On Noncrossing Partitions

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Outline

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Set Partitions

- **set partition** $\rightsquigarrow \Pi_n$

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$n = 16$

$$\Big\{\{1,6,7\},\{2,8,14,15\},\{3,4,5\},\{9,10,12,13\},\{11\},\{16\}\Big\}$$

Set Partitions

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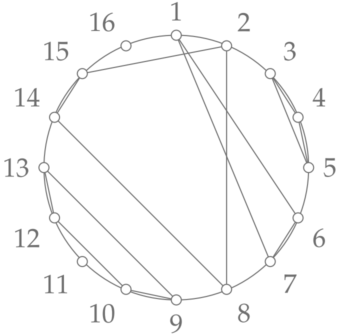

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Set Partitions

- **dual refinement order** $\rightsquigarrow \leq_{\mathrm{dref}}$

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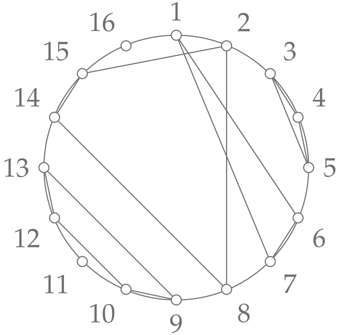

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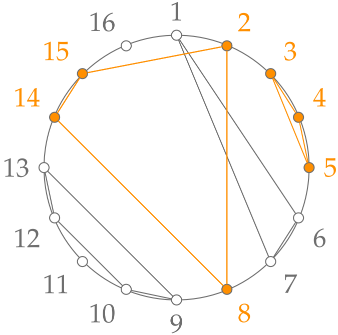

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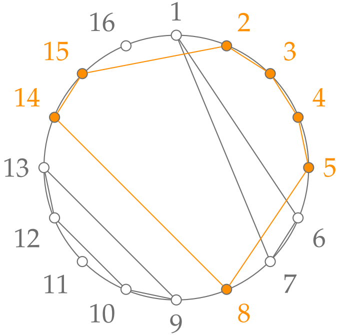

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Set Partitions

Proposition (Folklore)

For $n > 0$ the poset $(\Pi_n, \leq_{dref})$ is a graded lattice. We have

- *number of elements: $B(n)$*
- *number of elements of rank k: $S(n,k)$*
- *Möbius number: $(-1)^{n-1}(n-1)!$*
- *number of maximal chains: $\frac{n!(n-1)!}{2^{n-1}}$*

Set Partitions

- **Bell number**:

$$B(n) = \sum_{j=0}^{n-1} B_j \binom{n-1}{j}$$

1
2
5
15
52
203

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Set Partitions

- **Stirling number of second kind**:

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

1					
1	1				
1	3	1			
1	6	7	1		
1	10	25	15	1	
1	15	65	90	31	1

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Set Partitions

- **bounded poset**: poset with least and greatest element $\rightsquigarrow \hat{0}, \hat{1}$
- **Möbius number**: $\mu(\hat{0}, \hat{1})$

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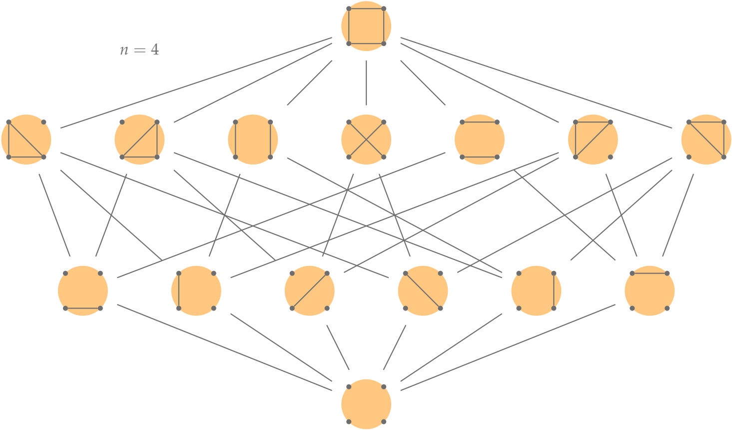
Example: $(\Pi_4, \leq_{\mathrm{dref}})$

$n = 4$



Noncrossing Set Partitions

- **noncrossing set partition** $\rightsquigarrow NC_n$

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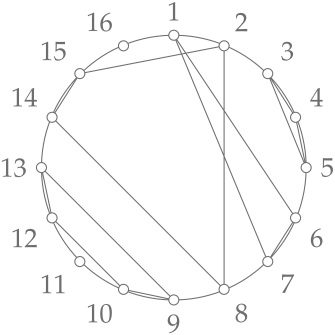

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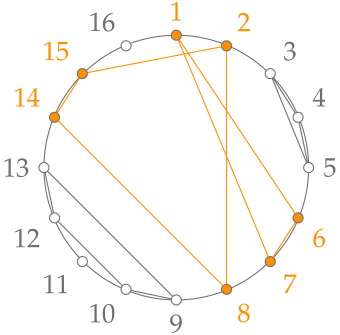

Nope!

$\Big\{\{1,6,7\},\{2,8,14,15\},\{3,4,5\},\{9,10,12,13\},\{11\},\{16\}\Big\}$

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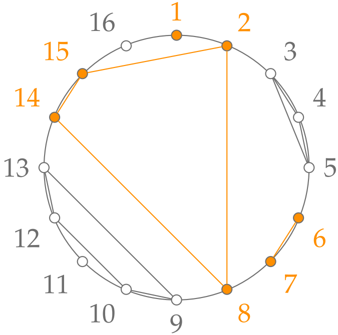

$$\big\{\{1\},\{2,8,14,15\},\{3,4,5\},\{6,7\},\{9,10,12,13\},\{11\},\{16\}\big\}$$

Noncrossing Set Partitions

Proposition (G. Kreweras, 1972)

For $n > 0$ the poset $(NC_n, \leq_{dref})$ is a graded, complemented lattice. We have

- *number of elements: $Cat(n)$*
- *number of elements of rank k: $Nar(n,k)$*
- *Möbius number: $(-1)^{n-1}Cat(n-1)$*
- *number of maximal chains: n^{n-2}*

Noncrossing Set Partitions

- **Catalan number**:

$$\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$$

1
2
5
14
42
132

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Noncrossing Set Partitions

- **Narayana number**:

$$\mathrm{Nar}(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

1					
1	1				
1	3	1			
1	6	6	1		
1	10	20	10	1	
1	15	50	50	15	1

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Example: $(NC_4, \leq_{dref})$

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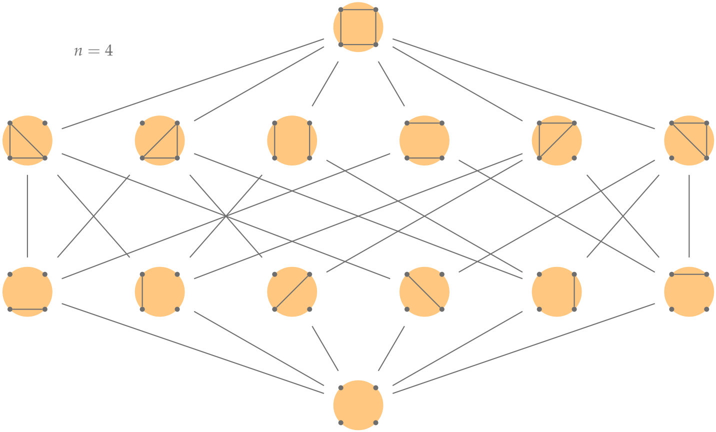
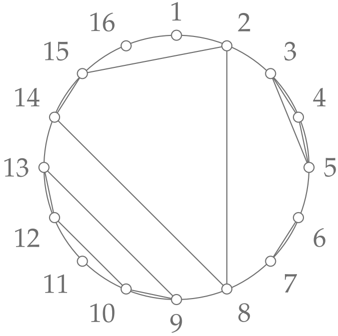


Kreweras Complement

$n = 16$



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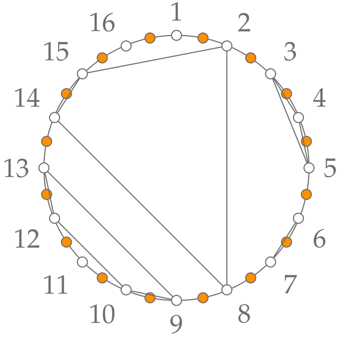

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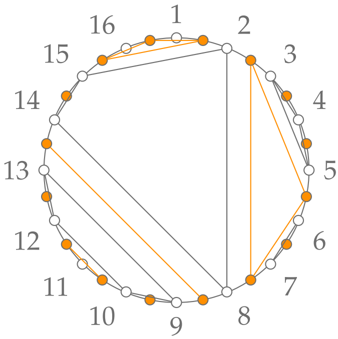

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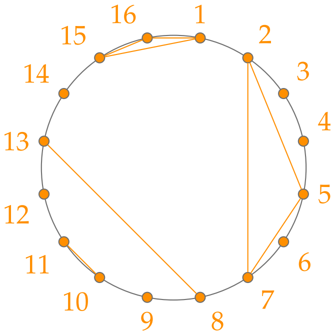

$\big\{\{1,15,16\},\{2,5,7\},\{3\},\{4\},\{6\},\{8,13\},\{9\},\{10,11\},\{12\},\{14\}\big\}$

Further Properties

- further properties of $(NC_n, \leq_{\text{dref}})$:
  - it is lexicographically shellable [A. Björner, P. Edelman, 1980]
  - it is self-dual [R. Simion, D. Ullman, 1991]
  - it admits a symmetric chain decomposition [R. Simion, D. Ullman, 1991]
  - it is strongly Sperner [R. Simion, D. Ullman, 1991]

Applications

- noncrossing partitions
  - determine the matrix of chromatic joins [W. Tutte, 1993]
  - index free cumulants in the moments of a non-commutative random variable [R. Speicher, 1997]
  - index connected components of positroids [F. Ardila, F. Rincón, L. Williams, 2016]
- the order complex of the noncrossing partition lattice
  - has a quotient with contractible universal cover and the braid group as fundamental group [D. Krammer, 2000; T. Brady, 2001]

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Guises

- the noncrossing partition lattice is (isomorphic to) the poset of
  - simple elements in the dual braid monoid [D. Bessis, 2003]
  - finitely-generated wide subcategories of representations of a directed path [C. Ingalls, H. Thomas, 2009]
  - certain shard intersections of the braid arrangement [N. Reading, 2011]

Outline

1. Noncrossing Set Partitions
2. A Symmetric Group Object
3. Reflection Groups
4. Combinatorial Models
5. Extensions

A Symmetric Group Object

- map parts to cycles $\rightsquigarrow \beta$

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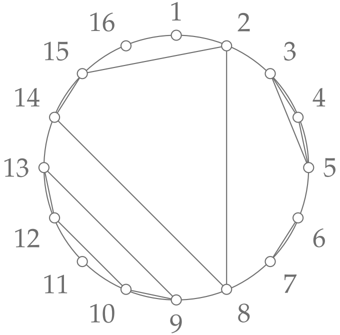

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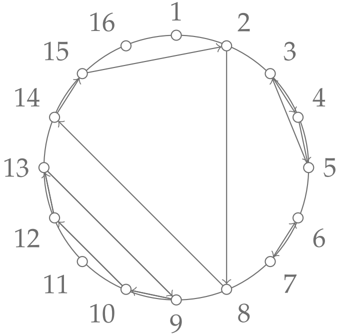

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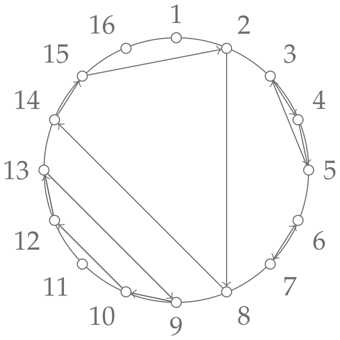

$(2\ 8\ 14\ 15)(3\ 4\ 5)(6\ 7)(9\ 10\ 12\ 13)$

A Symmetric Group Object

- map parts to cycles $\rightsquigarrow \beta$
- multiply by transpositions

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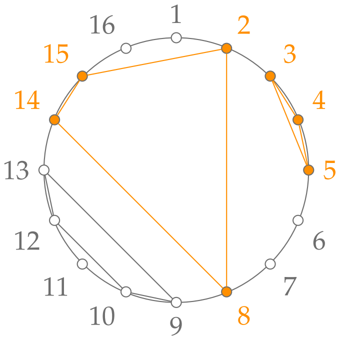

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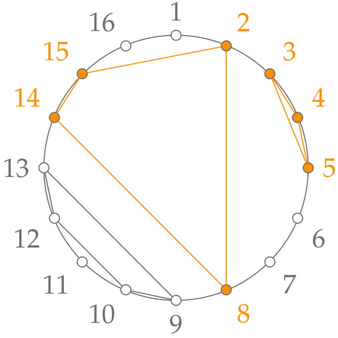

$(2\ 8\ 14\ 15)(3\ 4\ 5)(6\ 7)(9\ 10\ 12\ 13) \cdot (2\ 5)$

- map parts to cycles $\rightsquigarrow \beta$
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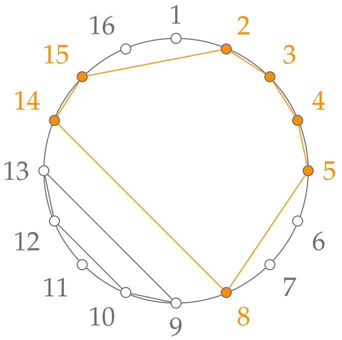

$(2\ 3\ 4\ 5\ 8\ 14\ 15)(6\ 7)(9\ 10\ 12\ 13)$

A Symmetric Group Object

- **absolute length**: $\ell_T(x) = n - \operatorname{cyc}(x)$
- **absolute order**: $u \leq_T v$ if and only if $$\ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v)$$
- $NC_n = \{x \in \mathfrak{S}_n \mid x \leq_T (1\,2\,\ldots\,n)\}$

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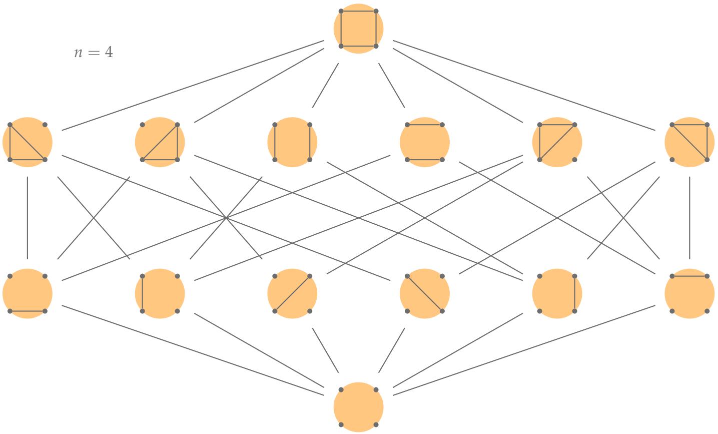
- $NC_n = \{x \in \mathfrak{S}_n \mid x \leq_T (1\,2\,\ldots\,n)\}$

Theorem (P. Biane, 1997)

For $x, y \in NC_n$ we have $x \leq_{dref} y$ if and only if $\beta(x) \leq_T \beta(y)$.

Example: $(NC_4, \leq_{\mathrm{dref}})$

$n = 4$



Example: $(NC_4, \leq_T)$

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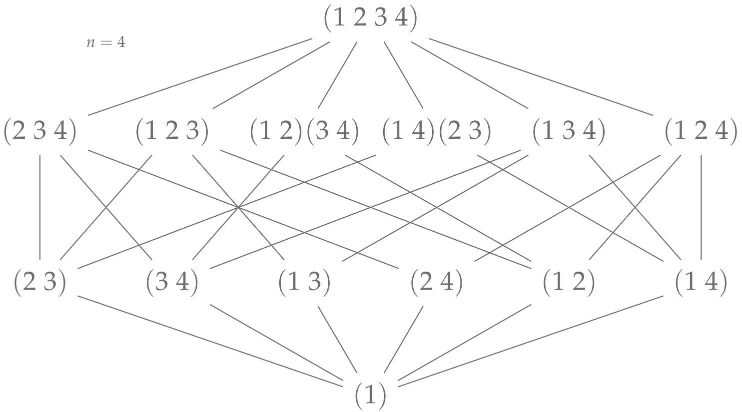

Outline

Reflection Groups

- V .. unitary complex vector space
- (complex) **reflection**: unitary transformation on V fixing a space of codimension 1 $\rightsquigarrow T$
- (complex) **reflection group**: finite subgroup of $U(V)$ generated by (complex) reflections $\rightsquigarrow W$
- **irreducible**: W does not fix a proper subspace of V
- **rank**: codimension of space fixed by W $\rightsquigarrow n$
- **well-generated**: minimal generating set has n elements

Reflection Groups

- $G(de, e, n)$ for $d, e, n \geq 1$ is the group consisting of
  - $n \times n$ matrices with a unique non-zero entry per row and column
  - the non-zero entries are $(de)^{\text{th}}$ roots of unity
  - the product of the non-zero entries is a d^{th} root of unity

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Theorem (G. C. Shephard, J. A. Todd, 1954)

The irreducible well-generated complex reflection groups are (isomorphic to) either

- $G(1,1,n)$ *for* $n \geq 1$,
- $G(d,1,n)$ *for* $d \geq 2, n \geq 1$,
- $G(d,d,n)$ *for* $d,n \geq 2$, *or*
- 26 *exceptional groups.*

Reflection Groups

- **Coxeter element**: a "well-behaved" element $\rightsquigarrow c$
- **absolute length**: minimum length of a factorization into reflections $\rightsquigarrow \ell_T$
- **absolute order**: $u \leq_T v$ if and only if
$$\ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v)$$

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Definition (T. Brady, C. Watt, 2002; D. Bessis, 2003)

Let W be an irreducible well-generated complex reflection group, T its set of reflections, and $c \in W$ a Coxeter element. The set of **W-noncrossing partitions** is

$$NC_W(c) = \{w \in W \mid w \leq_T c\}.$$

Reflection Groups

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$$\ell_T(v) = \ell_T(u) + \ell_T(u^{-1}v)$$

Theorem (V. Reiner, V. Ripoll, C. Stump, 2014)

Let W be an irreducible well-generated complex reflection group, and let $c, c' \in W$ be two Coxeter elements. The posets $(NC_W(c), \leq_T)$ and $(NC_W(c'), \leq_T)$ are isomorphic.

Reflection Groups

Theorem (C. Chevalley, 1955)

A finite group G is a complex reflection group if and only if its algebra of G-invariant polynomials is again a polynomial algebra.

Reflection Groups

- **degrees**: $d_1 \leq d_2 \leq \cdots \leq d_n$

Theorem (C. Chevalley, 1955)

A finite group G is a complex reflection group if and only if its algebra of G-invariant polynomials is again a polynomial algebra. The degrees of a homogeneous choice of generators of this algebra are group invariants.

Reflection Groups

- **degrees**: $d_1 \leq d_2 \leq \cdots \leq d_n$
- ***W*-Catalan number**:

$$\mathrm{Cat}(W) = \prod_{i=1}^{n} \frac{d_i + d_n}{d_i}$$

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Theorem (G. Kreweras, 1972; V. Reiner, 1997; D. Bessis, 2004–2016)

For every irreducible well-generated complex reflection group W the cardinality of NC_W is given by $Cat(W)$.

The Symmetric Group

- the degrees of $G(1,1,n)$ are $2,3,\ldots,n$
- we get

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$$\operatorname{Cat}(\mathfrak{S}_n) = \prod_{i=1}^{n-1} \frac{i+1+n}{i+1}$$

The Symmetric Group

- the degrees of $\mathfrak{S}_n$ are $2, 3, \ldots, n$
- we get

$$\operatorname{Cat}\bigl(\mathfrak{S}_n\bigr) = \frac{(2+n)(3+n)\cdots 2n}{2\cdot 3\cdots n}$$

The Symmetric Group

- the degrees of $\mathfrak{S}_n$ are $2, 3, \ldots, n$
- we get

$$\operatorname{Cat}(\mathfrak{S}_n) = \frac{(2n)!}{(n+1)n!n!}$$

The Symmetric Group

- the degrees of $\mathfrak{S}_n$ are $2, 3, \ldots, n$
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The Symmetric Group

- the degrees of $\mathfrak{S}_n$ are $2, 3, \ldots, n$
- we get

$$\mathrm{Cat}\big(\mathfrak{S}_n\big) = \mathrm{Cat}(n)$$

Coxeter-Catalan Numbers

W	$\mathrm{Cat}(W)$
$G(1,1,n)$	$\frac{1}{n+1}\binom{2n}{n}$
$G(2,1,n)$	$\binom{2n}{n}$
$G(2,2,n)$	$\frac{3n-2}{n}\binom{2n-2}{n-1}$
$G(d,1,n)$	$\binom{2n}{n}$
$G(d,d,n)$	$\frac{(d+1)n-d}{n}\binom{2n-2}{n-1}$

Coxeter-Catalan Combinatorics

- if W is a Weyl group, NC_W is in bijection with:
  - W-nonnesting partitions
  - c-sortable elements of W
  - facets of the c-cluster complex of W
  - finitely generated wide subcategories of finite-dimensional representations of the oriented Coxeter diagram

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Further Properties

- further properties of $(NC_W, \leq_T)$:
  - it is lexicographically shellable

    [A. Björner, P. Edelman, 1980; V. Reiner, 1997; C. A. Athanasiadis, T. Brady, C. Watt, 2007; , 2015]
  - it is self-dual
  - it admits a symmetric chain decomposition

    [R. Simion, D. Ullman, 1991; V. Reiner, 1997; , 2016]
  - it is strongly Sperner [R. Simion, D. Ullman, 1991; V. Reiner, 1997; , 2016]

Outline

$W = G(1,1,n)$

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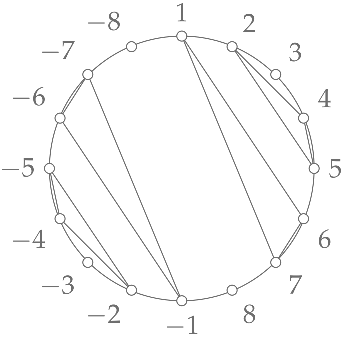
- $G(2,1,n)$: hyperoctahedral group
- $NC_{G(2,1,n)}$: noncrossing set partitions of $[n]^{\pm}$ invariant under 180° rotation

$W = G(2,1,n)$

$n = 8$

[V. Reiner, 1997]



$$\big\{\{1,6,7\},\{2,4,5\},\{3\},\{8\},\{-1,-6,-7\},\{-2,-4,-5\},\{-3\},\{-8\}\big\}$$

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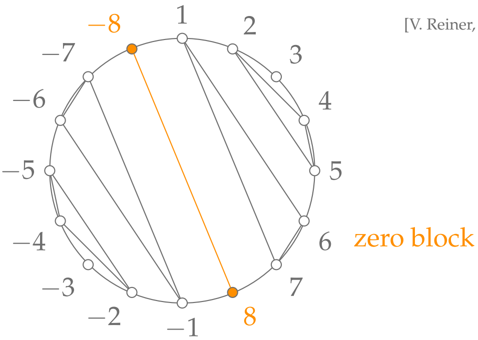

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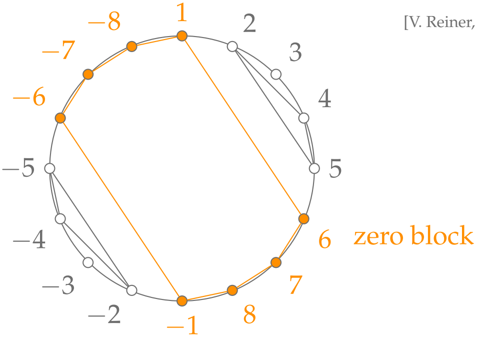

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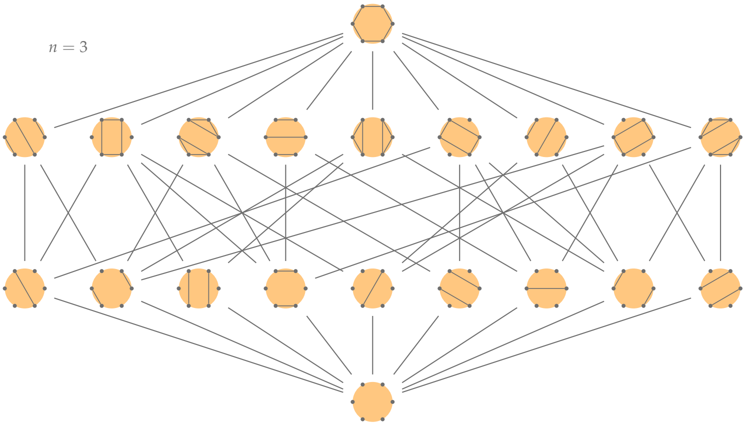

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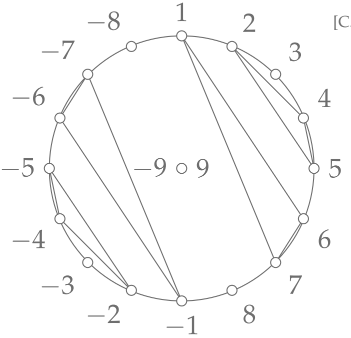
- $NC_{G(2,2,n)}$: centrally symmetric noncrossing set partitions of $[n]^{\pm}$ with zero block of cardinality $\neq 2$

$W = G(2,2,n)$

$n = 9$

[C. A. Athanasiadis, V. Reiner, 2004]



$$\Big\{\{1,6,7\},\{2,4,5\},\{3\},\{8\},\{9\},\{-1,-6,-7\},\{-2,-4,-5\},\{-3\},\{-8\},\{-9\}\Big\}$$

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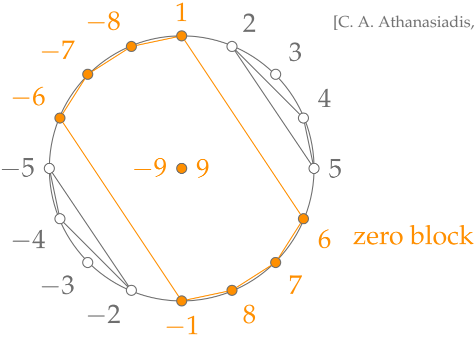

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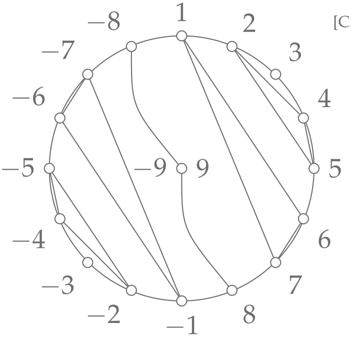

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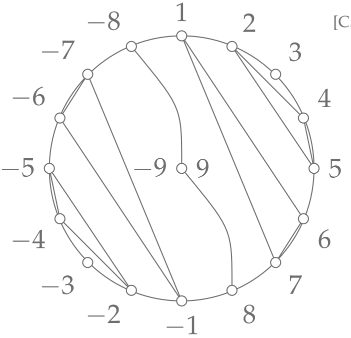

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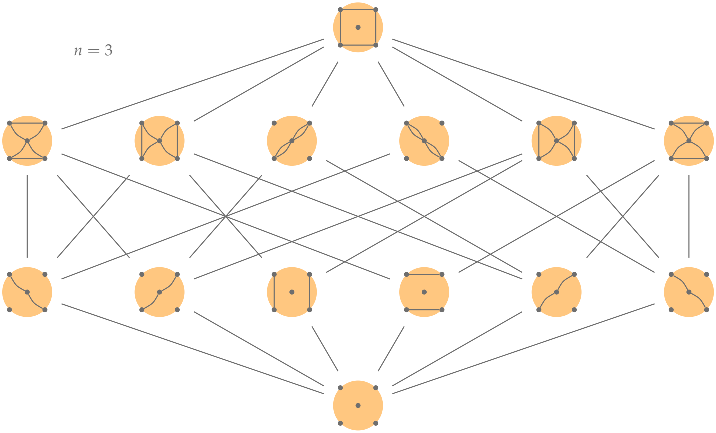
$W = G(2,2,n)$

$n = 3$



$W = G(d,1,n), d \geq 3$

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- $[n]^{(d)} = \{1^{(0)}, 2^{(0)}, \ldots, n^{(0)}, 1^{(1)}, 2^{(1)}, \ldots, n^{(d-1)}\}$
- ***d*-colored permutation**: $\pi : [n]^{(d)} \to [n]^{(d)}$ such that $\pi(i^{(s)}) = j^{(s+t_i)}$ for all i and s
- $G(d,1,n)$: group of d-colored permutations

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Proposition (D. Bessis, R. Corran, 2006)

For $d \geq 2$ we have $\left(NC_{G(d,1,n)}, \leq_T\right) \cong \left(NC_{G(2,1,n)}, \leq_T\right)$.

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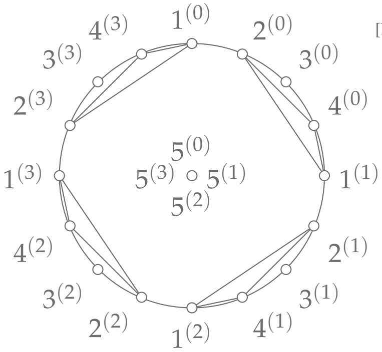
- $G(d,d,n)$: group of d-colored permutations, where the number of color-changes is divisible by d
- $NC_{G(d,d,n)}$: noncrossing set partitions of $[n]^{(d)}$ that are either
  - invariant under a (360/d)° rotation, or
  - have a unique asymmetric block

$W = G(d,d,n), d \geq 3$

$d = 4, n = 5$

[D. Bessis, R. Corran, 2006]



$$\Big\{\{2^{(0)},4^{(0)},1^{(1)}\},\{3^{(0)}\},\{5^{(0)}\},\{2^{(1)},4^{(1)},1^{(2)}\},\{3^{(1)}\},\{5^{(1)}\},\\ \{2^{(2)},4^{(2)},1^{(3)}\},\{3^{(2)}\},\{5^{(2)}\},\{2^{(3)},4^{(3)},1^{(0)}\},\{3^{(3)}\},\{5^{(3)}\}\Big\}$$

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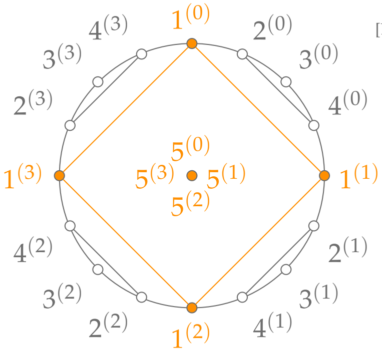

$$\Big\{\{1^{(0)},1^{(1)},1^{(2)},1^{(3)}\},\{2^{(0)},4^{(0)}\},\{3^{(0)}\},\{5^{(0)},5^{(1)},5^{(2)},5^{(3)}\},$$
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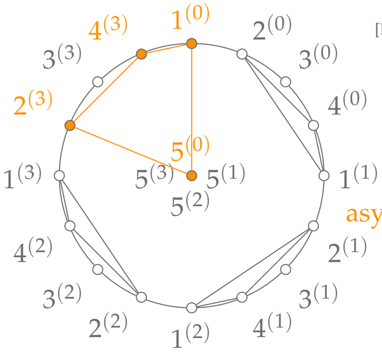

asymmetric block

$$\Big\{\{2^{(0)},4^{(0)},1^{(1)},5^{(1)}\},\{3^{(0)}\},\{2^{(1)},4^{(1)},1^{(2)},5^{(2)}\},\{3^{(1)}\},$$
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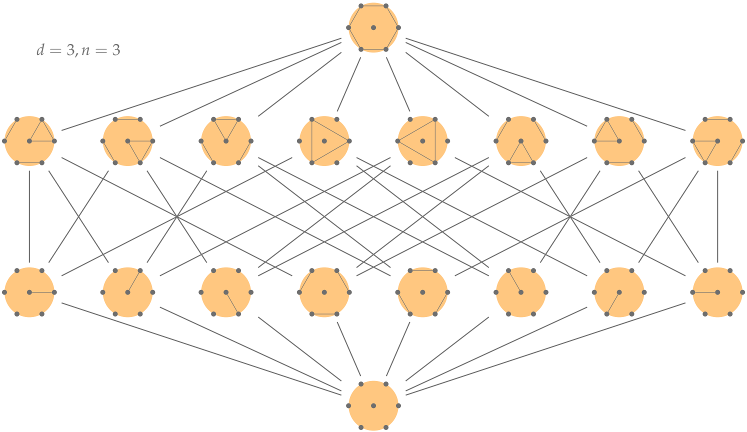

Outline

Extension: the Alternating Group

- recall: if $W = \mathfrak{S}_n$, then $T = \{(i\,j) \mid 1 \leq i < j \leq n\}$
- rework:
  - let $A = \{(i\,j\,k) \mid 1 \leq i,j,k \leq n, |\{i,j,k\}| = 3\}$
  - **alternating group**: $\langle A \rangle = \mathfrak{A}_n \subseteq \mathfrak{S}_n$

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Proposition (M. Herzog, K. Reid, 1976)

For $n \geq 3$ and $x \in \mathfrak{A}_n$ we have $\ell_A(x) = \frac{n - \mathrm{ocyc}(x)}{2}$, where $\mathrm{ocyc}(x)$ counts the odd cycles of x.

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  - **alternating group**: $\langle A \rangle = \mathfrak{A}_n \subseteq \mathfrak{S}_n$
- $ENC_n = \{x \in \mathfrak{A}_{2n+1} \mid x \leq_A (1\ 2\ \ldots\ 2n{+}1)\}$

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Extension: the Alternating Group

Proposition (, P. Nadeau, 2016)

For $n \geq 0$ the poset $(ENC_{2n+1}, \leq_A)$ is a graded, complemented, self-dual poset. We have

- *number of elements: $ECat(n)$*
- *number of elements of rank k: $ENar(n,k)$*
- *Möbius number: $(-1)^n \frac{1}{4n+1}\binom{4n+1}{n}$*
- *number of maximal chains: $(2n+1)^{n-1}$*

Extension: the Alternating Group

- **even Catalan number**:

$$\mathrm{ECat}(n) = \frac{1}{n+1}\binom{3n+1}{n}$$

1
2
7
30
143
728

Proposition (⚙, P. Nadeau, 2016)

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Extension: the Alternating Group

- **even Narayana number**:

$$\mathrm{ENar}(n,k)=\frac{2n+1}{(2n-2k+1)(2k+1)}\binom{2n-k}{k}\binom{n+k}{n-k}$$

1					
1	1				
1	5	1			
1	14	14	1		
1	30	81	30	1	
1	55	308	308	55	1

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Extension: the Alternating Group

Proposition (, P. Nadeau, 2016)

For $n \geq 0$, the poset $(\mathit{ENC}_{2n+1}, \leq_A)$ is an induced subposet of $(\mathit{NC}_{2n+1}, \leq_T)$.

Extension: larger Cycles

- we can also consider the subgroup $G \subseteq \mathfrak{S}_n$ generated by all k-cycles
- problem: for $k \geq 5$ the length function is not known
- however: the elements below $(1\;2\;\ldots\;kn+1)$ seem to behave nicely

Extension: Alternating Subgroups of Coxeter Groups

- consider the alternating subgroup of a Coxeter group
- $\mathfrak{A}(W) = \{x \in W \mid (-1)^{\ell_T(x)} \equiv 0 \pmod 2\}$
- it is generated by products of reflections
- there seem to be promising formulas...

Thank You.