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Harmonic analysis

– in Banach spaces –

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Contents

Chapter 1
Bochner-Lebesgue and Bochner-Sobolev spaces

1.1 The Bochner integral

Let X and Y be Banach spaces, and let $(\Omega,\mathcal{A},\mu)$ be a measure space. A function $f:\Omega\to X$ is called **step function**, if there exists a sequence $(A_n)\subseteq\mathcal{A}$ of mutually disjoint measurable sets and a sequence $(x_n)\subseteq X$ such that $f=\sum_n 1_{A_n}x_n$. A function $f:\Omega\to X$ is called **mesurable**, if there exists a sequence (f_n) of step functions $f_n:\Omega\to X$ such that $f_n\to f$ pointwise μ-almost everywhere.

Remark 1.1. Note that there may be a difference to the definition of mesurability of scalar valued functions. Measurability of a function is here depending on the measure μ. However, if the measure space $(\Omega,\mathcal{A},\mu)$ is **complete** in the sense that $\mu(A)=0$ and $B\subseteq A$ implies $B\in\mathcal{A}$, then the above definition of measurability and the classical definition of measurability coincide. Note that one may always consider complete measure spaces.

Lemma 1.2. *If $f:\Omega\to X$ is measurable, then $\|f\|:\Omega\to\mathbb{R}$ is measurable. More generally, if $f:\Omega\to X$ is measurable and if $g:X\to Y$ is continuous, then $g\circ f:\Omega\to Y$ is measurable.*

Proof. This is an easy consequence of the definition of measurability and the continuity of g. Note that in particular the norm $\|\cdot\|:X\to\mathbb{R}$ is continuous.

Lemma 1.3. *If $f:\Omega\to X$ and $g:\Omega\to\mathbb{K}$ are measurable, then $fg:\Omega\to X$ is measurable. Similarly, if $f:\Omega\to X$ and $g:\Omega\to X'$ are measurable, then $\langle g,f\rangle_{X',X}:\Omega\to\mathbb{K}$ is measurable.*

Proof. For the proof it suffices to use the definition of measurability and to show that the (duality) product of two step functions is again a step function. This is, however, straightforward.

Theorem 1.4 (Pettis). *A function $f:\Omega\to X$ is measurable if and only if $\langle x',f\rangle$ is measurable for every $x'\in X'$ (we say that f is* **weakly measurable***) and if there* *exists a μ-null set $N \in \mathcal{A}$ such that $f(\Omega \setminus N)$ is separable (we say that f is* **almost separably valued***).*

For the following proof of Pettis' theorem, see Hille & Phillips [Hille and Phillips (1957)].

Proof. Sufficiency. Assume that f is measurable. Then f is weakly measurable by Lemma 1.2. Moreover, by definition, there exists a sequence (f_n) of test functions and a μ-null set $N \in \mathcal{A}$ such that

$$f_n(t) \to f(t) \text{ for all } t \in \Omega \setminus N.$$

Hence,

$$f(\Omega \setminus N) \subseteq \overline{\bigcup_n f_n(\Omega)}.$$

Since for every step function f_n the range is countable, the set on the right-hand side of this inclusion is separable, and hence f is almost separably valued.

Necessity. Assume that f is weakly measurable and almost separably valued. We first show that $\|f\|$ is measurable. By assumption, there exists a μ-null set and a sequence (x_n) in X such that $D := \{x_n : n \in \mathbb{N}\}$ is dense in $f(\Omega \setminus N)$. By the Hahn-Banach theorem, there exists a sequence (x'_n) in X^* such that $\|x'_n\| = 1$ and $\langle x'_n, x_n \rangle = \|x_n\|$. Since f is weakly measurable, $|\langle x'_n, f\rangle|$ is measurable for every n. As a consequence, $\sup_n |\langle x'_n, f\rangle|$ is measurable. But $\sup_n |\langle x'_n, f\rangle| = \|f\|$ on $\Omega \setminus N$ by the choice of the sequence (x'_n) and the density of D in the $f(\Omega \setminus N)$. Since our measure space $(\Omega, \mathcal{A}, \mu)$ is supposed to be complete, we obtain that $\|f\|$ is measurable. In a similar way, one shows that $\|f - x\|$ is measurable for every $x \in X$, and in particular for $x = x_n$.

Now fix $m \in \mathbb{N}$ and define

$$\begin{aligned}
A_{m1} &:= \{\|f - x_1\| \le \inf_{1 \le k \le m} \|f - x_k\|\}, \\
A_{m2} &:= \{\|f - x_2\| \le \inf_{1 \le k \le m} \|f - x_k\|\} \setminus A_{m1}, \\
A_{m3} &:= \{\|f - x_3\| \le \inf_{1 \le k \le m} \|f - x_k\|\} \setminus (A_{m1} \cup A_{m2}), \\
&\vdots \quad \vdots \\
A_{mm} &:= \{\|f - x_m\| \le \inf_{1 \le k \le m} \|f - x_k\|\} \setminus (\bigcup_{k=1}^{m-1} A_{mk}).
\end{aligned}$$

Then $(A_{mn})_{1 \le n \le m}$ is a family of measurable, mutually disjoint sets such that $\bigcup_{n=1}^m A_{mn} = \Omega$. Define[1]

[1] We are grateful to Anton Claußnitzer for the definition of the sets A_{mn} and the functions f_m.

$$f_m := \sum_{n=1}^{m} 1_{A_{mn}} x_n.$$

Then (f_m) is a sequence of step functions, $(\|f_m - f\|)_m$ is decreasing pointwise everywhere, and since D is dense in $f(\Omega \setminus N)$,

$$\lim_{m\to\infty} \|f_m(t) - f(t)\| = 0 \text{ for every } t \in \Omega \setminus N.$$

that is, $f_m \to f$ μ-almost everywhere. As a consequence, f is measurable.

Corollary 1.5. *If (f_n) is a sequence of measurable functions $\Omega \to X$ such that $f_n \to f$ pointwise μ-almost everywhere, then f is measurable.*

Proof. We assume that this corollary is known in the scalar case, that is, when $X = \mathbb{K}$.

By Pettis's theorem (Theorem 1.4), for all n there exists a μ-null set $N_n \in \mathcal{A}$ such that $f_n(\Omega \setminus N_n)$ is separable. Moreover there exists a μ-null set $N_0 \in \Omega$ such that $f_n(t) \to f(t)$ for all $t \in \Omega \setminus N_0$. Let $N := \bigcup_{n\geq 0} N_n$; as a countable union of μ-null sets, N is a μ-null set.

Then f (restricted to $\Omega \setminus N$) is the pointwise limit everywhere of the sequence (f_n). In particular f is weakly measurable. Moreover, $f(\Omega \setminus N)$ is separable since

$$f(\Omega \setminus N) \subseteq \overline{\bigcup_n f_n(\Omega \setminus N)},$$

and since $f_n(\Omega \setminus N)$ is separable. The claim follows from Pettis' theorem.

A measurable function $f : \Omega \to X$ is called **integrable** if $\int_\Omega \|f\| \, \mathrm{d}\mu < \infty$.

Lemma 1.6. *For every integrable step function $f : \Omega \to X$, $f = \sum_n 1_{A_n} x_n$ the series $\sum_n x_n \mu(A_n)$ converges absolutely and its limit is independent of the representation of f.*

Proof. Let $f = \sum_n 1_{A_n} x_n$ be an integrable step function. The sets $(A_n) \subseteq \mathcal{A}$ are mutually disjoint and $(x_n) \subseteq X$. Then

$$\sum_n \|x_n\| \, \mu(A_n) = \int_\Omega \|f\| \, \mathrm{d}\mu < \infty.$$

Let $f : \Omega \to X$ be an integrable step function, $f = \sum_n 1_{A_n} x_n$. We define the **Bochner integral** (for integrable step functions) by

$$\int_\Omega f \, \mathrm{d}\mu := \sum_n x_n \, \mu(A_n).$$

Lemma 1.7. *(a) For every integrable function $f : \Omega \to X$ there exists a sequence (f_n) of integrable step functions $\Omega \to X$ such that $\|f_n\| \leq \|f\|$ and $f_n \to f$ pointwise*

μ-almost everywhere.
(b) Let $f : \Omega \to X$ be integrable. Let (f_n) be a sequence of integrable step functions such that $\|f_n\| \le \|f\|$ and $f_n \to f$ pointwise μ-almost everywhere. Then

$$x := \lim_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu \text{ exists}$$

and

$$\|x\| \le \int_\Omega \|f\| \, \mathrm{d}\mu.$$

Proof. (a) Let $f : \Omega \to X$ be integrable. Then $\|f\| : \Omega \to \mathbb{R}$ is integrable. Therefore there exists a sequence (g_n) of integrable step functions such that $0 \le g_n \le \|f\|$ and $g_n \to \|f\|$ pointwise μ-almost everywhere.

Since f is measurable, there exists a sequence $(\tilde{f}_n)$ of step functions $\Omega \to X$ such that $\tilde{f}_n \to f$ pointwise μ-almost everywhere.

Put

$$f_n := \frac{\tilde{f}_n \, g_n}{\|\tilde{f}_n\| + \frac{1}{n}}.$$

(b) For every integrable step function $g : \Omega \to X$ one has

$$\Big\| \int_\Omega g \, \mathrm{d}\mu \Big\| \le \int_\Omega \|g\| \, \mathrm{d}\mu.$$

Hence, for every n, m

$$\Big\| \int_\Omega f_n - f_m \, \mathrm{d}\mu \Big\| \le \int_\Omega \|f_n - f_m\| \, \mathrm{d}\mu,$$

and by Lebesgue's dominated convergence theorem the sequence $(\int_\Omega f_n \, \mathrm{d}\mu)$ is a Cauchy sequence. When we put $x = \lim_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu$ then

$$\|x\| \le \liminf_{n\to\infty} \int_\Omega \|f_n\| \, \mathrm{d}\mu = \int_\Omega \|f\| \, \mathrm{d}\mu.$$

Let $f : \Omega \to X$ be integrable. We define the **Bochner integral**

$$\int_\Omega f \, \mathrm{d}\mu := \lim_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu,$$

where (f_n) is a sequence of step functions $\Omega \to X$ such that $\|f_n\| \le \|f\|$ and $f_n \to f$ pointwise μ-almost everywhere. The definition of the Bochner integral for integrable functions is independent of the choice of the sequence (f_n) of step functions, by Lemma 1.7. Moreover, if f is a step function, then this definition of the Bochner integral and the previous definition coincide. Finally, by Lemma 1.7 (b),

$$\left\| \int_\Omega f \, \mathrm{d}\mu \right\| \leq \int_\Omega \|f\| \, \mathrm{d}\mu \quad \text{(triangle inequality).} \tag{1.1}$$

Remark 1.8. We will also use the following notation for the Bochner integral:

$$\int_\Omega f \text{ oder } \int_\Omega f(t) \, \mathrm{d}\mu(t),$$

and if $\Omega = (a, b)$ is an interval in $\mathbb{R}$:

$$\int_a^b f \text{ oder } \int_a^b f(t) \, \mathrm{d}\mu(t).$$

If $\mu = \lambda$ is the Lebesgue measure then we also write

$$\int_a^b f(t) \, dt.$$

Lemma 1.9. *Let $f : \Omega \to X$ be integrable and $T \in \mathcal{L}(X, Y)$. Then $Tf : \Omega \to Y$ is integrable and*

$$\int_\Omega Tf \, \mathrm{d}\mu = T \int_\Omega f \, \mathrm{d}\mu.$$

Proof. Exercise.

Theorem 1.10 (Lebesgue, dominated convergence). *Let (f_n) be a sequence of integrable functions. Suppose there exists an integrable function $g : \Omega \to \mathbb{R}$ and an (integrable) measurable function $f : \Omega \to X$ such that $\|f_n\| \leq g$ and $f_n \to f$ pointwise μ-almost everywhere. Then*

$$\int_\Omega f \, \mathrm{d}\mu = \lim_{n \to \infty} \int_\Omega f_n \, \mathrm{d}\mu.$$

Proof. By the triangle inequality and the classical Lebesgue dominated convergence theorem,

$$\left\| \int_\Omega f \, \mathrm{d}\mu - \int_\Omega f_n \, \mathrm{d}\mu \right\| \leq \int_\Omega \|f - f_n\| \, \mathrm{d}\mu \to 0 \text{ as } n \to \infty.$$

1.2 Bochner-Lebesgue spaces

Definition 1.11 ($\mathcal{L}^p$ spaces). For every $1 \leq p < \infty$ we define

$$\mathcal{L}^p(\Omega; X) := \{f : \Omega \to X \text{ measurable} : \int_\Omega \|f\|^p \, \mathrm{d}\mu < \infty\}.$$

We also define

$$\mathcal{L}^\infty(\Omega;X) := \{f : \Omega \to X \text{ measurable} : \exists C \geq 0 \text{ such that } \mu(\{\|f\| \geq C\}) = 0\}.$$

Lemma 1.12. *For every $1 \leq p < \infty$ we put*

$$\|f\|_p := \Big(\int_\Omega \|f\|^p \,\mathrm{d}\mu\Big)^{1/p}.$$

We also put

$$\|f\|_\infty := \inf\{C \geq 0 : \mu(\{\|f\| \geq C\}) = 0\}.$$

Then $\|\cdot\|_p$ is a seminorm on $\mathcal{L}^p(\Omega;X)$ ($1 \leq p \leq \infty$).

Remark 1.13. A function $\|\cdot\| : X \to \mathbb{R}_+$ on a real or complex vector space is called a *seminorm* if

(i) $x = 0 \Rightarrow \|x\| = 0$,

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for every $\lambda \in \mathbb{K}$ and all $x \in X$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Definition 1.14 (L^p spaces). For every $1 \leq p \leq \infty$ we put

$$\begin{aligned} N_p &:= \{f \in \mathcal{L}^p(\Omega;X) : \|f\|_p = 0\} \\ &= \{f \in \mathcal{L}^p(\Omega;X) : f = 0 \,\mu\text{-almost everywhere}\}. \end{aligned}$$

We define the quotient space

$$\mathrm{L}^p(\Omega;X) := \mathcal{L}^p(\Omega;X)/N_p,$$

which is the space of all equivalence classes

$$[f] := f + N_p, \quad f \in \mathcal{L}^p(\Omega;X).$$

Lemma 1.15. *For every $[f] \in \mathrm{L}^p(\Omega;X)$ ($f \in \mathcal{L}^p(\Omega;X)$) the value*

$$\|[f]\|_p := \|f\|_p$$

is well defined, i.e. independent of the representant f. The function $\|\cdot\|_p$ is a norm on $\mathrm{L}^p(\Omega;X)$. The space $\mathrm{L}^p(\Omega;X)$ is a Banach space when equipped with this norm.

Remark 1.16. As in the scalar case we will in the following identify *functions* $f \in \mathcal{L}^p(\Omega;X)$ with their *equivalence classes* $[f] \in \mathrm{L}^p(\Omega;X)$, and we say that L^p is a *function space* although we should be aware that it is only a space of equivalence classes of functions.

Remark 1.17. For $\Omega = (a,b)$ an interval in $\mathbb{R}$ and for $\mu = \lambda$ the Lebesgue measure we simply write

$$\mathrm{L}^p(a,b;X) := \mathrm{L}^p((a,b);X).$$

We can do so since the spaces $\mathrm{L}^p([a,b];X)$ and $\mathrm{L}^p((a,b);X)$ coincide since the end points $\{a\}$ and $\{b\}$ have Lebesgue measure zero and there is no danger of confusion.

Lemma 1.18. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. Then $\mathrm{C}(\bar{\Omega};X) \subseteq \mathrm{L}^p(\Omega;X)$ for every $1 \le p \le \infty$.*

Proof. Actually, for finite measure spaces, we have the more general inclusions

$$\mathrm{L}^\infty(\Omega;X) \subseteq \mathrm{L}^p(\Omega;X) \subseteq \mathrm{L}^q(\Omega;X) \subseteq \mathrm{L}^1(\Omega;X)$$

if $1 \le q \le p \le \infty$.

Lemma 1.19. *Let the measure space $(\Omega, \mathcal{A}, \mu)$ be such that $\mathrm{L}^p(\Omega)$ is separable for $1 \le p < \infty$ (e.g. $\Omega \subset \mathbb{R}^d$ be an open set with the Lebesgue measure). Let X be separable. Then $\mathrm{L}^p(\Omega;X)$ is separable for $1 \le p < \infty$.*

Proof. By assumption the spaces $\mathrm{L}^p(\Omega)$ and X are separable. Let $(h_n) \subseteq \mathrm{L}^p(\Omega;X)$ and $(x_n) \subseteq X$ be two dense sequences. Then the set

$$\mathcal{F} := \{f : \Omega \to X : f = h_n\, x_m\}$$

is countable. It suffices to shows that $\mathcal{F} \subseteq \mathrm{L}^p(\Omega;X)$ is total, i.e. $\operatorname{span}\mathcal{F}$ is dense in $\mathrm{L}^p(\Omega;X)$. This is an exercise.

Theorem 1.20. *Let Ω be as in lemma 1.19. Let $1 < p < \infty$ and assume that X is reflexive. Then the space $\mathrm{L}^p(\Omega;X)$ is reflexive and*

$$\mathrm{L}^p(\Omega;X)' \cong \mathrm{L}^{p'}(\Omega;X').$$

Proof. Without proof.

1.3 The convolution

Theorem 1.21 (Young's inequality). *Let $T \in \mathrm{L}^1(\mathbb{R}^N;\mathcal{L}(X,Y))$ and $f \in \mathrm{L}^p(\mathbb{R}^N;X)$ ($1 \le p \le \infty$). Then for almost every $x \in \mathbb{R}^N$ the integral*

$$T * f(x) := \int_{\mathbb{R}^N} T(x-y) f(y)\, \mathrm{d}y$$

converges absolutely, and for the function $T * f$ thus defined one has

$$T * f \in \mathrm{L}^p(\mathbb{R}^N;Y) \text{ and}$$
$$\|T * f\|_{\mathrm{L}^p} \le \|T\|_{\mathrm{L}^1}\, \|f\|_{\mathrm{L}^p}.$$

Proof. The case $p = \infty$ is almost trivial. Actually, the strong continuity of the shift semigroup on L^1 yields continuity (and thus measurability) of $T * f$ while the boundedness of $T * f$ and Young's inequality are immediate from the triangle inequality.

Assume now that $p = 1$. By Tonnelli's theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \|T(x-y)\| \|f(y)\| \,\mathrm{d}y \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \|T(x-y)\| \|f(y)\| \,\mathrm{d}x \,\mathrm{d}y \\
&= \|T\|_{\mathrm{L}^1} \|f\|_{\mathrm{L}^1},
\end{aligned}$$

and from this equality follows the claim.

Assume now $1 < p < \infty$. From the previous case we deduce that for almost all $x \in \mathbb{R}^N$

$$\|T(x-\cdot)\| \|f(\cdot)\|^p \in \mathrm{L}^1(\mathbb{R}^N),$$

and thus

$$\|T(x-\cdot)\|^{\frac{1}{p}} \|f(\cdot)\| \in \mathrm{L}^p(\mathbb{R}^N).$$

On the other hand, $\|T(x-\cdot)\|^{\frac{1}{p'}} \in \mathrm{L}^{p'}(\mathbb{R}^N)$ for every $x \in \mathbb{R}^N$. By Hölder's inequality, for almost every $x \in \mathbb{R}^N$,

$$\|T(x-\cdot)\| \|f(\cdot)\| \in \mathrm{L}^1(\mathbb{R}^N),$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \|T(x-y)\| \|f(y)\| \,\mathrm{d}y \right)^p \mathrm{d}x \\
&\le \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \|T(x-y)\| \,\mathrm{d}y \right)^{\frac{p}{p'}} \int_{\mathbb{R}^N} \|T(x-y)\| \|f(y)\|^p \,\mathrm{d}y \,\mathrm{d}x \\
&= \|T\|_{\mathrm{L}^1}^{p-1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \|T(x-y)\| \|f(y)\|^p \,\mathrm{d}x \,\mathrm{d}y \\
&= \|T\|_{\mathrm{L}^1}^p \|f\|_{\mathrm{L}^p}^p \\
&< \infty.
\end{aligned}$$

For every $T \in \mathrm{L}^1(\mathbb{R}^N; \mathcal{L}(X,Y))$ and every $f \in \mathrm{L}^1(\mathbb{R}^N; X)$ we call the function $T * f \in \mathrm{L}^p(\mathbb{R}^N; Y)$ the **convolution** of T and f. It is a fundamental tool in harmonic analysis and the theory of partial differential equations. One first property is the following regularizing effect of the convolution. We recall that we adopt multi-index notation. For example, for every **multi-index** $\alpha \in \mathbb{N}_0^N$ we define

$$|\alpha| := \sum_{k=1}^{N} \alpha_k,$$
$$\alpha! := \prod_{k=1}^{N} \alpha_k!, \text{ and}$$
$$x^\alpha := \prod_{k=1}^{N} x_k^{\alpha_k} \quad (x \in \mathbb{C}^N).$$

Moreover, we denote by ∂_k the partial derivative operator with respect to the k-th variable, and define the **α-th partial derivative**

$$\partial^\alpha := \partial_1^{\alpha_1} \dots \partial_N^{\alpha_N}.$$

Let $\Omega \subseteq \mathbb{R}^N$ be an open set. For every function $f \in \mathrm{C}(\Omega; X)$ we define the **support**

$$\operatorname{supp} f := \overline{\{x \in \Omega : f(x) \neq 0\}},$$

where the closure has to be taken in Ω! We then define for $k \in \mathbb{N}_0 \cup \{\infty\}$

$$\mathrm{C}_\mathrm{c}^k(\Omega; X) := \{f \in \mathrm{C}^k(\Omega; X) : \operatorname{supp} f \text{ is compact}\},$$

the space of compactly supported C^k-functions. In the special case $X = \mathbb{K}$ we define

$$\mathcal{D}(\Omega) := \mathrm{C}_\mathrm{c}^\infty(\Omega).$$

Elements of $\mathcal{D}(\Omega)$ are called **test functions**.

Lemma 1.22 (Regularization). *For every $f \in \mathrm{L}^1(\mathbb{R}^N; X)$ and every $\varphi \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^N)$ one has $f * \varphi \in \mathrm{C}^\infty(\mathbb{R}^N; X)$ and*

$$\partial^\alpha (f * \varphi) = f * \partial^\alpha \varphi.$$

Lemma 1.23 (Strong continuity of the shift-group). *For every $x \in \mathbb{R}^N$ and every $1 \leq p \leq \infty$ we define the* **shift operator** *$S(x) \in \mathcal{L}(\mathrm{L}^p(\mathbb{R}^N; X))$ by*

$$(S(x)f)(y) := f(x + y) \quad (f \in \mathrm{L}^p(\mathbb{R}^N; X),\ y \in \mathbb{R}^N).$$

Then $S(x)$ is an isometric isomorphism and, if $p < \infty$,

$$\lim_{x \to 0} \|S(x)f - f\|_{\mathrm{L}^p} = 0 \text{ for every } f \in \mathrm{L}^p(\mathbb{R}^N; X).$$

Proof. The first statement about $S(x)$ being an isometric isomorphism is easy (with $S(x)^{-1} = S(-x)$). Next, for every simple step function $f = 1_Q \otimes x$ with a cube $Q \subseteq \mathbb{R}^N$, the second statement follows easily from Lebesgue's dominated convergence theorem. By linearity, the second statement holds for every f in the dense subspace

$$D := \operatorname{span}\{1_Q \otimes x : Q \subseteq \mathbb{R}^N \text{ a cube}, x \in X\}.$$

Now fix $f \in \mathrm{L}^p(\mathbb{R}^N; X)$ and let $\varepsilon > 0$. Then there exists $g \in D$ such that $\|f - g\|_{\mathrm{L}^p} < \varepsilon$. Moreover, there exists $\delta > 0$ such that $\|S(x)g - g\|_{\mathrm{L}^p} < \varepsilon$ for every $x \in \mathbb{R}^N$ with $\|x\| < \delta$. Hence, for every $x \in \mathbb{R}^N$ with $\|x\| < \delta$

$$\begin{aligned}\|S(x)f - f\|_{\mathrm{L}^p} &\le \|S(x)f - S(x)g\|_{\mathrm{L}^p} + \|S(x)g - g\|_{\mathrm{L}^p} + \|g - f\|_{\mathrm{L}^p}\\ &\le 2\|g - f\|_{\mathrm{L}^p} + \|S(x)g - g\|_{\mathrm{L}^p}\\ &< 3\varepsilon.\end{aligned}$$

If $\varphi \in \mathrm{L}^1(\mathbb{R}^N)$ is such that $\int_{\mathbb{R}^N} \varphi = 1$, then we call the sequence $(\varphi_n)_n$ given by

$$\varphi_n(x) := n^N \varphi(nx) \quad (x \in \mathbb{R}^N, n \in \mathbb{N})$$

an **approximate identity** or an **approximate unit**. The reason for this notion follows from the following lemma.

Lemma 1.24 (Property of an approximate identity). *Let $f \in \mathrm{L}^p(\mathbb{R}^N; X)$ $(1 \le p < \infty)$ and let $(\varphi_n)_n$ be an approximate identity. Then*

$$\lim_{n\to\infty} f * \varphi_n = f \text{ in } \mathrm{L}^p(\mathbb{R}^N; X).$$

Proof. By Tonnelli's theorem, the Hölder inequality, by the strong continuity of the shift-group and by Lebesgue's dominated convergence theorem we have

$$\begin{aligned}\|f * \varphi_n - f\|_{\mathrm{L}^p}^p &= \int_{\mathbb{R}^N} \left\| \int_{\mathbb{R}^N} f(x-y)\varphi_n(y)\, \mathrm{d}y - f(x) \right\|^p \mathrm{d}x\\ &\le \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \|f(x-y) - f(x)\|\, |\varphi_n(y)|\, \mathrm{d}y \right)^p \mathrm{d}x\\ &\le \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \|f(x-y) - f(x)\|^p\, |\varphi_n(y)|\, \mathrm{d}y\, \|\varphi_n\|_{\mathrm{L}^1}^{p-1}\, \mathrm{d}x\\ &= \|\varphi_n\|_{\mathrm{L}^1}^{p-1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \|f(x-y) - f(x)\|\, \mathrm{d}x\, \varphi_n(y)\, \mathrm{d}y\\ &= \|\varphi_n\|_{\mathrm{L}^1}^{p-1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \|f(x - \frac{y}{n}) - f(x)\|\, \mathrm{d}x\, \varphi(y)\, \mathrm{d}y\\ &\to 0 \quad (n \to \infty).\end{aligned}$$

Corollary 1.25. *For every $1 \le p < \infty$ the space $\mathrm{C}_c^\infty(\mathbb{R}^N; X)$ is dense in $\mathrm{L}^p(\mathbb{R}^N; X)$.*

Proof (by regularization and truncation). Let $f \in \mathrm{L}^p(\mathbb{R}^N; X)$. In the first step, the regularization step, we choose an approximate identity (φ_n) starting with a test function $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^N)$. By Young's inequality, $f * \varphi_n \in \mathrm{L}^p(\mathbb{R}^N; X)$, by Lemma 1.22, $f * \varphi_n \in \mathrm{C}^\infty(\mathbb{R}^N)$, and by Lemma 1.24,

$$\lim_{n\to\infty} \|f * \varphi_n - f\|_{L^p} = 0.$$

In the second step, the truncation step, we choose a sequence $(\psi_m)_m$ of test functions satisfying $0 \le \psi_m \le 1$ and $\psi_m = 1$ on the ball $B(0,m)$ (such functions can be obtained by convolving characteristic functions $\chi_{B(0,2m)}$ with appropriate positive test functions, relying on Lemma 1.22). It is clear from Lebesgue's dominated convergence theorem, that for every $g \in L^p(\mathbb{R}^N;X)$ one has

$$\lim_{m\to\infty} \|g\psi_m - g\|_{L^p} = 0.$$

Combining the preceding two equalities, we find a sequence $(m_n)_n$ in $\mathbb{N}$ such that

$$\lim_{n\to\infty} \|(f * \varphi_n)\,\psi_{m_n} - f\|_{L^p} = 0,$$

and since $(f * \varphi_n)\,\psi_{m_n} \in C_c^\infty(\mathbb{R}^N;X)$, the claim is proved.

Corollary 1.26. *Let $f \in L^p(\mathbb{R}^N;X)$ be such that*

$$\int_{\mathbb{R}^N} f\varphi = 0 \textit{ for every } \varphi \in \mathcal{D}(\mathbb{R}^N).$$

Then $f = 0$.

Proof. The assumption implies that

$$f * \varphi(x) = \int_{\mathbb{R}^N} f(y)\varphi(x-y)\,\mathrm{d}y = 0 \text{ for every } x \in \mathbb{R}^N,\ \varphi \in \mathcal{D}(\mathbb{R}^N),$$

which just means that

$$f * \varphi = 0 \text{ for every } \varphi \in \mathcal{D}(\mathbb{R}^N).$$

The claim now follows upon choosing an approximate identity (φ_n) out of a test function φ and by applying Lemma 1.24.

1.4 Bochner-Sobolev spaces

Let $\Omega \subseteq \mathbb{R}^N$ be an open set, $1 \le p \le \infty$ and $k \in \mathbb{N}$. We define the **Bochner-Sobolev space**

$$W^{k,p}(\Omega;X) := \{u \in L^p(\Omega;X) : \forall \alpha \in \mathbb{N}_0^N \exists v_\alpha \in L^p(\Omega;X) \forall \varphi \in \mathcal{D}(\Omega)$$
$$\int_\Omega f\partial^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega v_\alpha \varphi\}$$

The functions v_α in this definition of the space $W^{k,p}(\Omega;X)$ are uniquely determined. We write $v_\alpha =: \partial^\alpha u$ and we call the function $\partial^\alpha u$ the **weak α-th partial derivative** of u. The space $W^{k,p}(\Omega;X)$ becomes a Banach space for the norm

$$\|u\|_{W^{k,p}} := \sum_{\substack{\alpha\in\mathbb{N}_0^N \\ |\alpha|\le k}} \|\partial^\alpha u\|_{L^p}.$$

Similarly as in the case of the L^p-spaces we write $W^{k,p}(a,b;X)$ instead of $W^{k,p}((a,b);X)$. In the special case when $p=2$ and $X=H$ is a Hilbert space, we also write

$$H^k(\Omega;H) := W^{k,2}(\Omega;H).$$

This space is a Hilbert space for the inner product

$$\langle u,v\rangle_{H^k} := \sum_{\substack{\alpha\in\mathbb{N}_0^N \\ |\alpha|\le k}} \langle \partial^\alpha u, \partial^\alpha v\rangle_{L^2}.$$

The resulting norm $\|\cdot\|_{H^k}$ is equivalent to the norm $\|\cdot\|_{W^{k,2}}$ defined above.

The main results about Sobolev spaces of scalar-valued functions remain true for Sobolev spaces of Banach space valued functions if interpreted properly. In particular, the Sobolev embedding theorem, a version of the product rule, the integration by parts formula and Poincaré's inequality remain true. Even a version of the Rellich-Kondrachev theorem remains true.

Lemma 1.27. *For every $-\infty < a < b < \infty$ and every $1 \le p \le \infty$ one has $W^{1,p}(a,b;X) \subseteq C^b(\overline{(a,b)};X)$. For every $u \in W^{1,p}(a,b;X)$ and every $s,\, t \in (a,b)$ one has*

$$u(t) - u(s) = \int_s^t u'(r)\,\mathrm{d}r.$$

Lemma 1.28. *Assume that the embedding $V \hookrightarrow H$ is continuous and let $u \in W^{1,2}(0,T;H) \cap L^\infty(0,T;V)$. Then u is weakly continuous with values in V, that is, for every $v \in V'$ the function $t \mapsto \langle v, u(t)\rangle_{V',V}$ is continuous on $[0,T]$.*

Proof. Since every function $u \in W^{1,2}(0,T;H)$ is continuous (and hence weakly continuous) with values in H, the claim follows from [Temam (1984), Lemma 1.4, page 263] .

Lemma 1.29. *Assume that the embedding $V \hookrightarrow H$ is continuous and let (u_n) be a sequence such that*

$$u_n \rightharpoonup u \text{ in } W^{1,2}(0,T;H) \text{ and}$$
$$u_n \overset{w*}{\to} u \text{ in } L^\infty(0,T;V).$$

Then there exists a subsequence of (u_n) (which we denote again by (u_n)) such that

$$u_n(t) \rightharpoonup u(t) \;\; \textit{in} \;\; V \;\; \textit{for every} \; t \in [0,T].$$

Proof. Using the fact that the point evaluation in $t \in [0,T]$ from $W^{1,2}(0,T;H)$ into H is bounded and linear, and maps weakly convergent sequences into weakly convergent sequences, the assumption implies that for every $t \in [0,T]$

$$u_n(t) \rightharpoonup u(t) \;\; \text{in} \;\; H.$$

Let now $w \in H'$ and $t \in [0,T]$. Then one has

$$\langle w, u_n(t) - u(t)\rangle_{V',V} = \langle w, u_n(t) - u(t)\rangle_{H',H} \longrightarrow 0.$$

Using the fact that H' is dense in V' and that the sequence $(u_n(t))$ is bounded in V, the claim follows from Lemma ??.

Chapter 2
The Fourier transform

2.1 The Fourier transform in L^1

Let X be a Banach space with norm $|\cdot| := |\cdot|_X$. For every $f \in L^1(\mathbb{R}^N; X)$ we define the **Fourier transform** $\mathcal{F} f$ and the **adjoint Fourier transform** $\bar{\mathcal{F}} f$ by

$$\begin{aligned} \mathcal{F} f(x) &:= \int_{\mathbb{R}^N} e^{-ixy} f(y)\,\mathrm{d}y \quad \text{and} \\ \bar{\mathcal{F}} f(x) &:= \int_{\mathbb{R}^N} e^{ixy} f(y)\,\mathrm{d}y \quad (x \in \mathbb{R}^N). \end{aligned}$$

The integrals are absolutely convergent, and we have the trivial estimates

$$|\mathcal{F} f(x)|, |\bar{\mathcal{F}} f(x)| \leq \|f\|_{L^1} \text{ for every } x \in \mathbb{R}^N.$$

In particular, the functions $\mathcal{F} f$ and $\bar{\mathcal{F}} f$ are bounded.

Theorem 2.1 (Riemann-Lebesgue). *For every $f \in L^1(\mathbb{R}^N; X)$ one has $\mathcal{F} f, \bar{\mathcal{F}} f \in C_0(\mathbb{R}^N; X)$.*

Proof. The fact that the Fourier transform $\mathcal{F} f$ is continuous follows easily from Lebesgue's dominated convergence theorem. Next, for every $x \in \mathbb{R}^N$, $x \neq 0$,

$$\begin{aligned} \mathcal{F} f(x) &= \frac{1}{2} \int_{\mathbb{R}^N} (e^{-ixy} - e^{-ixy}\, e^{i\pi \frac{x \cdot x}{|x|^2}}) f(y)\,\mathrm{d}y \\ &= \frac{1}{2} \int_{\mathbb{R}^N} e^{ixy} (f(y) - f(y + \frac{\pi x}{|x|^2}))\,\mathrm{d}y. \end{aligned}$$

Since the shift group on $L^1(\mathbb{R}^N; X)$ is strongly continuous, we thus obtain

$$\|\mathcal{F} f(x)\| \leq \frac{1}{2} \int_{\mathbb{R}^N} \|f(y) - f(y + \frac{\pi x}{|x|^2})\|\,\mathrm{d}y \to 0 \text{ as } |x| \to \infty.$$

The arguments for the adjoint Fourier transform are similar.

Corollary 2.2. *The Fourier transform $\mathcal{F}$ and the adjoint Fourier transform are bounded, linear operators from $L^1(\mathbb{R}^N;X)$ into $C_0(\mathbb{R}^N;X)$.*

We need the following basic lemma in order to prove the inversion formula for the Fourier transform.

Lemma 2.3 (Féjer kernel). *One has, for $a>0$,*

$$\int_{\mathbb{R}} \frac{\sin^2 ax}{x^2}\,\mathrm{d}x = a\pi.$$

Proof. We define

$$f(\lambda) := \int_0^\infty e^{-\lambda x}\frac{\sin^2 ax}{x^2}\,\mathrm{d}x \quad (\lambda\in(0,\infty)).$$

Then $f\in C^\infty((0,\infty))$ and

$$\begin{aligned}
&\lim_{\lambda\to 0+} f(\lambda) = \int_0^\infty \frac{\sin^2 ax}{x^2}\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}}\frac{\sin^2 ax}{x^2}\,\mathrm{d}x, \text{ and}\\
&\lim_{\lambda\to\infty} f(\lambda) = 0.
\end{aligned}$$

A simple computation shows

$$\begin{aligned}
f'(\lambda) &= -\int_0^\infty e^{-\lambda x}\frac{\sin^2 ax}{x}\,\mathrm{d}x, \text{ and}\\
f''(\lambda) &= \int_0^\infty e^{-\lambda x}\sin^2 ax\,\mathrm{d}x\\
&= \int_0^\infty e^{-\lambda x}\left(\frac{e^{iax}-e^{-iax}}{2i}\right)^2\,\mathrm{d}x\\
&= -\frac{1}{4}\left(\frac{1}{\lambda-2ia}-\frac{2}{\lambda}+\frac{1}{\lambda+2ia}\right)\\
&= \frac{1}{4}\left(\frac{2}{\lambda}-\frac{2\lambda}{\lambda^2+4a^2}\right).
\end{aligned}$$

As a consequence,

$$f'(\lambda) = \frac{1}{4}\log\frac{\lambda^2}{\lambda^2+4a^2}.$$

In order to integrate this function, we make the ansatz

$$f(\lambda) = \frac{1}{4}\left(\lambda\log\frac{\lambda^2}{\lambda^2+4a^2}+g(\lambda)\right),$$

which leads to the equation

$$g'(\lambda) = -\frac{8a^2}{\lambda^2 + 4a^2},$$

that is,

$$g(\lambda) = -4a \arctan \frac{\lambda}{2a} + C$$

Together with the condition $\lim_{\lambda\to\infty} f(\lambda) = 0$ we thus find

$$f(\lambda) = \frac{1}{4}\left(\lambda \log \frac{\lambda^2}{\lambda^2 + 4a^2} + 4a(\frac{\pi}{2} - \arctan \frac{\lambda}{2a})\right).$$

This yields

$$\lim_{\lambda\to 0} f(\lambda) = a\frac{\pi}{2},$$

which implies the claim.

Before stating the following theorem we define for every $r \in \mathbb{R}^N$ with $r_k \geq 0$ the set

$$Q_r := \bigtimes_{k=1}^{N} [-r_k, r_k].$$

Theorem 2.4 (Inversion formula for the Fourier transform I). *Let $f \in L^1(\mathbb{R}^N; X)$. For every $R > 0$ we put*

$$g_R(x) := \frac{1}{(2\pi R)^N} \int_{[0,R]^N} \int_{Q_r} e^{ixy} \mathcal{F} f(y)\, \mathrm{d}y\, \mathrm{d}r \quad (x \in \mathbb{R}^N).$$

Then $g_R \in L^1(\mathbb{R}^N; X)$ and

$$\lim_{R\to\infty} \|g_R - f\|_{L^1} = 0.$$

Proof. For every $R > 0$ and every $x \in \mathbb{R}^N$ we compute, using Fubini's theorem,

$$\begin{aligned}
&\frac{1}{(2\pi R)^N}\int_{[0,R]^N}\int_{Q_r} e^{ixy}\mathcal{F}f(y)\,\mathrm{d}y\,\mathrm{d}r\\
&=\frac{1}{(2\pi R)^N}\int_{[0,R]^N}\int_{\mathbb{R}^N}\int_{Q_r} e^{iy(x-z)}\,\mathrm{d}y f(z)\,\mathrm{d}z\,\mathrm{d}r\\
&=\frac{1}{(\pi R)^N}\int_{\mathbb{R}^N}\int_{[0,R]^N}\prod_{k=1}^N \frac{\sin(r_k(x_k-z_k))}{x_k-z_k}\,\mathrm{d}r f(z)\,\mathrm{d}z\\
&=\int_{\mathbb{R}^N}\prod_{k=1}^N \frac{1}{\sin^2(\frac{R}{2}(x_k-z_k))}\frac{R}{2}\pi(x_k-z_k)^2 f(z)\,\mathrm{d}z\\
&=\int_{\mathbb{R}} k_R(x-z)f(z)\,\mathrm{d}z\\
&=k_R*f(x)
\end{aligned}$$

where

$$k_R(x):=\prod_{k=1}^N \frac{\sin^2(\frac{R}{2}x_k)}{\frac{R}{2}\pi x_k^2}\quad (x\in\mathbb{R}^N)$$

is the **Féjer kernel**. Note that

$$\begin{aligned}
&k_R\in L^1(\mathbb{R}^N),\\
&k_R\geq 0,\\
&k_R(x)=R^N k_2(Rx)\text{ for every } x\in\mathbb{R}^N \text{ and}\\
&\int_{\mathbb{R}^N} k_R(x)\,\mathrm{d}x=1\quad\text{(Lemma 2.3)}
\end{aligned}$$

for every $R>0$. Hence, $(k_R)_{R\nearrow\infty}$ is an approximate identity, and the claim follows from Young's inequality and Lemma 1.24.

Corollary 2.5 (Inversion formula for the Fourier transform II). *Let $f\in L^1(\mathbb{R}^N;X)$ be such that $\mathcal{F}f\in L^1(\mathbb{R}^N;X)$. Then $\bar{\mathcal{F}}f\in L^1(\mathbb{R}^N;X)$ and*

$$\begin{aligned}
f&=\frac{1}{(2\pi)^N}\bar{\mathcal{F}}(\mathcal{F}f)\text{ and}\\
f&=\frac{1}{(2\pi)^N}\mathcal{F}(\bar{\mathcal{F}}f).
\end{aligned}$$

Proof. Since

$$\bar{\mathcal{F}}f(x)=\int_{\mathbb{R}^N} e^{ixy}f(y)\,\mathrm{d}y=\mathcal{F}f(-x)\text{ for every } x\in\mathbb{R}^N,$$

we immediately obtain $\bar{\mathcal{F}}f\in L^1(\mathbb{R}^N;X)$.

Now let g_R be defined as in the preceding theorem. For every $R > 0$ and every $x \in \mathbb{R}^N$ we then have

$$
\begin{aligned}
& g_R(x) - \frac{1}{(2\pi)^N} \bar{\mathcal{F}}(\mathcal{F} f)(x) = \\
&= \frac{1}{(2\pi)^N} \left[\frac{1}{R^N} \int_{[0,R]^N} \int_{Q_r} e^{ixy} \mathcal{F} f(y) \,\mathrm{d}y \,\mathrm{d}r - \int_{\mathbb{R}^N} e^{ixy} \mathcal{F} f(y) \, u dy \right] \\
&= \frac{1}{(2\pi)^N} \frac{1}{R^N} \int_{[0,R]^N} \int_{Q_r^c} e^{ixy} \mathcal{F} f(y) \,\mathrm{d}y \,\mathrm{d}r,
\end{aligned}
$$

and hence, for every $L > 0$

$$
\begin{aligned}
& \limsup_{R\to\infty} \left| g_R(x) - \frac{1}{(2\pi)^N} \bar{\mathcal{F}}(\mathcal{F} f)(x) \right| \leq \\
& \leq \frac{1}{(2\pi)^N} \Big[\limsup_{R\to\infty} \left| \frac{1}{R^N} \int_{[L,R]^N} \int_{Q_r^c} e^{ixy} \mathcal{F} f(y) \,\mathrm{d}y \,\mathrm{d}r \right| \\
& \quad + \limsup_{R\to\infty} \left| \frac{1}{R^N} \int_{[0,R]^N \setminus [L,R]^N} \int_{Q_r^c} e^{ixy} \mathcal{F} f(y) \,\mathrm{d}y \,\mathrm{d}r \right| \Big] \\
& \leq \frac{1}{(2\pi)^N} \Big[\limsup_{R\to\infty} \frac{(R-L)^N}{R^N} \int_{([-L,L]^N)^c} |\mathcal{F} f(y)| \,\mathrm{d}y + \limsup_{R\to\infty} \frac{NLR^{N-1}}{R^N} \|\mathcal{F} f\|_{L^1} \Big] \\
& \leq \frac{1}{(2\pi)^N} \int_{([-L,L]^N)^c} |\mathcal{F} f(y)| \,\mathrm{d}y
\end{aligned}
$$

Since $L > 0$ was arbitrary, and since

$$
\lim_{L\to\infty} \int_{([-L,L]^N)^c} |\mathcal{F} f(y)| \,\mathrm{d}y = 0,
$$

we thus obtain

$$
\lim_{R\to\infty} g_R(x) = \frac{1}{(2\pi)^N} \bar{\mathcal{F}}(\mathcal{F} f)(x) \text{ for every } x \in \mathbb{R}^N.
$$

Combining this with the first inversion formula, we obtain the first identity. The second identity is proved similarly.

Corollary 2.6. *The Fourier transforms $\mathcal{F}$, $\bar{\mathcal{F}} : L^1(\mathbb{R}^N; X) \to C_0(\mathbb{R}^N; X)$ are injective.*

Remark 2.7. The Fourier transform $\mathcal{F}$ on L^1 is not surjective onto C_0.

Lemma 2.8 (Fourier transform and convolution). *For every $T \in L^1(\mathbb{R}^N; \mathcal{L}(X,Y))$ and every $f \in L^1(\mathbb{R}^N; X)$ one has*

$$\mathcal{F}(T * f) = \mathcal{F}T\mathcal{F}f \textit{ and}$$
$$\bar{\mathcal{F}}(T * f) = \bar{\mathcal{F}}T\bar{\mathcal{F}}f.$$

Proof. For every $x \in \mathbb{R}^N$ we compute, using Fubini's theorem,

$$\begin{aligned}
\mathcal{F}(T * f)(x) &= \int_{\mathbb{R}^N} e^{-ixy} T * f(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-ixy} T(y-z) f(z)\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-ix(y+z)} T(y)\,\mathrm{d}y\, f(z)\,\mathrm{d}z \\
&= \int_{\mathbb{R}^N} e^{-ixy} T(y)\,\mathrm{d}y \int_{\mathbb{R}^N} e^{-ixz} f(z)\,\mathrm{d}z \\
&= \mathcal{F}T(x)\mathcal{F}f(x).
\end{aligned}$$

The second identity is proved similarly.

Lemma 2.9 (Fourier transforms of partial derivatives). *For every $f \in C_c^\infty(\mathbb{R}^N; X)$, every multi-index $\alpha \in \mathbb{N}_0^N$ and every $x \in \mathbb{R}^N$ one has*

$$\mathcal{F}(\partial^\alpha f)(x) = (ix)^\alpha \mathcal{F}f(x).$$

Proof. For every $k \in \{1, \ldots, N\}$ we obtain, using integration by parts,

$$\begin{aligned}
\mathcal{F}(\partial_k f)(x) &= \int_{\mathbb{R}^N} e^{-ixy} \partial_k f(y)\,\mathrm{d}y \\
&= -\int_{\mathbb{R}^N} (\partial_k e^{-ixy}) f(y)\,\mathrm{d}y \\
&= ix_k \int_{\mathbb{R}^N} e^{-ixy} f(y)\,\mathrm{d}y \\
&= ix_k \mathcal{F}f(x).
\end{aligned}$$

The general formula for higher derivatives follows by induction.

Corollary 2.10 (Fourier transforms of vector-valued test functions). *For every $f \in C_c^\infty(\mathbb{R}^N; X)$ we have $\mathcal{F}f \in L^1(\mathbb{R}^N; X)$.*

Proof. Let $p : \mathbb{C}^N \to \mathbb{C}$ be any polynomial. The preceding lemma implies

$$\mathcal{F}(p(\partial)f)(x) = p(ix)\mathcal{F}f(x) \text{ for every } x \in \mathbb{R}^N.$$

By the Lemma of Riemann-Lebesgue, the left-hand side of this equality is uniformly bounded in $x \in \mathbb{R}^N$. Hence, for every polynomial $p : \mathbb{C}^N \to \mathbb{C}$ we have

$$\sup_{x \in \mathbb{R}^N} |p(ix)\mathcal{F}f(x)| < \infty.$$

Choosing p such that $p(ix) = 1 + |x|^k$ for some $k \in \mathbb{N}$ large enough, we obtain the claim.

2.2 The Fourier transform on L^2

Theorem 2.11 (Parseval's identity).

a) *For every $T \in L^1(\mathbb{R}^N; \mathcal{L}(X,Y))$ and every $f \in L^1(\mathbb{R}^N; X)$ one has*

$$\int_{\mathbb{R}^N} \mathcal{F}T(x) f(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} T(x)\mathcal{F}f(x)\,\mathrm{d}x.$$

b) *For every $f, g \in L^1(\mathbb{R}^N)$ one has*

$$\int_{\mathbb{R}^N} \mathcal{F}f(x)\bar{g}(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} f(x)\overline{\bar{\mathcal{F}}g(x)}\,\mathrm{d}x.$$

c) *For every $f, g \in L^1(\mathbb{R}^N)$ such that $\mathcal{F}f, \mathcal{F}g \in L^1(\mathbb{R}^N)$ one has*

$$\int_{\mathbb{R}^N} f(x)\bar{g}(x)\,\mathrm{d}x = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} \mathcal{F}f(x)\overline{\mathcal{F}g(x)}\,\mathrm{d}x.$$

Similar identities hold if we replace everywhere $\mathcal{F}$ by $\bar{\mathcal{F}}$ and vice versa.

Proof. (a) We calculate, using Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}^N} \mathcal{F}T(x) f(x)\,\mathrm{d}x &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-ixy}T(y)\,\mathrm{d}y\, f(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} T(y)\int_{\mathbb{R}^N} e^{-ixy} f(x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^N} T(y)\mathcal{F}f(y)\,\mathrm{d}y.\end{aligned}$$

(b) is proved in a similar way and (c) follows from (b) by using the Inversion Formula II (Corollary 2.5).

Theorem 2.12 (Plancherel). *The Fourier transforms $\mathcal{F}, \bar{\mathcal{F}} : \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ extend uniquely to bounded, linear operators on $L^2(\mathbb{R}^N)$. The operators $\frac{1}{\sqrt{2\pi}^N}\mathcal{F}$, $\frac{1}{\sqrt{2\pi}^N}\bar{\mathcal{F}} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ are unitary and*

$$\Big(\frac{1}{\sqrt{2\pi}^N}\mathcal{F}\Big)^* = \frac{1}{\sqrt{2\pi}^N}\bar{\mathcal{F}}.$$

Proof. From Parseval's identity (Theorem 2.11 (c)) we obtain, that for every $f, g \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^N)$

$$\langle \mathcal{F} f, \mathcal{F} g\rangle_{L^2} = (2\pi)^N \langle f, g\rangle_{L^2},$$

and in particular,

$$\|\mathcal{F} f\|_{L^2}^2 = (2\pi)^N \|f\|_{L^2}^2.$$

As a consequence, since $C_c^\infty(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$, $\mathcal{F}$ extends in a unique way to a bounded, linear operator on $L^2(\mathbb{R}^N)$. Moreover, we see from the above equality that $\frac{1}{\sqrt{2\pi}^N}\mathcal{F}$ is isometric. As a consequence, this operator is injective and has closed range. However, from the inversion formula we see that $C_c^\infty(\mathbb{R}^N)$ is contained in the range. Hence, $\frac{1}{\sqrt{2\pi}^N}\mathcal{F}$ is surjective, and thus unitary.

The arguments for $\bar{\mathcal{F}}$ are similar.

Theorem 2.13 (Plancherel in Hilbert space). *Let H be a Hilbert space. Then the Fourier transforms $\mathcal{F}, \bar{\mathcal{F}} : C_c^\infty(\mathbb{R}^N;H) \to L^2(\mathbb{R}^N;H)$ extend to bounded, linear operators on $L^2(\mathbb{R}^N;H)$. The operators $\frac{1}{\sqrt{2\pi}^N}\mathcal{F}$, $\frac{1}{\sqrt{2\pi}^N}\bar{\mathcal{F}} : L^2(\mathbb{R}^N;H) \to L^2(\mathbb{R}^N;H)$ are unitary and*

$$\left(\frac{1}{\sqrt{2\pi}^N}\mathcal{F}\right)^* = \frac{1}{\sqrt{2\pi}^N}\bar{\mathcal{F}}.$$

Proof. The proof is very similar to the previous proof, once one has proved the following variant of Parseval's identity (Theorem 2.11 (c)) for every f, $g \in L^1(\mathbb{R}^N;H)$ such that $\mathcal{F} f$, $\mathcal{F} g \in L^1(\mathbb{R}^N;H)$:

$$\int_{\mathbb{R}^N} \langle f(x), g(x)\rangle_H \,\mathrm{d}x = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \langle \mathcal{F} f(x), \mathcal{F} g(x)\rangle_H \,\mathrm{d}x.$$

Remark 2.14. Kwapien has shown the following result: if the Fourier transfrom extends from $C_c^\infty(\mathbb{R}^N;X)$ to a bounded, linear operator on $L^2(\mathbb{R}^N;X)$ (X being a general Banach space), then X is already isomorphic to a Hilbert space, that is, there exists an inner product on X which induces an equivalent norm. We will not prove this result here.

2.3 The Fourier transform on $\mathcal{S}$

We define the space

$$\mathcal{S}(\mathbb{R}^N;X) := \{f \in C^\infty(\mathbb{R}^N;X) : \forall \alpha, \beta \in \mathbb{N}_0^N : \int_{\mathbb{R}^N} \|x^\beta \partial^\alpha f(x)\|^2 \,\mathrm{d}x < \infty\}.$$

Elements of $\mathcal{S}(\mathbb{R}^N)$ (that is, $X = \mathbb{C}$) are called the **rapidly decreasing functions** or **Schwartz (test) functions**. Clearly, the space of (classical) test functions $C_c^\infty(\mathbb{R}^N) = \mathcal{D}(\mathbb{R}^N)$ is a subspace of $\mathcal{S}(\mathbb{R}^N)$, but the function $f(x) = e^{-x^2}$ is an example of a Schwartz test function which does not have compact support.

It is an exercise to show that

$$\mathcal{S}(\mathbb{R}^N;X) = \{f \in C^\infty(\mathbb{R}^N;X) : \forall \alpha,\beta \in \mathbb{N}_0^N : \int_{\mathbb{R}^N} \|x^\beta \partial^\alpha f(x)\| \,\mathrm{d}x < \infty\}$$
$$= \{f \in C^\infty(\mathbb{R}^N;X) : \forall \alpha,\beta \in \mathbb{N}_0^N : \sup_{x\in\mathbb{R}^N} \|x^\beta \partial^\alpha f(x)\| < \infty\}.$$

The space $\mathcal{S}(\mathbb{R}^N;X)$ is equiped with the topology induced by the countable family of seminorms $(\|\cdot\|_{\alpha,\beta})_{\alpha,\beta\in\mathbb{N}_0^N}$, where

$$\|f\|_{\alpha,\beta} := \left(\int_{\mathbb{R}^N} \|x^\beta \partial^\alpha f(x)\| \,\mathrm{d}x\right)^{\frac{1}{2}}.$$

This countable family of seminorms induces in a natural way a metric d given by

$$d(f,g) := \sum_{\alpha,\beta\in\mathbb{N}_0^N} c_{\alpha,\beta} \frac{\|f-g\|_{\alpha,\beta}}{1+\|f-g\|_{\alpha,\beta}},$$

where the coefficients $c_{\alpha,\beta} > 0$ are fixed such that $\sum_{\alpha,\beta\in\mathbb{N}_0^N} c_{\alpha,\beta} < \infty$. We have

$$\begin{aligned} f_n \to f \text{ in } \mathcal{S}(\mathbb{R}^N;X) \quad &\Leftrightarrow \quad \forall \alpha,\beta \in \mathbb{N}_0^N : \|f_n - f\|_{\alpha,\beta} \to 0 \\ &\Leftrightarrow \quad d(f_n,f) \to 0, \end{aligned}$$

and the space $\mathcal{S}(\mathbb{R}^N;X)$ is complete. In other words, the countable family of seminorms turns $\mathcal{S}(\mathbb{R}^N;X)$ into a Fréchet space.

From the definition of the space $\mathcal{S}(\mathbb{R}^N;X)$ we immediately obtain the following lemma which is, however, worth of being stated separately.

Lemma 2.15. *For every $f \in \mathcal{S}(\mathbb{R}^N;X)$ and every polynomial $p : \mathbb{C}^N \to \mathbb{C}$ the product pf and the (sum of) partial derivative $p(\partial)f$ belong again to $\mathcal{S}(\mathbb{R}^N;X)$. In other words, the mappings*

$$f \mapsto pf \quad \text{and}$$
$$f \mapsto p(\partial)f$$

leave the space $\mathcal{S}(\mathbb{R}^N)$ invariant.

Lemma 2.16. *For every $f \in \mathcal{S}(\mathbb{R}^N;X)$ and every polynomial $p : \mathbb{C}^N \to \mathbb{C}$ one has $\mathcal{F}f, \bar{\mathcal{F}}f \in C^\infty(\mathbb{R}^N;X)$, and*

$$\begin{aligned} \mathcal{F}(p(\partial)f) &= p(i\cdot)\mathcal{F}f, \\ \mathcal{F}(p(-i\cdot)f) &= p(\partial)\mathcal{F}f, \\ \bar{\mathcal{F}}(p(\partial)f) &= p(-i\cdot)\bar{\mathcal{F}}f, \text{ and} \\ \bar{\mathcal{F}}(p(i\cdot)f) &= p(\partial)\bar{\mathcal{F}}f. \end{aligned}$$

Proof. Let $f \in \mathcal{S}(\mathbb{R}^N;X)$ and $k \in \{1,\dots,N\}$. Then

$$\begin{aligned}\mathcal{F}(-i\cdot_k f)(x) &= -\int_{\mathbb{R}^N} e^{-ixy} iy_k f(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^N} \frac{\partial}{\partial x_k}(e^{-ixy}) f(y)\,\mathrm{d}y \\ &= \partial_k \int_{\mathbb{R}^N} e^{-ixy} f(y)\,\mathrm{d}y \\ &= \partial_k \mathcal{F} f(x).\end{aligned}$$

Moreover, by an integration by parts,

$$\begin{aligned}\mathcal{F}(\partial_k f)(x) &= \int_{\mathbb{R}^N} e^{-ixy} \frac{\partial}{\partial y_k} f(y)\,\mathrm{d}y \\ &= -\int_{\mathbb{R}^N} \frac{\partial}{\partial y_k}(e^{-ixy}) f(y)\,\mathrm{d}y \\ &= -ix_k \int_{\mathbb{R}^N} e^{-ixy} f(y)\,\mathrm{d}y \\ &= -ix_k \mathcal{F} f(x).\end{aligned}$$

The first two equalities follow from these two identities and by induction. The proofs for the adjoint Fourier transform $\bar{\mathcal{F}}$ are similar.

Theorem 2.17. *For every $f \in \mathcal{S}(\mathbb{R}^N;X)$ one has $\mathcal{F} f$, $\bar{\mathcal{F}} f \in \mathcal{S}(\mathbb{R}^N;X)$ and the Fourier transforms $\mathcal{F}, \bar{\mathcal{F}} : \mathcal{S}(\mathbb{R}^N;X) \to \mathcal{S}(\mathbb{R}^N;X)$ are linear isomorphisms.*

Proof.

2.4 The Fourier transform on $\mathcal{S}'$

We call

$$\begin{aligned}\mathcal{S}'(\mathbb{R}^N;X) &:= \mathcal{L}(\mathcal{S}(\mathbb{R}^N);X) \\ &:= \{T : \mathcal{S}(\mathbb{R}^N) \to X : T \text{ is linear and continuous}\}\end{aligned}$$

the space of **(vector-valued) tempered distributions**. It is equiped with the following "topology": a sequence (T_n) of tempered distributions converges in $\mathcal{S}'(\mathbb{R}^N;X)$ to a tempered distribution T if

$$\lim_{n\to\infty} \langle T_n, \varphi\rangle = \langle T, \varphi\rangle \text{ for every } \varphi \in \mathcal{S}(\mathbb{R}^N).$$

Many classical function spaces are included in the space of tempered distributions. For example, the weighted spaces $L^1(\mathbb{R}^N, \frac{1}{1+|x|^k}\mathrm{d}x;X)$ $(k \in \mathbb{N}_0)$

are in a natural way contained in the space of tempered distributions via the mapping

$$L^1(\mathbb{R}^N, \frac{1}{1+|x|^k}\mathrm{d}x; X) \to \mathcal{S}'(\mathbb{R}^N; X),$$
$$f \mapsto T_f,$$

where

$$T_f\varphi = \int_{\mathbb{R}^N} f(x)\varphi(x)\,\mathrm{d}x \quad (\varphi \in \mathcal{S}(\mathbb{R}^N)).$$

Note that the integral is absolutely convergent.

Lemma 2.18. *Let $f, g \in L^1(\mathbb{R}^N, \frac{1}{1+|x|^k}\mathrm{d}x; X)$ be such that $T_f = T_g$. Then $f = g$.*

Proof. By linearity, it suffices to show that $T_f = 0$ implies $f = 0$. So let $f \in L^1(\mathbb{R}^N, \frac{1}{1+|x|^k}\mathrm{d}x; X)$ be such that $T_f = 0$. Then

$$\int_{\mathbb{R}^N} f(x)\varphi(x)\,\mathrm{d}x = 0 \text{ for every } \varphi \in \mathcal{S}(\mathbb{R}^N),$$

which implies

$$\int_{\mathbb{R}^N} f(x-y)\varphi(y)\,\mathrm{d}y = 0 \text{ for every } \varphi \in \mathcal{S}(\mathbb{R}^N),\ x \in \mathbb{R}^N.$$

Hence

$$f * \varphi = 0 \text{ for every } \varphi \in \mathcal{S}(\mathbb{R}^N).$$

Choosing an approximate identity (φ_n) out of a test function $\varphi \in \mathcal{S}(\mathbb{R}^N)$, we obtain $f = 0$.

Note that the classical space $L^p(\mathbb{R}^N; X)$ $(1 \le p \le \infty)$ is a subspace of $L^1(\mathbb{R}^N, \frac{1}{1+|x|^k}\mathrm{d}x; X)$ for some $k \in \mathbb{N}_0$ large enough. Hence, L^p functions are tempered distributions via the above embedding.

Conversely we say that a distribution $T \in \mathcal{S}'(\mathbb{R}^N; X)$ belongs to $L^p(\mathbb{R}^N, \frac{1}{1+|x|^k}\mathrm{d}x; X)$ $(1 \le p \le \infty,\ k \in \mathbb{N}_0)$ if there exists $f \in L^p(\mathbb{R}^N, \frac{1}{1+|x|^k}\mathrm{d}x; X)$ such that $T = T_f$. By the preceding lemma, the function f, if it exists, is uniquely determined. We simply write $T \in L^p(\mathbb{R}^N, \frac{1}{1+|x|^k}\mathrm{d}x; X)$ if the tempered distribution T belongs to this space.

For every tempered distribution $T \in \mathcal{S}(\mathbb{R}^N; X)$ and every multi-index $\alpha \in \mathbb{N}_0^N$ we define the **partial derivative** $\partial^\alpha T \in \mathcal{S}(\mathbb{R}^N; X)$ by

$$\langle \partial^\alpha T, \varphi\rangle := (-1)^{|\alpha|}\langle T, \partial^\alpha \varphi\rangle \quad (\varphi \in \mathcal{S}(\mathbb{R}^N)).$$

Moreover, we define the **Fourier transforms** $\mathcal{F}T, \bar{\mathcal{F}}T \in \mathcal{S}'(\mathbb{R}^N; X)$ by

$$\langle \mathcal{F} T, \varphi \rangle := \langle T, \mathcal{F} \varphi \rangle \text{ and}$$
$$\langle \bar{\mathcal{F}} T, \varphi \rangle := \langle T, \bar{\mathcal{F}} \varphi \rangle.$$

Finally, for every polynomial $p : \mathbb{C}^N \to \mathbb{C}$ we define the **product** pT by

$$\langle pT, \varphi \rangle := \langle T, p\varphi \rangle.$$

Lemma 2.19. *For every $f \in \mathcal{S}(\mathbb{R}^N; X)$, every multi-index $\alpha \in \mathbb{N}_0^N$ and every polynomial $p : \mathbb{C}^N \to \mathbb{C}$ one has*

$$\begin{aligned}
\partial^\alpha T_f &= T_{\partial^\alpha f}, \\
\mathcal{F} T_f &= T_{\mathcal{F} f}, \\
\bar{\mathcal{F}} T_f &= T_{\bar{\mathcal{F}} f}, \text{ and} \\
p\, T_f &= T_{pf},
\end{aligned}$$

that is, the distributional partial derivatives, Fourier transforms and products are consistent with the corresponding classical operators on $\mathcal{S}(\mathbb{R}^N; X)$ ($\subseteq \mathcal{S}'(\mathbb{R}^N; X)$).

Proof. For every $\varphi \in \mathcal{S}(\mathbb{R}^N)$ and every $\alpha \in \mathbb{N}_0^N$ one has, by definition of the distributional derivative and by integration by parts,

$$\begin{aligned}
\langle \partial^\alpha T_f, \varphi \rangle &= (-1)^{|\alpha|} \langle T_f, \partial^\alpha \varphi \rangle \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^N} f(x) \partial^\alpha \varphi(x) \, \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \partial^\alpha f(x) \varphi(x) \, \mathrm{d}x \\
&= \langle T_{\partial^\alpha f}, \varphi \rangle.
\end{aligned}$$

This proves the first equality. Using Parseval's identity, we obtain

$$\begin{aligned}
\langle \mathcal{F} T_f, \varphi \rangle &= \langle T_f, \mathcal{F} \varphi \rangle \\
&= \int_{\mathbb{R}^N} f(x) \mathcal{F} \varphi(x) \, \mathrm{d}x \\
&= \int_{\mathbb{R}^N} \mathcal{F} f(x) \varphi(x) \, \mathrm{d}x \\
&= \langle T_{\mathcal{F} f}, \varphi \rangle,
\end{aligned}$$

and this proves the second equality. The third one is proved similarly. The fourth equality uses only the associativity of the product:

$$
\begin{aligned}
\langle pT_f, \varphi\rangle &= \langle T_f, p\varphi\rangle \\
&= \int_{\mathbb{R}^N} f(x)p(x)\varphi(x)\,\mathrm{d}x \\
&= \langle T_{pf}, \varphi\rangle.
\end{aligned}
$$

Remark 2.20. In the above lemma, the function f can be replaced by any other function for which the formula of integration by parts (first equality) or Parseval's identity (second and third equality) still holds. For example, in the second and third equality, one may take $f \in L^1(\mathbb{R}^N;X)$ or, if X is a Hilbert space, $f \in L^2(\mathbb{R}^N;X)$.

Theorem 2.21. *The Fourier transforms $\mathcal{F}, \bar{\mathcal{F}} : \mathcal{S}'(\mathbb{R}^N;X) \to \mathcal{S}'(\mathbb{R}^N;X)$ are linear, bijective and continuous.*

Proof. This follows immediately from Theorem 2.17.

From Theorem 2.21, but also from the Riemann-Lebesgue Lemma (Theorem 2.1), Plancherel's Theorem (Theorem 2.12), the Hausdorff-Young Theorem and Theorem 2.17 we obtain the following picture for the Fourier transform. In the following diagram, a (horizontal) double arrow means that the Fourier transform is an isomorphism between the spaces in the same line. Vertical arrows mean inclusion / natural embeddings.

$$
\begin{tikzcd}
 & \mathcal{S}'(\mathbb{R}^N;X) \arrow[rrr, Leftrightarrow, "\mathcal{F}"] & & & \mathcal{S}'(\mathbb{R}^N;X) & \\
L^1(\mathbb{R}^N;X) \arrow[ur] \arrow[rrr, "\mathcal{F}"] & & & C_0(\mathbb{R}^N;X) \arrow[ur] & & \\
 & L^p(\mathbb{R}^N;X) \arrow[u] \arrow[rrr, "\mathcal{F}", "X\ \text{Fourier type}\ p"'] & & & L^{p'}(\mathbb{R}^N;X) \arrow[u] & \\
 & & L^2(\mathbb{R}^N;X) \arrow[uuul] \arrow[rrr, Leftrightarrow, "\mathcal{F}", "X\ \text{Hilbert}"'] & & & L^2(\mathbb{R}^N;X) \arrow[uuul] \\
 & \mathcal{S}(\mathbb{R}^N;X) \arrow[u] \arrow[uuul] \arrow[ur] \arrow[rrr, Leftrightarrow, "\mathcal{F}"] & & & \mathcal{S}(\mathbb{R}^N;X) \arrow[u] \arrow[uuul] \arrow[ur] &
\end{tikzcd}
$$

From Lemma 2.16 we immediately obtain the following analogon for tempered distributions.

Lemma 2.22. *For every $T \in \mathcal{S}'(\mathbb{R}^N;X)$ and every polynomial $p : \mathbb{C}^N \to \mathbb{C}$ one has*

$$\begin{aligned}
\mathcal{F}(p(\partial)T) &= p(i\cdot)\mathcal{F}T, \\
\mathcal{F}(p(-i\cdot)T) &= p(\partial)\mathcal{F}T, \\
\bar{\mathcal{F}}(p(\partial)T) &= p(-i\cdot)\bar{\mathcal{F}}T, \text{ and} \\
\bar{\mathcal{F}}(p(i\cdot)T) &= p(\partial)\bar{\mathcal{F}}T.
\end{aligned}$$

Theorem 2.23 (Fourier characterization of Sobolev spaces). *Let H be a Hilbert space and $k \in \mathbb{N}$. Then the Fourier transforms $\mathcal{F}$, $\bar{\mathcal{F}}$ map the Sobolev space $H^k(\mathbb{R}^N;H)$ isomorphically onto the weighted space $L^2(\mathbb{R}^N,(1+|x|^2)^k \mathrm{d}x;H)$.*

Proof. Let first $k = 1$. Then we have

$$\begin{aligned}
& f \in H^1(\mathbb{R}^N;H) \\
& \Leftrightarrow f, \partial_1 f, \dots, \partial_N f \in L^2(\mathbb{R}^N;H) \\
\text{(by Plancherel)}\ & \Leftrightarrow \mathcal{F}f, \mathcal{F}(\partial_1 f), \dots, \mathcal{F}(\partial_N f) \in L^2(\mathbb{R}^N;H) \\
\text{(by Lemma 2.22)}\ & \Leftrightarrow \mathcal{F}f, ix_1\mathcal{F}f, \dots, ix_N\mathcal{F}f \in L^2(\mathbb{R}^N;H) \\
& \Leftrightarrow \int_{\mathbb{R}^N} (1 + x_1^2 + \dots + x_N^2)\mathcal{F}f(x)\,\mathrm{d}x < \infty \\
& \Leftrightarrow \mathcal{F}f \in L^2(\mathbb{R}^N,(1+|x|^2)\,\mathrm{d}x;H).
\end{aligned}$$

The case $k \geq 2$ is proved by induction and the assertion for $\bar{\mathcal{F}}$ is proved similarly.

For the Sobolev spaces we thus have the following picture, in which again vertical arrows stand for inclusions and all (horizontal) double arrows mean that the Fourier transform is an isomorphism.

$$\begin{tikzcd}
L^2(\mathbb{R}^N;H) \arrow[r, Leftrightarrow, "\mathcal{F}"] & L^2(\mathbb{R}^N;H) \\
H^1(\mathbb{R}^N;H) \arrow[u] \arrow[r, Leftrightarrow, "\mathcal{F}"] & L^2(\mathbb{R}^N,(1+|x|^2)\,\mathrm{d}x;H) \arrow[u] \\
H^2(\mathbb{R}^N;H) \arrow[u] \arrow[r, Leftrightarrow, "\mathcal{F}"] & L^2(\mathbb{R}^N,(1+|x|^2)^2\,\mathrm{d}x;H) \arrow[u] \\
H^k(\mathbb{R}^N;H) \arrow[u, dotted, dash] \arrow[r, Leftrightarrow, "\mathcal{F}"] & L^2(\mathbb{R}^N,(1+|x|^2)^k\,\mathrm{d}x;H) \arrow[u, dotted, dash] \\
{} \arrow[u, dotted, dash] \arrow[r, Leftrightarrow, "\mathcal{F}"] & {} \arrow[u, dotted, dash]
\end{tikzcd}$$

Remark 2.24. This definition justifies to define the **fractional Sobolev space** $H^s(\mathbb{R}^N;H)$ for $s \in \mathbb{R}$ (thus including Sobolev spaces of negative order) by

$$H^s(\mathbb{R}^N;H) := \mathcal{F}^{-1}L^2(\mathbb{R}^N,(1+|x|^2)^s\,\mathrm{d}x;H).$$

Note that for negative $s \in \mathbb{R}$ the weighted space on the right-hand side of this definition is only included in $\mathcal{S}'(\mathbb{R}^N;H)$, but not in $L^2(\mathbb{R}^N;H)$. As a consequence, if $s \in \mathbb{R}$ is negative, then $H^s(\mathbb{R}^N;H)$ is a subspace of the space of tempered distributions which actually includes $L^2(\mathbb{R}^N;X)$.

2.5 Elliptic and parabolic equations in $\mathbb{R}^N$

2.6 The Marcinkiewicz multiplier theorem

Chapter 3
Singular integrals

3.1 The Marcinkiewicz interpolation theorem

Let (Ω, μ) be a measure space and $(X, |\cdot|_X)$ be a Banach space. Given a measurable function $f : \Omega \to X$ and a parameter $\lambda > 0$, we shortly write $\{|f|_X > \lambda\} := \{t \in \Omega : |f(t)|_X > \lambda\}$, and we define the **distribution function** $m_f : (0, \infty) \to [0, \infty]$ by

$$m_f(\lambda) := \mu(\{|f|_X > \lambda\}) \quad (\lambda > 0).$$

Lemma 3.1. *Let $\Phi : [0, \infty) \to [0, \infty)$ be differentiable, increasing and such that $\Phi(0) = 0$. Let $f : \Omega \to X$ be a measurable function. Then*

$$\int_\Omega \Phi(|f|_X)\, \mathrm{d}\mu = \int_0^\infty \Phi'(\lambda)\, m_f(\lambda)\, \mathrm{d}\lambda.$$

Proof. This follows from a simple application of Tonnelli's theorem:

$$\begin{aligned}\int_\Omega \Phi(|f|_X)\, \mathrm{d}\mu &= \int_\Omega \int_0^{|f(t)|_X} \Phi'(\lambda)\, \mathrm{d}\lambda\, \mathrm{d}\mu(t) \\ &= \int_0^\infty \Phi'(\lambda) \int_{\{|f|_X > \lambda\}} \mathrm{d}\mu\, \mathrm{d}\lambda \\ &= \int_0^\infty \Phi'(\lambda)\, m_f(\lambda)\, \mathrm{d}\lambda.\end{aligned}$$

Example 3.2. For $\Phi(\lambda) = \lambda^p$ $(p \geq 1)$ we obtain

$$\int_\Omega |f|_X^p\, \mathrm{d}\mu = p \int_0^\infty \lambda^p m_f(\lambda)\, \frac{\mathrm{d}\lambda}{\lambda},$$

and in particular, $f \in L^p(\Omega; X)$ if and only if

$$\lambda \mapsto \lambda m_f(\lambda)^{\frac{1}{p}} \in L^p_*(0,\infty),$$

where

$$L^p_*(0,\infty) := L^p((0,\infty); \frac{\mathrm{d}\lambda}{\lambda}).$$

Note that $f \in L^p(\Omega; X)$ thus implies that

$$\sup_{\lambda>0} \lambda^p m_f(\lambda) < \infty.$$

Denote by $M(\Omega; X)$ be the space of all measurable functions $\Omega \to X$. Let (Ω_1, μ_1) and (Ω_2, μ_2) be two measure spaces, and let $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ be two Banach spaces. We say that a (not necessarily linear) operator $T : L^p(\Omega_1, X) \to M(\Omega_2, Y)$ satisfies a **weak-(p,q) estimate**, or we say that T **is weak-(p,q)** if there exists a constant $C \geq 0$ such that, for every $f \in L^p(\Omega_1; X)$ and every $\lambda > 0$

$$m_{Tf}(\lambda) \leq \left(\frac{C\|f\|_{L^p}}{\lambda}\right)^q \quad (\text{if } q < \infty),$$

or

$$\|Tf\|_{L^\infty} \leq C\|f\|_{L^p} \quad (\text{if } q = \infty).$$

We say that an operator T on a subspace of $M(\Omega_1; X)$ with values in $M(\Omega_2, Y)$ **subadditive** if for every f, g in the domain of T

$$|T(f+g)|_Y \leq |Tf|_Y + |Tg|_Y \text{ almost everywhere},$$

and we say that it is **homogeneous** if for every $\alpha \in \mathbb{K}$ and every f in the domain of T

$$|T(\alpha f)|_Y = |\alpha||Tf|_Y \text{ almost everywhere}.$$

Finally, T is said to be **sublinear** if it is both subadditive and homogeneous.

Theorem 3.3 (Marcinkiewicz interpolation). *Let (Ω_1, μ_1), (Ω_2, μ_2) be two measure spaces, X, Y be two Banach spaces, $1 \leq p_0 < p_1 \leq \infty$, and let $T : L^{p_0} + L^{p_1}(\Omega_1; X) \to M(\Omega_2, Y)$ be a subadditive operator which is both weak-(p_0, p_0) and weak-(p_1, p_1). Then, for every $p_0 < p < p_1$ there exists $C \geq 0$ such that*

$$\|Tf\|_{L^p} \leq C\|f\|_{L^p} \text{ for every } f \in L^p(\Omega_1; X).$$

Proof. Let $f \in L^p(\Omega_1; X)$ and $c > 0$. For each $\lambda > 0$ we write $f = f_0 + f_1$ with

$$f_0 = f 1_{\{|f|_X > c\lambda\}},$$
$$f_1 = f 1_{\{|f|_X \leq c\lambda\}}.$$

Then $f_0 \in L^{p_0}(\Omega_1; X)$ and $f_1 \in L^{p_1}(\Omega_1; X)$, and, by the subadditivity of T,

$$|Tf(t)|_Y \leq |Tf_0(t)|_Y + |Tf_1(t)|_Y \text{ for every } t \in \Omega_2.$$

As a consequence, for every $\lambda' > 0$,

$$m_{Tf}(\lambda') \le m_{Tf_0}(\frac{\lambda'}{2}) + m_{Tf_1}(\frac{\lambda'}{2}).$$

By assumption, there exist constants $C_0, C_1 \ge 0$ independent of f, f_0, f_1, λ, λ' such that

$$\begin{aligned} m_{Tf_0}(\frac{\lambda'}{2}) &\le \left(\frac{2C_0\,\|f_0\|_{L^{p_0}}}{\lambda'}\right)^{p_0}, \text{ and} \\ m_{Tf_1}(\frac{\lambda'}{2}) &\le \left(\frac{2C_1\,\|f_1\|_{L^{p_1}}}{\lambda'}\right)^{p_1} \quad (p_1 < \infty), \\ \|Tf_1\|_{L^\infty} &\le C_1\,\|f\|_{L^\infty} \quad (p_1 = \infty). \end{aligned}$$

The case $p_1 = \infty$. Choose $c := (2C_1)^{-1}$ and $\lambda' = \lambda$. Then $m_{Tf_1}(\frac{\lambda}{2}) = 0$. Hence

$$\begin{aligned} \|Tf\|_{L^p}^p &= p\int_0^\infty \lambda^{p-1}\, m_{Tf}(\lambda)\,\mathrm{d}\lambda \\ &\le p\int_0^\infty \lambda^{p-1}\, m_{Tf_0}(\frac{\lambda}{2})\,\mathrm{d}\lambda \\ &\le p\int_0^\infty \lambda^{p-1-p_0}(2C_0)^{p_0}\int_{\{|f|_X > c\lambda\}} |f(t)|_X^{p_0}\,\mathrm{d}\mu(t)\,\mathrm{d}\lambda \\ &= p\,(2C_0)^{p_0}\int_{\Omega_1} |f(t)|_X \int_0^{|f(t)|_X/c} \lambda^{p-1-p_0}\,\mathrm{d}\lambda\,\mathrm{d}\mu(t) \\ &= \frac{p}{p-p_0}\,(2C_0)^{p_0}\,(2C_1)^{p-p_0}\,\|f\|_{L^p}^p. \end{aligned}$$

The case $p_1 < \infty$. Similarly as above we obtain, for every $c > 0$,

$$\begin{aligned} \|Tf\|_{L^p}^p &= p\int_0^\infty \lambda^{p-1}\, m_{Tf}(\lambda)\,\mathrm{d}\lambda \\ &\le p\int_0^\infty \lambda^{p-1}\, m_{Tf_0}(\frac{\lambda}{2})\,\mathrm{d}\lambda \\ &\quad + p\int_0^\infty \lambda^{p-1}\, m_{Tf_1}(\frac{\lambda}{2})\,\mathrm{d}\lambda \\ &\le p\int_0^\infty \lambda^{p-1-p_0}(2C_0)^{p_0}\int_{\{|f|_X > c\lambda\}} |f(t)|_X^{p_0}\,\mathrm{d}\mu(t)\,\mathrm{d}\lambda \\ &\quad + p\int_0^\infty \lambda^{p-1-p_1}(2C_1)^{p_1}\int_{\{|f|_X \le c\lambda\}} |f(t)|_X^{p_1}\,\mathrm{d}\mu(t)\,\mathrm{d}\lambda \\ &\le p\,(\frac{1}{p-p_0} + \frac{1}{p_1-p})\,(\frac{(2C_0)^{p_0}}{c^{p-p_0}} + \frac{(2C_0)^{p_0}}{c^{p-p_0}})\,\|f\|_{L^p}^p. \end{aligned}$$

This completes the proof.

Remark 3.4. By minimizing over $c > 0$, we obtain that the constant $C \geq 0$ in Theorem 3.3 can be chosen as

$$C = 2p^{\frac{1}{p}} \left(\frac{1}{p-p_0} + \frac{1}{p_1 - p}\right)^{\frac{1}{p}} C_0^{\theta} C_1^{1-\theta},$$

where the $C_0, C_1 \geq 0$ are the constants from the weak-(p_i, p_i) estimates, and $\theta \in (0,1)$ is chosen such that

$$\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}.$$

3.2 The Hardy-Littlewood maximal operator

Let $(X, |\cdot|_X)$ be a Banach space and $N \in \mathbb{N}$. For every $f \in M(\mathbb{R}^N; X)$ we define the **maximal function** $Mf : \mathbb{R}^N \to [0, \infty]$ by

$$Mf(x) := \sup_{Q \ni x} \fint_Q |f(y)|_X \, \mathrm{d}y \quad (x \in \mathbb{R}^N),$$

where Q is any cube with sides parallel to the axes, that is, ball with respect to the $|\cdot|_\infty$ norm, and where for every measurable set $B \subseteq \mathbb{R}^N$ we have set

$$\fint_B = \frac{1}{|B|} \int_B,$$

$|B|$ being the Lebesgue measure of B. By continuity, the definition does not change if one considers only the supremum over all cubes with rational centers and rational radii, so that one sees that Mf is measurable. The operator $M : M(\mathbb{R}^N; X) \to M(\mathbb{R}^N)$ is called the **Hardy-Littlewood maximal operator**. It is easily seen that M is sublinear.

Lemma 3.5 (Covering lemma in $\mathbb{R}$). *Let $K \subseteq \mathbb{R}$ be a compact set, and let $(I_\alpha)_{\alpha \in A}$ be a family of intervals such that $K = \bigcup_{\alpha \in A} I_\alpha$. Then there exists a finite subfamily $(I_{\alpha_j})_{1 \leq j \leq n}$ such that*

$$K \subseteq \bigcup_{j=1}^{n} I_{\alpha_j} \quad \textit{and}$$
$$\sum_{j=1}^{n} 1_{I_{\alpha_j}}(x) \leq 2 \textit{ for every } x \in \mathbb{R}.$$

Theorem 3.6. *The Hardy-Littlewood maximal operator M is weak* $(1,1)$ *and strong* (p,p) *for every* $1 < p \leq \infty$.

Proof (for the case $N = 1$). The estimate

$$\|Mf\|_{L^\infty} \leq \|f\|_{L^\infty}$$

follows immediately from the definition. Hence, M is strong (∞,∞) = weak (∞,∞). By the Marcinkiewicz interpolation theorem it suffices to show that M is weak $(1,1)$.

Now let $f \in L^1(\mathbb{R}^N;X)$. Let $\lambda > 0$ and let $K \subseteq \{Mf > \lambda\}$ be any compact subset. For every $x \in K$ there exists an interval I_x containing x such that

$$⨍_{I_x} |f|_X > \lambda.$$

Clearly, $K \subseteq \bigcup_{x\in K} I_x$, so that, by Lemma 3.5, there exists a finite subset $\{x_1,\dots,x_n\} \subseteq K$ such that

$$K \subseteq \bigcup_{j=1}^n I_{x_j} \quad \text{and}$$
$$\sum_{j=1}^n 1_{I_{x_j}}(x) \leq 2 \text{ for every } x \in \mathbb{R}.$$

Hence,

$$\begin{aligned} |K| &\leq \sum_{j=1}^n |I_{x_j}| \\ &\leq \sum_{j=1}^n \frac{1}{\lambda} \int_{I_{x_j}} |f|_X \\ &\leq \frac{1}{\lambda} \int_{\mathbb{R}} \sum_{j=1}^n 1_{I_{x_j}} |f|_X \\ &\leq \frac{2}{\lambda} \|f\|_{L^1}. \end{aligned}$$

Since this inequality holds for every compact subset $K \subseteq \{Mf > \lambda\}$, the inner regularity of the Lebesgue measure yields

$$m_{Mf}(\lambda) \leq \frac{2}{\lambda} \|f\|_{L^1}.$$

For the general case $N > 1$, we need the following covering lemma, which is a variant of Lemma 3.5.

Lemma 3.7 (Vitali). *Let $K \subseteq \mathbb{R}^N$ be a compact set, and let $(Q_\alpha)_{\alpha\in A}$ be a family of cubes in $\mathbb{R}^N$ such that*

$$K \subseteq \bigcup_{\alpha\in A} Q_\alpha.$$

Then there exist $\alpha_1, \ldots, \alpha_n \in A$ such that

$$K \subseteq \bigcup_{j=1}^n Q_{\alpha_j,5} \text{ and}$$

$$\sum_{j=1}^n 1_{Q_{\alpha_j}} \leq 1.$$

Proof.

Proof (of Theorem 3.6 for the general case $N \geq 1$). The beginning of the proof is the same as in the case $N = 1$. We only need to prove a weak (1,1) estimate. Let $f \in L^1(\mathbb{R}^N; X)$. Let $\lambda > 0$ and let $K \subseteq \{Mf > \lambda\}$ be any compact subset. For every $x \in K$ there exists a cube Q_x containing x such that

$$⨍_{Q_x} |f|_X > \lambda.$$

Clearly, $K \subseteq \bigcup_{x\in K} Q_x$, so that, by Lemma 3.7, there exists a finite subset $\{x_1, \ldots, x_n\} \subseteq K$ such that

$$K \subseteq \bigcup_{j=1}^n Q_{x_j,5} \quad \text{and}$$

$$\sum_{j=1}^n 1_{Q_{x_j}}(x) \leq 1 \text{ for every } x \in \mathbb{R}^N.$$

Hence,

$$\begin{aligned}
|K| &\leq \sum_{j=1}^{n} |Q_{x_j,5}| \\
&= \sum_{j=1}^{n} 5^N |Q_{x_j}| \\
&\leq 5^N \sum_{j=1}^{n} \frac{1}{\lambda} \int_{Q_{x_j}} |f|_X \\
&\leq \frac{5^N}{\lambda} \int_{\mathbb{R}^N} \sum_{j=1}^{n} 1_{Q_{x_j}} |f|_X \\
&\leq \frac{5^N}{\lambda} \|f\|_{L^1}.
\end{aligned}$$

Since this inequality holds for every compact subset $K \subseteq \{Mf > \lambda\}$, the inner regularity of the Lebesgue measure yields

$$m_{Mf}(\lambda) \leq \frac{5^N}{\lambda} \|f\|_{L^1}.$$

Lemma 3.8. *If $f \in L^1(\mathbb{R}^N; X)$ is not identically 0, then $Mf \notin L^1(\mathbb{R}^N)$.*

Proof. If $f \in L^1(\mathbb{R}^N; X)$ is not identically 0, then there exist $r > 0$ and $\varepsilon > 0$ such that

$$\int_{Q_r(0)} |f(y)|_X \, \mathrm{d}y \geq \varepsilon.$$

Now, for any $x \in \mathbb{R}^N$ with $|x| \geq r$ one has

$$Q_r(0) \subseteq Q_{2|x|}(x),$$

and hence

$$Mf(x) \geq \frac{1}{(2|x|)^N} \int_{Q_{2|x|}(x)} |f(y)|_X \, \mathrm{d}y \geq \frac{\varepsilon}{2^N |x|^N},$$

so that $Mf \notin L^1(\mathbb{R}^N)$.

A **weight** $w : \mathbb{R}^N \to \mathbb{R}$ is a measurable function which is strictly positive almost everywhere. Given a weight w, we denote also by w the weighted Lebesgue measure $w(x)\mathrm{d}\lambda(x)$, so that, for a measurable set $B \subseteq \mathbb{R}^N$

$$w(B) := \int_{\mathbb{R}^N} 1_B(x)\, w(x) \, \mathrm{d}\lambda(x).$$

If $f : \mathbb{R}^N \to X$ is in addition a measurable function, the we denote its distribution function with respect to the weighted Lebesgue measure by w_f, that is,

$$w_f(\lambda) := w(\{|f|_X > \lambda\}).$$

We denote by $L^p_w(\mathbb{R}^N;X)$ the weighted L^p space $L^p(\mathbb{R}^N, w(x)\,\mathrm{d}\lambda(x);X)$.

We say that a locally integrable weight w satisfies the **Muckenhoupt** A_p **condition** or that w is an A_p **weight** ($1 \le p < \infty$), and we write $w \in A_p$, if there exists a constant $C \ge 0$ such that

$$Mw(x) \le C\,w(x) \text{ for every } x \in \mathbb{R}^N, \text{ if } p = 1,$$

$$\Big(\fint_Q w\Big)\Big(\fint_Q w^{1-p'}\Big)^{p-1} \le C \text{ for every cube } Q \subseteq \mathbb{R}^N, \text{ if } p > 1.$$

The smallest possible constant $C \ge 0$ for which the above inequality (for $p = 1$ or for $p > 1$) holds is called the A_p**-constant** of the weight; it is denoted by $[w]_{A_p}$.

Theorem 3.9. *For every $1 \le p < \infty$ the weak (p,p) estimate*

$$w_{Mf}(\lambda) \le \frac{C}{\lambda^p} \int_{\mathbb{R}^N} |f|_X^p\, w \tag{3.1}$$

holds if and only if $w \in A_p$.

Proof. Necessity. Assume that there exists a constant $C \ge 0$ such that for every $f \in L^p(\Omega;X)$ the inequality (3.1) holds. Let $Q \subseteq \mathbb{R}^N$ be a cube such that $\int_Q |f|_X > 0$, and let $\lambda > 0$ be such that $\fint_Q |f|_X > \lambda$. Then

$$Q \subseteq \{M(f1_Q) > \lambda\}$$

and therefore

$$w(Q) \le w_{M(f1_Q)}(\lambda).$$

From inequality (3.1) follows

$$w(Q) \le \frac{C}{\lambda^p} \int_Q |f|_X^p\, w,$$

and since this inequality holds for every $\lambda > 0$ such that $\fint_Q |f|_X > \lambda$, we obtain

$$\frac{w(Q)}{|Q|^p} \le C\, \frac{\int_Q |f|_X^p\, w}{(\int_Q |f|_X)^p}. \tag{3.2}$$

The case $p = 1$. We choose any measurable subset $B \subseteq Q$ and $f = 1_B$ in the above inequality. Then this inequality becomes

$$\frac{w(Q)}{|Q|} \le C\frac{w(B)}{|B|}.$$

This inequality implies for almost every $x \in Q$

$$\frac{w(Q)}{|Q|} \leq C\, w(x),$$

and maximizing over all cubes Q containing x, we obtain

$$Mw(x) \leq C\, w(x),$$

that is, $w \in A_1$.

The case $1 < p < \infty$. We choose $f = w^{1-p'} 1_Q$ in inequality (3.2) which then becomes

$$w(Q) \left(\frac{1}{|Q|} \int_Q w^{1-p'} \right)^p \leq C \int_Q w^{1-p'},$$

or, since $Q \subseteq \mathbb{R}^N$ was arbitrary, $w \in A_p$.

Sufficiency. Let $f \in L^p_w(\mathbb{R}^N; X)$. For every cube $Q \subseteq \mathbb{R}^N$ one has, by Hölder's inequality,

$$\begin{aligned} \left(\fint_Q |f|_X \right)^p &= \left(\frac{1}{|Q|} \int_Q |f|_X \, w^{\frac{1}{p}} \, w^{-\frac{1}{p}} \right)^p \\ &\leq \left(\frac{1}{|Q|} \int_Q |f|_X^p w \right) \left(\frac{1}{|Q|} \int_Q w^{1-p'} \right)^{p-1} \\ &\leq [w]_{A_p} \left(\frac{1}{|Q|} \int_Q |f|_X^p w \right) \frac{|Q|}{w(Q)}. \end{aligned} \tag{3.3}$$

Now let $\lambda > 0$ and let $K \subseteq \{Mf > \lambda\}$ be any compact subset. For every $x \in K$ there exists a cube Q_x containing x such that

$$\fint_{Q_x} |f|_X > \lambda.$$

Clearly, $K \subseteq \bigcup_{x \in K} Q_x$.

We consider noew the case $N = 1$. By the covering lemma (3.5, there exists a finite subset $\{x_1, \dots, x_n\} \subseteq K$ such that

$$K \subseteq \bigcup_{j=1}^n Q_{x_j} \quad \text{and}$$

$$\sum_{j=1}^n 1_{Q_{x_j}}(x) \leq 2 \text{ for every } x \in \mathbb{R}.$$

Hence, if we combine this with inequality (3.3),

$$\begin{aligned}
w(K) &\le \sum_{j=1}^{n} w(Q_{x_j}) \\
&\le \sum_{j=1}^{n} \frac{[w]_{A_p}}{\lambda^p} \int_{Q_{x_j}} |f|_X^p w \\
&\le \frac{[w]_{A_p}}{\lambda^p} \int_{\mathbb{R}} \sum_{j=1}^{n} 1_{Q_{x_j}} |f|_X^p w \\
&\le \frac{2\,[w]_{A_p}}{\lambda^p} \|f\|_{L^p_w}^p .
\end{aligned}$$

Since this inequality holds for every compact subset $K \subseteq \{Mf > \lambda\}$, the inner regularity of the weighted Lebesgue measure yields

$$w_{Mf}(\lambda) \le \frac{2\,[w]_{A_p}}{\lambda^p} \|f\|_{L^p_w}^p .$$

Remark 3.10. The proof of Theorem 3.9 shows in the necessity part that if $C \ge 0$ is a constant such that weak (p,p)-inequality (3.1) holds, then

$$[w]_{A_p} \le C.$$

On the other hand, the sufficiency part shows that in the case $N = 1$

$$w_{Mf}(\lambda) \le \frac{2\,[w]_{A_p}}{\lambda^p} \|f\|_{L^p_w}^p ,$$

that is, C can be chosen equal to $2\,[w]_{A_p}$ in (3.1), if $N = 1$. In the general case $N \ge 1$ we obtain that $C = 5^N\,[w]_{A_p}$ is a possible constant in (3.1).

We list a few properties of Muckenhoupt weights. The first few properties are mainly simple consequences of the Hölder inequality.

Lemma 3.11. a) *For $1 \le p \le q < \infty$ one has $A_p \subseteq A_q$, and for every $w \in A_p$*

$$[w]_{A_q} \le [w]_{A_p} .$$

b) *For $1 < p < \infty$ one has $w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$, and then*

$$[w^{1-p'}]_{A_{p'}} = [w]_{A^p}^{\frac{1}{p-1}} .$$

c) *If $w, v \in A_1$ and $1 < p < \infty$, then $wv^{1-p} \in A_p$ and*

$$[wv^{1-p}]_{A_p} \le [w]_{A_1}\,[v]_{A_1}^{1-p} .$$

d) *If $1 \le p < q$, $w \in A_q$ and $v \in A_1$, then $u := (w^{q-1} u^{q-p})^{\frac{1}{q-1}} \in A_p$ and*

$$[u]_{A_p} \le ([w]_{A_q}^{p-1} [v]_{A_1}^{q-p})^{\frac{1}{p-1}}.$$

Proof. (a) For $p = 1$ and $q > 1$ one has

$$\left(\fint_Q w^{1-q'} \right)^{q-1} \le \sup_{x \in Q} w(x)^{-1} = (\inf_{x \in Q} w(x))^{-1} \le [w]_{A_1} \left(\fint_Q w \right)^{-1}.$$

For $p > 1$, the inclusion follows immediately from Hölder's inequality.
(b) The $A_{p'}$ condition for $w^{1-p'}$ is

$$\sup_Q \left(\fint_Q w^{1-p'} \right) \left(\fint_Q w^{(1-p')(1-p)} \right)^{p'-1} < \infty,$$

but since $(p'-1)(p-1) = 1$, the left-hand side is equal to $[w]_{A_p}^{\frac{1}{p-1}}$.
(c) We compute, using a similar inequality as in the proof of (a),

$$\begin{aligned}
&\left(\fint_Q w v^{1-p} \right) \left(\fint_Q w^{1-p'} v^{(1-p)(1-p')} \right)^{p-1} \\
&\le \left(\fint_Q w [v]_{A_1}^{p-1} (\fint_Q v)^{1-p} \right) \left(\fint_Q v [w]_{A_1}^{p'-1} (\fint_Q w)^{1-p'} \right)^{p-1} \\
&= [w]_{A_1} [v]_{A_1}^{p-1}.
\end{aligned}$$

Theorem 3.12 (Reverse Hölder inequality). *Let $w \in A_p$ for some $1 < p < \infty$. Then there exist constants $\varepsilon > 0$, $C \ge 0$ depending only on $[w]_{A_p}$ such that*

$$\left(\fint_Q w^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}} \le C \fint_Q w \text{ for every cube } Q \subseteq \mathbb{R}^N.$$

Corollary 3.13. a) *For every $w \in A_p$ $(1 < p < \infty)$ there exists $\varepsilon > 0$ depending only on $[w]_{A_p}$ such that $w \in A_{p-\varepsilon}$. In other words,*

$$A_p = \bigcup_{1 \le q < p} A_q.$$

b) *For every $w \in A_p$ $(1 < p < \infty)$ there exist $\varepsilon > 0$ such that $w^{1+\varepsilon} \in A_p$.*

c) *If $w \in A_p$ for some $1 \le p < \infty$, then there exists $\delta > 0$, $C \ge 0$ such that*

$$\frac{w(S)}{w(Q)} \le C \left(\frac{|S|}{|Q|} \right)^{\delta} \textit{ for every cube } Q \subseteq \mathbb{R}^N \textit{ and every } S \subseteq Q.$$

Proof. (a) Let $w \in A_p$ for some $1 < p < \infty$. By Lemma 3.11 (b), $w^{1-p'} \in A_{p'}$. By the reverse Hölder inequality (Theorem 3.12), there exists $\varepsilon > 0$ and $C \ge 0$ such that

$$\left(⨍_Q w^{(1-p')(1+\varepsilon)} \right)^{\frac{1}{1+\varepsilon}} \le C ⨍_Q w^{1-p'} \text{ for every cube } Q \subseteq \mathbb{R}^N.$$

Fix $q \in (1, p)$ such that $1 - q' = (1-p')(1-\varepsilon)$. Then the preceding inequality gives

$$\left(⨍_Q w^{1-q'} \right)^{q-1} \le C^{p-1} \left(⨍_Q w^{1-p'} \right)^{p-1},$$

which finally yields $w \in A_q$.
(b) Since $w \in A_p$ and $w^{1-p'} \in A_{p'}$, we can choose, by the reverse Hölder inequality and by Lemma 3.11 (a), a common $\varepsilon > 0$ such

$$\left(⨍_Q w^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}} \le C ⨍_Q w \text{ and}$$
$$\left(⨍_Q w^{(1+\varepsilon)(1-p')} \right)^{\frac{1}{1+\varepsilon}} \le C ⨍_Q w^{1-p'} \text{ for every cube } Q \subseteq \mathbb{R}^N.$$

From both inequalities together follows $w^{1+\varepsilon} \in A_p$.
(c) Fix a cube $Q \subseteq \mathbb{R}^N$ and $S \subseteq Q$. Let $\varepsilon > 0$ be such that w satisfies the reverse Hölder inequality with exponent $1 + \varepsilon$. Then, by Hölder's inequality and by the reverse Hölder inequality,

$$\begin{aligned} w(S) &= \int_Q 1_S \, w \\ &\le \left(\int_Q w^{1+\varepsilon} \right)^{\frac{1}{\varepsilon}} |S|^{\frac{\varepsilon}{1+\varepsilon}} \\ &\le C \, w(Q) \left(\frac{|S|}{|Q|} \right)^{\frac{\varepsilon}{1+\varepsilon}}. \end{aligned}$$

We say that a weight satisfies the **Muckenhoupt** A_∞ **condition** or that w is an A_∞ **weight**, and we write $w \in A_\infty$, if w satisfies the property in Corollary 3.13 (c). By Corollary 3.13 (c), one has

$$\bigcup_{1 \le p < \infty} A_p \subseteq A_\infty,$$

and one can show that one actually has equality (compare with Corollary 3.13 (a)).

Theorem 3.14 (Muckenhoupt). *For every $1 < p < \infty$ and every Muckenhoupt weight $w \in A_p$ there exists a constant $C \geq 0$ (depending only on $[w]_{A_p}$) such that*

$$\|Mf\|_{L^p_w} \leq C\|f\|_{L^p_w} \text{ for every } f \in L^p_w(\Omega;X).$$

In other words, the Hardy-Littlewood maximal operator M is strong (p,p) on the weighted Lebesgue space L^p_w.

Proof. Let w be a Muckenhoupt A_p-weight for some $1 < p < \infty$. By the corollary to the reverse Hölder inequality (Corollary 3.13 (a)), there exists $q < p$ such that $w \in A_q$. By Theorem 3.9, the Hardy-Littlewood maximal operator M is weak (q,q) on L^q_w. On the other hand, M satisfies the strong = weak (∞,∞) estimate on $L^\infty_w = L^\infty_w$. By Marcinkiewicz' interpolation theorem (Theorem 3.3), M satisfies therefore a strong (p,p)-estimate on L^p_w.

We conclude this section by noting that in some cases it is also useful to consider the **centered maximal operator**

$$M_c f(x) := \sup_{r>0} \fint_{Q_r(x)} |f|_X \quad (x \in \mathbb{R}^N),$$

in which the supremum is only taken over all cubes centered at x, and the **dyadic maximal operator**

$$M_d f(x) := \sup_{k\in\mathbb{Z}} \fint_{Q_{2^k}(x)} |f|_X \quad (x \in \mathbb{R}^N)$$

in which the supremum is only taken over all **dyadic cubes** centered at x, that is, cubes with radius $= 2^k$ for some $k \in \mathbb{Z}$. Clearly, for every $f \in M(\mathbb{R}^N;X)$, $M_d f \leq M_c f \leq Mf$ pointwise everywhere. These trivial inequalities show that the centered and the dyadic maximal operators also satisfy strong (p,p) estimates on L^p_w whenever $1 < p < \infty$ and $w \in A_p$. However, one actually has a sort of equivalence between the maximal operators in the sense that, for every $f \in M(\mathbb{R}^N;X)$,

$$Mf \leq 2^N M_c f \leq 4^N M_d f \text{ pointwise everywhere.}$$

We may exploit this fact later on. We also define the **sharp maximal operator**

$$M^\sharp f(x) := \sup_{Q\ni x} \fint_Q |f - f_Q|_X \quad (x \in \mathbb{R}^N),$$

where the supremum is taken over all cubes containing x, and where $f_Q := \fint_Q f$ is the mean of f over Q. We say that a function f has **bounded mean**

oscillation if $M^\sharp f \in L^\infty(\mathbb{R}^N)$, and we consider the space of all functions of bounded mean oscillation

$$BMO(\mathbb{R}^N; X) := \{f \in M(\mathbb{R}^N; X) : M^\sharp f \in L^\infty(\mathbb{R}^N)\},$$

which is equipped with the seminorm

$$\|f\|_{BMO} := \|M^\sharp f\|_{L^\infty}.$$

Note that constant functions have mean oscillation equal to 0, and one would have to take the quotient of *BMO* with respect to the constant functions in order to obtain a Banach space.

3.3 The Rubio de Francia extrapolation theorem

by Sebastian Król

The aim of the section is the proof of a variant of the following boundedness principle by Rubio de Francia [**?**]:

The boundedness properties of a linear operator depend only on the weighted L^2 inequalities that it satisfies.

This is a final version of the extrapolation theorem Muckenhoupt's A_p weights.

The first variant of the principle is was given by Rubio de Francia in 1982 [Rubio de Francia (1982)]. Below, we shall present the proof of the extrapolation theorem of Rubio de Francia in its first formulation.

The underlying philosophy of Rubio de Francia's extrapolation theory has been summarized by A. Cordoba [Córdoba (1988)]:

There are no L_p spaces only weighted L_2.

This was the basic idea in the original (non constructive) proof of the extrapolation theorem for Muckenhoupt's A_p weights. Although originally given for operators, it was realized that the operators do not play any role and all the statements can be given in therms of families of nonnegative measurable functions. It is a setting observed by Cruz-Uribe and Perez [Cruz-Uribe and Pérez (2000)].

Below we follow a presentation given by Duoandikoetxea [Duoandikoetxea (2011)] - a version of the extrapolation theorem with sharp bounds - it is of main interest in studying the sharp dependence of the norms of operators in therms of the A_p-constant of the weights.

Subsequently, given $p \in [1, \infty)$, a weight w, and a family $\mathcal{F}$ of pairs of nonnegative, measurable functions, we put

$$\mathcal{F}_{p,w} = \{(f,g) \in \mathcal{F} : \int_{\mathbb{R}^n} g^p w dx < \infty\}.$$

Theorem 3.15. *Let $p_0 \in (1,\infty)$ and $\mathcal{F}$ be a family of pairs of nonnegative, measurable functions. Assume that there exists an increasing function $N : (0,\infty) \to (0,\infty)$ such that for every Muckenhoupt weight $w \in A_{p_0}$ we have:*

$$\left(\int_{\mathbb{R}^n} g^{p_0} w dx\right)^{1/p_0} \le N([w]_{p_0})\left(\int_{\mathbb{R}^n} f^{p_0} w dx\right)^{1/p_0} \qquad \big((f,g) \in \mathcal{F}_{p_0,w}\big). \tag{3.4}$$

Then for every $p \in (1,\infty)$ and every Muckenhoupt weight $w \in A_p$ we have

$$\left(\int_{\mathbb{R}^n} g^{p} w dx\right)^{1/p} \le CN_p([w]_{p})\left(\int_{\mathbb{R}^n} f^{p} w dx\right)^{1/p} \qquad \big((f,g) \in \mathcal{F}_{p,w}\big),$$

where C does not depend on w, and $N_p([w]_{A_p})$ is given by

$$N_p([w]_{A_p}) := \begin{cases} N\Big([w]_{A_p}(2\|M\|_{L_p(w)})^{p-p_0}\Big), & \text{if } p \le p_0, \\ N\left([w]_{A_p}^{\frac{p_0-1}{p-1}}\Big(2\|M\|_{L_{p'}(w^{1-p'})}\Big)^{\frac{p-p_0}{p-1}}\right), & \text{if } p > p_0. \end{cases}$$

The proof of the Theorem 3.15 is based on the following results: the factorization of Muckenhoupt's A_p weights and the construction of A_1 weights via Rubio de Francia's iteration algorithm.

Lemma 3.16 (Factorization).

a) *Let $1 \le p < p_0 < \infty$. If $w \in A_p$ and $u \in A_1$, then $v := wu^{p-p_0} \in A_{p_0}$ and $[v]_{A_{p_0}} \le [w]_{A_p}[u]_{A_1}^{p_0-p}$.*

b) *Let $1 < p_0 < p < \infty$. If $w \in A_p$ and $u \in A_1$, then $v := (w^{p_0-1}u^{p-p_0})^{\frac{1}{p-1}} \in A_{p_0}$ and $[v]_{A_{p_0}} \le [w]_{A_p}^{\frac{p_0-1}{p-1}}[u]_{A_1}^{\frac{p-p_0}{p-1}}$.*

Proof. The statements of Lemma 3.16 follow directly from the definition of Muckenhoupt's A_p classes and Hölder's inequality. Here, we provide only the proof of the statement (b).

Fix a cube $Q \subseteq \mathbb{R}^n$. Note that $\frac{p-1}{p_0-1} > 1$, and $(\frac{p-1}{p_0-1})' = \frac{p-1}{p_0-p}$. By Hölder's inequality, we easily get

$$\frac{1}{|Q|}\int_Q v dx = \frac{1}{|Q|}\int_Q w^{\frac{p_0-1}{p-1}} u^{\frac{p-p_0}{p-1}} dx \le \left(\frac{1}{|Q|}\int_Q w dx\right)^{\frac{p_0-1}{p-1}} \left(\frac{1}{|Q|}\int_Q u dx\right)^{\frac{p-p_0}{p-1}}. \tag{3.5}$$

On the other hand, since $u(x)^{-1} \le [u]_{A_1}(\frac{1}{|Q|}\int_Q u\,dx)^{-1}$ for almost every $x \in Q$, and $(p_0-1)(p_0'-1) = 1$, we have

$$\left(\frac{1}{|Q|}\int_Q v^{1-p_0{}'}\,dx\right)^{p_0-1} \le \left(\frac{1}{|Q|}\int_Q w^{1-p'}\,dx\right)^{p_0-1} [u]_{A_1}^{\frac{p-p_0}{p-1}} \left(\frac{1}{|Q|}\int_Q u\,dx\right)^{-\frac{p-p_0}{p-1}}. \quad (3.6)$$

Therefore, combining (3.5) and (3.6) we obtain the desired conclusion.

Recall that, by Muckenhoupt's theorem (Theorem 3.14, the Hardy-Littlewood maximal operator M is bounded on L^p_w for every $p \in (1,\infty)$ and every Muckenhoupt weight $w \in A_p$. For every $p \in (1,\infty)$ and every Muckenhoupt weight $w \in A_p$ we define therefore define the **Rubio de Francia operator** $R = R_{p,w}$ as follows

$$Rf := \sum_{k=0}^{\infty} \frac{M^k f}{(2\|M\|_{L^p_w})^k} \qquad (f \in L^p_w),$$

where M^k denotes the k-th iterate of M, and $\|M\|_{L^p_w}$ is the operator norm of M on L^p_w, that is, $\|M\|_{L^p_w} = \sup_{\|f\|_{L^p_w} \le 1} \|Mf\|_{L^p_w}$.

Lemma 3.17 (Rubio de Francia's iteration algorithm). *Let R be the Rubio de Francia extrapolation operator, $1 < p < \infty$, $w \in A_p$, and $h \in L^p_w$. Then:*

a) *One has*

$$|h| \le Rh \textit{ pointwise almost everywhere, and}$$
$$\|Rh\|_{L^p_w} \le 2\,\|h\|_{L^p_w}.$$

b) *If $h \ge 0$, then*

$$M(Rh) \le 2\,\|M\|_{L^p_w}\, Rh \textit{ pointwise almost everywhere,}$$

that is, $Rh \in A_1$ and

$$[Rh]_{A_1} \le 2\,\|M\|_{L^p_w}.$$

Proof. The proof is immediate.

Proof (of Theorem 3.15). **The case** $p < p_0$**.** Fix $w \in A_p$ and $(f,g) \in \mathcal{F}_{p,w}$ with $f \in L^p_w$ and $g \neq 0$. Let

$$h := f + \frac{\|f\|_{L^p_w}}{\|g\|_{L^p_w}} g.$$

Note that $\|h\|_{L^p_w} \le 2\|f\|_{L^p_w}$. By Lemma 3.17,

$$Rh \in A_1 \text{ with } [Rh]_{A_1} \le 2\|M\|_{L^p_w}.$$

Consequently, Lemma 3.16 yields

$$v := w(Rh)^{p-p_0} \in A_{p_0} \text{ with } [v]_{A_{p_0}} \leq [w]_{A_p}[Rh]_{A_1}^{p_0-p}.$$

Furthermore, since $g \leq \frac{\|g\|_{L^p_w}}{\|f\|_{L^p_w}} h \leq \frac{\|g\|_{L^p_w}}{\|f\|_{L^p_w}} Rh$ (see Lemma 3.17), note that $g \in L^{p_0}_v$, that is, $(f,g) \in \mathcal{F}_{p_0,v}$.

Thus, combining Hölder's inequality (with respect to the weighted measure $w\,\mathrm{d}x$ - note also $\frac{p_0}{p} > 1$ and $(\frac{p_0}{p})' = \frac{p_0}{p-p_0}$,), our assumption (3.4), and Lemma 3.17, we easily get:

$$\begin{aligned}
\int_{\mathbb{R}^n} g^p w \,\mathrm{d}x &= \int_{\mathbb{R}^n} g^p w (Rh)^{\frac{p}{p_0}(p-p_0)} (Rh)^{\frac{p}{p_0}(p_0-p)} \,\mathrm{d}x \\
&\leq \left(\int_{\mathbb{R}^n} g^{p_0} w (Rh)^{p-p_0} dx\right)^{\frac{p}{p_0}} \left(\int_{\mathbb{R}^n} (Rh)^p w \,\mathrm{d}x\right)^{1-\frac{p}{p_0}} \\
&\leq N\Big([w]_{A_p}[Rh]_{A_1}^{p_0-p}\Big)^p \left(\int_{\mathbb{R}^n} f^{p_0} w (Rh)^{p-p_0} \,\mathrm{d}x\right)^{\frac{p}{p_0}} \left(\int_{\mathbb{R}^n} (Rh)^p w \,\mathrm{d}x\right)^{1-\frac{p}{p_0}} \\
&\leq N\Big([w]_{A_p}(2\|M\|_{L^p_w})^{p_0-p}\Big)^p \int_{\mathbb{R}^n} (Rh)^p w \,\mathrm{d}x \\
&\leq N\Big([w]_{A_p}(2\|M\|_{L^p_w})^{p_0-p}\Big)^p 2^p \int_{\mathbb{R}^n} h^p w \,\mathrm{d}x \\
&\leq 2^{2p} N\Big([w]_{A_p}(2\|M\|_{L^p_w})^{p_0-p}\Big)^p \int_{\mathbb{R}^n} f^p w \,\mathrm{d}x.
\end{aligned}$$

The case $p > p_0$**.** Fix $w \in A_p$ and $(f,g) \in \mathcal{F}_{p,w}$ with $f \in L^p_w$ and $g \neq 0$. By a duality argument we can write

$$\left(\int_{\mathbb{R}^n} g^p w dx\right)^{\frac{p_0}{p}} := \sup\left\{\int_{\mathbb{R}^n} g^{p_0} \varphi w dx : \varphi \in L^{(p/p_0)'}(w), \|\varphi\| = 1, \varphi \geq 0\right\}.$$

Fix such a function φ. Let h stand for the function given by

$$h^{p'} w^{1-p'} = \left(\varphi + \left(\frac{f}{\|f\|_{L^p_w}}\right)^{p-p_0} + \left(\frac{g}{\|g\|_{L^p_w}}\right)^{p-p_0}\right)^{\frac{p}{p-p_0}} w.$$

Note that $h \in L^{p'}_{w^{1-p'}}$, and $w^{1-p'} \in A_{p'}$. Therefore, by Rubio de Francia's iteration algorithm (Lemma 3.17), we obtain that the operator $R = R_{p',w^{1-p'}}$ is bounded on $L^{p'}_{w^{1-p'}}$, with norm less than or equal to 2, $h \leq Rh$, and $Rh \in A_1$ with $[Rh]_{A_1} \leq 2\|M\|_{L^{p'}_{w^{1-p'}})}$. Lemma 3.16 shows that $v := \Big(w^{p_0-1}(Rh)^{p-p_0}\Big)^{\frac{1}{p-1}} \in A_{p_0}$. Moreover, since

$$g^{p_0} \le \|g\|_{L^p_w}^{\frac{p_0}{p-p_0}} h^{\frac{p_0}{p-1}} w^{-\frac{p_0}{p-1}} \le \|g\|_{L^p_w}^{\frac{p_0}{p-p_0}} (Rh)^{\frac{p_0}{p-1}} w^{-\frac{p_0}{p-1}},$$

note that $(f, g) \in \mathcal{F}_{p_0,v}$.

Finally, since $\varphi w \le h^{\frac{p-p_0}{p-1}} w^{\frac{p_0-1}{p-1}} \le (Rh)^{\frac{p-p_0}{p-1}} w^{\frac{p_0-1}{p-1}} = v$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} g^{p_0} \varphi w \,\mathrm{d}x &\le \int_{\mathbb{R}^n} g^{p_0} v dx \le N([v]_{A_{p_0}})^{p_0} \int_{\mathbb{R}^n} f^{p_0} v \,\mathrm{d}x \\
&= N([v]_{A_{p_0}})^{p_0} \int_{\mathbb{R}^n} f^{p_0} (Rh)^{\frac{p-p_0}{p-1}} w^{\frac{p_0-1}{p-1}} w^{-1} w \,\mathrm{d}x \\
&\le N([v]_{A_{p_0}})^{p_0} \left(\int_{\mathbb{R}^n} f^p w \,\mathrm{d}x \right)^{\frac{p_0}{p}} \left(\int_{\mathbb{R}^n} (Rh)^{p'} w^{1-p'} \,\mathrm{d}x \right)^{1-\frac{p_0}{p}} \\
&\le 2^{\frac{p-p_0}{p-1}} N([v]_{A_{p_0}})^{p_0} \left(\int_{\mathbb{R}^n} f^p w \,\mathrm{d}x \right)^{\frac{p_0}{p}} \left(\int_{\mathbb{R}^n} h^{p'} w^{1-p'} \,\mathrm{d}x \right)^{1-\frac{p_0}{p}} \\
&\le 2^{\frac{p-p_0}{p-1}} 3N([v]_{A_{p_0}})^{p_0} \left(\int_{\mathbb{R}^n} f^p w \,\mathrm{d}x \right)^{\frac{p_0}{p}},
\end{aligned}$$

where we applied Hölder's inequality. Therefore, the proof is complete.

The example considered below well illustrates the underlying ideas of Rubio de Francia's extrapolation theory, which are not so transparent in the setting of pairs of functions presented above. This theory was summarized by A. Cordoba as follows:

There are no L^p spaces only L^2;

compare with (3.7) below.

Example 3.18. Let X be a Banach space. Let T be a sublinear operator, which is bounded on $L^2_w(X)$ for every Muckenhoupt weight $w \in A_2$. Assume that, for every $C > 0$,

$$\sup\{\|T\|_{L^2_w(X)} : w \in A_2, [w]_{A_2} \le C\} < \infty.$$

How can we apply Rubio de Francia's extrapolation principle to show that T extends *to a bounded operator on $L^p_w(X)$ for every $p \in (1, \infty)$ and every Muckenhoupt weight $w \in A_p$?*

Set

$$\mathcal{F} := \{(|f|_X, |Tf|_X) : f \in \bigcup_{w \in A_2} L^2_w(X)\},$$

and

$$N(t) := \sup\{\|T\|_{L^2_w(X)} : w \in A_2, [w]_{A_2} \le t\} \qquad (t > 0).$$

Then,

$$\mathcal{F}_{2,w} \supseteq \{(|f|_X, |Tf|_X) : f \in L^2_w(X)\}.$$

By Rubio de Francia's extrapolation theorem (Theorem 3.15), for every $p \in (1, \infty)$ and every Muckenhoupt weight $w \in A_p$ there exists a constant $C_{p,w}$ such that

$$\int_{\mathbb{R}^N} |Tf|_X^p w \, \mathrm{d}x \le C_{p,w} \int_{\mathbb{R}^N} |f|_X^p w \, \mathrm{d}x \quad \text{for every } f \in \bigcup_{v \in A_2} L^2_v(X) \text{ with } Tf \in L^p_w(X).$$

Therefore, if

$$\mathcal{D}_{p,w} := \{f \in \bigcup_{v \in A_2} L^2_v(X) : Tf \in L^p_w(X)\}$$

is dense in $L^p_w(X)$, then T admits a unique *extension* to a bounded operator on $L^p_w(X)$, and $\|T\|_{L^p_w(X)} \le C_{p,w}$.

In fact, we show that $\mathcal{D}_{p,w} \supseteq L^p_w(X)$. Note first that

$$\bigcup_{\substack{p \in (1,\infty) \\ v \in A_p}} L^p_v(X) = \bigcup_{v \in A_2} L^2_v(X). \tag{3.7}$$

Indeed, let $0 \neq f \in L^2_w(X)$. In the case $p < 2$, following the lines of the corresponding part of the proof of Theorem 3.15, it is easy to check that $f \in L^2_v(X)$ with $v := wu^{p-2} \in A_2$, where $u := Rh$ for $h := |f|_X$. For $p > 2$, $f \in L^2_v(X)$ with $v := (wu^{p-2})^{\frac{1}{p-1}} \in A_2$, where $u := Rh$ for h given by $h^{p'} w^{1-p'} = |f|_X^p w$.

In particular, Tf is well-defined for all $f \in L^p_w(X)$. Therefore, it is sufficient to show that $|Tf|_X \in L^p_w$ for all $f \in L^p_w(X)$.

For this purpose, consider the (formal) adjoint operator M' to the Hardy-Littlewood maximal operator M (as an operator on L^p_w), that is,

$$M'h := \frac{M(hw)}{w} \qquad (f \in L^{p'}_w).$$

Note that M' is bounded on $L^{p'}_w$. Indeed, we first note that $w^{1-p'} \in A_{p'}$ with $[w^{1-p'}]_{A_{p'}} \le [w]_{A_p}^{p'-1}$ - it follows immediately from the definition of the $A_{p'}$ class. Therefore, since $fw \in L^{p'}_{w^{1-p'}}$ if and only if $f \in L^{p'}_w$, and $\|f\|_{L^{p'}_{w^{1-p'}}} = \|f\|_{L^{p'}_w}$, we have

$$\left(\int_{\mathbb{R}^N} (M'f)^{p'} w \, \mathrm{d}x \right)^{\frac{1}{p'}} = \left(\int_{\mathbb{R}^N} (M(fw))^{p'} w^{1-p'} \, \mathrm{d}x \right)^{\frac{1}{p'}} \le \|M\|_{L^{p'}_{w^{1-p'}}} \|f\|_{L^{p'}_w}.$$

Consequently,

$$\|M'\|_{L^{p'}_w} \le \|M\|_{L^{p'}_{w^{1-p'}}}.$$

Consider also the (formal) adjoint operator R' to R (see Lemma 3.17) given by:

$$R'h := \sum_{k=0}^{\infty} \frac{M'(h)}{(2\|M'\|_{L^{p'}_w})^k} \qquad (f \in L^{p'}_w).$$

The counterpart of Lemma 3.17 holds for the operator R', namely:

a) For $h \in L^{p'}_w$ one has

$$|h| \le R'h \text{ pointwise almost everywhere, and}$$
$$\|R'h\|_{L^{p'}_w} \le 2\,\|h\|_{L^{p'}_w}.$$

b) If, in addition, $h \ge 0$, then

$$M(wR'h) \le 2\,\|M\|_{L^p_w}\, wR'h \text{ pointwise almost everywhere,}$$

that is, $wR'h \in A_1$ and

$$[wR'h]_{A_1} \le 2\,\|M\|_{L^{p'}_w}.$$

Let $0 \ne f \in L^p_w(X)$ and $h \in L^{p'}_w$ with $h \ge 0$. Set $v := R(|f|_X)^{-1} R'hw$. Note that, by Lemma 3.16 (a), $v \in A_2$ with $[v]_{A_2} \le [R(\|f\|_X)]_{A_1}[R'h]_{A_1}$. Then, by Hölder's inequality, we obtain

$$\begin{aligned}
&\|T\|_{L^2_v(X)} \|R(\|f\|_X)\|_{L^p_w} \|R'h\|_{L^{p'}_w} \\
&\qquad \ge \|T\|_{L^2(v;X)} \int_{\mathbb{R}^N} R(\|f\|_X)\, R'(h) w \,\mathrm{d}x \\
&\qquad \ge \|T\|_{L^2_v(X)} \left(\int_{\mathbb{R}^N} |f|_X^2 R(|f|_X)^{-1}\, R'(h)\, w \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} R(|f|_X)\, R'(h)\, w \,\mathrm{d}x \right)^{\frac{1}{2}} \\
&\qquad \ge \left(\int_{\mathbb{R}^N} |Tf|_X^2\, R(|f|_X)^{-1}\, R'(h)\, w \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} R(|f|_X)\, R'(h)\, w \,\mathrm{d}x \right)^{\frac{1}{2}} \\
&\qquad \ge \int_{\mathbb{R}^N} |Tf|_X\, R'(h)\, w \,\mathrm{d}x \\
&\qquad \ge \int_{\mathbb{R}^N} |Tf|_X\, h w \,\mathrm{d}x.
\end{aligned}$$

This implies that $|Tf|_X \in (L^{p'}_w)^* = L^p_w$. Therefore, our claim holds.

3.4 Calderon-Zygmund operators

Let X and Y be two Banach spaces. We say that a measurable kernel $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathcal{L}(X,Y)$ satisfies the **standard conditions** if there exist constants $C \geq 0$ and $\delta > 0$ such that

$$|K(x,y)|_{\mathcal{L}(X,Y)} \leq \frac{C}{|x-y|^N} \qquad \text{if } |x-y| > 0, \tag{S1}$$

$$|K(x,y) - K(x,y')|_{\mathcal{L}(X,Y)} \leq C\frac{|y-y'|^\delta}{|x-y|^{N+\delta}} \qquad \text{if } 0 < |y-y'| \leq \frac{1}{2}|x-y|, \tag{S2}$$

$$|K(x,y) - K(x',y)|_{\mathcal{L}(X,Y)} \leq C\frac{|x-x'|^\delta}{|x-y|^{N+\delta}} \qquad \text{if } 0 < |x-x'| \leq \frac{1}{2}|x-y|. \tag{S3}$$

Moreover, we call a bounded, linear operator $T : L^p(\mathbb{R}^N;X) \to L^p(\mathbb{R}^N;Y)$ a **(generalized) Calderon-Zygmund operator** ($1 \leq p \leq \infty$ fixed) if there exists a kernel satisfying the standard conditions such that

$$\begin{aligned} &Tf(x) = \int_{\mathbb{R}^N} K(x,y)f(y)\,dy \\ &\text{for every } f \in C_c(\mathbb{R}^N;X) \text{ and almost every } x \notin \operatorname{supp} f. \end{aligned} \tag{3.8}$$

Theorem 3.19 (Weak $(1,1)$ estimate for Calderon-Zygmund operators). *Every Calderon-Zygmund operator $T : L^p(\mathbb{R}^N;X) \to L^p(\mathbb{R}^N;Y)$ ($1 < p < \infty$ fixed) is weak $(1,1)$ in the sense that there exists a constant $C \geq 0$ such that for every $\lambda > 0$ and every $f \in L^p \cap L^1(\mathbb{R}^N;X)$ one has*

$$m_{Tf}(\lambda) = m(\{x \in \mathbb{R}^N : |Tf(x)|_Y > \lambda\}) \leq \frac{C}{\lambda}\int_{\mathbb{R}^N} |f(x)|_X\,dx.$$

Proof. Fix $\lambda > 0$ and $f \in L^p \cap L^1(\mathbb{R}^N;X)$. Applying the corollary of [Stein (1970), Theorem 4, Chapter I.3.4] to the function $|f(\cdot)|_X$, we obtain a decomposition $\mathbb{R}^n = F \cup \Omega$, $F \cap \Omega = \emptyset$, such that

$$|f(x)|_X \leq \lambda \quad \text{for almost every } x \in F,$$

$$\Omega = \bigcup_j Q_j \text{ for cubes } Q_j \text{ such that } m(Q_j \cap Q_k) = 0 \text{ for } j \neq k,$$

$$\text{and such that } Q_{j,2} \cap F = \emptyset,$$

$$m(\Omega) \leq \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(x)|_X\,dx, \text{ and}$$

$$\frac{1}{|Q_j|}\int_{Q_j} |f(x)|_X\,dx \leq C\lambda.$$

Here $Q_{j,2}$ denotes, similarly as before, the double cube which has the same center as Q_j but whose sides are twice as long as those of Q. We set

$$g(x) := \begin{cases} f(x) & \text{for } x \in F, \\ \frac{1}{|Q_j|} \int_{Q_j} f(x)\, dx & \text{for } x \in Q_j, \end{cases}$$

and we set $b(x) = f(x) - g(x)$. Then

$$\begin{aligned} & b(x) = 0 \quad \text{for } x \in F, \text{ and} \\ & \int_{Q_j} b(x)\, dx = 0 \quad \text{for each cube } Q_j. \end{aligned}$$

Moreover, $g \in L^1 \cap L^\infty(\mathbb{R}^N; X)$, $\|g\|_{L^1(X)} \leq \|f\|_{L^1(X)}$ and $\|g\|_{L^\infty(X)} \leq C\lambda$.

Since $Tf = Tg + Tb$, it follows that

$$\begin{aligned} m(\{x \in \mathbb{R}^n : |Tf(x)|_Y > \lambda\}) \leq \\ \leq m(\{x \in \mathbb{R}^n : |Tg(x)|_Y > \frac{\lambda}{2}\}) + m(\{x \in \mathbb{R}^n : |Tb(x)|_Y > \frac{\lambda}{2}\}), \end{aligned}$$

and it suffices to estimate both terms on the right hand side separately.

First, we estimate Tg. First of all, $g \in L^p(X)$ and

$$\|g\|_{L^p(X)}^p = \int_{\mathbb{R}^n} |g(x)|_X^p\, dx \leq C^{p-1} \lambda^{p-1} \|g\|_{L^1(X)} \leq C^{p-1} \lambda^{p-1} \|f\|_{L^1(X)}$$

By using in addition the assumption of boundedness of T,

$$\|Tg\|_{L^p(Y)}^p \leq C^p \, \|g\|_{L^p(X)}^p \leq C \, \lambda^{p-1} \, \|f\|_{L^1(X)},$$

and this implies

$$m(\{x \in \mathbb{R}^n : |Tg(x)|_Y > \frac{\lambda}{2}\}) \leq \frac{C}{\lambda} \, \|f\|_{L^1(X)}.$$

Second, we estimate Tb. Let $b_j = b\chi_{Q_j}$. Then $b = \sum_j b_j$ and it suffices to estimate Tb_j.

Fix $x \in F$ and fix j. Since $\int_{Q_j} b = 0$, we have

$$Tb_j(x) = \int_{Q_j} \Big(K(x,y) - K(x,x_j)\Big) b(y)\, dy,$$

where x_j is the center of the cube Q_j. In particular,

$$\begin{aligned}
\int_F |Tb_j(x)|_Y\,dx &\le \int_F \int_{Q_j} |K(x,y)-K(x,x_j)|_{\mathcal{L}(X,Y)}\,|b(y)|_X\,dy\,dx \\
&= \int_{Q_j} \int_F |K(x,y)-K(x,x_j)|_{\mathcal{L}(X,Y)}\,dx\,|b(y)|_X\,dy \\
&\le \int_{Q_j} \int_{Q_{j,2}^c} |K(x,y)-K(x,x_j)|_{\mathcal{L}(X,Y)}\,dx\,|b(y)|_X\,dy \\
&\le C_K \int_{Q_j} |b(y)|_X\,dy,
\end{aligned}$$

where we have used the fact that F is a subset of the complement of the double cube $Q_{j,2}$. Of course, we also used that K satisfies the second standard condition (S2). From the preceding estimate we obtain

$$\begin{aligned}
\int_F |Tb(x)|_Y\,dx &\le \sum_j \int_F |Tb_j(x)|_Y\,dx \\
&\le C_K \int_\Omega |b(y)|_X\,dy \\
&\le 2C_K \|f\|_{L^1(X)}.
\end{aligned}$$

This estimate implies

$$m(\{x \in F : |Tb(x)|_Y > \frac{\lambda}{2}\}) \le \frac{C}{\lambda}\|f\|_{L^1(X)}.$$

On the other hand,

$$m(\{x \in \Omega : |Tb(x)|_Y > \frac{\lambda}{2}\}) \le m(\Omega) \le \frac{C}{\lambda}\|f\|_{L^1(X)}.$$

The preceding two estimates together give the estimate for Tb.

Lemma 3.20 (Good-λ inequality). *Fix a weight $w \in A_\infty$. Then there exist $C \ge 0$ and $\delta > 0$ such that for every $\lambda > 0$, and every $\gamma > 0$ small enough the inequality*

$$w(\{x \in \mathbb{R}^n : \bar{T}^* f(x) > 2\lambda \text{ and } Mf(x) \le \gamma\lambda\}) \le C\gamma^\delta w(\{x \in \mathbb{R}^n : \bar{T}^* f(x) > \lambda\}) \tag{3.9}$$

holds.

Proof. First, we can assume that $w(\{x \in \mathbb{R}^n : \bar{T}^* f(x) > \lambda\}) \ne 0$, for otherwise the above inequality is clearly satisfied. Since the measure $w(x)\,dx$ is outer regular, there exists an open set U_λ such that

$$\begin{aligned}
&\{x \in \mathbb{R}^n : \bar{T}^* f(x) > \lambda\} \subseteq U_\lambda \quad \text{and} \\
&w(U_\lambda) \le 2w(\{x \in \mathbb{R}^n : \bar{T}^* f(x) > \lambda\}).
\end{aligned} \tag{3.10}$$

By Whitney's lemma (Lemma 3.30), there exists a sequence (Q_k) of mutually disjoint cubes with sides parallel to the coordinate axes such that $U_\lambda = \bigcup_k \bar{Q}_k$ and such that $Q_{k,4}$ (the cube which has the same center as the cube Q_k and satisfies $\operatorname{diam} Q_{k,4} = 4 \operatorname{diam} Q_k$) has nonempty intersection with U_λ^c. The cubes Q_k are of the form $B(\bar{x}_k, r_k)$ for some center $\bar{x}_k$ and some radius $r_k > 0$, and hence $Q_{k,4} = B(\bar{x}_k, 4r_k)$.

We prove that there exists a constant $C \geq 0$ which is independent of f such that for every $\gamma > 0$ small enough and every k,

$$m(\{x \in Q_k : \bar{T}^* f(x) > 2\lambda \text{ and } Mf(x) \leq \gamma\lambda\}) \leq C\gamma\, m(Q_k). \tag{3.11}$$

We may in fact assume that γ is small, since the above inequality is trivial for $\gamma \geq C^{-1}$.

Fix k. We may assume that there exists $\xi_k \in Q_k$ such that $Mf(\xi_k) \leq \gamma\lambda$, because otherwise the inequality (3.11) is obviously satisfied. Moreover, since $Q_{k,4} \cap U_\lambda^c$ is nonempty, there exists $x_k \in Q_{k,4}$ such that

$$\bar{T}^* f(x_k) \leq \lambda.$$

Now let $\tilde{Q}_k := B(x_k, 16r_k)$ be the cube centered at x_k and satisfying $\operatorname{diam} \tilde{Q}_k = 16 \operatorname{diam} Q_k$. Define $f_1 = f\chi_{\tilde{Q}_k}$ and $f_2 = f\chi_{\tilde{Q}_k^c}$, so that $f = f_1 + f_2$. By subadditivity of the maximal operator $\bar{T}^*$, we have

$$\begin{aligned} m(\{x \in Q_k : \bar{T}^* f(x) > 2\lambda \text{ and } Mf(x) \leq \gamma\lambda\}) &\leq \\ \leq m(\{x \in Q_k : \bar{T}^* f_1(x) > \frac{\lambda}{2} \text{ and } Mf(x) \leq \gamma\lambda\}) &+ \\ + m(\{x \in Q_k : \bar{T}^* f_2(x) > \frac{3\lambda}{2} \text{ and } Mf(x) \leq \gamma\lambda\}), & \end{aligned} \tag{3.12}$$

and it suffices to estimate the two terms on the right-hand side of this inequality.

Since $\xi_k \in Q_k \subseteq \tilde{Q}_k$, it follows that

$$\frac{1}{m(Q_k)} \int_{\mathbb{R}^n} |f_1(y)|_X \, dy = \frac{1}{m(Q_k)} \int_{\tilde{Q}_k} |f(y)|_X \, dy \leq 16^n Mf(\xi_k) \leq 16^n \gamma\lambda,$$

so that the weak (1,1) estimates from Theorem 3.19 and Corollary **??** yield

$$m(\{x \in \mathbb{R}^n : \bar{T}^* f_1 > \frac{\lambda}{2}\}) \leq \frac{2C}{\lambda} \int_{\mathbb{R}^n} |f_1(y)|_X \, dy \leq C\gamma\, m(Q_k) \tag{3.13}$$

for some constant $C \geq 0$ which is independent of f, γ, k and λ. Next, we shall estimate, for small $\gamma > 0$, the second term on the right-hand side of (3.12). Fix $x \in Q_k = B(\bar{x}_k, r_k)$ and $\varepsilon > 0$. Then

$$\begin{aligned}
|T_\varepsilon f_2(x)|_Y &= \Big|\int_{B(x,\varepsilon)^c} K(x,y) f_2(y)\,dy\Big|_Y \\
&\le \Big|\int_{B(x_k,\varepsilon)^c} K(x_k,y) f_2(y)\,dy\Big|_Y \\
&\quad + \int_{B(x_k,\varepsilon)^c} |K(x_k,y)-K(x,y)|_{\mathcal{L}(X,Y)}\, |f_2(y)|_X\,dy \\
&\quad + \int_{B(x,\varepsilon)\triangle B(x_k,\varepsilon)} |K(x,y)|_{\mathcal{L}(X,Y)}\, |f_2(y)|_X\,dy \\
&=: I_1 + I_2 + I_3,
\end{aligned}$$

where $B(x,\varepsilon) \triangle B(x_k,\varepsilon)$ denotes the symmetric difference of $B(x,\varepsilon)$ and $B(x_k,\varepsilon)$. We have

$$I_1 = \Big|\int_{(B(x_k,\varepsilon)\cup B(x_k,8r_k))^c} K(x_k,y) f(y)\,dy\Big|_Y \le \bar{T}^* f(x_k) \le \lambda$$

by the choice of x_k, since $B(x_k,\varepsilon)\cup B(x_k,8r_k) = B(x_k,\sup\{\varepsilon,8r_k\})$ is a cube centered at x_k, and by the definition of $\bar{T}^*$. Furthermore, by using $\xi_k \in Q_k \subseteq \tilde{Q}_k$ again, and by proceeding similarly as in the estimates (**??**), one obtains

$$I_2 \le C Mf(\xi_k) \le C\gamma\lambda.$$

Note that $I_3 = 0$ whenever $\varepsilon \le 16 r_k$. On the other hand, for $\varepsilon \ge 16 r_k$ one has $B(x,\varepsilon) \triangle B(x_k,\varepsilon) \subseteq B(x,2\varepsilon)\setminus B(x,\varepsilon/2)$, and therefore,

$$\begin{aligned}
I_3 &\le \int_{B(x,2\varepsilon)\setminus B(x,\varepsilon/2)} \frac{C_K}{|x-y|^n}\, |f(y)|_X\,dy \\
&\le \frac{C_K 2^n}{\varepsilon^n} \int_{B(x,2\varepsilon)} |f(y)|_X\,dy \\
&\le C Mf(\xi_k) \quad \le \quad C\gamma\lambda.
\end{aligned}$$

We note that the preceding estimates can also be made for $\varepsilon = 0$ if one interpretes $T_0 = T$. Taking all the above estimates together, and taking the supremum over $\varepsilon \ge 0$, we find that there exists a constant $C \ge 0$ which is independent of f, λ, k, γ such that for every $x \in Q_k$,

$$|\bar{T}^* f_2(x)|_Y \le C\gamma\lambda + \lambda.$$

Taking $\gamma > 0$ so small that $C\gamma < \frac{1}{2}$, it follows that

$$m(\{x \in Q_k : \bar{T}^* f_2(x) > \frac{3\lambda}{2} \text{ and } Mf(x) \le \gamma\lambda\}) = 0.$$

As a consequence, we have proved (3.11). Now, since the weight w belongs to A_∞, there exist constants $\delta > 0$ such that

$$\frac{w(E)}{w(Q)} \le C\Big(\frac{m(E)}{m(Q)}\Big)^{\delta} \text{ for every cube } Q \text{ and every measurable } E \subseteq Q.$$

Hence, the estimate (3.11) implies that, for every k,

$$w(\{x \in Q_k : \bar{T}^* f(x) > 2\lambda \text{ and } Mf(x) \le \gamma\lambda\}) \le C\gamma^{\delta}\, w(Q_k).$$

Summing up in k and recalling the inequality (3.10) yields the estimate (3.9).

Theorem 3.21 (Coifman-Fefferman). *Let $T : L^p(\mathbb{R}^N;X) \to L^p(\mathbb{R}^N;Y)$ be a Calderon-Zygmund operator. Then, for every Muckenhoupt weight $w \in A_p$ the operator T extends to a bounded, linear operator from $L^p_w(\mathbb{R}^N;X)$ into $L^p_w(\mathbb{R}^N;Y)$.*

Proof. By the good-λ inequality (3.9) from Lemma 3.20, for every $f \in L^1 \cap L^p(\mathbb{R}^N;X)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \bar{T}^* f(x)^p w(x)\, dx = \\
&= C \int_0^\infty \lambda^{p-1}\, w(\{\bar{T}^* f > 2\lambda\})\, d\lambda \\
&\le C \int_0^\infty \lambda^{p-1} w(\{Mf > \gamma\lambda\})\, d\lambda + C\gamma^{\delta} \int_0^\infty \lambda^{p-1} w(\{\bar{T}^* f > \lambda\})\, d\lambda \\
&= C \int_{\mathbb{R}^n} Mf(x)^p w(x)\, dx + C\gamma^{\delta} \int_{\mathbb{R}^n} \bar{T}^* f(x)^p w(x)\, dx.
\end{aligned}$$

Taking $\gamma > 0$ so small that $C\gamma^{\delta} \le \frac{1}{2}$, we obtain the claim.

From the preceding theorem and the Rubio de Francia extrapolation theorem (Theorem 3.15, we immediately obtain the following main result of this section.

Theorem 3.22. *Let $T : L^p(\mathbb{R}^N;X) \to L^p(\mathbb{R}^N;Y)$ be a Calderon-Zygmund operator. Then, for every $1 < q < \infty$ and every Muckenhoupt weight $w \in A_q$ the operator T extends to a bounded, linear operator from $L^q_w(\mathbb{R}^N;X)$ into $L^q_w(\mathbb{R}^N;Y)$.*

Lemma 3.23. *If T is a Calderon-Zygmund operator, then, for each $s > 1$,*

$$M^{\sharp}(Tf)(x) \le C_s\, M(|f|_X^s)(x)^{\frac{1}{s}} \qquad (f \in L^p(\mathbb{R}^N;X),\, x \in \mathbb{R}^N),$$

where $M^{\sharp}$ is the sharp maximal operator.

Proof. Fix $s > 1$. Let $f \in L^p(\mathbb{R}^N;X)$. Given a cube $Q \subseteq \mathbb{R}^N$ and given $x \in Q$, we decompose $f = f_1 + f_2$, where $f_1 = f\,1_{2Q}$ and $f_2 = f\,1_{(2Q)^c}$, and we let $a := Tf_2(x)$. Then

$$\begin{aligned}
&\fint_Q |Tf(y)-a|_Y \\
&\le \fint_Q |Tf_1|_Y + \fint_Q |Tf_2 - Tf_2(x)|_Y.
\end{aligned}$$

Since T is bounded on L^p, the first term on the right-hand side of this inequality can be estimated by

$$\begin{aligned}
\fint_Q |Tf_1|_Y &\le \left(\fint_Q |Tf_1|_Y^s\right)^{\frac{1}{s}} \\
&\le C\left(\fint_{2Q} |f|_X^s\right)^{\frac{1}{s}} \\
&\le 2^{\frac{N}{s}} CM(|f|_X^s)(x)^{\frac{1}{s}}.
\end{aligned}$$

For the second term on the right-hand side of the above inequality one has, if L denotes the length of one side of the cube Q,

$$\begin{aligned}
&\fint_Q |Tf_2 - Tf_2(x)|_Y \\
&\le \fint_Q \left| \int_{\mathbb{R}^N \setminus 2Q} (K(y,z)-K(x,z))\, f(z)\,\mathrm{d}z \right|_Y \mathrm{d}y \\
&\le C \fint_Q \int_{\mathbb{R}^N \setminus 2Q} \frac{|y-x|^\delta}{|x-z|^{N+\delta}} |f(z)|_X \,\mathrm{d}z\,\mathrm{d}y \\
&\le CL^\delta \fint_Q \sum_{k=-1}^{\infty} \int_{2^kL<|x-z|<2^{k+1}L} \frac{|f(z)|_X}{|x-z|^{N+\delta}}\,\mathrm{d}z\,\mathrm{d}y \\
&\le CL^\delta \sum_{k=-1}^{\infty} \frac{1}{(2^kL)^{N+\delta}} \int_{|x-z|<2^{k+1}L} |f(z)|_X\,\mathrm{d}z \\
&\le C \sum_{k=1}^{\infty} \frac{2^N}{2^{k\delta}} \fint_{|x-z|<2^{k+1}L} |f(z)|_X\,\mathrm{d}z \\
&\le \frac{2^N C}{1-2^\delta} Mf(x) \\
&\le \tilde{C} M(|f|_X^s)(x)^{\frac{1}{s}}
\end{aligned}$$

Lemma 3.24. *Fix $1 \le p < \infty$ and $w \in A_p$. If $f \in M(\mathbb{R}^N; X)$ is such that $M_d f \in L^p_w(\mathbb{R}^N)$, then*

$$\int_{\mathbb{R}^N} |M_d f|^p\, w \le C \int_{\mathbb{R}^N} |M^\sharp f|^p\, w.$$

Proof.

3.5 The Hilbert transform

Let X be a Banach space. For every $f \in \mathcal{S}(\mathbb{R};X)$ we define the **Hilbert transform** $Hf : \mathbb{R} \to X$ by

$$\begin{aligned} Hf(x) &:= \text{P.V.} \int_{\mathbb{R}} \frac{1}{y} f(x-y) \, \mathrm{d}y \\ &= \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \frac{1}{y} f(x-y) \, \mathrm{d}y \quad (x \in \mathbb{R}). \end{aligned}$$

The limit actually exists for every Schwartz test functions f and every $x \in \mathbb{R}$, since

$$\begin{aligned} \int_{|y| \geq \varepsilon} \frac{1}{y} f(x-y) \, \mathrm{d}y &= [\ln|y| f(x-y)]_{\varepsilon}^{\infty} + [\ln|y| f(x-y)]_{-\infty}^{-\varepsilon} + \int_{|y|\geq\varepsilon} \ln|y| f'(x-y) \, \mathrm{d}y \\ &= \ln\varepsilon \, (f(x+\varepsilon) - f(x-\varepsilon)) + \int_{|y|\geq\varepsilon} \ln|y| f'(x-y) \, \mathrm{d}y \\ &\to \int_{\mathbb{R}} \ln|y| f'(x-y) \, \mathrm{d}y \text{ as } \varepsilon \to 0. \end{aligned}$$

In particular, if we take $x = 0$ and $X = \mathbb{C}$, we see that the **principal value**

$$\text{P.V.} \int_{\mathbb{R}} \frac{1}{y} f(y) \, \mathrm{d}y$$

is well defined as tempered distribution.

Theorem 3.25. *If X is a Hilbert space, then the Hilbert transform extends to a bounded, linear operator on $L^2(\mathbb{R};X)$.*

We give two different proofs for this fact.

Proof (First proof of Theorem 3.25). Our first proof uses the Fourier transform. We prove that for every $f \in \mathcal{S}(\mathbb{R};H)$ one has

$$Hf = \mathcal{F}^{-1} M_{\text{sgn}} \mathcal{F} f,$$

where M_{sgn} is the multiplication operator associated with the sign function, that is, $M_{\text{sgn}} g := \text{sgn} \cdot g$ for appropriate functions g. If the above equality is true, then

Lemma 3.26 (Cotlar). *Let $(T_n)_{n\in\mathbb{Z}}$ be a sequence of bounded, linear operators on a Hilbert space H. Assume that*

$$|T_n T_m^*|_{\mathcal{L}(H)}, |T_n^* T_m|_{\mathcal{L}(H)} \leq a_{n-m},$$

where

$$\sum_{n\in\mathbb{Z}} a_n^{\frac{1}{2}} =: A < \infty.$$

Then:

a) *For every finite $I \subseteq \mathbb{Z}$*

$$|\sum_{n\in I} T_n|_{\mathcal{L}(H)} \le A.$$

b) *For every finite family $(I_\alpha)_{1\le\alpha\le m}$ of finite, mutually disjoint $I_\alpha \subseteq \mathbb{Z}$ one has*

$$|\sum_{\alpha=1}^{m} T_{I_\alpha}^* T_{I_\alpha}|_{\mathcal{L}(H)} \le A^2,$$

where

$$T_{I_\alpha} := \sum_{n\in I_\alpha} T_n.$$

c) *The series*

$$\sum_{n=-\infty}^{\infty} T_n$$

is strongly convergent.

Proof. (a) Recall that for every operator $T \in \mathcal{L}(H)$ and every $k \in \mathbb{N}$ one has

$$|T|^2_{\mathcal{L}(H)} = |TT^*|_{\mathcal{L}(H)} \text{ and } |T|^{2k}_{\mathcal{L}(H)} = |(T^*T)^k|_{\mathcal{L}(H)}.$$

In fact, using the Cauchy-Schwarz inequality in its full form (including the equality), we obtain

$$\begin{aligned}
|T^*T|_{\mathcal{L}(H)} &= \sup_{|x|_H\le 1} |T^*Tx|_H \\
&= \sup_{|x|_H,|y|_H\le 1} \langle T^*Tx, y\rangle_H \\
&= \sup_{|x|_H,|y|_H\le 1} \langle Tx, Ty\rangle_H \\
&= \sup_{|x|_H\le 1} |Tx|_H^2 \\
&= |T|^2_{\mathcal{L}(H)},
\end{aligned}$$

which is the above equality for $k = 1$. Iterating this equality, and using that $(T^*T)^* = T^*T$, we obtain first

$$|T|^4_{\mathcal{L}(H)} = |T^*T|^2_{\mathcal{L}(H)} = |(T^*T)^2|_{\mathcal{L}(H)}$$

and then the above equality for all powers $k = 2^m$ ($m \in \mathbb{N}$). The full claim follows by writing $k \in \mathbb{N}$ in its binary extension. Having proved the above equal-

ity, one sees that assertion (a) follows from (b), namely when one chooses $m = 1$.
(b) Take now a finite family $(I_\alpha)_{1\le\alpha\le m}$ of finite, mutually disjoint $I_\alpha \subseteq \mathbb{Z}$. Let

$$\Delta := \bigcup_{\alpha=1}^{m} I_\alpha \times I_\alpha \subseteq \mathbb{Z}\times\mathbb{Z},$$

the union being disjoint. Then

$$S := \sum_{\alpha=1}^{m} T_{I_\alpha}^* T_{I_\alpha} \qquad\qquad = \sum_{(i,j)\in\Delta} T_i^* T_j$$

is selfadjoint, nonnegative and, for every $k \in \mathbb{N}$ one has

$$|S^k|_{\mathcal{L}(H)} = |S|_{\mathcal{L}(H)}^k.$$

However,

$$S^k = \sum_{(i_1,j_1),\dots,(i_k,j_k)\in\Delta} T_{i_1}^* T_{j_1} \dots T_{i_k}^* T_{j_k}$$

We have the two estimates

$$|T_{i_1}^* T_{j_1} \dots T_{i_k}^* T_{j_k}|_{\mathcal{L}(H)} \le a(i_1 - j_1)\cdot\dots\cdot a(i_k - j_k)$$

and

$$|T_{i_1}^* T_{j_1} \dots T_{i_k}^* T_{j_k}|_{\mathcal{L}(H)} \le a(0)^{\frac12}\cdot a(j_1 - i_2)\cdot\dots\cdot a(j_{k-1} - i_k)\cdot a(0)^{\frac12}.$$

Hence

$$\begin{aligned}
|S|_{\mathcal{L}(H)}^k &\le a(0) \sum_{(i_1,j_1),\dots,(i_k,j_k)\in\Delta} a(i_1 - j_1)^{\frac12}\cdot a(j_1 - i_2)^{\frac12}\cdot\dots\cdot a(j_{k-1} - i_k)^{\frac12}\cdot a(i_k - j_k)^{\frac12} \\
&\le a(0)\,|I| \left(\sum_{j\in\mathbb{Z}} a(j)^{\frac12}\right)^{2k-1},
\end{aligned}$$

where $I = \bigcup_{\alpha=1}^{m}$. Taking the k-th root and letting k tend to infinity, we obtain the claim.
(c) Assume that there exists $x \in H$ such that the series $\sum_{n=-\infty}^{\infty} T_n x$ does not converge. Then there exists $\varepsilon > 0$ and a sequence $(I_\alpha)_{\alpha\in\mathbb{N}}$ of mutually disjoint, finite subsets $I_\alpha \subseteq \mathbb{Z}$ such that, using the notation from (b),

$$|T_{I_\alpha} u|_H \ge \varepsilon.$$

Then we can choose $m \in \mathbb{N}$ large enough so that

$$\sum_{\alpha=1}^{m} |T_{I\alpha}x|_H^2 > A^2 \|x\|^2,$$

which is a contradiction to

$$\sum_{\alpha=1}^{m} |T_{I\alpha}x|_H^2 = \sum_{\alpha=1}^{m} \langle T_{I_\alpha}^* T_{I_\alpha} x, x\rangle_H \\ \leq A^2 |x|_H^2,$$

which follows from (b).

Proof (Second proof of Theorem 3.25). For every $n \in \mathbb{Z}$ we define

$$\begin{aligned} &\Delta_n := \{x \in \mathbb{R} : 2^n \leq |x| \leq 2^{n+1}\}, \\ &k_n \in L^1(\mathbb{R}) \text{ by } k_n(x) := \frac{1}{x} 1_{\Delta_n}(x), \text{ and} \\ &T_n \in \mathcal{L}(L^2(\mathbb{R}; H)) \text{ by } T_n f := k_n * f. \end{aligned}$$

From the equality $T_n^* = -T_n$, and from Young's inequality 1.21 we obtain

$$|T_n T_m^*|_{\mathcal{L}(H)} = |T_n^* T_m|_{\mathcal{L}(H)} = |T_n T_m|_{\mathcal{L}(H)} \leq \|k_n * k_m\|_{L^1}.$$

Assume that $n \leq m$, and let $x \in \mathbb{R}$ be positive. Then we can estimate

$$\begin{aligned} |k_n * k_m(x)| &= \left| \int_{2^n \leq |y| \leq 2^{n+1}} \frac{1}{y} \frac{1}{x-y} 1_{\Delta_m}(x-y)\, \mathrm{d}y \right| \\ &\leq \begin{cases} 0 & \text{if } x \notin [2^m - 2^{n+1}, 2^{m+1} + 2^{n+1}], \\ 4 \cdot 2^{-m} & \text{if } x \in [2^m - 2^{n+1}, 2^m + 2^{n+1}], \\ 2 \cdot 2^n \cdot 2^{-2m} & \text{if } x \in [2^m + 2^{n+1}, 2^{m+1} - 2^{n+1}], \\ 4 \cdot 2^{-m} & \text{if } x \in [2^{m+1} - 2^{n+1}, 2^{m+1} + 2^{n+1}]. \end{cases} \end{aligned}$$

While the second and fourth estimate follow directly from crude estimates in the integral, the third estimate uses the equality $\int_{\mathbb{R}} k_n = 0$, which allows one to write

$$\begin{aligned} k_n * k_m(x) &\int_{2^n \leq |y| \leq 2^{n+1}} \frac{1}{y} \frac{1}{x-y}\, \mathrm{d}y \\ &= \int_{2^n \leq |y| \leq 2^{n+1}} \frac{1}{y} \left(\frac{1}{x-y} - \frac{1}{x}\right) \mathrm{d}y \\ &= \int_{2^n \leq |y| \leq 2^{n+1}} \frac{1}{x-y} \frac{1}{x}\, \mathrm{d}y. \end{aligned}$$

As a consequence,

$$\begin{aligned}\|k_n * k_m\|_{L^1} &\le 2\cdot(8\cdot 2^{n-m} + 2\cdot 2^{n-m})\\ &\le 36\cdot 2^{-|n-m|}.\end{aligned}$$

The sequence $(T_n)_{n\in\mathbb{Z}}$ thus satisfies the hypotheses of Cotlar's lemma, so that, for every $f\in L^2(\mathbb{R};H)$ the limit

$$\lim_{N\to\infty}\sum_{n=-N}^{N-1} T_n f = \lim_{N\to\infty}\left(\sum_{n=-N}^{N-1} k_n\right) * f$$

exists in $L^2(\mathbb{R};H)$. Since for every $f\in\mathcal{S}(\mathbb{R};H)$ and every $x\in\mathbb{R}$

$$\begin{aligned}\lim_{N\to\infty}\left(\sum_{n=-N}^{N-1} k_n\right) * f(x) &= \int_{2^{-n}\le|y|\le 2^n}\frac{1}{y}f(x-y)\,\mathrm{d}y\\ &= Hf(x),\end{aligned}$$

we have proved the boundedness of the Hilbert transform in $L^2(\mathbb{R};H)$.

Corollary 3.27. *If X is a Hilbert space, then for every $1<p<\infty$ and every Muckenhoupt weight $w\in A_p$ the Hilbert transform extends to a bounded, linear operator on $L^p_w(\mathbb{R};X)$.*

We say that a Banach space X has the **Hilbert transform property** if, for some $1<p<\infty$, the Hilbert transform extends to a bounded, linear operator on $L^p(\mathbb{R};X)$. By Theorem 3.25, Hilbert spaces have the Hilbert transform property. From Theorem 3.22, we have the following general result.

Corollary 3.28. *A Banach space X has the Hilbert transform property if and only if for every $1<p<\infty$ and every Muckenhoupt weight $w\in A_p$ the Hilbert transform extends to a bounded, linear operator on $L^p_w(\mathbb{R};X)$.*

Corollary 3.29. a) *Hilbert spaces have the Hilbert transform property.*

b) *If X has the Hilbert transform property and if $1<p<\infty$, then $L^p(\Omega;X)$ has the Hilbert transform property ($(\Omega,\mathcal{A},\mu)$ being any measure space).*

c) *The spaces $L^p(\Omega)$ have the Hilbert transform property if $1<p<\infty$.*

Proof. (a) We already remarked that Hilbert spaces have the Hilbert transform property by Theorem 3.25.
(b) This follows from the boundedness of the Hilbert transform on the scalar-valued $L^p(\mathbb{R})$ (Corollary 3.27) and Tonnelli's theorem. In fact, for every $f\in L^p(\mathbb{R};L^p(\Omega))$ (this space can be identified with $L^p(\mathbb{R}\times\Omega)$ and $L^p(\Omega;L^p(\mathbb{R}))$),

$$\begin{aligned}
\|Hf\|^p_{L^p(\mathbb{R};L^p(\Omega))} &= \int_{\mathbb{R}} \|Hf(x)\|^p_{L^p(\Omega)}\, udx \\
&= \int_{\mathbb{R}}\int_{\Omega} |Hf(x,\omega)|^p \,\mathrm{d}\omega\,\mathrm{d}x \\
&= \int_{\Omega}\int_{\mathbb{R}} |Hf(x,\omega)|^p \,\mathrm{d}x\,\mathrm{d}\omega \\
&\leq C \int_{\Omega}\int_{\mathbb{R}} |f(x,\omega)|^p \,\mathrm{d}x\,\mathrm{d}\omega \\
&= C \int_{\mathbb{R}}\int_{\Omega} |f(x,\omega)|^p \,\mathrm{d}\omega\,\mathrm{d}x \\
&= C\|f\|^p_{L^p(\mathbb{R};L^p(\Omega))},
\end{aligned}$$

where C is the operator norm of the Hilbert transform on $L^p(\mathbb{R})$.

3.6 Covering and decomposition

This section contains several covering and decomposition results which where useful in this chapter.

Lemma 3.30 (Whitney decomposition of open sets). *Let $U \subsetneq \mathbb{R}^N$ be an open, proper subset. Then there exists a sequence (Q_k) of mutually disjoint open (dyadic) cubes such that*

$$U = \bigcup_k \bar{Q}_k \text{ and}$$
$$\operatorname{diam} Q_k \leq \operatorname{dist}(Q_k, U^c) \leq 4 \operatorname{diam} Q_k.$$

Proof. Let
$$Q := (0,1)^N$$
be a reference cube and consider for each $k \in \mathbb{Z}$ the collection

$$\mathcal{D}_k := \{2^k(x+Q) : x \in \mathbb{Z}^N\}$$

of dyadic cubes of side length equal to 2^k, and

$$\mathcal{D} := \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$$

the collection of all dyadic cubes.

Let now, for every $k \in \mathbb{Z}$,

$$U_k := \{x \in U : 2^k < \operatorname{dist}(x, U^c) \leq 2^{k+1}\},$$

so that

$$U = \dot{\bigcup}_{k\in\mathbb{Z}} U_k.$$

We now put

$$\mathcal{Q}_k := \{Q \in \mathcal{D}_k : \bar{Q} \cap U_k \neq \emptyset\}$$

and

$$\mathcal{Q} := \bigcup_{k\in\mathbb{Z}} \mathcal{Q}_k.$$

One easily observes, by definition of the $\mathcal{Q}_k$, that for every $Q \in \mathcal{Q}$

$$\operatorname{diam} Q_k \leq \operatorname{dist}(Q_k, U^c) \leq 4\sqrt{N} \operatorname{diam} Q_k.$$

Moreover,

$$U = \bigcup_{Q\in\mathcal{Q}} \bar{Q},$$

but it may happen that the cubes in the collection $\mathcal{Q}$ are not mutually disjoint. In order to obtain a mutually disjoint union, we choose for every $Q \in \mathcal{Q}$ the unique cube $Q_{max} \in \mathcal{Q}$ which contains Q and which has maximal diameter (such a cube exists and is uniquely determined due to the fact that U is a proper subset of $\mathbb{R}^N$, that is, the complement is nonempty). So if we take the collection $\mathcal{Q}' := \{Q_{max} : Q \in \mathcal{Q}\}$, then this collection satisfies all required properties.

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