters

- metric in terms of new parameters is given by

$$
\begin{aligned}
h^{(2)}_{00}\left(c\tau, \boldsymbol{\xi}\right) &= \frac{2}{c^2} \sum_{l=0}^{\infty} \frac{\left(-1\right)^l}{l!} \, \hat{M}_L \, \hat{\partial}_L \frac{1}{r_{\rm N}} \\
h^{(3)}_{0i}\left(c\tau, \boldsymbol{\xi}\right) &= \frac{4}{c^3} \sum_{l=1}^{\infty} \frac{\left(-1\right)^l \, l}{(l+1)!} \, \epsilon_{iab} \, \hat{S}_{bL-1} \, \hat{\partial}_{aL-1} \frac{1}{r_{\rm N}} \\
h^{(2)}_{ij}\left(c\tau, \boldsymbol{\xi}\right) &= \frac{2}{c^2} \, \delta_{ij} \sum_{l=0}^{\infty} \frac{\left(-1\right)^l}{l!} \, \hat{M}_L \, \hat{\partial}_L \frac{1}{r_{\rm N}}
\end{aligned}
$$

- with the STF differential operator

$$
\begin{aligned}
\hat{\partial}_L &= {\rm STF}_{i_1 \dots i_l} \sum_{p=0}^{l} \frac{l!}{(l-p)! \, p!} \, \sigma_{i_1} \dots \sigma_{i_p} \\
&\quad \times P^{j_{p+1}}_{i_{p+1}} \dots P^{j_l}_{i_l} \, \frac{\partial}{\partial \xi^{j_{p+1}}} \dots \frac{\partial}{\partial \xi^{j_l}} \left(\frac{\partial}{\partial c\tau}\right)^p
\end{aligned}
$$

2.6 Geodesic equation in terms of new parameters

- geodesic equation is given by

$$\frac{\ddot{x}^i(\tau)}{c^2} = P^{ij}\frac{\partial h_{00}^{(2)}}{\partial \xi^j} - \frac{\partial h_{00}^{(2)}}{\partial c\tau}\,\sigma^i - \frac{\partial h_{0i}^{(3)}}{\partial c\tau} + P^{ik}\frac{\partial h_{0j}^{(3)}}{\partial \xi^k}\,\sigma^j$$

- It is **not difficult to integrate the geodesic equation**, because one may first perform the integrations and afterwards one may perform the differentiations

- first and second integration are given by [4]

$$\frac{\dot{\boldsymbol{x}}(\tau)}{c} = \boldsymbol{\sigma} + \sum_{l=0}^{\infty}\frac{\Delta\dot{\boldsymbol{x}}_{\rm 1PN}^{M_L}(\tau)}{c} + \sum_{l=1}^{\infty}\frac{\Delta\dot{\boldsymbol{x}}_{\rm 1.5PN}^{S_L}(\tau)}{c}$$

$$\boldsymbol{x}(\tau) = \boldsymbol{x}_{\rm N}(\tau) + \sum_{l=0}^{\infty}\Delta\boldsymbol{x}_{\rm 1PN}^{M_L}(\tau,\tau_0) + \sum_{l=1}^{\infty}\Delta\boldsymbol{x}_{\rm 1.5PN}^{S_L}(\tau,\tau_0)$$

2.7 Tangent vector of light ray for arbitrary body

- multipole decomposition of tangent vector at future infinity

$$\boldsymbol{\nu} = \boldsymbol{\sigma} + \sum_{l=0}^{\infty} \boldsymbol{\nu}_{1\mathrm{PN}}^{M_L} + \sum_{l=1}^{\infty} \boldsymbol{\nu}_{1.5\mathrm{PN}}^{S_L} + \mathcal{O}\left(c^{-4}\right)$$

- mass-multipole term can be obtained from Eq. (34) in [4]

$$\nu_{1\mathrm{PN}}^{i\,M_L} = -\frac{4G}{c^2} P^{ij} \frac{\partial}{\partial \xi^j} \frac{(-1)^l}{l!} \hat{M}_L \, \hat{\partial}_L \, \ln|\boldsymbol{\xi}|$$

- spin-multipole term can be obtained from Eq. (37) in [4]

$$\nu_{1.5\mathrm{PN}}^{i\,S_L} = -\frac{8G}{c^3} P^{ij} \frac{\partial}{\partial \xi^j} \sigma^c \, \epsilon_{i_l bc} \frac{(-1)^l \, l}{(l+1)!} \hat{S}_{bL-1} \, \hat{\partial}_L \, \ln|\boldsymbol{\xi}|$$

2.8 Total light deflection for arbitrary body

- multipole decomposition of total light deflection

$$\delta\left(\boldsymbol{\sigma},\boldsymbol{\nu}\right)=\sum_{l=0}^{\infty}\delta\Big(\boldsymbol{\sigma},\boldsymbol{\nu}_{1\mathrm{PN}}^{M_L}\Big)+\sum_{l=1}^{\infty}\delta\Big(\boldsymbol{\sigma},\boldsymbol{\nu}_{1.5\mathrm{PN}}^{S_L}\Big)+\mathcal{O}\left(c^{-4}\right)$$

- mass-multipole term $(l\geq 2)$; cf. Eq. (47) in [4]

$$\delta\left(\boldsymbol{\sigma},\boldsymbol{\nu}_{1\mathrm{PN}}^{M_L}\right)=-\frac{4G}{c^2}\,\frac{1}{|\boldsymbol{\xi}|}\,\frac{(-1)^l}{(l-1)!}\,\hat{M}_L\,\hat{\partial}_L\;\ln|\boldsymbol{\xi}|$$

- spin-multipole term $(l\geq 1)$; cf. Eq. (48) in [4]

$$\delta\left(\boldsymbol{\sigma},\boldsymbol{\nu}_{1.5\mathrm{PN}}^{S_L}\right)=-\frac{8G}{c^3}\,\frac{1}{|\boldsymbol{\xi}|}\,\epsilon_{abc}\,\sigma^c\,\frac{(-1)^l\;l^2}{(l+1)!}\,\hat{S}_{bL-1}\hat{\partial}_{aL-1}\ln|\boldsymbol{\xi}|$$

- the STF differential operation is given by

$$\hat{\partial}_L \ \ln|\boldsymbol{\xi}| = \mathrm{STF}_{i_1 \ldots i_l} P^{j_{p+1}}_{i_{p+1}} \ \ldots \ P^{j_l}_{i_l} \frac{\partial}{\partial \xi^{j_1}} \ \ldots \ \frac{\partial}{\partial \xi^{j_l}} \ \ln|\boldsymbol{\xi}|$$

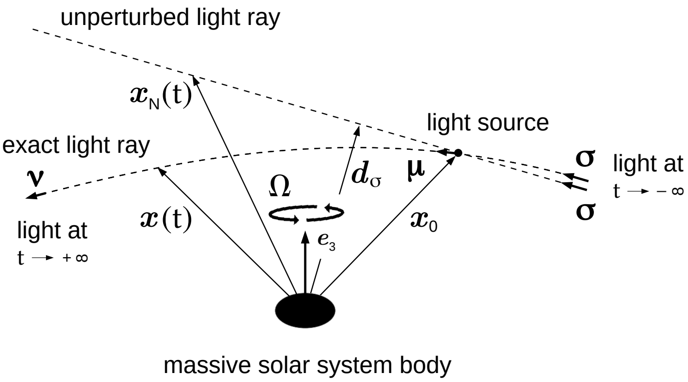
- considerable amount of algebra yields [9]:

$$\hat{\partial}_L \ln|\boldsymbol{\xi}| = \frac{(-1)^{l+1}}{|\boldsymbol{\xi}|^l} \ \mathrm{STF}_{i_1 \ldots i_l} \sum_{n=0}^{[l/2]} G^l_n P_{i_1 i_2} \ldots P_{i_{2n-1} i_{2n}} \frac{\xi_{i_{2n+1}} \ldots \xi_{i_l}}{|\boldsymbol{\xi}|^{l-2n}}$$

- where the coefficients [9]

$$G^l_n = (-1)^n \ 2^{l-2n-1} \ \frac{l!}{n!} \ \frac{(l-n-1)!}{(l-2n)!}$$

3. Total light deflection for axisymmetric body



- body rotates with constant Ω around its symmetry axis e_3

3.1 Mass and Spin multipoles of axisymmetric body

- mass monopole

$$\boxed{\hat{M}_0 = -M\,(P)^0\;J_0}$$

- mass multipoles [5]

$$\boxed{\hat{M}_L = -M\,(P)^l\;J_l\,\delta^{\,3}_{<i_1}\;\ldots\;\delta^3_{i_l>}}$$

- spin dipole (moment of inertia $\kappa^2 = I/MP^2$)

$$\boxed{\hat{S}_a = -\kappa^2\,M\,\Omega\,(P)^2\;J_0\,\delta^3_a}$$

- spin multipoles [5]

$$\boxed{\hat{S}_L = -M\,\Omega\,(P)^{l+1}\;J_{l-1}\,\frac{l+1}{l+4}\,\delta^{\,3}_{<i_1}\;\ldots\;\delta^3_{i_l>}}$$

- zonal harmonic coefficients [6]

$$J_l = -\frac{1}{M\ (P)^l} \int d^3x\, r^l\, \Sigma\, P_l\,(\cos\theta)$$

- STF tensor

$$\delta^{\,3}_{<i_1}\ \ldots\ \delta^3_{i_l>} = \sum_{p=0}^{[l/2]} H^l_p\, \delta_{\{i_1 i_2}\ \ldots\ \delta_{i_{2p-1} i_{2p}}\, \delta^3_{i_{2p+1}}\ \ldots\ \delta^3_{i_l\}}$$

- with the scalar coefficients

$$H^l_p = (-1)^p\ \frac{(2l-2p-1)!!}{(2l-1)!!}$$

3.2 Tangent vector and light deflection: M_L

- mass-multipole term of tangent vector [9]:

$$\nu^{i\,M_L}_{1\mathrm{PN}} = -\frac{4GM}{c^2}\,\frac{J_l}{l}\,P^{ij}\,\frac{\partial}{\partial \xi^j}\left(\frac{P}{|\boldsymbol{\xi}|}\right)^l\,F^l_M$$

- mass-multipole term of total light deflection [9]:

$$\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}^{M_L}_{1\mathrm{PN}}\right) = -\,\frac{4GM}{c^2}\,\frac{J_l}{d_\sigma}\left(\frac{P}{d_\sigma}\right)^l\,F^l_M$$

- where the scalar function is given by

$$F^l_M = \frac{1}{(l-1)!}\delta^{\,3}_{<i_1}\ldots\delta^3_{i_l>}\sum_{n=0}^{[l/2]} G^l_n\,P_{i_1 i_2}\,\ldots\,P_{i_{2n-1}i_{2n}}\frac{\xi_{i_{2n+1}}\ldots\xi_{i_l}}{|\boldsymbol{\xi}|^{l-2n}}$$

- by algebraic calculations [9]

  in agreement with time-transfer function approach [8]

$$F^l_M = \frac{1}{(l-1)!} \sum_{n=0}^{[l/2]} G^l_n \left(1 - (\boldsymbol{\sigma} \cdot \boldsymbol{e}_3)^2\right)^n \left(\frac{\boldsymbol{d}_\sigma \cdot \boldsymbol{e}_3}{d_\sigma}\right)^{l-2n}$$

- by introducing the variable $-1 \leq x \leq +1$

$$x = \left(1 - (\boldsymbol{\sigma} \cdot \boldsymbol{e}_3)^2\right)^{-1/2} \left(\frac{\boldsymbol{d}_\sigma \cdot \boldsymbol{e}_3}{d_\sigma}\right)$$

  one may write the scalar function in the form [9]:

$$F^l_M = \left[1 - (\boldsymbol{\sigma} \cdot \boldsymbol{e}_3)^2\right]^{[l/2]} \frac{l}{2} \sum_{n=0}^{[l/2]} \frac{(-1)^n}{n!} \frac{(l-n-1)!}{(l-2n)!} (2x)^{l-2n}$$

3.2.1 Chebyshev polynomials of first kind

$$T_l\left(x\right) = \frac{l}{2} \sum_{n=0}^{[l/2]} \frac{\left(-1\right)^n}{n!} \frac{\left(l-n-1\right)!}{\left(l-2n\right)!} \left(2x\right)^{l-2n}$$

- scalar function (mass multipoles) (for $l \geq 0$) [9]:

$$F_M^l = \left[1 - \left(\boldsymbol{\sigma} \cdot \boldsymbol{e}_3\right)^2\right]^{[l/2]} T_l\left(x\right)$$

- tangent vector $\boldsymbol{\nu}_{1.\mathrm{PN}}^{M_L}$ and total light deflection $\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\mathrm{PN}}^{M_L}\right)$ caused by mass-multipoles M_L are given in terms of Chebyshev polynomials of first kind $T_l(x)$

3.2.2 Example: mass quadrupole

- Chebyshev polynomial of first kind and second degree:

$$\boxed{T_2\,(x) = -1 + 2\,x^2}$$

- mass-quadrupole term of total light deflection [7]:

$$\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\rm 1PN}^{M_2}\right) = + \frac{4\,GM}{c^2}\,\frac{J_2}{d_\sigma}\left(\frac{P}{d_\sigma}\right)^2 \times \left[1 - \left(\boldsymbol{\sigma}\cdot\boldsymbol{e}_3\right)^2 - 2\left(\frac{\boldsymbol{d}_\sigma\cdot\boldsymbol{e}_3}{d_\sigma}\right)^2\right]$$

3.2.3 Example: mass octupole

- Chebyshev polynomial of first kind and fourth degree:

$$T_4\,(x) = 1 - 8\,x^2 + 8\,x^4$$

- mass-octupole term of total light deflection:

$$\delta\left(\boldsymbol{\sigma},\boldsymbol{\nu}_{\rm 1PN}^{M_4}\right) = -\frac{4\,GM}{c^2}\,\frac{J_4}{d_\sigma}\,\left(\frac{P}{d_\sigma}\right)^4 \times \left[\left(1-\left(\boldsymbol{\sigma}\cdot\boldsymbol{e}_3\right)^2\right)^2 - 8\left(\frac{\boldsymbol{d}_\sigma\cdot\boldsymbol{e}_3}{d_\sigma}\right)^2\left(1-\left(\boldsymbol{\sigma}\cdot\boldsymbol{e}_3\right)^2\right) + 8\left(\frac{\boldsymbol{d}_\sigma\cdot\boldsymbol{e}_3}{d_\sigma}\right)^4\right]$$

3.2.4 Upper limits of total light deflection: mass multipoles

- The upper limit of Chebyshev polynomial of first kind:

$$T_l\left(x\right) = \cos\left(l\ \arccos x\right) \quad \Longrightarrow \quad \left|T_l\left(x\right)\right| \le 1$$

- Upper limits of total light deflection for mass multipoles [9]:

$$\left|\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\rm 1PN}^{M_L}\right)\right| \le \frac{4GM}{c^2}\,\frac{|J_l|}{d_\sigma}\left(1 - \left(\boldsymbol{\sigma}\cdot\boldsymbol{e}_3\right)^2\right)^{[l/2]}\left(\frac{P}{d_\sigma}\right)^l$$

$$\left|\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\rm 1PN}^{M_L}\right)\right| \le \frac{4GM}{c^2}\,\frac{|J_l|}{d_\sigma}\left(\frac{P}{d_\sigma}\right)^l$$

$$\left|\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{\rm 1PN}^{M_L}\right)\right| \le \frac{4GM}{c^2}\,\frac{|J_l|}{d_\sigma}$$

3.3 Tangent vector and light deflection: S_L

- spin-multipole term of tangent vector [9]:

$$\nu^{i\,S_L}_{1.5\mathrm{PN}} = -\frac{8GM}{c^3}\,\Omega\,P\,\frac{J_{l-1}}{l+4}\,P^{ij}\,\frac{\partial}{\partial\xi^j}\left(\frac{P}{|\boldsymbol{\xi}|}\right)^l F_S^l$$

- spin-multipole term of total light deflection [9]:

$$\delta\left(\boldsymbol{\sigma},\boldsymbol{\nu}^{S_L}_{1.5\mathrm{PN}}\right) = -\,\frac{8GM}{c^3}\,\Omega\,J_{l-1}\,\left(\frac{P}{d_\sigma}\right)^{l+1} F_S^l$$

- where the scalar function is given by

$$F_S^l = \frac{1}{(l-1)!}\,\frac{l}{l+4}\,\epsilon_{i_l bc}\,\sigma^c\,\delta^3_{<b}\,\delta^3_{i_1}\,\ldots\,\delta^3_{i_{l-1}>}$$
$$\times\ \mathrm{STF}_{i_1\ldots i_l}\sum_{n=0}^{[l/2]} G_n^l\,P_{i_1 i_2}\,\ldots\,P_{i_{2n-1}i_{2n}}\,\frac{\xi_{i_{2n+1}}\,\ldots\,\xi_{i_l}}{|\boldsymbol{\xi}|^{l-2n}}$$

- by algebraic calculations [9]:

$$F_S^l = + \frac{(\boldsymbol{\sigma} \times \boldsymbol{d}_\sigma) \cdot \boldsymbol{e}_3}{d_\sigma} \frac{l}{l+4} \frac{1}{(l-1)!} \times \sum_{n=0}^{[l/2]} G_n^l \frac{l-2n}{l} \left(1 - (\boldsymbol{\sigma} \cdot \boldsymbol{e}_3)^2\right)^n \left(\frac{\boldsymbol{d}_\sigma \cdot \boldsymbol{e}_3}{d_\sigma}\right)^{l-2n-1}$$

- by introducing the variable x as defined above one obtains

$$F_S^l = \frac{(\boldsymbol{\sigma} \times \boldsymbol{d}_\sigma) \cdot \boldsymbol{e}_3}{d_\sigma} \frac{l}{l+4} \left[1 - (\boldsymbol{\sigma} \cdot \boldsymbol{e}_3)^2\right]^{[l/2]} \times \sum_{n=0}^{[l/2]} (l-2n) \frac{(-1)^n}{n!} \frac{(l-n-1)!}{(l-2n)!} (2x)^{l-2n-1}$$

- using

$$\boxed{(l-2n)\,(2x)^{l-2n-1} = \frac{1}{2}\,\frac{d}{dx}\,(2x)^{l-2n}}$$

- one may write the scalar function in the form [9]:

$$\boxed{F_S^l = \frac{(\boldsymbol{\sigma}\times \boldsymbol{d}_\sigma)\cdot \boldsymbol{e}_3}{d_\sigma}\,\frac{1}{l+4}\left[1-(\boldsymbol{\sigma}\cdot \boldsymbol{e}_3)^2\right]^{[l/2]}\frac{d\,T_l\,(x)}{dx}}$$

- derivative of Chebyshev polynomial of first kind related to Chebyshev polynomial of second kind

$$\boxed{\frac{d\,T_l\,(x)}{dx} = l\,U_{l-1}\,(x)}$$

3.3.1 Chebyshev polynomials of second kind

$$U_l\left(x\right) = \sum_{n=0}^{[l/2]} \frac{\left(-1\right)^n}{n!} \, \frac{\left(l-n\right)!}{\left(l-2n\right)!} \, \left(2x\right)^{l-2n}$$

- spin-dipole term of total light deflection:

$$\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\mathrm{PN}}^{S_1}\right) = -\,\frac{4GM}{c^3}\,\Omega\,\kappa^2\,J_0\,\left(\frac{P}{d_\sigma}\right)^2\,\frac{\left(\boldsymbol{\sigma}\times\boldsymbol{d}_\sigma\right)\cdot\boldsymbol{e}_3}{d_\sigma}$$

- scalar function (spin multipoles) (for $l \geq 3$) [9]:

$$F_S^l = \frac{\left(\boldsymbol{\sigma}\times\boldsymbol{d}_\sigma\right)\cdot\boldsymbol{e}_3}{d_\sigma}\left[1-\left(\boldsymbol{\sigma}\cdot\boldsymbol{e}_3\right)^2\right]^{[l/2]} U_{l-1}\left(x\right)$$

- tangent vector $\boldsymbol{\nu}_{1.5\mathrm{PN}}^{S_L}$ and total light deflection $\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\mathrm{PN}}^{S_L}\right)$ caused by spin-multipoles S_L are given in terms of Chebyshev polynomials of second kind $U_l(x)$

3.3.2 Example: spin hexapole

- Chebyshev polynomial of second kind and second degree:

$$U_2\,(x) = -1 + 4\,x^2$$

- spin-hexapole term of total light deflection:

$$\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}^{S_3}_{1.5\mathrm{PN}}\right) = +\frac{24}{7}\,\frac{GM}{c^3}\,\Omega\,J_2\left(\frac{P}{d_\sigma}\right)^4\,\frac{(\boldsymbol{\sigma}\times\boldsymbol{d}_\sigma)\cdot\boldsymbol{e}_3}{d_\sigma}$$
$$\times\left[1-(\boldsymbol{\sigma}\cdot\boldsymbol{e}_3)^2 - 4\left(\frac{\boldsymbol{d}_\sigma\cdot\boldsymbol{e}_3}{d_\sigma}\right)^2\right]$$

3.3.3 Example: spin decapole

- Chebyshev polynomial of second kind and fourth degree:

$$U_4\left(x\right) = 1 - 12\,x^2 + 16\,x^4$$

- spin-decapole term of total light deflection:

$$\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}^{S_5}_{1.5\mathrm{PN}}\right) = -\frac{40}{9}\,\frac{GM}{c^3}\,\Omega\,J_4\left(\frac{P}{d_\sigma}\right)^6\,\frac{\left(\boldsymbol{\sigma}\times\boldsymbol{d}_\sigma\right)\cdot\boldsymbol{e}_3}{d_\sigma}$$
$$\times\left[\left(1-\left(\boldsymbol{\sigma}\cdot\boldsymbol{e}_3\right)^2\right)^2 - 12\left(\frac{\boldsymbol{d}_\sigma\cdot\boldsymbol{e}_3}{d_\sigma}\right)^2\left(1-\left(\boldsymbol{\sigma}\cdot\boldsymbol{e}_3\right)^2\right)\right.$$
$$\left. + 16\left(\frac{\boldsymbol{d}_\sigma\cdot\boldsymbol{e}_3}{d_\sigma}\right)^4\right]$$

3.3.4 Upper limits of total light deflection: spin multipoles

- The upper limit of Chebyshev polynomial of second kind:

$$U_{l-1}\left(x\right) = \frac{1}{\sqrt{1-x^2}}\;\sin\left(l\;\arccos x\right) \quad \Longrightarrow \quad \left|U_{l-1}\left(x\right)\right| \le l$$

- Upper limits of total light deflection for spin multipoles [9]:

$$\left|\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\mathrm{PN}}^{S_L}\right)\right| \le \frac{8GM}{c^3}\,\frac{\Omega l^2}{l+4}\left|J_{l-1}\right|\left(\frac{P}{d_\sigma}\right)^{l+1}\left(1-\left(\boldsymbol{\sigma}\cdot\boldsymbol{e}_3\right)^2\right)^{\frac{l-1}{2}}$$

$$\left|\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\mathrm{PN}}^{S_L}\right)\right| \le \frac{8GM}{c^3}\,\Omega\,\frac{l^2}{l+4}\,\left|J_{l-1}\right|\,\left(\frac{P}{d_\sigma}\right)^{l+1}$$

$$\left|\delta\left(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\mathrm{PN}}^{S_L}\right)\right| \le \frac{8GM}{c^3}\,\Omega\,\frac{l^2}{l+4}\,\left|J_{l-1}\right|$$

4. Numerical values of total light deflection

Light deflection	Sun	Jupiter	Saturn
$\lvert\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\mathrm{PN}}^{M_0})\rvert$	1.75×10^6	16.3×10^3	5.8×10^3
$\lvert\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\mathrm{PN}}^{M_2})\rvert$	0.35	239	94
$\lvert\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\mathrm{PN}}^{M_4})\rvert$	1.72	9.6	5.41
$\lvert\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\mathrm{PN}}^{M_6})\rvert$	0.07	0.55	0.50
$\lvert\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\mathrm{PN}}^{M_8})\rvert$	0.007	0.04	0.06
$\lvert\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1\mathrm{PN}}^{M_{10}})\rvert$	–	0.003	0.01

Light deflection	Sun	Jupiter	Saturn
$\lvert\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\mathrm{PN}}^{S_1})\rvert$	0.7	0.17	0.04
$\lvert\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\mathrm{PN}}^{S_3})\rvert$	–	0.026	0.008
$\lvert\delta(\boldsymbol{\sigma}, \boldsymbol{\nu}_{1.5\mathrm{PN}}^{S_5})\rvert$	–	0.001	–

5. Summary

- mass-multipole terms of tangent vector and total light deflection are given in terms of Chebyshev polynomials of first kind.
- spin-multipole terms of tangent vector and total light deflection are given in terms of Chebyshev polynomials of second kind.
- strict upper limits for mass-multipole terms of total light deflection.
- strict upper limits for spin-multipole terms of total light deflection.

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