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Systèmes gradients

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CHAPTER 1

Introduction

A *gradient system* in finite dimension is an ordinary differential equation of the form

$$\dot{u} + \nabla\varphi(u) = 0, \tag{0.1}$$

where $\varphi \in C^1(\mathbb{R}^n;\mathbb{R})$ is a function and $\nabla\varphi$ is its euclidean gradient: $\nabla\varphi = (\partial_1\varphi, \dots, \partial_n\varphi)$.

Every solution u of the gradient system (0.1) has the important property that φ is decreasing along u, that is, the function $\varphi(u)$ is decreasing. This follows simply from derivating:

$$\frac{d}{dt}\varphi(u(t)) = \langle\nabla\varphi(u(t)), \dot{u}(t)\rangle = -\|\dot{u}(t)\|^2 \leq 0.$$

In this equation, $\langle\cdot,\cdot\rangle$ denotes the euclidean scalar product and $\|\cdot\|$ the corresponding euclidean norm. In fact, the equation (0.1) just says that the time derivative of u is opposite to the gradient $\nabla\varphi(u)$ which shows into the direction into which the function φ has largest directional derivative with respect to a unit vector in the euclidean norm. A solution of the gradient system (0.1) thus always tries to minimize the value $\varphi(u)$ as fast as possible in the given geometry (here, the euclidean geometry).

The gradient system (0.1) admits φ as *Lyapunov function* or *energy function*: by this we just mean the fact that $\varphi(u)$ is decreasing along every solution. Having a Lyapunov/energy function is very natural in examples of ordinary differential equations arising in physics when the quantity $\varphi(u)$ has an interpretation of a real energy. In the following, we will often call φ an energy function.

In this course we will study gradient systems in finite and infinite dimension, with an emphasis on the infinite dimensional case.

Examples of infinite dimensional gradient systems include the linear heat equation

$$u_t - \Delta u = f$$

and the semilinear heat equation

$$u_t - \Delta u + f(u) = 0$$

on an open subset of $\mathbb{R}^n$. These parabolic partial differential equations can in fact be rewritten as ordinary differential equations on infinite dimensional Hilbert spaces spaces. The resulting ordinary differential equations are gradient systems.

Other examples of gradient systems are given by certain geometric evolution equations like the mean curvature flow (curve shortening flow in one dimension), the surface diffusion flow, the Willmore flow, and the Ricci flow. The latter has recently played an important role in the (very probable) solution of the Poincaré conjecture, one of seven millenium problems. An introduction to at least one of these flows will be aim of this course.

In a first place, however, we will have to provide some basic material necessary for studying gradient systems in infinite dimensions. After these preliminaries, we will study existence and uniqueness of solutions of linear gradient systems. The next step will bring us to prove existence and uniqueness of solutions of certain nonlinear evolution equations. Eventually, we will also study their regularity properties. While we will always test our abstract results in concrete examples, we will only at the end of this course be able to turn to geometric evolution equations.

CHAPTER 2

Linear gradient systems

Throughout we denote by X and Y (real) Banach spaces and by H, K, V (real) Hilbert spaces. The norm on a Banach space X is usually denoted by $\|\cdot\|_X$ or $\|\cdot\|$, and the inner product on a Hilbert space H is usually denoted by $(\cdot,\cdot)_H$ or $(\cdot,\cdot)$.

Recall that a linear operator $T : X \to Y$ is continuous if and only if it is bounded, i.e. if and only if $\|T\|_{\mathcal{L}(X,Y)} := \sup_{\|x\|_X \le 1} \|Tx\|_Y$ is finite. Instead of speaking of continuous linear operators we will in the following speak of bounded linear operators. The space of all bounded linear operators from X into Y is denoted by $\mathcal{L}(X,Y)$. It is a Banach space for the norm $\|\cdot\|_{\mathcal{L}(X,Y)}$.

1. Definition of gradient systems

Let V be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_V$ and let $\varphi : V \to \mathbb{R}$ be a function of class C^1. At every point $u \in V$ the derivative $\varphi'(u)$ is by definition an element of V' and therefore φ' is a function $V \to V'$.

Let H be a second real Hilbert space with inner product $\rangle\cdot,\cdot\rangle_H$ and suppose that V is a dense subspace of H and that the embedding of V into H is bounded, i.e. there exists a constant $C \ge 0$ such that $\|u\|_H \le C\|u\|_V$ for every $u \in V$. We will write $V \hookrightarrow H$ for this situation.

In the following, we denote by V' the dual space of V, i.e. $V' = \mathcal{L}(V,\mathbb{R})$. The duality between V' and V is denoted by the bracket $\langle\cdot,\cdot\rangle_{V',V}$.

As soon as V is densely and continuously embedded into H, the dual H' is densely and continuously embedded into the dual V'. In fact, the restriction to V of a bounded linear functional $u' \in H'$ defines a bounded linear functional in V'. The resulting operator $H' \to V'$ is clearly linear and bounded, and it is injective by the fact that V is dense in H. Using reflexivity of Hilbert spaces, one can even show that the embedding of H' into V' is dense. Hence, if $V \hookrightarrow H$, then $H' \hookrightarrow V'$.

We recall the theorem of Riesz-Fréchet which says that for every bounded linear functional $u' \in H'$ there exists a unique element $u \in H$ such that

$$\langle u', v\rangle_{H',H} = \langle u, v\rangle_H \text{ for every } v \in H.$$

On the other hand, it is clear from the bilinearity of the inner product that for every $u \in H$ the functional $u' : v \mapsto \langle u, v\rangle_H$ is linear and bounded, i.e. it belongs to H'. So the theorem of Riesz-Fréchet allows us to identify the spaces H and H' via the linear isomorphism $u \mapsto u'$. This isomorphism is even isometric as one easily verifies. It will be a *convention* in the following that we will always identify H and H' via this isomorphism. We write $H = H'$, but we have in mind that this equality does not hold in the set theoretic sense and that the isomorphism behind this equality depends on the choice of the inner product in H.

By our assumption that $V \hookrightarrow H$ and by our convention that $H = H'$, we thus obtain the following picture

$$V \hookrightarrow H = H' \hookrightarrow V', \tag{1.1}$$

and in particular V is densely and continuously embedded into V' (but we can not identify V and V' once we have identified H and H'! The space V is only a subspace of V'). The above chain implies that for every $u \in V$ and every $v \in H$

$$\langle u, v\rangle_{V,V'} = \langle u, v\rangle_H = \langle u, v\rangle_{H,H'}. \tag{1.2}$$

By a *gradient system* we will understand an evolution equation of the form

$$\dot{u} + \varphi'(u) = 0. \tag{1.3}$$

Classical solutions of this gradient system will be continuously differentiable functions $u : [0, T] \to V$ for which the equality (1.3) holds in the space V': recall that $\varphi'(u)$ is an element of V', that $\dot{u}$ is an element of V and that V is a subspace of V' by our convention. We emphasize the fact that this evolution equation depends on the choice of the Hilbert space H and in particular on the choice of the inner product in H. Sometimes, it will therefore be convenient to write $\nabla_H\varphi(u)$ instead of the derivative $\varphi'(u)$. If the Hilbert space H is clear from the context, it suffices to write $\nabla\varphi(u)$.

Example 1.1. Let $V = \mathbb{R}^n$ and $H = \mathbb{R}^n$ equipped with the euclidean inner product. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be of class C^1. By Riesz-Fréchet, for every $u \in \mathbb{R}^n$ there exists $\nabla_{\mathbb{R}^n}\varphi(u)$ such that

$$\langle \varphi'(u), v\rangle_{(\mathbb{R}^n)',\mathbb{R}^n} = \langle \nabla_{\mathbb{R}^n}\varphi(u), v\rangle_{\mathbb{R}^n} \text{ for every } v \in \mathbb{R}^n.$$

It is easy to verify that $\nabla_{\mathbb{R}^n}\varphi = \nabla\varphi$ is the euclidean gradient of φ, i.e. $\nabla_{\mathbb{R}^n}\varphi(u) = (\partial_1\varphi(u), \dots, \partial_n\varphi(u))$. The resulting gradient system is the system (0.1) from the Introduction.

Example 1.2. We let again $V = \mathbb{R}^n$ and $H = \mathbb{R}^n$ but we equip H with the inner product

$$\langle u, v\rangle_H := \langle Qu, v\rangle_{\mathbb{R}^n},$$

where Q is a symmetric and positive definite matrix. By Riesz-Fréchet, for every $u \in \mathbb{R}^n$ there exists $\nabla_H\varphi(u)$ such that

$$\langle \varphi'(u), v\rangle_{(\mathbb{R}^n)',\mathbb{R}^n} = \langle \nabla_H\varphi(u), v\rangle_H \text{ for every } v \in \mathbb{R}^n.$$

On the other hand, by the definition of the scalar product in H and by the previous example

$$\langle \nabla_H \varphi(u), v\rangle_H = \langle Q\nabla_H\varphi(u), v\rangle_{\mathbb{R}^n} = \langle \nabla_{\mathbb{R}^n}\varphi(u), v\rangle_{\mathbb{R}^n}.$$

Since this equality holds for every $v \in \mathbb{R}^n$, we obtain $\nabla_H\varphi(u) = Q^{-1}\nabla_{\mathbb{R}^n}\varphi(u)$. The resulting gradient system is

$$\dot{u} + Q^{-1}\nabla\varphi(u) = 0,$$

where $\nabla\varphi$ denotes the euclidean gradient.

2. Operators associated with bilinear forms

In this section, V will be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_V$.

Definition 2.1 (Bilinear form). A function $a : V \times V \to \mathbb{R}$ is called a *bilinear form* if it is linear in each variable, i.e.

$$\begin{aligned} a(\alpha u + \beta v, w) &= \alpha a(u, w) + \beta a(v, w) \text{ and} \\ a(u, \alpha v + \beta w) &= \alpha a(u, v) + \beta a(u, w) \end{aligned}$$

for every $u, v, w \in V$ and every $\alpha, \beta \in \mathbb{K}$.

There are some simple but important examples of bilinear forms.

Example 2.2. Every inner product on V is a bilinear form!

Example 2.3. Let $V = H_0^1(\Omega)$ ($\Omega \subset \mathbb{R}^n$ open) be the Sobolev space which is obtained by taking the closure of $\mathcal{D}(\Omega)$ (the test functions on Ω) in $H^1(\Omega)$. The space V is equipped with the inner product

$$(u, v)_{H_0^1} := \int_\Omega uv + \int_\Omega \nabla u\nabla v,$$

and the corresponding norm

$$\|u\|_{H_0^1} = \left(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right)^{\frac{1}{2}}.$$

On this Sobolev space the equality

$$a(u, v) := \int_\Omega \nabla u\,\nabla v, \quad u,\ v \in V,$$

defines a bilinear form.

Example 2.4. More generally, if $A \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ is a bounded, measurable, matrix valued function, then the equality

$$a(u, v) := \int_\Omega A(x)\nabla u\,\overline{\nabla v}, \quad u,\ v \in V,$$

defines a bilinear form on the Sobolev space $V = H_0^1(\Omega)$.

Definition 2.5 (Boundedness, coercivity, symmetry). Let a be a bilinear form on a Hilbert space V.

(a) We say that a is *bounded* if there exists a constant $C \geq 0$ such that

$$|a(u,v)| \leq C\,\|u\|_V\,\|v\|_V \text{ for every } u,\ v \in V.$$

(b) We say that a is *coercive* if there exists a constant $\eta > 0$ such that

$$\operatorname{Re} a(u,u) \geq \eta\,\|u\|_V^2 \text{ for every } u \in V.$$

(c) We say that a is *symmetric* if

$$a(u,v) = a(v,u) \text{ for every } u,\ v \in V.$$

Definition 2.6 (Operator associated with a bounded bilinear form). Given a bounded, bilinear form a on V, we define the linear operator $A : V \to V'$ associated with this form by

$$\langle Au, \varphi\rangle_{V',V} := a(u,\varphi), \quad u,\ \varphi \in V.$$

It follows from the boundedness of a that the operator A is well-defined and bounded. In fact, let $C \geq 0$ be the constant from Definition 2.5 (a). Then

$$\begin{aligned} \|Au\|_{V'} &= \sup_{\|\varphi\|_V \leq 1} |\langle Au, \varphi\rangle_{V',V}| \\ &= \sup_{\|\varphi\|_V \leq 1} |a(u,\varphi)| \\ &\leq \sup_{\|\varphi\|_V \leq 1} C\,\|u\|_V\,\|\varphi\|_V = C\,\|u\|_V. \end{aligned}$$

The following theorem says something about the solvability of the equation $Au = f$ for given $f \in V'$. As one can see from the statement, coercivity of a implies invertibility of A.

Theorem 2.7 (Lax-Milgram). *Let a be a bounded, coercive, bilinear form on V. Then for every $f \in V'$ there exists a unique $u \in V$ such that*

$$a(u,\varphi) = \langle f, \varphi\rangle_{V',V} \textit{ for every } \varphi \in V.$$

Proof. We have to prove that the bounded linear operator $A \in \mathcal{L}(V,V')$ associated with a is bijective. By coercivity, for every $u \in V \setminus \{0\}$,

$$\begin{aligned} \|Au\|_{V'} &= \sup_{\|v\|_V \leq 1} |\langle Au, v\rangle_{V',V}| \\ &\geq |\langle Au, \frac{u}{\|u\|_V}\rangle_{V',V}| \\ &= \frac{1}{\|u\|_V}\,a(u,u) \\ &\geq \eta\,\|u\|_V. \end{aligned}$$

This proves on the one hand injectivity of A, but also that $\operatorname{Rg} A$ is closed in V'.

If $\operatorname{Rg} A \neq V'$, then there exists $v \in V \setminus \{0\}$ such that $\langle Au, v\rangle_{V',V} = 0$ for every $u \in V$. If we take $u = v$, then we obtain

$$0 = \langle Av, v\rangle_{V',V} = a(v,v) \geq \eta\,\|v\|_V^2 > 0,$$

a contradiction. Hence, $\operatorname{Rg} A = V'$, i.e. A is surjective. □

Definition 2.8. Let V and H be two Hilbert spaces such that (1.1) holds. We call a form $a : V \times V \to \mathbb{R}$ *H-elliptic* if there exists $\omega \in \mathbb{R}$ such that the form $a_\omega : V \times V \to \mathbb{R}$ defined by $a_\omega(u, v) := a(u, v) + \omega(u, v)_H$ is coercive, i.e. if there exists $\eta > 0$ such that

$$a(u, u) + \omega \|u\|_H^2 \geq \eta \|u\|_V^2 \text{ for every } u \in V.$$

Definition 2.9. We call a matrix $A \in \mathbb{R}^{n\times n}$ *elliptic* if there exists a constant $\eta > 0$ such that

$$A\xi\,\overline{\xi} \geq \eta\,|\xi|^2 \text{ for every } \xi \in \mathbb{C}^n.$$

We call a matrix-valued function $A \in L^\infty(\Omega, \mathbb{R}^{n\times n})$ *uniformly elliptic* if there exists a constant $\eta > 0$ such that

$$\operatorname{Re} A(x)\xi\,\overline{\xi} \geq \eta\,|\xi|^2 \text{ for every } \xi \in \mathbb{C}^n,\ x \in \Omega.$$

In the above definition, if the matrix A is symmetric then ellipticity of A is equivalent to saying that A is positive definite.

Example 2.10. Take the bilinear form a from Example 2.4 and assume that $A \in L^\infty(\Omega, \mathbb{R}^{n\times n})$ is uniformly elliptic. Then a is bounded and elliptic. Indeed,

$$\begin{aligned} a(u, u) + \eta \|u\|_{L^2}^2 &= \int_\Omega A(x)\nabla u\, \nabla u + \eta \|u\|_{L^2}^2 \\ &\geq \eta \int_\Omega |\nabla u|^2 + \eta \|u\|_{L^2}^2 \\ &= \eta \|u\|_{H_0^1}^2. \end{aligned}$$

We define a second operator associated with a form a.

Definition 2.11 (Operator associated with a bilinear form). Let a be a bounded, bilinear form on V, and let H be a second Hilbert space such that (1.1) holds. We define the operator $A_H : H \supset D(A_H) \to H$ associated with a by

$$\begin{aligned} D(A_H) &:= \{u \in V : \exists v \in H\, \forall \varphi \in V\ :\ a(u, \varphi) = (v, \varphi)_H\}, \\ A_H u &= v. \end{aligned}$$

The operator A_H is well-defined in the sense that the element $v \in H$ is uniquely determined if it exists. Indeed, assume that there are two elements v_1, $v_2 \in H$ such that

$$(v_1, \varphi)_H = a(u, \varphi) = (v_2, \varphi)_H \text{ for every } \varphi \in V.$$

Then $(v_1 - v_2, \varphi)_H = 0$ for every $\varphi \in V$, and since V is dense in H (here the density of the embedding is used!), this already implies $v_1 = v_2$.

Lemma 2.12. *The operator A_H is the restriction of A to the space H, i.e.*

$$D(A_H) = \{u \in V : Au \in H\} \text{ and } A_H u = Au \text{ for } u \in D(A_H).$$

Proof. Exercise. ◻

EXAMPLE 2.13 (Dirichlet-Laplace operator). Let $V = H_0^1(0,1)$, $H = L^2(0,1)$ and consider the form $a : H_0^1 \times H_0^1 \to \mathbb{R}$ defined by

$$a(u,v) = \int_0^1 u'v'.$$

The dual space of H_0^1 is denoted by H^{-1}. Let $A : H_0^1 \to H^{-1}$ be the operator associated with the form a and let A_{L^2} be its restriction to L^2. We show that

$$\begin{aligned} D(A_{L^2}) &= H^2(0,1) \cap H_0^1(0,1) \text{ and} \\ A_{L^2}u &= -u''. \end{aligned}$$

This operator is called the Dirichlet-Laplace operator on the interval $(0,1)$.

Let $u \in D(A_{L^2})$ and let $f = Au \in L^2$. Then, for every $\varphi \in H_0^1$ one has

$$\begin{aligned} \langle f, \varphi \rangle_H &= \langle Au, \varphi \rangle_H \\ &= \langle Au, \varphi \rangle_{H^{-1}, H_0^1} \\ &= a(u, \varphi), \end{aligned}$$

or

$$\int_0^1 f\varphi = \int_0^1 u'\varphi' \text{ for every } \varphi \in H_0^1(0,1) \tag{2.1}$$

By the definition of H^1, this means that $u' \in H^1(0,1)$ and $u'' := (u')' = -f$. In other words, $u \in H^2(0,1)$ and $Au = -u''$.

On the other hand, let $u \in H^2 \cap H_0^1$ and let $f = -u'' \in L^2$. One easily shows that (2.1) holds, so that $\langle f, \varphi \rangle_H = a(u, \varphi)$ for every $\varphi \in H_0^1$. By definition, this implies $u \in D(A_{L^2})$ and $Au = -u''$.

3. The theorem of J.-L. Lions

Throughout this section, V and H are two real separable Hilbert spaces such that

$$V \hookrightarrow H = H' \hookrightarrow V',$$

with dense injections. Moreover, we let $a : V \times V \to \mathbb{R}$ be a bounded bilinear form, and we let $A : V \to V'$ be the operator associated with A. Then we consider the evolution equation

$$\dot{u}(t) + Au(t) = f(t),\ t \in [0,T], \quad u(0) = u_0. \tag{3.1}$$

This evolution equation is a gradient system if the form a is in addition symmetric. The underlying energy is then $\varphi : V \to \mathbb{R}$, $\varphi(u) = \frac{1}{2}a(u,u)$. In fact, the derivative of this quadratic form can be calculated very easily using the definition of the Fréchet derivative and the bilinearity and the symmetry of the form a:

$$\begin{aligned}\varphi(u+h) &= \frac{1}{2}(a(u,u)+a(u,h)+a(h,u)+a(h,h)) \\ &= \varphi(u)+a(u,h)+\frac{1}{2}a(h,h) \\ &= \varphi(u)+\langle\varphi'(u),h\rangle_{V',V}+o(h),\end{aligned}$$

i.e. $\varphi'(u)=Au$.

Theorem 3.1 (J.-L. Lions). *Let $a : V\times V\to\mathbb{R}$ be a bilinear, bounded, elliptic form and let $A : V\to V'$ be the associated operator. Let $T>0$. Then for every $f\in L^2(0,T;V')$ and every $u_0\in H$ there exists a unique solution $u\in W^{1,2}(0,T;V')\cap L^2(0,T;V)$ of the problem* (3.1).

We will prove this theorem in several steps. First, we study the maximal regularity space $MR_2(a,b;V',V)$.

Lemma 3.2. *For every $u\in W^{1,2}(\mathbb{R};V')\cap L^2(\mathbb{R};V)$ the function $t\mapsto\frac{1}{2}\|u(t)\|_H^2$ is differentiable almost everywhere and*

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2=\langle\dot u(t),u(t)\rangle_{V',V}. \tag{3.2}$$

Proof. One shows by regularisation that the space $C_c^1(\mathbb{R};V)$ is dense in $W^{1,2}(\mathbb{R};V')\cap L^2(\mathbb{R};V)$. Then one verifies that for functions $u\in C_c^1(\mathbb{R};V)$ the equality (3.2) is true, using also the equality

$$\frac{d}{dt}\frac{1}{2}\|u(t)\|_H^2=(\dot u(t),u(t))_H=\langle\dot u(t),u(t)\rangle_{V',V}.$$

The claim then follows by an approximation argument. □

Lemma 3.3. *One has*

$$W^{1,2}(\mathbb{R};V')\cap L^2(\mathbb{R};V)\hookrightarrow C_0(\mathbb{R};H). \tag{3.3}$$

Proof. We use again the fact that the space $C_c^1(\mathbb{R};V)$ is dense in $W^{1,2}(\mathbb{R};V')\cap L^2(\mathbb{R};V)$. For every $u\in C_c^1(\mathbb{R};V)$ and every $t\in\mathbb{R}$ one has

$$\begin{aligned}\|u(t)\|_H^2 &= \int_{-\infty}^t\frac{d}{ds}\|u(s)\|_H^2\,ds \\ &= 2\int_{-\infty}^t(\dot u(s),u(s))_H\,ds \\ &= 2\int_{-\infty}^t\langle\dot u(s),u(s)\rangle_{V',V}\,ds \\ &\le 2\,\|\dot u\|_{L^2(\mathbb{R};V')}\,\|u\|_{L^2(\mathbb{R};V)} \\ &\le \|\dot u\|_{L^2(\mathbb{R};V')}^2+\|u\|_{L^2(\mathbb{R};V)}^2 \\ &\le 2\,\|u\|_{MR_2(\mathbb{R};V',V)}^2.\end{aligned}$$

Hence, the embedding operator

$$(C_c^1(\mathbb{R};V), \|\cdot\|_{MR_2(\mathbb{R};V',V)}) \to (C_0(\mathbb{R};H), \|\cdot\|_{C_0(\mathbb{R};H)})$$

is bounded. Since $C_c^1(\mathbb{R};V)$ is dense in $W^{1,2}(\mathbb{R};V') \cap L^2(\mathbb{R};V)$, the embedding (3.3) follows. □

LEMMA 3.4. *For every $-\infty < a < b < \infty$ one has*

$$W^{1,2}(a,b;V') \cap L^2(a,b;V) \hookrightarrow C([a,b];H). \tag{3.4}$$

PROOF. Let a, b be arbitrary, but finite. There exists a linear bounded extension operator

$$E : W^{1,2}(a,b;V') \cap L^2(a,b;V) \to W^{1,2}(\mathbb{R};V') \cap L^2(\mathbb{R};V)$$

with the property that Eu restricted to the interval (a,b) equals u (exercice!). Using that the restriction operator

$$C_0(\mathbb{R};H) \to C([a,b];H), \quad u \mapsto u|_{[a,b]}$$

is linear and bounded, too, the claim follows by considering the composition of the extension operator E, the embedding (3.3), and this restriction operator. □

LEMMA 3.5 (Uniqueness). *Let A be as in Theorem 3.1. Then for every $f \in L^2(0,T;V')$ and every $u_0 \in H$ there exists at most one solution $u \in W^{1,2}(0,T;V') \cap L^2(0,T;V)$ of the problem* (3.1).

PROOF. By linearity, it suffices to prove that if $u \in W^{1,2}(0,T;V') \cap L^2(0,T;V)$ is a solution of

$$\dot{u}(t) + Au(t) = 0, \; t \in [0,T], \quad u(0) = 0,$$

then $u = 0$. So let u be a solution of this problem. Then, by ellipticity of the form a, and by Lemma 3.2,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u(t)\|_H^2 &= \langle \dot{u}(t), u(t)\rangle_{V',V} \\ &= -\langle Au(t), u(t)\rangle_{V',V} \\ &\leq \omega\,\|u(t)\|_H^2. \end{aligned}$$

As a consequence,

$$\|u(t)\|_H^2 \leq e^{2\omega t}\|u(0)\|_H^2 = 0 \text{ for every } t \in [0,T].$$

Hence, $u = 0$. □

PROOF OF THEOREM 3.1. By Lemma 3.5 it remains only to prove existence of a solution. The proof of existence will be done by a Galerkin approximation.

Let $(w_n) \subset V$ be a linearly independent sequence such that $\operatorname{span}\{w_n\}$ is dense in V (here we use that V is separable in order to ensure existence of such a sequence). Let $V_m := \operatorname{span}\{w_n : 1 \leq n \leq m\}$. As a finite dimensional vector space, the space V_m is a closed subspace of V, H and V'. It will be equipped with the norms coming from these three spaces. Note that the three norms are equivalent on V_m.

The restriction of the form a to the space V_m (i.e. the form $a_m : V_m \times V_m \to \mathbb{R}$ defined by $a_m(u,v) := a(u,v)$) is a bilinear, bounded and elliptic form. Hence, there exists an operator $A_m : (V_m, \|\cdot\|_V) \to (V_m, \|\cdot\|_{V'})$ such that

$$\langle A_m u, v\rangle_{V',V} = a_m(u,v) = a(u,v) \text{ for every } u,\ v \in V_m.$$

Consider the ordinary differential equation

$$\dot{u}_m(t) + A_m u_m(t) = f_m(t),\ t \in [0,T], \quad u_m(0) = u_0^m, \tag{3.5}$$

where $u_0^m := P_m u_0$, $P_m : H \to H$ being the orthogonal projection in H onto V_m, and where $f_m(t) = P_m f(t)$ (note that the orthogonal projection P_m extends to a bounded projection $V' \to V'$ and that $\|P_m\|_{V'} = 1$).

The problem (3.5) is a linear inhomogeneous ordinary differential equation in a finite dimensional Hilbert/Banach space and we know from the theory of ordinary differential equations that (3.5) admits a unique solution $u_m \in C^1([0,T]; V_m)$.

Multiplying the equation (3.5) with u_m, we obtain

$$(\dot{u}_m(t), u_m(t))_H + (A_m u_m(t), u_m(t))_H = (f_m(t), u_m(t))_H$$

and hence, by ellipticity of a,

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|u_m(t)\|_H^2 + \eta\,\|u_m(t)\|_V^2 \\
\leq\ & \langle \dot{u}_m(t), u_m(t)\rangle_{V',V} + \langle A_m u_m(t), u_m(t)\rangle_{V',V} + \omega\,\|u_m(t)\|_H^2 \\
=\ & \langle f_m(t), u_m(t)\rangle_{V',V} + \omega\,\|u_m(t)\|_H^2 \\
\leq\ & C_\eta \|f_m(t)\|_{V'}^2 + \frac{\eta}{2}\|u_m(t)\|_V^2 + \omega\,\|u_m(t)\|_H^2.
\end{aligned}$$

As a first consequence, we obtain the inequality

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_H^2 \leq C_\eta \|f_m(t)\|_{V'}^2 + \omega\,\|u_m(t)\|_H^2.$$

By Gronwall's lemma, this implies for every $t \in [0,T]$,

$$\begin{aligned}
\|u_m(t)\|_H^2 &\leq e^{2\omega t}\|u_0^m\|_H^2 + C_\eta \int_0^t e^{2\omega(t-s)}\|f_m(s)\|_{V'}^2\,ds \\
&\leq C(\|u_0\|_H^2 + \int_0^T \|f(s)\|_{V'}^2\,ds),
\end{aligned}$$

where $C \geq e^{2\omega T}(C_\eta + 1)$. When we plug this inequality into the above inequality, then we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_m(t)\|_H^2 + \frac{\eta}{2}\,\|u_m(t)\|_V^2 \leq C_\eta\,\|f_m(t)\|_{V'}^2 + C(\|u_0\|_H^2 + \int_0^T \|f(s)\|_{V'}^2\,ds).$$

This implies, when integrating over $(0,T)$,

$$\frac{\eta}{2}\int_0^T \|u_m(t)\|_V^2\,dt + \frac{1}{2}\|u_m(T)\|_H^2 \leq C(\|u_0\|_H^2 + \int_0^T \|f(s)\|_{V'}^2\,ds).$$

This and the equation (3.5) imply

$$\begin{aligned}\int_0^T \|\dot u_m(t)\|^2_{V'}\,dt &\le \int_0^T \|A_m u_m(t)\|^2_{V'}\,dt + \int_0^T \|f_m(t)\|^2_{V'}\,dt\\ &\le M\int_0^T \|u_m(t)\|^2_V\,dt + \int_0^T \|f(t)\|^2_{V'}\,dt\\ &\le \frac{2(MC+1)}{\eta}\Big(\int_0^T \|f(t)\|^2_{V'}\,dt + \|u_0\|^2_H\Big).\end{aligned}$$

Summing up, we see that there exists a constant $C \ge 0$ such that for every $m \ge 1$

$$\int_0^T \|u_m(t)\|^2_V\,dt + \int_0^T \|\dot u_m(t)\|^2_{V'}\,dt \le C\left(\int_0^T \|f(t)\|^2_{V'}\,dt + \|u_0\|^2_H\right).$$

The right-hand side is finite by assumption and does not depend on $m \ge 1$.

As a consequence, (u_m) is bounded in $L^2(0,T;V)$ and $W^{1,2}(0,T;V')$. By reflexivity, we can thus extract a subsequence (which we denote again by (u_m)) such that

$$\begin{aligned}&u_m \rightharpoonup u \text{ in } L^2(0,T;V) \text{ and}\\ &\dot u_m \rightharpoonup v \text{ in } L^2(0,T;V').\end{aligned}$$

This means that for every $\varphi \in L^2(0,T;V')$

$$\lim_{m\to\infty}\int_0^T \langle \varphi(t), u_m(t)\rangle_{V',V} = \int_0^T \langle \varphi(t), u(t)\rangle_{V',V}$$

and for every $\varphi \in L^2(0,T;V)$

$$\lim_{m\to\infty}\int_0^T \langle \dot u_m(t), \varphi(t)\rangle_{V',V} = \int_0^T \langle v(t), \varphi(t)\rangle_{V',V}.$$

Let $w \in V$ be any fixed vector and let $\varphi \in \mathcal{D}(0,T)$ be a scalar test function. Then an integration by parts yields

$$\begin{aligned}\langle \int_0^T u(t)\dot\varphi(t)\,dt, w\rangle_{V',V} &= \int_0^T \langle u(t), \dot\varphi(t)w\rangle_{V',V}\\ &= \lim_{m\to\infty}\int_0^T \langle u_m(t), \dot\varphi(t)w\rangle_{V',V}\\ &= -\lim_{m\to\infty}\int_0^T \langle \dot u_m(t), \varphi(t)w\rangle_{V',V}\\ &= -\int_0^T \langle v(t), \varphi(t)w\rangle_{V',V}\\ &= -\langle \int_0^T v(t)\varphi(t), w\rangle_{V',V}.\end{aligned}$$

Since this equality is true for every $w \in V$, we find that

$$\int_0^T u\dot\varphi = -\int_0^T v\varphi \text{ in } V',$$

for every test function $\varphi \in \mathcal{D}(0,T)$. Hence, by definition of the Sobolev space, the function u belongs to $W^{1,2}(0,T;V')$ and $\dot u = v$.

Since $A : V \to V'$ is a bounded linear operator, we find that

$$Au_m \rightharpoonup Au \text{ in } L^2(0,T;V'),$$

i.e. for every $\varphi \in L^2(0,T;V)$

$$\lim_{m\to\infty} \int_0^T \langle Au_m(t), \varphi(t)\rangle_{V',V} = \int_0^T \langle Au(t), \varphi(t)\rangle_{V',V}.$$

Note that also

$$A_m u_m \rightharpoonup Au \text{ in } L^2(0,T;V').$$

In order to see this, let $w \in V_n$ for some $n \geq 1$ and let $\varphi \in L^2(0,T)$. Then, for every $m \geq n$,

$$\begin{aligned}
\int_0^T \langle A_m u_m(t), \varphi(t)w\rangle_{V',V} &= \int_0^T a_m(u_m(t), \varphi(t)w) \\
&= \int_0^T a(u_m(t), \varphi(t)w) \\
&= \int_0^T \langle Au_m(t), \varphi(t)w\rangle_{V',V} \\
&\to \int_0^T \langle Au(t), \varphi(t)w\rangle_{V',V} \quad (m \to \infty).
\end{aligned}$$

Since $\bigcup_n V_n$ is dense in V, and since therefore the set $\{\varphi(\cdot)v : \varphi \in L^2(0,T),\ v \in \bigcup_n V_n\}$ is total in $L^2(0,T;V)$, the last claim follows.

Note also that $f_m \to f$ in $L^2(0,T;V')$. We thus obtain for every $\varphi \in L^2(0,T;V)$

$$\begin{aligned}
\int_0^T \langle \dot u(t), \varphi(t)\rangle_{V',V} &= \lim_{m\to\infty} \int_0^T \langle \dot u_m(t), \varphi(t)\rangle_{V',V} \\
&= \lim_{m\to\infty} \int_0^T \langle f_m(t) - A_m u_m(t), \varphi(t)\rangle_{V',V} \\
&= \int_0^T \langle f(t) - Au(t), \varphi(t)\rangle_{V',V}.
\end{aligned}$$

Since this equality holds for every $\varphi \in L^2(0,T;V)$, we find that

$$\dot u(t) + Au(t) = f(t) \text{ for a.e. } t \in [0,T],$$

i.e. u is a solution of our differential equation.

It remains to show that u verifies also the initial condition. Let $w \in V$ and let $\varphi \in C^1([0,T])$ be such that $\varphi(0) = 1$ and $\varphi(T) = 0$. Then an integration by parts yields on the one hand

$$\int_0^T \langle u, \dot\varphi w\rangle_{V',V} = -\langle u(0), w\rangle_{V',V} - \int_0^T \langle \dot u, \varphi w\rangle_{V',V}.$$

On the other hand, since $u_0^m \to u_0$ in H,

$$\begin{aligned}\int_0^T \langle u, \dot\varphi w\rangle_{V',V} &= \lim_{m\to\infty} \int_0^T \langle u_m, \dot\varphi w\rangle_{V',V} \\ &= \lim_{m\to\infty} \big(-\langle u_m(0), w\rangle_{V',V} - \int_0^T \langle \dot u_m, \varphi w\rangle_{V',V} \\ &= -\lim_{m\to\infty} \langle u_0^m, w\rangle_{V',V} - \int_0^T \langle \dot u, \varphi w\rangle_{V',V} \\ &= -\langle u_0, w\rangle_{V',V} - \int_0^T \langle \dot u, \varphi w\rangle_{V',V}.\end{aligned}$$

Comparing both equalities, we obtain

$$\langle u(0), w\rangle_{V',V} = \langle u_0, w\rangle_{V',V}$$

for every $w \in V$. Hence, $u(0) = u_0$. □

REMARK 3.6. Lions' Theorem says that the operator $A : V \to V'$, considered as a closed, unbounded operator on V' with domain $D(A) = V$, has L^2-maximal regularity. This follows when regarding the inhomogeneous problem with initial value $u(0) = 0$.

Moreover, it follows from Lions' Theorem, especially the solvability of the initial value problem, that $H \subset Tr_2(V', V)$. Together with Lemma 3.4 this implies the identity

$$Tr_2(V', V) = H,$$

i.e. a complete description of the trace space in this special situation.

EXAMPLE 3.7. We consider the linear heat equation with Dirichlet boundary conditions and initial condition

$$\begin{aligned} &u_t(t,x) - \Delta u(t,x) = f(t,x) && (t,x) \in \Omega_T, \\ &u(t,x) = 0 && x \in \partial\Omega, \\ &u(0,x) = u_0(x) && x \in \Omega, \end{aligned} \tag{3.6}$$

where $\Omega \subset \mathbb{R}^n$ is any open set and $\Omega_T = (0,T) \times \Omega$. This heat equation can be abstractly rewritten as a linear Cauchy problem

$$\dot u(t) + Au(t),\ t \in [0,T], \quad u(0) = u_0,$$

where $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ is the Dirichlet-Laplace operator associated with the form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by

$$a(u,v) = \int_\Omega \nabla u \nabla v.$$

It follows from Lions' Theorem that for every $f \in L^2(0,T;H^{-1}(\Omega))$ and every $u_0 \in L^2(\Omega)$ there exists a unique solution

$$u \in W^{1,2}(0,T;H^{-1}(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$$

of this problem.

A particular situation arises when $\Omega = \mathbb{R}^n$ (in this case the boundary conditions are obsolete) and when $f = 0$, because in this case we have an explicit formula for the solution. Using the heat kernel, one has for every $u_0 \in L^2(\mathbb{R}^n)$ the solution u of the heat equation is given by

$$u(t,x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{|x-y|^2/(4t)} u_0(y)\, dy.$$

Lions' Theorem implies that this solution belongs to the space

$$u \in W^{1,2}(0,T;H^{-1}(\mathbb{R}^n)) \cap L^2(0,T;H^1(\mathbb{R}^n)) \cap C([0,T];L^2(\mathbb{R}^n)).$$

4. * L^p-maximal regularity

Let $A : X \supset D(A) \to X$ be a closed, linear, densely defined operator on X.

We consider the abstract linear inhomogeneous Cauchy problem

$$\dot{u}(t) + Au(t) = f(t),\ t \in [0,T], \quad u(0) = 0. \tag{4.1}$$

Here, $f \in L^p(0,T;X)$ for some $1 \le p \le \infty$.

Definition 4.1. (a) A function $u \in C^1([0,T];X) \cap C([0,T];D(A))$ is called a *classical solution* if $u(0) = 0$ and if u satisfies the differential equation (4.1) for every $t \in [0,T]$.

(b) A function $u \in W^{1,p}(0,T;X) \cap L^p(0,T;D(A))$ is called a *(L^p) strong solution* if $u(0) = 0$ and if u satisfies the differential equation (4.1) for almost every $t \in [0,T]$.

Definition 4.2. We say that A has *L^p-maximal regularity* (on $(0,T)$) if for every $f \in L^p(0,T;X)$ there exists a unique strong solution $u \in W^{1,p}(0,T;X) \cap L^p(0,T;D(A))$ of the problem (4.1).

By definition, if A has L^p-maximal regularity, then the Cauchy problem (4.1) is uniquely solvable in the space $W^{1,p}(0,T;X) \cap L^p(0,T;D(A))$, for every $f \in L^p(0,T;X)$. It will be convenient to introduce the *maximal regularity space*

$$MR_p(a,b;X,D(A)) := W^{1,p}(a,b;X) \cap L^p(a,b;D(A)) \quad (-\infty \le a < b \le \infty)$$

which is naturally endowed with the norm

$$\|u\|_{MR_p} := \|u\|_{W^{1,p}(a,b;X)} + \|u\|_{L^p(a,b;D(A))}.$$

Since $W^{1,p}(a,b;X)$ and $L^p(a,b;D(A))$ are Banach spaces, $MR_p(a,b;X,D(A))$ is also a Banach space. If there is no danger of confusion, we will write $MR_p(a,b)$ instead of $MR_p(a,b;X,D(A))$.

We will first show that the definition of L^p-maximal regularity is independent of $T > 0$, so that it suffices in fact to speak only of L^p-maximal regularity. On the way we will also show that the initial value problem is uniquely solvable in the maximal regularity space, at least for certain initial values. For this, we first need the following locality lemma.

LEMMA 4.3. *Assume that A has L^p-maximal regularity on* $(0, T)$. *If* $f \in L^p(0, T; X)$ *is zero on the interval* $(0, T')$ *(with $0 < T' \leq T$), and if* $u \in MR_p(0, T; X, D(A))$ *is the corresponding solution of* (4.1), *then* $u = 0$ *on* $(0, T')$.

PROOF. Define the function

$$g(t) := \begin{cases} f(t+T') & \text{if } 0 \leq t \leq T - T', \\ 0 & \text{if } T - T' < t \leq T. \end{cases}$$

Then $g \in L^p(0, T; X)$. By definition of L^p-maximal regularity, there exists a unique $v \in MR_p(0, T; X, D(A))$ solution of (4.1).

Now define

$$w(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq T', \\ v(t - T') & \text{if } T' < t \leq T. \end{cases}$$

Then the function w restricted to the two intervals $[0, T']$ and $[T', T]$ belongs to the maximal regularity spaces $MR_p(0, T')$ and $MR_p(T', T)$, respectively. Since w is also continuous in T' (note that $v(0) = 0$!), we actually have $w \in MR_p(0, T)$.

It follows easily from the definition of w (the definition of g and v), that w solves the problem (4.1) for the function f. Since (4.1) is uniquely solvable, $u = w$, and therefore $u = 0$ on $[0, T']$. □

We also have to define the *trace space*

$$Tr_p(X, D(A)) := \{u(0) : u \in MR_p(0, 1)\},$$

which is naturally a Banach space for the norm

$$\|u_0\|_{Tr_p} := \inf\{\|u\|_{MR_p(0,1)} : u \in MR_p(0, 1) \text{ and } u(0) = u_0\}.$$

If there is no danger of confusion, we simply write Tr_p instead of $Tr_p(X, D(A))$. The space Tr_p is called trace space since it contains all *traces* in $t = 0$ of functions $u \in MR_p(0, 1)$. Note that we can evaluate $u(0)$ for every function u in the maximal regularity space $MR_p(0, T; X, D(A))$ since $W^{1,p}(0, T; X)$ is contained in the space of all continuous functions (see vector-valued Sobolev spaces in one dimension). Clearly, by definition, Tr_p is contained in X, and since for every $u_0 \in D(A)$ the constant function $u \equiv u_0$ belongs to $MR_p(0, 1)$, one has the inclusions

$$D(A) \hookrightarrow Tr_p \hookrightarrow X.$$

It turns out that Tr_p is a strictly contained between $D(A)$ and X (see below). For the moment, however, we need not to know this.

LEMMA 4.4. *The following are true:*
(a) For every $T > 0$ and every $0 \leq t \leq T$ one has

$$Tr_p = \{u(t) : u \in MR_p(0, T)\}.$$

(b) One has the inclusion

$$MR_p(0, T) \subset C([0, T]; Tr_p)$$

and there exists a constant $C \geq 0$ (depending on $T > 0$) such that

$$\|u\|_{C([0,T];Tr_p)} \leq C\, \|u\|_{MR_p} \text{ for every } u \in MR_p.$$

Proof. The spaces $MR_p(0,T)$ and $MR_p(0,1)$ are isomorphic via the isomorphism $u \mapsto u(\cdot T)$. Hence, for every $T > 0$,

$$Tr_p = \{u(0) : u \in MR_p(0,T)\},$$

and

$$\|u_0\|_{Tr_p,T} := \inf\{\|u\|_{MR_p(0,T)} : u \in MR_p(0,T) \text{ and } u(0) = u_0\}$$

defines an equivalent norm on Tr_p.

Given $u \in MR_p(0,T)$ we may define the extension $v \in MR_p(0,2T)$ by

$$v(t) := \begin{cases} u(t) & \text{if } 0 \leq t \leq T, \\ u(2T-t) & \text{if } T < t \leq 2T. \end{cases}$$

We define next the functions $u_t \in MR_p(0,T)$ by

$$u_t(s) := v(t+s), \quad 0 \leq s,\, t \leq T.$$

Then one sees that $u(t) = u_t(0) \in Tr_p$ for every $0 \leq t \leq T$ and since

$$t \mapsto u_t, \quad [0,T] \to MR_p[0,T]$$

is continuous, one obtains from the definition of the norm on Tr_p that

$$t \mapsto u(t), \quad [0,T] \to Tr_p$$

is continuous. Moreover,

$$\sup_{t\in[0,T]} \|u(t)\|_{Tr_p} \leq C \sup_{t\in[0,T]} \|u(t)\|_{Tr_p,2T} \leq C\, \|v\|_{MR_p(0,2T)} = 2C\, \|u\|_{MR_p(0,T)}.$$

□

Theorem 4.5 (Initial value problem). *Assume that A has L^p-maximal regularity on $(0,T)$. Then for every $u_0 \in Tr_p$ there exists a unique $u \in MR_p(0,T)$ solution of the problem*

$$\dot{u}(t) + Au(t) = 0,\ t \in [0,T], \quad u(0) = u_0.$$

Proof. *Existence:* Let $u_0 \in Tr_p$. By definition of Tr_p and Lemma 4.4, there exists $v \in MR_p(0,T)$ such that $v(0) = u_0$. By definition of L^p-maximal regularity, there exists $w \in MR_p(0,T)$ solution of

$$\dot{w}(t) + Aw(t) = \dot{v}(t) + Av(t),\ t \in [0,T], \quad w(0) = 0.$$

Now put $u := v - w$.

Uniqueness: Let u and v be two solutions of the initial value problem. Then $u - v$ is a solution of the same initial value problem with initial value u_0 replaced by 0. The solution for that problem, however, is unique by definition of L^p-maximal regularity. Hence, $u = v$. □

Theorem 4.6 (Independence of $T > 0$). *Assume that A has L^p-maximal regularity on $(0,T)$. Then A has L^p-maximal regularity on $(0,T')$ for every $T' > 0$.*

Proof. Fix $T' \in (0, T]$. Let $f \in L^p(0, T'; X)$ and extend f by zero on $(T', T]$. The resulting function is denoted by $\tilde{f}$. Let $\tilde{u} \in MR_p(0, T)$ be the unique solution of (4.1). Let u be restriction of $\tilde{u}$ to the interval $[0, T']$. Then $u \in MR_p(0, T')$ is a solution of (4.1) with T replaced by T'. Hence, we have proved existence of strong solutions.

In order to prove uniqueness, by linearity, it suffices to show that $u = 0$ is the only solution of (4.1) with T replaced by T' and with $f = 0$. So let u be some solution of the problem

$$\dot{u}(t) + Au(t) = 0, \; t \in [0, T'], \quad u(0) = 0.$$

□

Remark 4.7. Theorem 4.6 allows us just to speak of L^p-maximal regularity of an operator A or of the Cauchy problem (4.1) without making the $T > 0$ precise.

5. * Interpolation and L^p-maximal regularity

The aim of this section is to study interpolation results for maximal regularity. In particular, as a corollary, we will prove that the operator $A_H : D(A_H) \to H$ associated with a bounded, elliptic bilinear form $a : V \times V \to \mathbb{R}$ has L^2-maximal regularity.

Given two Banach spaces X, Y such that $Y \hookrightarrow X$, and given $T > 0$, $p \in [1, \infty]$, we define the *maximal regularity space*

$$MR_p(0, T; X, Y) := W^{1,p}(0, T; X) \cap L^p(0, T; Y)$$

and the trace space

$$Tr_p(X, Y) := \{u(0) : u \in MR_p(0, T; X, Y)\}$$

with usual norms. The maximal regularity space and the trace space used up to now was obtained for $Y = D(A)$. The definition of $Tr_p(X, Y)$ is independent of $T > 0$.

Lemma 5.1 (Interpolation of a bounded linear operator). *Let X_1, X_2, Y_1, Y_2 be four Banach spaces such that $Y_i \hookrightarrow X_i$ for $i = 1, 2$. Let $S : X_1 \to X_2$ be a bounded linear operator such that its restriction to Y_1 is a bounded linear operator $S : Y_1 \to Y_2$. Then, for every $p \in [1, \infty]$, the restriction of S to $Tr_p(X_1, Y_1)$ is a bounded linear operator $S : Tr_p(X_1, Y_1) \to Tr_p(X_2, Y_2)$ and*

$$\|S\|_{\mathcal{L}(Tr_p(X_1,Y_1),Tr_p(X_2,Y_2))} \le \max\{\|S\|_{\mathcal{L}(X_1,X_2)}, \|S\|_{\mathcal{L}(Y_1,Y_2)}\}.$$

Moreover, if $S : X_1 \to X_2$ and $S : Y_1 \to Y_2$ are invertible, then $S : Tr_p(X_1, Y_1) \to Tr_p(X_2, Y_2)$ is invertible, too.

Proof. Let $u_0 \in Tr_p(X_1, Y_1)$. By definition of the trace space, and by definition of the norm on the trace space, for every $\varepsilon > 0$ there exists $u \in MR_p(0, T; X_1, Y_1)$ such that $\|u\|_{MR_p} \le (1 + \varepsilon)\,\|u_0\|_{Tr_p}$. Put $v(t) := S\,u(t)$. Then $v \in MR_p(0, T; X_2, Y_2)$ and

$$\begin{aligned} \|v\|_{MR_p} &= \|v\|_{W^{1,p}(0,T;X_2)} + \|v\|_{L^p(0,T;Y_2)} \\ &\le \|S\|_{\mathcal{L}(X_1,X_2)}\,\|u\|_{W^{1,p}(0,T;X_1)} + \|S\|_{\mathcal{L}(Y_1,Y_2)}\,\|u\|_{L^p(0,T;Y_1)} \\ &\le \max\{\|S\|_{\mathcal{L}(X_1,X_2)}, \|S\|_{\mathcal{L}(Y_1,Y_2)}\}\,\|u\|_{MR_p} \quad < \quad \infty. \end{aligned}$$

In particular, $v(0) = Su(0) = Su_0 \in Tr_p(X_2, Y_2)$ and

$$\|Su_0\|_{Tr_p(X_2,Y_2)} \leq \|v\|_{MR_p} \leq \max\{\|S\|_{\mathcal{L}(X_1,X_2)}, \|S\|_{\mathcal{L}(Y_1,Y_2)}\} \|u\|_{MR_p} \leq$$
$$\leq (1+\varepsilon) \max\{\|S\|_{\mathcal{L}(X_1,X_2)}, \|S\|_{\mathcal{L}(Y_1,Y_2)}\} \|u_0\|_{Tr_p(X_1,Y_1)}.$$

Since $\varepsilon > 0$ was arbitrary, the first claim follows.

If $S : X_1 \to X_2$ and $S : Y_1 \to Y_2$ are invertible, then one applies the above argument to the operator $S^{-1} : X_2 \to X_1$ whose restriction to Y_2 is a bounded linear operator $S^{-1} : Y_2 \to Y_1$. □

Remark 5.2. The situation in the interpolation lemma. The boundedness of $S : X_1 \to X_2$ and $S : Y_1 \to Y_2$ is assumed, the boundedness of S in the interpolation spaces is a consequence:

$$\begin{array}{ccc} X_1 & \stackrel{S}{\longrightarrow} & X_2 \\ \uparrow & & \uparrow \\ Tr_p(X_1,Y_1) & \stackrel{S}{\longrightarrow} & Tr_p(X_2,Y_2) \\ \uparrow & & \uparrow \\ Y_1 & \stackrel{S}{\longrightarrow} & Y_2 \end{array}$$

The following lemma will not be proved.

Lemma 5.3. *Let X, Y be two Banach spaces such that $Y \hookrightarrow X$. Then, for every $p \in [1, \infty]$,*

$$Tr_p(L^p(0,T;X), L^p(0,T;Y)) = L^p(0,T;Tr_p(X,Y)).$$

Let $A : D(A) \to X$ be a closed linear operator on X. This implies that the domain $D(A)$ equipped with the graph norm is a Banach space. We can define the restriction of A to the space $D(A)$ by

$$\begin{aligned} D(A_1) &:= \{x \in D(A) : Ax \in D(A)\}, \\ A_1 x &:= Ax. \end{aligned}$$

This restriction is again a closed linear operator (exercice!).

Lemma 5.4. *Let $A : D(A) \to X$ be a closed linear operator on X and define $A_1 : D(A_1) \to D(A)$ as above. Assume that $A + \omega I$ is invertible and that A has L^p-maximal regularity. Then A_1 has L^p-maximal regularity.*

Proof. The operator $A + \omega I$ is an isomorphism between the Banach spaces $D(A)$ and X, and also between the Banach spaces $D(A_1)$ and $D(A)$.

Let $f \in L^p(0,T;D(A))$. Then $(A+\omega I)f \in L^p(0,T;X)$ and by L^p-maximal regularity there exists a unique solution $u \in MR_p(0,T;X,D(A))$ of the problem

$$\dot{u} + Au = (A+\omega I)f, \; t \in [0,T], \quad u(0) = 0.$$

Multiply this differential equation by $(A+\omega I)^{-1}$ and put $v(t) := (A+\omega I)^{-1}u(t)$. Then $v \in MR_p(0,T;D(A),D(A_1))$ is solution of the problem

$$\dot{v} + Av = f, \; t \in [0,T], \quad v(0) = 0.$$

This solution is unique since every solution in $MR_p(0,T;D(A),D(A_1))$ is also a solution in $MR_p(0,T;X,D(A))$ of the same problem and the solution in the latter space is unique by L^p-maximal regularity.

As a consequence, $A_1 : D(A_1) \to D(A)$ has L^p-maximal regularity. □

Remark 5.5. One can repeat the above argument and restrict the operator A to the space $D(A_1)$ which is also a Banach space. This restriction is given by

$$\begin{aligned} D(A_2) &:= \{x \in D(A_1) : Ax \in D(A_1)\}, \\ A_2 x &:= Ax, \end{aligned}$$

and it is also a closed linear operator. By iteration, one can define closed linear operators

$$\begin{aligned} D(A_k) &:= \{x \in D(A_{k-1}) : Ax \in D(A_{k-1})\}, \\ A_k x &:= Ax, \end{aligned}$$

and one obtains the following picture:

$$\begin{array}{ccccc} D(A) & \overset{A}{\longrightarrow} & X & =: X_0 \\ \uparrow & & \uparrow & \\ D(A_1) & \overset{A}{\longrightarrow} & D(A) & =: X_1 \\ \uparrow & & \uparrow & \\ D(A_2) & \overset{A}{\longrightarrow} & D(A_1) & =: X_2 \\ \vdots & & \vdots & \end{array}$$

If $A + \omega I$ is invertible and if A has L^p-maximal regularity, then each operator A_k has L^p-maximal regularity.

Even more is true: we know from the interpolation lemma (Lemma 5.1) that A is also a closed linear operator on the interpolation spaces between X and $D(A)$. In the following theorem we prove that if $A + \omega I$ is invertible and if A has L^p-maximal regularity, then also the restriction of A to $Tr_p(X, D(A))$ has L^p-maximal regularity.

Theorem 5.6. *Let $A : D(A) \to X$ be a closed linear operator on X and define its restriction to $Tr_p(X, D(A))$ by*

$$\begin{aligned} D(A_{Tr_p}) &:= Tr_p(D(A), D(A_1)), \\ A_{Tr_p} x &:= Ax. \end{aligned}$$

Assume that $A + \omega I$ is invertible and that A has L^p-maximal regularity. Then A_{Tr_p} has L^p-maximal regularity.

Proof. Define

$$MR_p^0(0,T;X,D(A)) := \{u \in MR_p(0,T;X,D(A)) : u(0) = 0\}$$

and define the operator

$$\begin{array}{rcl} S\,:\, MR_p^0(0,T;X,D(A)) & \to & L^p(0,T;X), \\ u & \mapsto & \dot{u}+Au. \end{array}$$

The operator S is clearly bounded. Moreover, the operator A has L^p-maximal regularity if and only if the operator S is invertible. Hence, by assumption, S is invertible.

The restriction of S to the space $MR_p(0,T;D(A),D(A_1))$ is a bounded operator with values in $L^p(0,T;D(A))$, and, by Lemma 5.4, this restriction is also invertible.

By Lemma 5.3, we have

$$Tr_p(L^p(0,T;D(A)),L^p(0,T;D(A_1))) = L^p(0,T;Tr_p(D(A),D(A_1))),$$

and

$$Tr_p(W^{1,p}(0,T;X);W^{1,p}(0,T;D(A))) = W^{1,p}(0,T;Tr_p(X,D(A))).$$

It then follows that

$$\begin{aligned} &Tr_p(MR_p(0,T;X,D(A)),MR_p(0,T;D(A),D(A_1))) = \\ &\qquad = MR_p(0,T;Tr_p(X,D(A)),Tr_p(D(A),D(A_1))). \end{aligned}$$

By the Interpolation Lemma (Lemma 5.1), the restriction

$$\begin{array}{rcl} S\,:\, MR_p^0(0,T;Tr_p(X,D(A)),Tr_p(D(A),D(A_1))) & \to & L^p(0,T;Tr_p(X,D(A))), \\ u & \mapsto & \dot{u}+Au. \end{array}$$

is bounded and invertible. This means that the operator A_{Tr_p} has L^p-maximal regularity. □

REMARK 5.7. One has the equality

$$Tr_p(D(A),D(A_1)) = \{x\in D(A) : Ax\in Tr_p(X,D(A))\},$$

so that A_{Tr_p} is really the restriction of A to the space $Tr_p(X,D(A))$.

In order to prove this equality, let $u_0\in Tr_p(D(A),D(A_1))\subset D(A)$. Then there exists $u\in MR_p(0,T;D(A),D(A_1))$ such that $u(0)=u_0$. Put $v(t):=(A+\omega I)u(t)$. Then $v\in MR_p(0,T;X,D(A))$ and thus $(A+\omega I)u_0\in Tr_p(X,D(A))$. Hence, $u_0\in D(A_{Tr_p})$. The other inclusion is proved similarly, using the invertibility of $A+\omega I$.

REMARK 5.8. As before, the procedure of considering restrictions to intermediate spaces can be repeated on the smaller spaces $X_1=D(A)$, $X_2=D(A_1)$, etc.. One thus obtains the following picture:

$$\begin{array}{ccccc}
D(A) & \overset{A}{\longrightarrow} & X & =: X_0 \\
\uparrow & & \uparrow & \\
Tr_p(D(A), D(A_1)) & \overset{A}{\longrightarrow} & Tr_p(X, D(A)) & =: X_{1/p'} \\
\uparrow & & \uparrow & \\
D(A_1) & \overset{A}{\longrightarrow} & D(A) & =: X_1 \\
\uparrow & & \uparrow & \\
Tr_p(D(A_1), D(A_2)) & \overset{A}{\longrightarrow} & Tr_p(D(A), D(A_1)) & =: X_{1+1/p'} \\
\uparrow & & \uparrow & \\
D(A_2) & \overset{A}{\longrightarrow} & D(A_1) & =: X_2 \\
\vdots & & \vdots &
\end{array}$$

If $A + \omega I$ is invertible and if A has L^p-maximal regularity, then each operator in this picture has L^p-maximal regularity.

Corollary 5.9. *Let $a : V \times V \to \mathbb{R}$ be a bilinear, bounded, elliptic form and let $A_H : D(A_H) \to H$ be the associated operator on H. Then A_H has L^2-maximal regularity. In particular, for every $f \in L^2(0, T; H)$ and every $u_0 \in V$ there exists a unique solution $u \in W^{1,2}(0, T; H) \cap L^2(0, T; D(A_H))$ of the problem*

$$\dot{u}(t) + Au(t) = f(t),\ t \in [0, T], \quad u(0) = u_0.$$

Proof. By Lions' Theorem (Theorem 3.1, see also Remark 3.6), the operator $A : V \to V'$ associated with the form a has L^2-maximal regularity.

By ellipticity of the form a and by the theorem of Lax-Milgram, the operator $A + \omega I$ is invertible. Hence, by Theorem 5.6, the restriction of A to the trace space $Tr_2(V', V)$ has L^2-maximal regularity, too.

But by Remark 3.6, this trace space is equal to H and the restriction of A to the space H is nothing else than A_H. Hence, A_H has L^2-maximal regularity.

For the second statement, one has to prove that $Tr_2(H, D(A_H)) = V$. □

Example 5.10. We consider again the linear heat equation (3.6) with Dirichlet boundary conditions and initial condition from Example 3.7. From the results in this section follows that for every $u_0 \in H_0^1(\Omega)$ and every $f \in L^2(0, T; L^2(\Omega))$ the heat equation (3.6) admits a unique solution

$$u \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; D(\Delta_{L_2})) \cap C([0, T]; H_0^1(\Omega)),$$

where $D(\Delta_{L^2})$ is the domain in $L^2(\Omega)$ of the Laplace operator with Dirichlet boundary conditions. If $\Omega = (a, b)$ is a bounded interval, then $D(\Delta_{L^2}) = H^2(a, b) \cap H_0^1(a, b)$ (exercice). One also has $D(\Delta_{L^2}) = H^2(\Omega) \cap H_0^1(\Omega)$ if $\Omega \subset \mathbb{R}^N$ has smooth boundary, but this result is more difficult to prove and will be omitted.

Note that one can identify the spaces $L^2(0,T;L^2(\Omega))$ and $L^2((0,T)\times\Omega)$ in a natural way so that the inhomogeneity f is actually a real valued function on the product $(0,T)\times\Omega$ as suggested in the heat equation (3.6).

CHAPTER 3

Nonlinear gradient systems

1. Quasilinear equations: existence and uniqueness of local solutions

Let X and D be two Banach spaces such that D is densely and continuously embedded into X. Fix $1 < p < \infty$. Let $u_0 \in Tr_p(X, D)$ and $f \in L^p(0,1;X)$.

Let, moreover, $A : D \to \mathcal{L}(D, X)$ and $F : D \to X$ be two functions having the property that

$$(H) \qquad \begin{array}{l} \text{for every } T > 0 \text{ and every } u,\, v \in MR_p(0,T;X,D) \text{ one has} \\ A(u)v \in L^p(0,T;X) \text{ and } F(u) \in L^p(0,T;X). \end{array}$$

In this section, we consider the *quasilinear problem*

$$\begin{cases} \dot{u} + A(u)u + F(u) = f, \quad t \geq 0, \\ u(0) = u_0. \end{cases} \tag{1.1}$$

A *local solution* of this problem will be a function $u \in MR_p(0,T;X,D)$ which satisfies the differential equation almost everywhere on $[0,T]$ and which satisfies the initial condition.

Theorem 1.1 (Existence and uniqueness). *Assume that there exists $z \in MR_p(0,1;X,D)$ such that $z(0) = u_0$ and*

(i) *there exists $r > 0$, $L \geq 0$ such that for every $0 < T \leq 1$ and every u, v, $w \in MR_p(0,T;X,D)$ satisfying $u(0) = v(0) = w(0) = u_0$ and $\|u - z\|_{MR_p}$, $\|v - z\|_{MR_p}$, $\|w - z\|_{MR_p} \leq r$ one has*

$$\|(A(u) - A(v))w\|_{L^p(0,T;X)} \leq L\, \|u - v\|_{MR_p(0,T)}\, \|w\|_{MR_p(0,T)},$$

(ii) *there exists $r > 0$ and $L_T \geq 0$ such that $\lim_{T\to 0} L_T = 0$ and for every $0 < T \leq 1$ and every $u, v \in MR_p(0,T;X,D)$ satisfying $u(0) = v(0) = u_0$ and $\|u - z\|_{MR_p}$, $\|v - z\|_{MR_p} \leq r$ one has*

$$\|F(u) - F(v)\|_{L^p(0,T;X)} \leq L_T\, \|u - v\|_{MR_p(0,T)},$$

(iii) *for every $0 < T \leq 1$, every $g \in L^p(0,T;X)$ and every $v_0 \in Tr_p$ the linear problem*

$$\dot{v} + A(z)v = g,\ t \in [0,T], \quad v(0) = v_0$$

admits a unique solution $v \in MR_p(0,T;X,D)$.

Then the quasilinear problem (1.1) *admits a unique local solution $u \in MR_p(0,T;X,D)$.*

Proof. For every $T > 0$ we set

$$\tilde{M}_T := \{u \in MR_p(0,T;X,D) : u(0) = u_0\}.$$

The set $\tilde{M}_T$ will be equiped with the metric induced by the norm in MR_p. Functions in $\tilde{M}$ already satisfy the initial condition from (1.1).

Consider the nonlinear map

$$\begin{aligned} R : \tilde{M} &\to L^p(0,T;X), \\ u &\mapsto (A(z) - A(u))u - F(u) + f. \end{aligned}$$

By the hypothesis (H), the map R is well defined.

Consider also the solution map

$$\begin{aligned} S : L^p(0,T;X) &\to \tilde{M}, \\ g &\mapsto Sg := u, \end{aligned}$$

which assigns to every function $g \in L^p(0,T;X)$ the unique solution in $\tilde{M}$ of the problem

$$\dot{v} + A(z)v = g(t),\ t \in [0,T], \quad v(0) = u_0.$$

By assumption (iii), this solution map is well defined, too.

By definition of the two maps above, a function $u \in MR_p(0,T;X,D)$ is a solution of the quasilinear problem (1.1) if and only if $u \in \tilde{M}$ and $SRu = u$, i.e. if u is a fixed point of SR. We have thus reduced the problem of existence to a fixed point problem which we will solve by using Banach's fixed point theorem.

Let $S_0 : L^p(0,T;X) \to MR_p(0,T;X,D)$ be the solution operator which assigns to every function $g \in L^p(0,T;X)$ the unique solution $v := S_0 g$ of the problem

$$\dot{v} + A(z)v = g,\ t \in [0,T], \quad v(0) = 0.$$

There exists a constant $C_S \geq 0$ independent of $0 < T \leq 1$ such that $\|S_0\| \leq C_S$ for every $0 < T \leq 1$.

We may assume that the constant $r > 0$ from assumptions (i) and (ii) is the same. Let $r' > 0$ be such that

$$r' \leq \min\{r, \frac{1}{10\, C_S L}\},$$

choose $0 < T \leq 1$ sufficiently small so that

$$\begin{aligned} & L_T \leq \frac{1}{5\, C_S}, \\ & \|z\|_{MR_p(0,T)} \leq r', \text{ and} \\ & \|F(z)\|_{L^p(0,T;X)} + \|\dot{z} + A(z)z\|_{L^p(0,T;X)} + \|f\|_{L^p(0,T;X)} \leq \frac{3}{5\, C_S}\, r'. \end{aligned} \tag{1.2}$$

Such a parameter T clearly exists, by the assumption that $\lim_{T\to 0} L_T = 0$ and by the properties of the norms in L^p and $W^{1,p}$. Set

$$M := \{u \in \tilde{M}_T : \|u - z\|_{MR_p(0,T)} \leq r'\}.$$

The set M is a complete metric space for the metric induced by the norm $\|\cdot\|_{MR_p} = \|\cdot\|_{MR_p(0,T)}$. For every $u \in M$ one has

$$\|u\|_{MR_p} \leq \|u - z\|_{MR_p} + \|z\|_{MR_p} \leq 2r'.$$

We prove that SR maps the set M into itself. In order to see this, let $u \in M$. Then

$$\begin{aligned}
\|SRu - z\|_{MR_p} &= \|SRu - S(\dot z + A(z)z)\|_{MR_p} \\
&= \|S_0(Ru - \dot z + A(z)z)\|_{MR_p} \\
&\leq C_S \big(\|(A(z) - A(u))u\|_{L^p(0,T;X)} + \|F(z) - F(u)\|_{L^p(0,T;X)} + \\
&\qquad + \|F(z)\|_{L^p(0,T;X)} + \|\dot z + A(z)z\|_{L^p(0,T;X)} + \|f\|_{L^p(0,T;X)}\big) \\
&\leq C_S \,(L\|z - u\|_{MR_p}\, \|u\|_{MR_p} + L_T\|z - u\|_{MR_p} + \frac{3}{5\,C_S}\, r') \\
&\leq C_S \,(2Lr'\, r' + L_T r' + \frac{3}{5\,C_S}\, r') \\
&\leq r'\,(\frac{1}{5} + \frac{1}{5} + \frac{3}{5}) \quad = \quad r'.
\end{aligned}$$

This proves that $SRu \in M$.

We next prove that SR is a strict contraction. In order to see this, let $u, v \in M$. Then

$$\begin{aligned}
\|SRu - SRv\|_{MR_p} &= \|S_0(Ru - Rv)\|_{MR_p} \\
&\leq C_S \big(\|(A(z) - A(u))(u - v)\|_{L^p(0,T;X)} + \|(A(u) - A(v))v\|_{L^p(0,T;X)} + \\
&\qquad + \|F(u) - F(v)\|_{L^p(0,T;X)}\big) \\
&\leq C_S \,(L\,\|u - z\|_{MR_p} + L\,\|v\|_{MR_p} + L_T)\,\|u - v\|_{MR_p} \\
&\leq C_S \,(L\,r' + L\,2r' + L_T)\,\|u - v\|_{MR_p} \\
&\leq (\frac{3}{10} + \frac{1}{5})\,\|u - v\|_{MR_p} \\
&= \frac{1}{2}\,\|u - v\|_{MR_p}.
\end{aligned}$$

Hence, $SR : M \to M$ is a strict contraction. By Banach's fixed point theorem, there exists a unique fixed point $u \in M$ which by construction of S and R is a solution of the quasilinear problem (1.1). □

Remark 1.2. It follows from the proof of Theorem 1.1 that one could actually also study non-autonomous (i.e. time-dependent) quasi-linear problems of the form

$$\dot u + A(t,u)u + F(t,u) = 0,\ t \geq 0, \quad u(0) = u_0. \tag{1.3}$$

Here $A : [0,T] \times D \to \mathcal{L}(D,X)$ and $F : [0,T] \times D \to X$ are two functions such that

$$\begin{aligned}
&\text{for every } T > 0 \text{ and every } u,\, v \in MR_p(0,T) \text{ one has} \\
&A(t,u)v \in L^p(0,T;X) \text{ and } F(t,u) \in L^p(0,T;X).
\end{aligned} \tag{1.4}$$

By this hypothesis, for every $T > 0$, the operators $A : MR_p(0,T) \times MR_p(0,T) \to L^p(0,T;X)$ and $F : MR_p(0,T) \to L^p(0,T;X)$ given respectively by $(u,v) \mapsto A(t,u)v$

and $(u,v) \mapsto F(t,u)$ are well-defined. Theorem 1.1, with an obvious small change in condition (iii), holds then true for the non-autonomous problem 1.3, too.

The following lemma gives sufficient conditions when the conditions (i) and (ii) of Theorem 1.1 are satisfied.

Lemma 1.3. *Fix $1 < p < \infty$. The following are true:*

(i) *If $A : Tr_p \to \mathcal{L}(D,X)$ is Lipschitz continuous in a neighbourhood of u_0 (with respect to the topology in Tr_p), then the assumption (i) of Theorem 1.1 is satisfied.*
(ii) *If $F : Tr_p \to X$ is Lipschitz continuous in a neighbourhood of u_0 (with respect to the topology in Tr_p), then the assumption (i) of Theorem 1.1 is satisfied.*

In order to prove this lemma, we need the following lemma.

Lemma 1.4. *For every $T > 0$ and every $u \in MR_p(0,T;X,D)$ satisfying $u(0) = 0$ one has*

$$\|u\|_{C([0,T];Tr_p)} \leq 2\,\|u\|_{MR_p^0(0,T;X,D)}. \tag{1.5}$$

Proof. Every function $u \in MR_p(0,T;X,D)$ satisfying $u(0) = 0$ can be extended to a function $\bar{u} \in MR_p(0,\infty;X,D)$ by setting

$$\bar{u}(t) := \begin{cases} u(t) & \text{if } 0 \leq t \leq T, \\ u(2T-t) & \text{if } T \leq t \leq 2T, \\ 0 & \text{if } 2T \leq t, \end{cases}$$

and for this particular extension one has

$$\|\bar{u}\|_{MR_p(0,\infty;X,D)} \leq 2\,\|u\|_{MR_p(0,T;X,D)}.$$

Note that in this reasoning it is important that $u(0) = 0$! As a consequence, by definition of the norm in the trace space, for every $t \geq 0$,

$$\|u(t)\|_{Tr_p} \leq \|\bar{u}(t+\cdot)\|_{MR_p(0,1;X,D)} \leq 2\,\|u\|_{MR_p(0,T;X,D)},$$

and the inequality (1.5) follows. □

Proof of Lemma 1.3. (i) By assumption, there exists $r > 0$ and $L \geq 0$ such that

$$\|A(u) - A(v)\|_{\mathcal{L}(D,X)} \leq L\,\|u-v\|_{Tr_p},$$

whenever $u, v \in Tr_p$ are such that $\|u-u_0\|_{Tr_p} \leq r$ and $\|v-u_0\|_{Tr_p} \leq r$.

Let $z \in MR_p(0,1)$ be such that $\sup_{t\in[0,1]} \|z(t)-u_0\|_{Tr_p} \leq r/3$. For every u, v, $w \in MR_p(0,T)$ with $u(0) = v(0) = w(0) = u_0$ and $\|u-z\|_{MR_p} \leq r/3$, $\|v-z\|_{MR_p} \leq r/3$ and $\|w-z\|_{MR_p} \leq r/3$ we have, by Lemma 1.4,

$$\|u-z\|_{C([0,1];Tr_p)},\ \|v-z\|_{C([0,1];Tr_p)} \leq 2r/3,$$

and therefore, for every $t \in [0,1]$,

$$\|u(t)-u_0\|_{Tr_p},\ \|v(t)-u_0\|_{Tr_p} \leq r.$$

As a consequence,

$$\begin{aligned} \|(A(u)-A(v))w\|_{L^p(0,T;X)} &\leq \sup_{t\in[0,T]} \|A(u(t))-A(v(t))\|_{\mathcal{L}(D,X)} \|w\|_{L^p(0,T;D)} \\ &\leq L\|u-v\|_{C([0,T];Tr_p)} \|w\|_{MR_p} \\ &\leq 2L\,\|u-v\|_{MR_p} \|w\|_{MR_p}. \end{aligned}$$

Hence, A satisfies the condition (i) of Theorem 1.1.

(ii) By assumption, there exists $r > 0$ and $L \geq 0$ such that

$$\|F(u)-F(v)\|_X \leq L\,\|u-v\|_{Tr_p},$$

whenever $u, v \in Tr_p$ are such that $\|u-u_0\|_{Tr_p} \leq r$ and $\|v-u_0\|_{Tr_p} \leq r$. Similarly as above, for every $u, v \in MR_p(0,T)$ with $u(0) = v(0) = u_0$ and $\|u-z\|_{MR_p} \leq r/3$ and $\|v-z\|_{MR_p} \leq r/3$ we obtain

$$\begin{aligned} \|F(u)-F(v)\|_{L^p(0,T;X)} &\leq T^{\frac{1}{p}}\,\|F(u)-F(v)\|_{C([0,T];X)} \\ &\leq T^{\frac{1}{p}}\,L\,\|u-v\|_{C([0,T];Tr_p)} \\ &\leq 2T^{\frac{1}{p}}L\,\|u-v\|_{MR_p}. \end{aligned}$$

Hence, F satisfies the condition (ii) of Theorem 1.1. □

A special case of the quasilinear equation (1.1) is obtained when the function A is *constant*. In this case, we call the quasilinear equation *semilinear*. We will formulate the local existence and uniqueness of solutions in this special case. For this, we assume again that D and X are two Banach spaces such that D is densely and continuously embedded into X. Let $A : D \to X$ be a fixed bounded linear operator.

Fix $1 < p < \infty$ and let $F : D \to X$ be a function such that

(1.6) for every $T > 0$ and every $u \in MR_p(0,T;X,D)$ one has $F(u) \in L^p(0,T;X)$.

We consider the *semilinear problem*

$$\begin{cases} \dot{u} + Au + F(u) = f, & t \geq 0, \\ u(0) = u_0, \end{cases} \tag{1.7}$$

where $f \in L^p(0,T;X)$ and $u_0 \in Tr_p = Tr_p(D,X)$. A *local solution* of this problem will be a function $u \in MR_p(0,T;X,D)$ which satisfies the differential equation almost everywhere on $[0,T]$ and which satisfies the initial condition.

Note that the linear operator A gives rise to the constant function $A : D \to \mathcal{L}(D,X)$ which assigns to every $u \in D$ the operator A. A constant function A clearly satisfies condition (i) from Theorem 1.1. Hence, the following result is an immediate corollary to Theorem 1.1. Note that condition (i) in the following corollary is nothing else than condition (ii) from Theorem 1.1.

Corollary 1.5 (Existence and uniqueness). *Assume that there exists* $z \in MR_p(0,1;X,D)$ *such that* $z(0) = u_0$ *and*

(i) *there exists* $r > 0$ *and* $L_T \geq 0$ *such that* $\lim_{T\to 0} L_T = 0$ *and for every* $0 < T \leq 1$ *and every* $u, v \in MR_p(0, T; X, D)$ *satisfying* $u(0) = v(0) = u_0$ *and* $\|u - z\|_{MR_p}, \|v - z\|_{MR_p} \leq r$ *one has*

$$\|F(u) - F(v)\|_{L^p(0,T;X)} \leq L_T \, \|u - v\|_{MR_p(0,T)},$$

(ii) *for every* $g \in L^p(0, 1; X)$ *and every* $v_0 \in Tr_p$ *the linear problem*

$$\dot{v} + Av = g, \; t \in [0, 1], \quad v(0) = v_0$$

admits a unique solution $v \in MR_p(0, 1; X, D)$*, that is, the operator* A *has* L^p*-maximal regularity.*

Then the semilinear problem (1.7) *admits a unique local solution* $u \in MR_p(0, T; X, D)$*.*

Example 1.6 (Semilinear heat equation). Let $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) be open, $T > 0$, $\Omega_T = (0, T)\times\Omega$, $f \in C^1(\mathbb{R})$, and consider the semilinear heat equation with Dirichlet boundary conditions and initial condition:

$$\begin{cases} u_t(t, x) - \Delta u(t, x) + f(u(t, x)) = 0 & (t, x) \in \Omega_T, \\ u(t, x) = 0 & x \in \partial\Omega, \\ u(0, x) = u_0(x) & x \in \Omega, \end{cases} \tag{1.8}$$

Assume in addition that there exists some constant $C \geq 0$ such that

$$|f'(s)| \leq |s|^{\frac{2}{N-2}} \text{ for every } s \in \mathbb{R}.$$

Corollary 1.7. *For every* $u_0 \in H_0^1(\Omega)$ *there exists a unique local solution* $u \in W^{1,2}(0, T'; L^2(\Omega)) \cap L^2(0, T'; D(\Delta_{L^2})) \cap C([0, T']; H_0^1(\Omega))$ *of the problem* (3.6).

Proof. In fact, we may apply Corollary 1.5, where $A = -\Delta_{L^2}$ is the Dirichlet-Laplace operator associated with the form $a : H_0^1(\Omega)\times H_0^1(\Omega) \to \mathbb{R}$ given by $a(u, v) = \int_\Omega \nabla u \nabla v$ (which has L^2-maximal regularity on $L^2(\Omega)$ by Corollary 5.9) and where $F : H_0^1(\Omega) \to L^2(\Omega)$ is the *Nemytski operator* associated with the function f: $F(u)(x) := f(u(x))$. By Lemma 1.3, it suffices to show that this Nemytski operator is locally Lipschitz continuous. We will need the Sobolev embedding

$$H_0^1(\Omega) \hookrightarrow L^q(\Omega),$$

which is true for $q = \frac{2N}{N-2}$. From this embedding, the growth condition on f, the mean value theorem and Hölder's inequality we deduce that for every $u, v \in H_0^1(\Omega)$

$$\begin{aligned} \|F(u) - F(v)\|_{L^2}^2 &= \int_\Omega |f(u) - f(v)|^2 \\ &= \int_\Omega |f'(\xi(x))(u(x) - v(x))|^2 \\ &\leq \int_\Omega |\xi(x)|^{\frac{4}{N-2}} |u - v|^2 \\ &\leq \int_\Omega ||u| \vee |v||^{\frac{4}{N-2}} |u - v|^2 \\ &\leq \Big(\int_\Omega ||u| \vee |v||^{\frac{2N}{N-2}}\Big)^{\frac{2}{N}} \Big(\int_\Omega |u - v|^{\frac{2N}{N-2}}\Big)^{\frac{N-2}{N}} \\ &\leq \max\{\|u\|_{H_0^1}, \|v\|_{H_0^1}\}^{\frac{4}{N-2}} \|u - v\|_{H_0^1}^2. \end{aligned}$$

Hence, for every $R > 0$ there exists a Lipschitz constant $L \geq 0$ such that for every u, $v \in H_0^1(\Omega)$ with norms less than R one has

$$\|F(u) - F(v)\|_{L^2} \leq L \, \|u - v\|_{H_0^1}.$$

In fact, one may take $L := R^{\frac{2}{N-2}}$. In other words, F is Lipschitz continuous on bounded subsets of $H_0^1(\Omega)$. □

Example 1.8 (Cahn-Hilliard equation). Let $\Omega \subset \mathbb{R}^3$ be a bounded domain which is regular in the send that the domain of the Dirichlet-Laplace operator $A = A_{L^2}$ which is associated with the form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$, $a(u, v) = \int_\Omega \nabla u \nabla v$, is given by

$$D(A) = H^2 \cap H_0^1(\Omega).$$

In this example, we consider the Cahn-Hilliard equation

$$\tag{1.9} \begin{cases} u_t(t, x) + \Delta(\Delta u(t, x) - f(u(t, x))) = 0 & (t, x) \in \Omega_T, \\ u(t, x) = \Delta u(t, x) = 0 & x \in \partial\Omega, \\ u(0, x) = u_0(x) & x \in \Omega, \end{cases}$$

where as before $\Omega_T = (0, T) \times \Omega$, and where the nonlinearity f belongs to $C^3(\mathbb{R})$. No growth restrictions on f are imposed. We will apply Corollary 1.5 in order to prove existence and uniqueness of solutions of the Cahn-Hilliard equation, at least for initial values $u_0 \in H^2 \cap H_0^1(\Omega)$. For this, we start by the following lemma.

Lemma 1.9. *The bilinear form $b : H^2 \cap H_0^1(\Omega) \times H^2 \cap H_0^1(\Omega) \to \mathbb{R}$ defined by*

$$b(u, v) = \int_\Omega \Delta u \Delta v$$

is bounded and elliptic.

PROOF. The Dirichlet-Laplace operator on $L^2(\Omega)$ is a closed linear operator. In particular, its domain $D(A)$ is a Banach space for the graph norm

$$\|u\|_{D(A)} = \|u\|_{L^2} + \|\Delta u\|_{L^2}.$$

By assumption, the domain $D(A)$ coincides with $H^2 \cap H^1_0(\Omega)$. Hence, by the bounded inverse theorem, the graph norm is equivalent to the usual norm in $H^2 \cap H^1_0$, which is the norm induced from H^2. This implies that there exists $\eta > 0$ such that

$$b(u,u) + \|u\|^2_{L^2} = \|\Delta u\|^2_{L^2} + \|u\|^2_{L^2} \geq \eta \, \|u\|^2_{H^2\cap H^1_0}.$$

Hence, the form b is elliptic. Boundedness of b is straightforward. □

LEMMA 1.10. *Let $B = B_{L^2}$ be the operator on L^2 which is associated with the form b from Lemma 1.9. Then*

$$D(B) = \{u \in H^2 \cap H^1_0(\Omega) : \Delta u \in H^2 \cap H^1_0(\Omega)\}$$

and $Bu = \Delta^2 u$.

PROOF. Let A be the Dirichlet-Laplace operator on $L^2(\Omega)$. By assumption, $D(A) = H^2 \cap H^1_0(\Omega)$. Moreover, by definition of the form b, $b(u,v) = (Au, Av)_{L^2}$ and thus

$$D(B) = \{u \in D(A) : \exists v \in L^2(\Omega)\, \forall \varphi \in D(A) : (Au, A\varphi)_{L^2} = (v,\varphi)_{L^2}\}.$$

Define the adjoint A^* of A by

$$\begin{aligned} D(A^*) &= \{u \in L^2(\Omega) : \exists v \in L^2(\Omega)\,, \forall \varphi \in D(A) : (u, A\varphi)_{L^2} = (v,\varphi)_{L^2}\}, \\ A^* u &= v. \end{aligned}$$

For every $u \in D(A)$ and every $\varphi \in D(A)$ the symmetry of a and the scalar product in L^2 imply

$$\begin{aligned} (u, A\varphi)_{L^2} &= (A\varphi, u)_{L^2} \\ &= a(\varphi, u) \\ &= a(u, \varphi) \\ &= (Au, \varphi)_{L^2}. \end{aligned}$$

This identity implies $u \in D(A^*)$ and $A^*u = Au$, i.e. A^* is an extension of A.

Since A is already surjective, the operator A^* is surjective. On the other hand, A^* is also injective: if $A^*u = 0$, then for every $\varphi \in D(A)$

$$0 = (A^*u, \varphi) = (u, A\varphi).$$

Since A is surjective, this implies $u = 0$, i.e. A^* is injective. Since A^* is a bijective extension of A which is itself already bijective, we obtain $D(A^*) = D(A)$ and $A^* = A$.

This, the definition of A^* and the above characterization of $D(B)$ imply the claim. □

LEMMA 1.11. *Let $f \in C^3(\mathbb{R})$ be such that $f(0) = 0$. Then the Nemytski operator $F : H^2(\Omega) \to H^2(\Omega)$ given by $F(u)(x) := f(u(x))$ is well-defined and Lipschitz continuous on bounded subsets of $H^2(\Omega)$.*

PROOF. We use the Sobolev embeddings

$$H^2(\Omega) \hookrightarrow L^\infty(\Omega) \text{ and } H^1(\Omega) \hookrightarrow L^4(\Omega),$$

which are valid since Ω is a subset of $\mathbb{R}^3$ (in fact, the above embeddings are also true for open subsets of $\mathbb{R}$ and $\mathbb{R}^2$, but they are not true in dimension 4 and higher dimensions).

Let $u \in H^2(\Omega)$. Then, by the Sobolev embeddings and by the fact that f and its derivatives are bounded on bounded intervals one obtains

$$f(u) \in L^2(\Omega),$$

$$\frac{\partial}{\partial x_i} f(u) = f'(u)\frac{\partial u}{\partial x_i} \in L^\infty \cdot L^2 \subset L^2(\Omega)$$

and

$$\frac{\partial^2}{\partial x_j \partial x_i} f(u) = f''(u)\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i} + f'(u)\frac{\partial^2 u}{\partial x_j \partial x_i} \in L^\infty \cdot L^4 \cdot L^4 + L^\infty \cdot L^2 \subset L^2(\Omega),$$

where in the last inclusion we also used Hölder's inequality. This proves that F is well-defined from $H^2(\Omega)$ into $H^2(\Omega)$.

We next show that F is Lipschitz continuous on bounded sets of $H^2(\Omega)$. Let $R > 0$ and let $u, v \in H^2(\Omega)$ such that $\|u\|_{H^2}, \|v\|_{H^2} \leq R$. By the Sobolev embeddings, there exists a constant $C \geq 0$ such that

$$\|u\|_{L^\infty}, \; \|v\|_{L^\infty}, \; \|\nabla u\|_{L^4}, \; \|\nabla v\|_{L^4} \leq CR.$$

Let $M = M(R) \geq 0$ be a constant such that

$$\|f^{(k)}\|_{L^\infty(-CR,CR)} \leq M \text{ for every } k \in \{0, 1, 2, 3\}.$$

Then

$$\begin{aligned}
\|f(u) - f(v)\|_{L^2}^2 &= \int_\Omega |f(u) - f(v)|^2 \\
&\leq \|f'\|_{L^\infty(-CR,CR)}^2 \|u - v\|_{L^2}^2 \\
&\leq M^2 \, \|u - v\|_{H^2}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|\frac{\partial}{\partial x_i}(f(u) - f(v))\|_{L^2}^2 &\leq \int_\Omega |f'(u) - f'(v)|^2 |\frac{\partial u}{\partial x_i}|^2 + \int_\Omega |f'(v)|^2 |\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}|^2 \\
&\leq \|f''\|_{L^\infty(-CR,CR)}^2 \|u - v\|_{L^\infty}^2 \|u\|_{H^2}^2 + \\
&\qquad + \|f'\|_{L^\infty(-CR,CR)}^2 \|u - v\|_{H^2}^2 \\
&\leq (M^2 C^2 R^2 + M^2) \, \|u - v\|_{H^2}^2 \\
&= L_2(R)^2 \, \|u - v\|_{H^2}^2.
\end{aligned}$$

Finally,

$$
\begin{aligned}
\|\frac{\partial^2}{\partial x_j \partial x_i}(f(u)-f(v))\|_{L^2}^2 \quad &\leq \quad \int_\Omega |f''(u)-f''(v)|^2 |\frac{\partial u}{\partial x_j}|^2 |\frac{\partial u}{\partial x_i}|^2 + \\
&\qquad + \int_\Omega |f''(v)|^2 |\frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial x_j}|^2 |\frac{\partial u}{\partial x_i}|^2 + \\
&\qquad + \int_\Omega |f''(v)|^2 |\frac{\partial v}{\partial x_j}|^2 |\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}|^2 + \\
&\qquad + \int_\Omega |f'(u)-f'(v)|^2 |\frac{\partial^2 u}{\partial x_j \partial x_i}|^2 + \\
&\qquad + \int_\Omega |f'(v)|^2 |\frac{\partial^2 u}{\partial x_j \partial x_i} - \frac{\partial^2 v}{\partial x_j \partial x_i}|^2 \\
&\leq \quad M^2 \|u-v\|_{L^\infty}^2 \|\frac{\partial u}{\partial x_j}\|_{L^4}^2 \|\frac{\partial u}{\partial x_i}\|_{L^4}^2 + \\
&\qquad + M^2 \|\frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial x_j}\|_{L^4}^2 \|\frac{\partial u}{\partial x_i}\|_{L^4}^2 + \\
&\qquad + M^2 \|\frac{\partial v}{\partial x_j}\|_{L^4}^2 \|\frac{\partial u}{\partial x_i} - \frac{\partial v}{\partial x_i}\|_{L^4}^2 + \\
&\qquad + M^2 \|u-v\|_{L^\infty}^2 \|\frac{\partial^2 u}{\partial x_j \partial x_i}\|_{L^2}^2 + \\
&\qquad + M^2 \|\frac{\partial^2 u}{\partial x_j \partial x_i} - \frac{\partial^2 v}{\partial x_j \partial x_i}\|_{L^2}^2 \\
&\leq \quad L_3(R)^2 \, \|u-v\|_{H^2}^2 .
\end{aligned}
$$

Putting the last three estimates together we have thus proved that F is Lipschitz continuous on bounded subsets of $H^2(\Omega)$. ◻

Theorem 1.12. *Assume that $f \in C^3(\mathbb{R})$ satisfies $f(0) = 0$ and that Ω is regular in the sense described above. Then for every $u_0 \in H^2 \cap H_0^1(\Omega)$ there exists a unique local solution*

$$u \in W^{1,2}(0, T; L^2(\Omega)) \cap L^2(0, T; D(B))$$

of the Cahn-Hilliard equation (1.9). *Here, $D(B)$ is as in Lemma 1.10.*

Proof. Apply Corollary 1.5. ◻

Example 1.13 (Quasilinear diffusion equation). More generally, if $\Omega \subset \mathbb{R}^N$ is open and bounded (!), we may also solve the following quasilinear problem:

$$
\begin{cases}
u_t - \operatorname{div}(a(x,u)\nabla u) + f(x,u) = 0 & (t,x) \in \Omega_T, \\
u(t,x) = 0 & x \in \partial\Omega, \\
u(0,x) = u_0(x) & x \in \Omega,
\end{cases}
\tag{1.10}
$$

where a, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are of class C^1. We assume that there exist constants C, L, $\eta > 0$ such that for every $x \in \Omega$, u, $v \in \mathbb{R}$

$$\begin{aligned} &(H1) \qquad C \geq a(x,u) \geq \eta > 0, \\ &(H2) \qquad |a(x,u) - a(x,v)| \leq L|u - v|, \\ &(H3) \qquad |f'(u)| \leq C\,|u|^{\frac{2}{N-2}}. \end{aligned}$$

Corollary 1.14. *For every $u_0 \in L^2(\Omega)$ there exists a unique local solution $u \in W^{1,2}(0,T';H^{-1}(\Omega)) \cap L^2(0,T';H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$ of the problem* (3.6).

Proof. We apply Theorem 1.1, working in the Hilbert space $H^{-1}(\Omega) =: X$. The Nemytski operator $F : L^2(\Omega) \to H^{-1}(\Omega)$ defined by $F(u)(x) := f(x,u(x))$ is locally Lipschitz continuous by hypothesis (H3); the proof is very similar to that in the preceeding example, but we now use that $L^{\frac{2N}{N+2}}(\Omega)$ embeds continuously into $H^{-1}(\Omega)$ and we actually show that the Nemytski operator F is locally Lipschitz continuous from $L^2(\Omega)$ into $L^{\frac{2N}{N+2}}(\Omega)$.

Next, for every $u \in H_0^1(\Omega)$ we define the bilinear form $a(u) : H_0^1 \times H_0^1 \to \mathbb{R}$ by

$$a(u)(w_1,w_2) = \int_\Omega a(x,u(x))\nabla w_1 \nabla w_2 .$$

By hypothesis (H1), for every $u \in H_0^1(\Omega)$. □

2. Regularity of solutions

In this section we want to study the regularity of solutions of the nonlinear equation

$$\dot{u} + F(u) = 0,\ t \geq 0, \quad u(0) = u_0. \tag{2.1}$$

Here, $F : D \to X$ is a function satisfying hypothesis (1.6), and $u_0 \in Tr_p(X,D)$. This problem includes as special cases the quasilinear problem (1.1) (we do not assume a vanishing Lipschitz condition as in condition (ii) of Theorem 1.1) and the semilinear problem (1.7). Our regularity result will be true in this general situation, but it will really be applied in the situations considered before.

For the proof of our regularity theorem we need the following classical theorem from calculus (which is in fact also proved by using Banach's fixed point theorem, like our theorem of existence and uniqueness for the quasilinear problem).

Theorem 2.1 (Implicit function theorem). *Let X, Y, Z be three Banach spaces and let $G : X \times Y \to Z$ be of class C^k for some $k \geq 1$. Assume that $G(\bar{x},\bar{y}) = 0$ and assume that the partial derivative $\frac{\partial G}{\partial y}(\bar{x},\bar{y}) : Y \to Z$ is boundedly invertible.*

Then there exists a neighbourhood $U \subset X$ of $\bar{x}$, a neighbourhood $V \subset Y$ of $\bar{y}$, and a function $g : U \to Y$ of class C^k such that

$$\{(x,y) \in U \times V : G(x,y) = 0\} = \{(x,g(x)) : x \in U\}.$$

If, in addition, the function G is analytic, then the implicit function g is analytic, too.

THEOREM 2.2 (Regularity for the quasilinear problem). *In the nonlinear problem* (2.1)*, assume that the induced operator* $F : MR_p(0,T) \to L^p(0,T;X)$ *is of class* C^k *for some* $k \geq 1$*. Let* $u \in MR_p(0,T;X,D(A))$ *be a local solution of* (2.1) *and assume that the linear problem*

$$\dot{v} + F'(u)v = g,\ t \in [0,T], \quad v(0) = v_0,$$

has L^p*-maximal regularity in the sense that it admits for every* $g \in L^p(0,T;X)$ *and every* $v_0 \in Tr_p$ *a unique solution* $v \in MR_p(0,T)$*. Then, for every* $\tau > 0$*,*

$$\begin{aligned} u &\in W^{k+1,p}(\tau,T;X) \cap W^{k,p}(\tau,T;D) \text{ and} \\ u &\in C^k(]0,T];X) \cap C^{k-1}(]0,T];D). \end{aligned}$$

If F *is of class* C^∞*, then in fact* $u \in C^\infty(]0,T];D)$*, and if* F *is analytic, then* u *is analytic in a local sector around the positive real axis.*

PROOF OF THEOREM 2.2. Let $\varepsilon > 0$ be sufficiently small so that for every $\lambda \in (-\varepsilon,\varepsilon)$ the function

$$u_\lambda(t) := u((1+\lambda)t), \quad t \in [0,T],$$

is well-defined. For every $\lambda \in (-\varepsilon,\varepsilon)$ the function $u_\lambda \in MR_p(0,T;X,D)$ is the unique solution of the nonlinear problem

$$\dot{u} + (1+\lambda)F(u) = 0,\ t \geq 0, \quad u(0) = u_0.$$

Consider the nonlinear operator

$$\begin{aligned} G : \mathbb{R} \times MR_p(0,T;X,D) &\to L^p(0,T;X) \times Tr_p(X,D), \\ (\lambda, v) &\mapsto (\dot{v} + (1+\lambda)F(v), v(0) - u_0). \end{aligned}$$

Since F is of class C^k, the operator G is also of class C^k as one easily verifies.

Moreover, by definition of G and the functions u_λ, one has

$$G(\lambda, u_\lambda) = (0,0) \text{ for every } \lambda \in (-\varepsilon,\varepsilon).$$

We show that G actually satisfies the assumptions of the implicit function theorem in $(0,u)$. For this, we have to consider the partial derivative $\frac{\partial G}{\partial u}(0,u)$ which is the linear operator given by

$$\begin{aligned} \frac{\partial G}{\partial u}(0,u) : MR_p(0,T;X,D) &\to L^p(0,T;X) \times Tr_p(X,D), \\ v &\mapsto (\dot{v} + F'(u(t))v, v(0)). \end{aligned}$$

Hence, by our assumption on the linear problem $\dot{v} + F'(u)v = g$, and by the bounded inverse theorem, the partial derivative $\frac{\partial G}{\partial u}(0,u)$ is boundedly invertible.

By the implicit function theorem, there exists $\varepsilon' \in (0,\varepsilon)$, a neighbourhood $U \subset MR_p(0,T;X,D)$ and an implicit function $g : (-\varepsilon',\varepsilon') \to U$ of class C^k such that

$$G(\lambda, g(\lambda)) = (0,0),$$

and all solutions in $(-\varepsilon',\varepsilon') \times U$ of the equation $G(\lambda,v) = (0,0)$ are of the form $(\lambda, g(\lambda))$. Since the elements (λ, u_λ) are solutions of this equation, we obtain $u_\lambda = g(\lambda)$.

Moreover, the function

$$\begin{aligned} g : (-\varepsilon', \varepsilon') &\to MR_p(0, T; X, D), \\ \lambda &\mapsto u_\lambda = u((1+\lambda)\cdot), \end{aligned}$$

is of class C^k. When calculating the consecutive derivates of g at $\lambda = 0$, we see that

$$\begin{aligned} &t \mapsto t\dot{u}(t) \in MR_p, \\ &\vdots \\ &t \mapsto t^k u^{(k)}(t) \in MR_p, \end{aligned}$$

and this yields the stated regularity of the solution u. □

The above regularity theorem is already interesting in the linear case, that is, when the operator $F : D \to X$ is bounded and linear. In this case, the induced operator $F : MR_p(0, T; X, D) \to L^p(0, T; X)$ is also linear, hence analytic.

Corollary 2.3. *Assume that $A : D(A) \to X$ is a closed linear operator on a Banach space X. Assume that A has L^p-maximal regularity. Then for every $u_0 \in Tr_p(X, D(A))$ the unique solution u of the linear problem*

$$\dot{u} + Au = 0, \ t \geq 0, \quad u(0) = u_0,$$

satisfies for every $k \geq 1$

$$u \in MR_p(0, T; X, D(A)) \cap C^\infty(]0, T]; D(A^k)).$$

In fact, the solution u is analytic in a neighbourhood of $(0, \infty)$.

Proof. Applying Theorem 2.2, one obtains $u \in C^\infty([\tau, T]; D(A))$ for every $0 < \tau < T$. From this and the Cauchy problem one obtains then $u \in L^p(\tau, T; D(A^2))$ and derivating in the Cauchy problem one successively obtains first $u \in C^\infty([\tau, T]; D(A^2))$ and then by induction $u \in C^\infty([\tau, T]; D(A^k))$ for every $k \geq 1$. □

Remark 2.4 (C_0-semigroups). A family $(S(t))_{t\geq 0}$ of bounded linear operators on a Banach space X is called a C_0-*semigroup* if

(i) $S(0) = I$,
(ii) $S(t+s) = S(t)S(s)$ for every $t, s \geq 0$, and
(iii) for every $x \in X$ the function $t \mapsto S(t)x$ is continuous.

A closed linear operator $A : D(A) \to X$ is called the *generator* of a C_0-semigroup $(S(t))_{t\geq 0}$ if for every $x \in X$ and $t \geq 0$ one has $\int_0^t S(s)x\, ds \in D(A)$ and $A \int_0^t S(s)x\, ds = S(t)x - x$.

Let $A : D(A) \to X$ be a closed linear operator on a Banach space X and assume that A has L^p-maximal regularity. Then for every $x \in Tr_p = Tr_p(X, D(A))$ there exists a unique solution $u \in MR_p$ of the problem

$$\dot{u} + Au = 0, \ t \geq 0, \quad u(0) = x.$$

If one puts $S(t)x := u(t)$ (u being the unique solution for the initial value $x \in Tr_p$), then $(S(t))_{t\geq 0}$ is a C_0-semigroup on the trace space Tr_p. The orbits $S(\cdot)x$ being by definition the solutions of the above Cauchy problem, this semigroup is even *analytic* by Corollary 2.3.

In the special situation when $A : V \to V'$ is the operator associated with a bilinear, bounded and coercive form $a : V \times V \to \mathbb{R}$ ($V \hookrightarrow H = H' \hookrightarrow V$), then we obtain an analytic semigroup $(S(t))_{t\geq 0}$ on the space H whose generator is actually the operator A_H.

Theorem 2.5. *Let $A : D(A) \to X$ be a closed, linear operator on a Banach space X, and let $B \in \mathcal{L}(Tr_p, X)$. Assume that A has L^p-maximal regularity. Then $A + B$ has L^p-maximal regularity.*

Proof. Saying that the operator A has L^p-maximal regularity is equivalent to saying that the operator

$$\begin{aligned} S_0 : MR_p^0(0,T;X,D(A)) &\to L^p(0,T;X), \\ u &\mapsto \dot{u} + Au, \end{aligned}$$

is invertible for some (for all) $T > 0$. Note that the norm of the inverse $\|S_0^{-1}\|$ is uniformly bounded in $T \in (0,1]$.

We have to prove that the operator

$$\begin{aligned} S : MR_p^0(0,T;X,D(A)) &\to L^p(0,T;X), \\ u &\mapsto \dot{u} + Au + Bu, \end{aligned}$$

is invertible for some (for all) $T > 0$. There exists a constant $C \geq 0$ such that for every $T \in (0,1]$

$$\begin{aligned} \|S_0^{-1}B\|_{\mathcal{L}(MR_p^0)} &\leq C\,\|B\|_{\mathcal{L}(MR_p^0,L^p)} \\ &\leq C \sup_{\|u\|_{MR_p^0}\leq 1} \|Bu\|_{L^p(0,T;X)} \\ &\leq CT^{1/p} \sup_{\|u\|_{MR_p^0}\leq 1} \|Bu\|_{C([0,T];X)} \\ &\leq CT^{1/p}\|B\|_{\mathcal{L}(Tr_p,X)} \sup_{\|u\|_{MR_p^0}\leq 1} \|u\|_{C([0,T];Tr_p)} \\ &\leq 2CT^{1/p}\|B\|_{\mathcal{L}(Tr_p,X)}. \end{aligned}$$

Hence, if $T > 0$ is small enough, then, by the Neumann series, $I + S_0^{-1}B$ is invertible in $MR_p^0(0,T;X,D)$. As a consequence, the operator $S = S_0(I + S_0^{-1}B)$ is invertible for $T > 0$ small enough. Hence, $A + B$ has L^p-maximal regularity. □

Example 2.6. Let $\Omega = (0,1)$, $f \in C^\infty(\mathbb{R})$, and consider the semilinear heat equation

$$\begin{cases} u_t - u_{xx} + f(u) = 0, & (t,x) \in \mathbb{R}_+ \times (0,1), \\ u(t,0) = u(t,1) = 0, & t \in \mathbb{R}_+, \\ u(0,x) = u_0(x), & x \in (0,1). \end{cases} \tag{2.2}$$

Corollary 2.7. *For every $u_0 \in H^1_0(0,1)$ there exists a unique local solution $u \in W^{1,2}(0,T;L^2(0,1)) \cap L^2(0,T;H^2 \cap H^1_0(0,1))$ of the semilinear heat equation* (2.2) *satisying*

$$u \in C^\infty((0,T) \times (0,1)).$$

Proof. The negative Dirichlet-Laplace operator on $L^2(0,1)$ given by

$$-\Delta_{L^2} : H^2 \cap H^1_0(0,1) \to L^2(0,1), \quad u \mapsto -u_{xx},$$

has L^2-maximal regularity by Lions' theorem and by interpolation.

The Nemytski operator

$$F : H^1_0(0,1) \to L^2(0,1), \quad u \mapsto f(u),$$

is of class C^∞. In order to see this, one may use the embedding of $H^1_0(0,1)$ into $C_0(0,1)$. In particular, F is locally Lipschitz continuous.

By Theorem 1.1, the heat equation (2.2) admits a unique local solution in the maximal regularity space. By Theorem 2.2, this solution even satisfies

$$u \in C^\infty(]0,T];H^2(0,1)). \tag{2.3}$$

Note that for every $k \in \mathbb{N}$ one has

$$F(H^k(0,1)) \subset H^k(0,1),$$

and the restriction of the Nemytski operator F to $H^k(0,1)$ is again of class C^∞. In particular, by (2.3),

$$\partial_t u,\ f(u) \in C^\infty(]0,T];H^2(0,1)),$$

which by the heat equation implies

$$u_{xx} \in C^\infty(]0,T];H^2(0,1))$$

or

$$u \in C^\infty(]0,T];H^4(0,1))$$

By induction, one shows that for every $k \in \mathbb{N}$

$$u \in C^\infty(]0,T];H^k(0,1)) \hookrightarrow C^\infty(]0,T];C^{k-1}([0,1])).$$

In particular, all the partial derivatives of u exist and are continuous. The claim follows. □

3. * Navier-Stokes equations: local existence of regular solutions

In this section we are looking for (regular) solutions

$$\begin{aligned} u : [0,T] \times \mathbb{R}^n &\to \mathbb{R}^n, \\ p : [0,T] \times \mathbb{R}^n &\to \mathbb{R} \end{aligned}$$

of the Navier-Stokes equation

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, & (t,x) \in [0,T] \times \mathbb{R}^n, \\ \operatorname{div} u = 0, & (t,x) \in [0,T] \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.1}$$

The Navier-Stokes equation is actually a system of n equations which we may also write in the form

$$\begin{cases} \partial_t u_i - \Delta u_i + \sum_{j=1}^n u_j \partial_{x_j} u_i + \partial_{x_i} p = 0, & (t,x) \in [0,T] \times \mathbb{R}^n, \\ \operatorname{div} u = 0, & (t,x) \in [0,T] \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

with $u = (u_1, \dots, u_n)$.

The first step in solving the Navier-Stokes equation will be to rewrite it in an abstract functional analytic setting and to obtain an abstract nonlinear evolution equation of parabolic type.

We introduce the following spaces. First the Sobolev space of all *solenoidal* (i.e. divergence free) vector fields

$$H^1_\sigma := H^1_\sigma(\mathbb{R}^n; \mathbb{R}^n) := \{u \in H^1(\mathbb{R}^n; \mathbb{R}^n) : \operatorname{div} u = 0\},$$

and also the Lebesgue space of all solenoidal vector fields

$$L^2_\sigma := L^2_\sigma(\mathbb{R}^n; \mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n; \mathbb{R}^n) : \forall \varphi \in H^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} u \cdot \nabla \varphi = 0\}.$$

Lemma 3.1. *The spaces H^1_σ and L^2_σ are closed subspaces of H^1 and L^2, respectively. The space H^1_σ is a dense subspace of L^2_σ.*

Proof. The closedness of the two spaces is straightforward. Let $u \in H^1_\sigma$ and let $\varphi \in H^1(\mathbb{R}^n)$. Then an integration by parts yields

$$0 = \int_{\mathbb{R}^n} \operatorname{div} u \varphi = \int_{\mathbb{R}^n} u \cdot \nabla \varphi,$$

and hence $u \in L^2_\sigma$. The density of H^1_σ in L^2_σ follows from a usual regularization argument. □

Lemma 3.2. *Let*

$$L^2_\nabla := L^2_\nabla(\mathbb{R}^n; \mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n; \mathbb{R}^n) : \exists v \in H^1(\mathbb{R}^n) \text{ s.t. } u = \nabla v\}$$

be the space of all gradient vector fields. Then

$$L^2_\sigma \perp L^2_\nabla,$$

i.e. the two spaces are orthogonal in $L^2(\mathbb{R}^n;\mathbb{R}^n)$.

Proof. This follows from the very definition of L^2_σ. □

Since L^2_σ is a closed subspace of L^2, there exists the orthogonal projection $P : L^2 \to L^2$ onto the space L^2_σ. This projection is called the *Helmholtz projection*. By Lemma 3.2, $PL^2_\nabla = \{0\}$.

Next, we need the following form $a : H^1_\sigma \times H^1_\sigma \to \mathbb{R}$ which is defined by

$$a(u,v) := \int_{\mathbb{R}^n} \nabla u \nabla v = \sum_{i,j=1}^n \int_{\mathbb{R}^n} \partial_{x_j} u_i \partial_{x_j} v_i .$$

The following lemma is straightforward.

Lemma 3.3. *The form a defined above is bilinear, bounded and elliptic.*

Let $A = A_{L^2_\sigma}$ be the operator on L^2_σ associated with the form a. The operator A is called the *Stokes operator*.

For every $u \in H^2_\sigma = H^2 \cap H^1_\sigma$ and every $v \in H^1_\sigma$ an integration by parts yields

$$\begin{aligned} a(u,v) &= \int_{\mathbb{R}^n} \nabla u \nabla v \\ &= -\int_{\mathbb{R}^n} \Delta u v \\ &= -\int_{\mathbb{R}^n} \Delta u P v \\ &= \int_{\mathbb{R}^n} (-P\Delta u) v, \end{aligned}$$

where we have used that $Pv = v$ since P is the identity on L^2_σ. Hence, for every $u \in H^2_\sigma$ one has $u \in D(A)$ and

$$Au = -P\Delta u.$$

We will use in the following (without proof) that $D(A) = H^2_\sigma$. This can be shown by using the Fourier transform on $L^2(\mathbb{R}^n)$ and the Plancherel theorem.

Lemma 3.4. *The Stokes operator* $A : H^2_\sigma \to L^2_\sigma$ *has* L^2*-maximal regularity.*

Proof. This lemma is a direct consequence of Lions' theorem (Theorem 3.1) and interpolation. See in particular Corollary 5.9. □

Assume for the moment, that the Navier-Stokes equation admits a solution (u, p) such that $u(t,\cdot) \in L^2$, $p(t,\cdot) \in H^1(\mathbb{R}^n)$ and such that the members $\partial_t u(t,\cdot)$ and $\Delta u(t,\cdot)$ belong to L^2 (so that necessarily also the nonlinear term belongs to L^2). Then for each t we can apply the Helmholtz projection to each member and we obtain the following equation

$$\begin{cases} \partial_t Pu - P\Delta u + P((u\cdot\nabla)u) = 0, & (t,x) \in [0,T]\times\mathbb{R}^n, \\ \operatorname{div} u = 0, & (t,x) \in [0,T]\times\mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases}$$

Here we have used that the Helmholtz projection applied to gradient vector fields gives 0. The resulting equation is only an equation in the unknown function u. Since $\operatorname{div} u = 0$, one has $u \in L^2_\sigma$, and the above equation can abstractly be rewritten in the space L^2_σ:

$$\begin{cases} \dot u - Au + P((u\cdot\nabla)u) = 0, & t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{3.2}$$

This is the equation which we will solve by abstract methods. Let

$$MR_2 = W^{1,2}(0,T;L^2_\sigma) \cap L^2(0,T;H^2_\sigma).$$

LEMMA 3.5. *Assume that $n = 2$ or $n = 3$. Then the operator*

$$\begin{aligned} B : MR_2 \times MR_2 &\to L^2(0,T;L^2_\sigma), \\ (u,v) &\mapsto P((u\cdot\nabla)v) \end{aligned}$$

is well-defined, bilinear and bounded (i.e. continuous).

PROOF. We will use the Sobolev embeddings

$$H^1_\sigma \hookrightarrow L^4 \text{ and } H^2_\sigma \hookrightarrow L^\infty$$

which hold true if $n = 2$ or $n = 3$.

These Sobolev embeddings imply the embeddings

$$MR_2 \hookrightarrow C([0,T];H^1_\sigma) \hookrightarrow L^\infty(0,T;L^4)$$

and for every $u \in MR_2$ one has

$$\nabla u \in L^2(0,T;H^1_\sigma) \hookrightarrow L^2(0,T;L^4).$$

By Hölder's inequality, this implies for every $u, v \in MR_2$

$$(u\cdot\nabla)v \in L^2(0,T;L^2),$$

and

$$\|(u\cdot\nabla)v\|_{L^2(0,T;L^2)} \leq C\,\|u\|_{MR_2}\,\|v\|_{MR_2},$$

for some constant $C \geq 0$ independent of u and v. Hence, $P((u\cdot\nabla)v) \in L^2(0,T;L^2_\sigma)$, i.e. B is well-defined. Moreover, by the preceeding inequality, B is also bounded. □

THEOREM 3.6. *Assume that $n = 2$ or $n = 3$. For every $u_0 \in H^1_\sigma$ the equation* (3.2) *admits a unique local solution*

$$u \in W^{1,2}(0,T;L^2_\sigma) \cap L^2(0,T;H^2_\sigma).$$

Sketch of the proof. Let

$$\tilde{M} := \{u \in MR_2 : u(0) = u_0\}.$$

Define the nonlinear operator

$$\begin{aligned} R : \tilde{M} &\to L^2(0,T;L^2_\sigma), \\ u &\mapsto -P((u\cdot\nabla)u). \end{aligned}$$

This operator is well-defined by Lemma 3.5. Actually, for every $u \in \tilde{M}$ one has $Ru = -B(u,u)$.

Define in addition the operator

$$\begin{aligned} S : L^2(0,T;L^2_\sigma) &\to \tilde{M}, \\ f &\mapsto Sf, \end{aligned}$$

which assigns to every f the unique solution $u = Sf$ of the problem

$$\dot{u} + Au = f,\ t \in [0,T], \quad u(0) = u_0,$$

where A is the Stokes operator. The operator S is well-defined by L^2-maximal regularity of the Stokes operator (Lemma 3.4).

Then $u \in MR_2$ is a solution of the abstract Navier-Stokes equation (3.2) if and only if $u \in \tilde{M}$ is a fixed point of $SR : \tilde{M} \to \tilde{M}$.

The rest of the proof is very similar to the proof of Theorem 1.1 on existence and uniqueness of local solutions of the quasilinear problem. In particular, existence and uniqueness for the abstract Navier-Stokes equation follows from Banach's fixed point theorem. We omit this part of the proof. □

Corollary 3.7. *Assume that $n = 2$ or $n = 3$. Then for every $u_0 \in H^1_\sigma$ there exists a unique local solution of the Navier-Stokes equation:*

$$\begin{aligned} u &\in W^{1,2}(0,T;L^2_\sigma) \cap L^2(0,T;H^2_\sigma) \cap C([0,T];H^1_\sigma) \text{ and} \\ u, \nabla p &\in C^\infty(]0,T] \times \mathbb{R}^n;\mathbb{R}^n). \end{aligned}$$

Proof. □

4. * Diffusion equations: comparison principle

In this section, we want to study order preservingness of semilinear diffusion equations of the form

$$\begin{cases} u_t - Lu + f(u) = 0, & (t,x) \in \mathbb{R}_+ \times \Omega, \\ u(t,x) = 0, & t \in \mathbb{R}_+ \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{4.1}$$

where L is a second order elliptic operator of the form

$$Lu = \sum_{i,j} \partial_i a_{ij}(x)\partial_j u + \sum_i (b_i(x)\partial_i u + \partial_i(c_i(x)u)) + d(x)u.$$

We will work on the Hilbert space $H = L^2(\Omega)$. We say that a function $u \in L^2(\Omega)$ is *positive* (and we write $u \geq 0$) if $u(x) \geq 0$ almost everywhere. We also write $u \geq v$ if $u - v \geq 0$. For every pair of functions $u, v \in L^2(\Omega)$ we define the *supremum* $u \vee v$ and the *infimum* $u \wedge v$ respectively by

$$u \vee v(x) := \sup\{u(x), v(x)\} \text{ and } u \wedge v(x) := \inf\{u(x), v(x)\}..$$

Note that $u \vee v$ and $u \wedge v$ belong to $L^2(\Omega)$. For a function $u \in L^2(\Omega)$ we define the *positive part* u^+, the *negative part* u^- and the *absolute value* $|u|$ respectively by

$$u^+ := u \vee 0,\ u^- := (-u) \vee 0 \text{ and } |u| = u^+ + u^-.$$

Note that u^+, u^- and $|u|$ are positive, and $u = u^+ - u^-$. Note also that

$$\|\,|u|\,\|_{L^2} = \|u\|_{L^2}.$$

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set. For every $u \in H_0^1(\Omega)$ one has u^+, $u^- \in H_0^1(\Omega)$ and*

$$\nabla u^+ = 1_{u \geq 0} \nabla u \text{ and } \nabla u^- = 1_{u \leq 0} \nabla u.$$

Proof. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(t) := \begin{cases} 0, & t \leq 0, \\ t, & t \geq 0. \end{cases}$$

Moreover, for every $\varepsilon > 0$ we put

$$g_\varepsilon(t) := \begin{cases} 0, & t \leq 0, \\ \frac{t^2}{2\varepsilon}, & 0 < t < \varepsilon, \\ t - \frac{\varepsilon}{2}, & t \geq \varepsilon. \end{cases}$$

Note that $g_\varepsilon \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ with $\|g_\varepsilon'\|_\infty \leq 1$.

We first show that for every $u \in H_0^1(\Omega)$ one has $g_\varepsilon \circ u \in H_0^1(\Omega)$ and $\nabla g_\varepsilon \circ u = g_\varepsilon'(u)\nabla u$. In fact, given $u \in H_0^1(\Omega)$, there exists a sequence $(u_n) \in \mathcal{D}(\Omega)$ such that $u_n \to u$ in $H_0^1(\Omega)$. By the classical chain rule, for every $n \in \mathbb{N}$, $g_\varepsilon \circ u_n \in H_0^1(\Omega)$ (even $C_c^1(\Omega)$) and $\nabla(g_\varepsilon \circ u_n) = g_\varepsilon'(u_n)\nabla u_n$. In particular, if $\varphi \in \mathcal{D}(\Omega)$, then

$$\int_\Omega g_\varepsilon \circ u_n \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_\varepsilon'(u_n) \frac{\partial u_n}{\partial x_i} \varphi.$$

Letting $n \to \infty$ and using Lebesgue's dominated convergence theorem, we obtain

$$\int_\Omega g_\varepsilon \circ u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_\varepsilon'(u) \frac{\partial u}{\partial x_i} \varphi.$$

This implies the first claim about $g_\varepsilon \circ u$. But in this equation we may now let tend $\varepsilon \to 0$ and use that

$$g_\varepsilon(u) \to u^+ \text{ and } g_\varepsilon'(u) \to 1_{u>0}.$$

By Lebesgue's dominated convergence theorem again, we find that

$$\int_\Omega u^+ \frac{\partial \varphi}{\partial x_i} = -\int_\Omega 1_{u>0} \frac{\partial u}{\partial x_i} \varphi.$$

The claim is proved. □

REMARK 4.2. Lemma 4.1 remains true for functions in $H^1(\Omega)$, but then one should use the fact that $C^1(\Omega) \cap H^1(\Omega)$ is dense in $H^1(\Omega)$, at least if one wants to copy the proof above.

LEMMA 4.3. *For every $u, v \in H_0^1(\Omega)$ one has $u \vee v$, $u \wedge v \in H_0^1(\Omega)$.*

PROOF. Note that

$$u \vee v = v + (u-v)^+ \text{ and } u \wedge v = v - (u-v)^-,$$

and use Lemma 4.1. □

Let V be a Hilbert space which is densely and continuously embedded into $H = L^2(\Omega)$. Let $a : V \times V \to \mathbb{R}$ be a bounded, bilinear, elliptic form. Let $A = A_{L^2}$ be the operator on $L^2(\Omega)$ which is associated with the form a. Let $F : V \to L^2(\Omega)$ be a nonlinear locally Lipschitz continuous operator. Consider the semilinear evolution problem

$$\dot{u} + Au + F(u) = 0, \; t \geq 0, \quad u(0) = u_0. \tag{4.2}$$

By Theorem 1.1 we know that for every $u_0 \in V$ there exists a unique local solution $u \in MR_2(0, T; L^2(\Omega), V)$ of this problem. Let u and v be two solutions of (4.2) and assume that $u_0 \leq v_0$. We prove the following comparison principle under additional assumptions on V, a and F.

THEOREM 4.4 (Comparison principle). *Assume that there exists $\omega \in \mathbb{R}$, $L \geq 0$ such that for every $u \in V$ one has $u^+ \in V$ and*

$$a(u, u^+) + \omega \|u^+\|_{L^2}^2 \geq 0$$

and for every $u, v \in V$,

$$\left|(F(u) - F(v))(u-v)^+\right| \leq L((u-v)^+)^2.$$

Let $u_0, v_0 \in V$, and let u and v be two local solutions of (4.2) (both existing on $[0,T]$) corresponding to the initial condition u_0 and v_0, respectively. Assume that $u_0 \leq v_0$.

Then $u(t) \leq v(t)$ for every $t \in [0,T]$.

PROOF. Since u and v are solutions of (4.2), we have

$$(\dot{u} - \dot{v}) + A(u-v) + F(u) - F(v) = 0.$$

Multiplying this equation scalarly in $H = L^2(\Omega)$ by $(u-v)^+ \in V$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega ((u-v)^+)^2 + a(u-v, (u-v)^+) + \int_\Omega (F(u) - F(v))(u-v)^+ = 0.$$

By hypothesis on a and F, this implies

$$\frac{1}{2}\frac{d}{dt}\|(u-v)^+\|_{L^2}^2 \leq (\omega + L)\|(u-v)^+\|_{L^2}^2.$$

By assumption on u_0 and v_0 one has $(u-v)^+(0) = (u_0 - v_0)^+ = 0$. Hence, by Gronwall's lemma,

$$\|(u(t) - v(t))^+\|_{L^2}^2 = 0 \text{ for every } t \in [0,T],$$

i.e. $(u(t) - v(t))^+ = 0$ or $u(t) \le v(t)$ for every $t \in [0,T]$. □

REMARK 4.5. The condition on a from Theorem 4.4 is satisfied if for every $u \in V$ one has u^+, $u^- \in V$ and

$$a(u^-, u^+) \ge 0.$$

EXAMPLE 4.6. Let $V = H_0^1(\Omega)$ so that V is densely and continuously embedded into $H = L^2(\Omega)$. Let $a : V \times V \to \mathbb{R}$ be the bounded, bilinear, elliptic form given by

$$a(u,v) = \int_\Omega A(x)\nabla u \nabla v + \int_\Omega (b(x)\nabla u v + c(x) u \nabla v) + \int_\Omega d(x) uv.$$

Here, the coefficient matrix $A \in L^\infty(\Omega;\mathbb{R}^{N\times N})$ is uniformly elliptic, and b_i, c_i, $d \in L^\infty(\Omega)$, $b = (b_i)$, $c = (c_i)$.

Let $A = A_{L^2}$ be the operator on $L^2(\Omega)$ which is associated with the form a. Formally, A is a realization of the elliptic operator from the diffusion equation (4.1).

Let $f \in C^1(\mathbb{R})$ be globally Lipschitz continuous and let $F : L^2(\Omega) \to L^2(\Omega)$ be the Nemytski operator associated with f (note that F is also Lipschitz continuous). Consider the semilinear evolution problem (4.1) from the beginning of this section. We know that for every $u_0 \in H_0^1(\Omega)$ there exists a unique solution $u \in MR_2(0,T;L^2(\Omega),D(A))$ of (4.1). By Theorem 4.4, if u and v are two solutions of (4.1), and if $u(0) \le v(0)$, then $u(t) \le v(t)$ for every t in the common interval of existence. In fact, we have to prove the two conditions from Theorem 4.4.

First, if $u \in H_0^1(\Omega)$, then u^+, $u^- \in H_0^1(\Omega)$ by Lemma 4.1 and by Lemma 4.1 one also obtains

$$\begin{aligned}
a(u^-, u^+) &= \int_\Omega A(x)\nabla u^- \nabla u^+ + \int_\Omega (b(x)\nabla u^- u^+ + c(x) u^- \nabla u^+) + \int_\Omega d(x) u^- u^+ \\
&= \int_\Omega A(x)|\nabla u|^2 1_{\{u<0\}} 1_{\{u>0\}} + \int_\Omega b(x)\nabla u u 1_{\{u<0\}} 1_{\{u>0\}} + \\
&\quad + \int_\Omega c(x) u \nabla u 1_{\{u<0\}} 1_{\{u>0\}} + \int_\Omega d(x) uu 1_{\{u<0\}} 1_{\{u>0\}} \\
&= 0.
\end{aligned}$$

Hence, by Remark 4.5, the condition on a is satisfied.

Next, since f is globally Lipschitz continuous, for every u, $v \in H_0^1(\Omega)$,

$$\begin{aligned}
\int_\Omega (f(u) - f(v))(u-v)^+ &= \int_\Omega (f(u) - f(v))(u-v)\, 1_{\{u>v\}} \\
&\ge -L \int_\Omega (u-v)^2 1_{\{u>v\}} \\
&= -L \int_\Omega ((u-v)^+)^2,
\end{aligned}$$

where $L \geq 0$ is the Lipschitz constant of f.

If one assumes in addition that $f(0) = 0$, so that the constant function $u \equiv 0$ is a solution of the diffusion equation (4.1), then the comparison principle yields the following form of the maximum principle. If u is a solution of the diffusion equation (4.1) such that $u(0) \geq 0$, then $u \geq 0$.

Example 4.7. One may also consider the heat equation with Neumann boundary conditions, i.e. the problem

$$\begin{cases} u_t - \Delta u + f(u) = 0, & (t,x) \in \mathbb{R}_+ \times \Omega, \\ \frac{\partial u}{\partial \nu}(t,x) = 0, & t \in \mathbb{R}_+ \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{4.3}$$

Assume that f is globally Lipschitz continuous. The negative Laplacian with Neumann boundary conditions is realized (at least for regular Ω such as intervals or smooth domains) on $L^2(\Omega)$ by the operator $A = A_{L^2}$ which is associated with the form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ given by

$$a(u,v) = \int_\Omega \nabla u \nabla v.$$

It follows from the previous results that for every $u_0 \in H^1(\Omega)$ there exists a unique local solution $u \in MR_2(0,T; L^2(\Omega), D(A))$ of (4.3). Moreover, the comparison principle can be applied.

Assume that $f(0) = 0$ and $f(\bar{u}) = 0$ for some $\bar{u} > 0$. Then the constant functions $u \equiv 0$ and $u \equiv \bar{u}$ are global solutions of the heat equation (4.3). Hence, if an initial value u_0 satisfies $0 \leq u_0 \leq \bar{u}$, and if $u \in MR_2(0,T; L^2(\Omega), D(A))$ is a local solution of (4.3), then the comparison principle implues $0 \leq u \leq \bar{u}$. In particular, the solution u is bounded uniformly in time and space. We will see in the next section that the solution can be extended to a global solution (i.e. existing for all $t \geq 0$).

Example 4.8. Also in this example we show how the comparison principle may be applied in order to prove global existence of solutions. We consider the semilinear heat equation

$$\begin{cases} u_t - u_{xx} - |u|^{p-2}u = 0, & (t,x) \in \mathbb{R}_+ \times (0,1), \\ u(t,0) = u(t,1) = 0, & t \in \mathbb{R}_+, \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{4.4}$$

where $p \geq 2$ is a real parameter. Again, we know that for every $u_0 \in H^1_0(0,1)$ there exists a unique local solution

$$u \in MR_2(0,T; L^2(0,1), H^2 \cap H^1_0(0,1))$$

of this heat equation.

We calculate the *stationary solutions* of (4.4), i.e. the solutions $\varphi \in H^2 \cap H^1_0(0,1)$ of the ordinary differential boundary value problem

$$\begin{cases} -\varphi_{xx} - |\varphi|^{p-2}\varphi = 0, & x \in (0,1), \\ \varphi(0) = \varphi(1) = 0. \end{cases} \tag{4.5}$$

Assume that φ is a solution of this stationary problem. Then a multiplcation of (4.5) by φ_x implies that

$$\frac{d}{dx}(\varphi_x^2 + \frac{1}{p}|\varphi|^p) = 0,$$

or

$$\varphi_x^2 + \frac{1}{p}|\varphi|^p = C \tag{4.6}$$

for a constant C which is necessarily positive. The constant C represents an energy of the solution.

Clearly, every solution φ of the stationary problem (4.5) is necessarily also a solution of the initial value problem

$$\begin{cases} -\varphi_{xx} - |\varphi|^{p-2}\varphi = 0, & x \in (0,1), \\ \varphi(0) = 0, \\ \varphi_x(0) = c. \end{cases} \tag{4.7}$$

The theory of ordinary differential equations implies that for every $c \in \mathbb{R}$ the initial value problem (4.7) admits a unique solution φ existing for all $x \in \mathbb{R}$ and in particular on the interval $[0,1]$. For every such solution φ, the identity (4.6) holds, but $\varphi(1)$ is not necessarily equal to 0.

By identity (4.6), for every solution φ of the initial value problem (4.7) one has

$$\varphi_x(0) = \pm\sqrt{C} = c,$$

and if $x_0 \in \mathbb{R}$ is a local extremum, then

$$\varphi(x_0) = \pm(pC)^{\frac{1}{p}}.$$

Assume that $\varphi_x(0) > 0$ (so that φ is positive on some interval $[0,\varepsilon]$) and let $x_0 \in (0,\infty)$ be the first maximum of φ. The function φ is thus positive and increasing on $[0,x_0]$. By the identity (4.6),

$$\varphi_x = \sqrt{C - \frac{1}{p}\varphi^p},$$

which implies

$$\int_0^{x_0} \frac{\varphi_x}{\sqrt{C - \frac{1}{p}\varphi^p}}\, dx = x_0.$$

After substitution, one obtains

$$\begin{aligned} x_0 &= \int_0^{(pC)^{\frac{1}{p}}} \frac{1}{\sqrt{C - \frac{1}{p} s^p}}\, ds \\ &= \frac{1}{\sqrt{C}} \int_0^{p^{\frac{1}{p}}} \frac{1}{\sqrt{1 - \frac{1}{p} s^p}}\, ds. \end{aligned}$$

The above calculation shows that if we define x_0 as in this last equality, then x_0 is a maximum of φ. Solutions of the initial value problem are then obtained by taking a solution on $[0, x_0]$, extending it by reflection to $[0, 2x_0]$ and $[0, 4x_0]$, and then to extend the thus obtained function $4x_0$-periodically.

Hence, if φ is a solution of the boundary value problem (4.5), then $x_0 = \frac{1}{2n}$ for some $n \in \mathbb{N}$. This shows that solutions of (4.5) exist for a discrete set of energies C_n. For every $n \in \mathbb{N}$, there exist two stationary solutions φ_n and φ_{-n} which have their first extremum in $x_0 = \frac{1}{2n}$. The function φ_n is positive in x_0, the function φ_{-n} is negative in x_0. To this set of stationary solutions one has to add the solution $\varphi_0 \equiv 0$. We see that the stationary problem (4.5) admits a countable number of solutions which form a discrete subset of $H^2 \cap H_0^1(0, 1)$.

Let φ_1 and φ_{-1} be the solutions of (4.5) which have exactly one maximum (resp. minimum) in $(0, 1)$. Then φ_1 is positive on $(0, 1)$ and φ_{-1} is negative.

The functions u_1 and u_{-1} defined by

$$u_1(t, x) = \varphi_1(x) \text{ and } u_{-1}(t, x) = \varphi_{-1}(x)$$

are global solutions of the heat equation (4.4). If $u_0 \in H_0^1(0, 1)$ is an initial value such that $\varphi_{-1} \leq u_0 \leq \varphi_1$, and if u is the corresponding solution of (4.4), then the comparison principle implies

$$\varphi_{-1}(x) \leq u(t, x) \leq \varphi_1(x).$$

In particular, the solution u remains uniformly bounded in time and space, as long as it exists. We will see in the next section that this implies that for every initial value as above the corresponding solution can be extended to a global solution (i.e. existing for every $t \geq 0$).

5. * Energy methods and stability

CHAPTER 4

Appendix

1. Closed linear operators

For the following, we will have to consider a larger class of linear operators. Whenever X and Y are two Banach spaces, a *linear operator* is a linear mapping $A : D(A) \to Y$ defined on a linear subspace $D(A)$ of X. The space $D(A)$ is called *domain* of A. Note that the domain $D(A)$ need not be a closed linear subspace of X.

Definition 1.1. Let X and Y be two Banach spaces. A linear operator $A : D(A) \to Y$ is called *closed* if its graph

$$G(A) := \{(x, Ax) : x \in D(A)\} \subset X \times Y$$

is closed in the product space $X \times Y$.

Lemma 1.2. *A linear operator $A : D(A) \to Y$ is closed if and only if the following property holds:*

$$\left.\begin{array}{l} D(A) \ni x_n \to x \text{ in } X \text{ and} \\ Ax_n \to y \text{ in } Y \end{array}\right\} \Rightarrow x \in D(A) \text{ and } Ax = y.$$

Proof. It suffices to note that $D(A) \ni x_n \to x$ in X and $Ax_n \to y$ in Y if and only if $G(A) \ni (x_n, Ax_n) \to (x, y)$ in the product space $X \times Y$, by definition of the product topology.

If A is closed and if $G(A) \ni (x_n, Ax_n) \to (x, y)$ then $(x, y) \in G(A)$ by the closedness of A and thus $x \in D(A)$ and $y = Ax$.

Conversely, if $G(A) \ni (x_n, Ax_n) \to (x, y)$ implies necessarily $x \in D(A)$ and $y = Ax$, then $(x, y) \in G(A)$, i.e. $G(A)$ is closed, and thus A is closed. □

Lemma 1.3. *A linear operator $A : D(A) \to Y$ is closed if and only if its domain $D(A)$ equipped with the graph norm*

$$\|x\|_{D(A)} := \|x\|_X + \|Ax\|_Y, \quad x \in D(A),$$

is a Banach space.

Proof. If A is closed, then, by definition, $G(A)$ is a closed subspace of the product space $X \times Y$. Since $X \times Y$ is a Banach space, the graph $G(A)$ is a Banach space. Now note that $D(A)$ equipped with the graph norm and $G(A)$ equipped with the product norm are isometrically isomorphic under the isometry $D(A) \to G(A)$, $x \mapsto (x, Ax)$. Hence $D(A)$ equipped with the graph norm is a Banach space.

Conversely, assume that $D(A)$ equipped with the graph norm is a Banach space. Then $G(A)$ (equipped with the product norm from $X \times Y$) is a Banach space by the same argument as before. In particular, $G(A)$ is a closed subspace of $X \times Y$. Hence, A is closed. □

LEMMA 1.4. *Every bounded linear operator $T : X \to Y$ (with domain $D(T) = X$) is closed.*

PROOF. Let $T \in \mathcal{L}(X, Y)$. The norms $\|\cdot\|_X$ and $\|\cdot\|_{D(T)}$ are equivalent norms on X which is a Banach space for the norm $\|\cdot\|_X$. Hence $X = D(T)$ is a Banach space for the norm $\|\cdot\|_{D(T)}$. By Lemma 1.3, T is closed. □

The following theorem is a fundamental theorem in functional analysis. It is a consequence of Baire's theorem, but it will not be proved here.

THEOREM 1.5 (Closed graph theorem). *Let X and Y be Banach spaces and let $T : X \to Y$ (with domain $D(T) = X$) be closed. Then T is bounded.*

EXAMPLE 1.6. Let $X = Y = C([0, 1])$ be the space of continuous functions on $[0, 1]$ with norm $\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|$. Define the *derivation operator D* by

$$D(D) := C^1([0, 1]) \text{ and } Df := f' \text{ for } f \in D(D).$$

Then D is closed. In fact, the space $C^1([0, 1])$ is a Banach space for the graph norm $\|f\|_{D(D)} = \|f\|_\infty + \|f'\|_\infty$ (exercice).

EXAMPLE 1.7. Let $X = Y = L^p(\mathbb{R})$ $(1 \le p \le \infty)$ with norm $\|\cdot\|_p$. Define the *multiplication operator M* by

$$D(M) := \{f \in L^p(\mathbb{R}) : xf(x) \in L^p(\mathbb{R})\} \text{ and } (Mf) := xf(x) \text{ for } f \in D(M).$$

Then M is closed. In fact,

$$D(M) = L^p(\mathbb{R}; (1 + |x|^p)\, dx),$$

and the graph norm $\|\cdot\|_{D(M)}$ is equivalent to the norm

$$\|f\|_{L^p(\mathbb{R};(1+|x|^p)\, dx)} := \left(\int_{\mathbb{R}} |f|^p (1 + |x|^p)\, dx \right)^{1/p},$$

which makes $L^p(\mathbb{R}; (1 + |x|^p)\, dx)$ a Banach space.

2. Vector-valued L^p spaces

As before X denotes a Banach space. In this section $(\Omega, \mathcal{A}, \mu)$ is a measure space.

DEFINITION 2.1. (a) A function $f : \Omega \to X$ is called *step function*, if there exists a sequence $(A_n) \subset \mathcal{A}$ of mutually disjoint measurable sets and a sequence $(x_n) \subset X$ such that $f = \sum_n 1_{A_n} x_n$.
(b) A function $f : \Omega \to X$ is called *mesurable*, if there exists a sequence (f_n) of step functions $f_n : \Omega \to X$ such that $f_n \to f$ pointwise μ-almost everywhere.

REMARK 2.2. Note that there may be a difference to the definition of mesurability of a scalar valued functions. Measurability of a function is here depending on the measure μ. However, if the measure space $(\Omega, \mathcal{A}, \mu)$ is *complete* in the sense that $\mu(A) = 0$ and $B \subset A$ implies $B \in \mathcal{A}$, then the above definition of measurability and the classical definition of measurability coincide. Note that one may always consider complete measure spaces.

LEMMA 2.3. *If $f : \Omega \to X$ is measurable, then $\|f\| : \Omega \to \mathbb{R}$ is measurable. More generally, if $f : \Omega \to X$ is measurable and if $g : X \to Y$ is continuous, then $g \circ f : \Omega \to Y$ is measurable.*

PROOF. This is an easy consequence of the definition of measurability and the continuity of g. Note that in particular the norm $\|\cdot\| : X \to \mathbb{R}$ is continous. □

LEMMA 2.4. *If $f : \Omega \to X$ and $g : \Omega \to \mathbb{K}$ are measurable, then $fg : \Omega \to X$ is measurable.*

Similarly, if $f : \Omega \to X$ and $g : \Omega \to X'$ are measurable, then $\langle g, f\rangle_{X',X} : \Omega \to \mathbb{K}$ is measurable.

THEOREM 2.5 (Pettis). *A function $f : \Omega \to X$ is measurable if and only if $\langle x', f\rangle$ is measurable for every $x' \in X'$ (we say that f is* weakly measurable*) and if there exists a μ-null set $N \in \mathcal{A}$ such that $f(\Omega \setminus N)$ is separable.*

For a proof of Pettis' theorem, see HILLE & PHILLIPS [**11**].

COROLLARY 2.6. *If (f_n) is a sequence of measurable functions $\Omega \to X$ such that $f_n \to f$ pointwise μ-almost everywhere, then f is measurable.*

PROOF. We assume that this corollary is known in the scalar case, i.e. when $X = \mathbb{K}$.

By Pettis's theorem, for all n there exists a μ null set $N_n \in \mathcal{A}$ such that $f_n(\Omega \setminus N_n)$ is separable. Moreover there exists a μ null set $N_0 \in \Omega$ such that $f_n(t) \to f(t)$ for all $t \in \Omega \setminus N_0$. Let $N := \bigcup_{n \geq 0} N_n$; as a countable union of μ null sets, N is a μ null set.

Then f (restricted to $\Omega \setminus N$) is the pointwise limit everywhere of the sequence (f_n). In particular f is weakly measurable. Moreover, $f(\Omega \setminus N)$ is separable since

$$f(\Omega \setminus N) \subset \overline{\bigcup_n f_n(\Omega \setminus N)},$$

and since $f_n(\Omega \setminus N)$ is separable. The claim follows from Pettis' theorem. □

DEFINITION 2.7. A measurable function $f : \Omega \to X$ is called *integrable* if $\int_\Omega \|f\| \, d\mu < \infty$.

LEMMA 2.8. *For every integrable step function $f : \Omega \to X$, $f = \sum_n 1_{A_n} x_n$ the series $\sum_n x_n \mu(A_n)$ converges absolutely and it is independent of the representation of f.*

PROOF. Let $f = \sum_n 1_{A_n} x_n$ be an integrable step function. The sets $(A_n) \subset \mathcal{A}$ are mutually disjoint and $(x_n) \subset X$. Then

$$\sum_n \|x_n\| \, \mu(A_n) = \int_\Omega \|f\| \, d\mu < \infty.$$

$\square$

DEFINITION 2.9 (Bochner integral for integrable step functions). Let $f : \Omega \to X$ be an integrable step function, $f = \sum_n 1_{A_n} x_n$. We define

$$\int_\Omega f \, d\mu := \sum_n x_n \, \mu(A_n).$$

LEMMA 2.10. *(a) For every integrable function $f : \Omega \to X$ there exists a sequence (f_n) of integrable step functions $\Omega \to X$ such that $\|f_n\| \le \|f\|$ and $f_n \to f$ pointwise μ-almost everywhere.*
(b) Let $f : \Omega \to X$ be integrable. Let (f_n) be a sequence of integrable step functions such that $\|f_n\| \le \|f\|$ and $f_n \to f$ pointwise μ-almost everywhere. Then

$$x := \lim_{n\to\infty} \int_\Omega f_n \, d\mu \text{ exists}$$

and

$$\|x\| \le \int_\Omega \|f\| \, d\mu.$$

PROOF. (a) Let $f : \Omega \to X$ be integrable. Then $\|f\| : \Omega \to \mathbb{R}$ is integrable. Therefore there exists a sequence (g_n) of integrable step functions such that $0 \le g_n \le \|f\|$ and $g_n \to \|f\|$ pointwise μ-almost everywhere.

Since f is measurable, there exists a sequence $(\tilde{f}_n)$ of step functions $\Omega \to X$ such that $\tilde{f}_n \to f$ pointwise μ-almost everywhere.

Put

$$f_n := \frac{\tilde{f}_n \, g_n}{\|\tilde{f}_n\| + \frac{1}{n}}.$$

(b) For every integrable step function $g : \Omega \to X$ one has

$$\Big\| \int_\Omega g \, d\mu \Big\| \le \int_\Omega \|g\| \, d\mu.$$

Hence, for every n, m

$$\Big\| \int_\Omega f_n - f_m \, d\mu \Big\| \le \int_\Omega \|f_n - f_m\| \, d\mu,$$

and by Lebesgue's dominated convergence theorem the sequence $(\int_\Omega f_n \ d\mu)$ is a Cauchy sequence. When we put $x = \lim_{n\to\infty} \int_\Omega f_n \, d\mu$ then

$$\|x\| \le \liminf_{n\to\infty} \int_\Omega \|f_n\| \, d\mu = \int_\Omega \|f\| \, d\mu.$$

$\square$

DEFINITION 2.11 (Bochner integral for integrable functions). Let $f : \Omega \to X$ be integrable. We define

$$\int_\Omega f \, d\mu := \lim_{n\to\infty} \int_\Omega f_n \, d\mu,$$

where (f_n) is a sequence of step functions $\Omega \to X$ such that $\|f_n\| \le \|f\|$ and $f_n \to f$ pointwise μ-almost everywhere.

REMARK 2.12. The definition of the Bochner integral for integrable functions is independent of the choice of the sequence (f_n) of step functions, by Lemma 2.10.

REMARK 2.13. We will also use the follwing notation for the Bochner integral:

$$\int_\Omega f \text{ oder } \int_\Omega f(t) \, d\mu(t),$$

and if $\Omega = (a, b)$ is an interval in $\mathbb{R}$:

$$\int_a^b f \text{ oder } \int_a^b f(t) \, d\mu(t).$$

If $\mu = \lambda$ is the Lebesgue measure then we also write

$$\int_a^b f(t) \, dt.$$

LEMMA 2.14. *Let $f : \Omega \to X$ be integrable and $T \in \mathcal{L}(X, Y)$. Then $Tf : \Omega \to Y$ is integrable and*

$$\int_\Omega Tf \, d\mu = T \int_\Omega f \, d\mu.$$

PROOF. Exercise. □

THEOREM 2.15 (Lebesgue, dominates convergence). *Let (f_n) be a sequence of integrable functions. Suppose there exists an integrable function $g : \Omega \to \mathbb{R}$ and an (integrable) measurable function $f : \Omega \to X$ such that $\|f_n\| \le g$ and $f_n \to f$ pointwise μ-almost everywhere. Then*

$$\int_\Omega f \, d\mu = \lim_{n\to\infty} \int_\Omega f_n \, d\mu.$$

PROOF. Exercise. □

DEFINITION 2.16 ($\mathcal{L}^p$ spaces). For every $1 \le p < \infty$ we define

$$\mathcal{L}^p(\Omega; X) := \{f : \Omega \to X \text{ measurable} : \int_\Omega \|f\|^p \, d\mu < \infty\}.$$

We also define

$$\mathcal{L}^\infty(\Omega; X) := \{f : \Omega \to X \text{ measurable} : \exists C \ge 0 \text{ such that } \mu(\{\|f\| \ge C\}) = 0\}.$$

Lemma 2.17. *For every* $1 \leq p < \infty$ *we put*

$$\|f\|_p := \Big(\int_\Omega \|f\|^p \, d\mu\Big)^{1/p}.$$

We also put

$$\|f\|_\infty := \inf\{C \geq 0 : \mu(\{\|f\| \geq C\}) = 0\}.$$

Then $\|\cdot\|_p$ *is a seminorm on* $\mathcal{L}^p(\Omega; X)$ *(*$1 \leq p \leq \infty$*).*

Remark 2.18. A function $\|\cdot\| : X \to \mathbb{R}_+$ on a real or complex vector space is called a *seminorm* if

(i) $x = 0 \Rightarrow \|x\| = 0$,
(ii) $\|\lambda x\| = |\lambda| \, \|x\|$ for every $\lambda \in \mathbb{K}$ and all $x \in X$,
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Definition 2.19 (L^p spaces). For every $1 \leq p \leq \infty$ we put

$$\begin{aligned} N_p &:= \{f \in \mathcal{L}^p(\Omega; X) : \|f\|_p = 0\} \\ &= \{f \in \mathcal{L}^p(\Omega; X) : f = 0 \, \mu\text{-almost everywhere}\}. \end{aligned}$$

We define the quotient space

$$L^p(\Omega; X) := \mathcal{L}^p(\Omega; X)/N_p,$$

which is the space of all equivalence classes

$$[f] := f + N_p, \quad f \in \mathcal{L}^p(\Omega; X).$$

Lemma 2.20. *For every* $[f] \in L^p(\Omega; X)$ *(*$f \in \mathcal{L}^p(\Omega; X)$*) the value*

$$\|[f]\|_p := \|f\|_p$$

is well defined, i.e. independent of the representant f*. The function* $\|\cdot\|_p$ *is a norm on* $L^p(\Omega; X)$*. The space* $L^p(\Omega; X)$ *is a Banach space when equipped with this norm.*

Remark 2.21. As in the scalar case we will in the following identify *functions* $f \in \mathcal{L}^p(\Omega; X)$ with their *equivalence classes* $[f] \in L^p(\Omega; X)$, and we say that L^p is a *function space* although we should be aware that it is only a space of equivalence classes of functions.

Remark 2.22. For $\Omega = (a, b)$ an interval in $\mathbb{R}$ and for $\mu = \lambda$ the Lebesgue measure we simply write

$$L^p(a, b; X) := L^p((a, b); X).$$

We can do so since the spaces $L^p([a, b]; X)$ and $L^p((a, b); X)$ coincide since the end points $\{a\}$ and $\{b\}$ have Lebesgue measure zero and there is no danger of confusion.

Lemma 2.23. *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded. Then* $C(\overline{\Omega}; X) \subset L^p(\Omega; X)$ *for every* $1 \leq p \leq \infty$*.*

Proof. Actually, for finite measure spaces, we have the more general inclusions

$$L^\infty(\Omega; X) \subset L^p(\Omega; X) \subset L^q(\Omega; X) \subset L^1(\Omega; X)$$

if $1 \leq q \leq p \leq \infty$. $\square$

LEMMA 2.24. *Let the measure space* $(\Omega, \mathcal{A}, \mu)$ *be such that* $L^p(\Omega)$ *is separable for* $1 \leq p < \infty$ *(e.g.* $\Omega \subset \mathbb{R}^n$ *be an open set with the Lebesgue measure). Let* X *be separable. Then* $L^p(\Omega; X)$ *is separable for* $1 \leq p < \infty$.

PROOF. By assumption the spaces $L^p(\Omega)$ and X are separable. Let $(h_n) \subset L^p(\Omega; X)$ and $(x_n) \subset X$ be two dense sequences. Then the set

$$\mathcal{F} := \{f : \Omega \to X : f = h_n \, x_m\}$$

is countable. It suffices to shows that $\mathcal{F} \subset L^p(\Omega; X)$ is total, i.e. $\operatorname{span} \mathcal{F}$ is dense in $L^p(\Omega; X)$. This is an exercise. □

THEOREM 2.25. *Let* Ω *be as in lemma 2.24. Let* $1 < p < \infty$ *and assume that* X *is reflexive. Then the space* $L^p(\Omega; X)$ *is reflexive and*

$$L^p(\Omega; X)' \cong L^{p'}(\Omega; X').$$

PROOF. Without proof. □

3. Vector-valued Sobolev spaces

DEFINITION 3.1 (Sobolev spaces). Let $-\infty \leq a < b \leq \infty$ and $1 \leq p \leq \infty$. We define

$$W^{1,p}(a, b; X) := \{u \in L^p(a, b; X) \ : \ \exists v \in L^p(a, b; X) \, \forall \varphi \in \mathcal{D}(a, b) \ \int_a^b u\varphi' = -\int_a^b v\varphi\}.$$

Notation: $v =: u'$.

LEMMA 3.2. *For every* $-\infty \leq a < b \leq \infty$ *and every* $1 \leq p \leq \infty$ *one has* $W^{1,p}(a, b; X) \subset C^b(\overline{(a, b)}; X)$*. For every* $u \in W^{1,p}(a, b; X)$ *and every* $s, t \in (a, b)$ *one has*

$$u(t) - u(s) = \int_s^t u'(r) \, dr.$$

Bibliographie

1. H. Amann, *Linear and Quasilinear Parabolic Problems: Abstract Linear Theory*, Monographs in Mathematics, vol. 89, Birkhäuser Verlag, Basel, 1995.
2. S. B. Angenent, *Nonlinear analytic semiflows*, Proc. Roy. Soc. Edinburgh **115A** (1990), 91–107.
3. H. Brézis, *Analyse fonctionnelle*, Masson, Paris, 1992.
4. R. Dautray and J.-L. Lions, *Analyse mathématique et calcul numérique pour les sciences et les techniques. Vol. I*, INSTN: Collection Enseignement, Masson, Paris, 1985.
5. ______, *Analyse mathématique et calcul numérique pour les sciences et les techniques. Vol. VIII*, INSTN: Collection Enseignement, Masson, Paris, 1987.
6. ______, *Analyse mathématique et calcul numérique pour les sciences et les techniques. Vol. VI*, INSTN: Collection Enseignement, Masson, Paris, 1988.
7. J. Escher, J. Prüss, and G. Simonett, *A new approach to the regularity of solutions of parabolic equations*, Evolution Equations: Proceedings in Honor of J. A. Goldstein's 60th Birthday, Marcel Dekker, New York, 2003, pp. 167–190.
8. E. Feireisl and F. Simondon, *Convergence for semilinear degenerate parabolic equations in several space dimensions*, J. Dynam. Differential Equations **12** (2000), 647–673.
9. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer Verlag, Berlin, Heidelberg, New York, 2001.
10. A. Haraux, *Systèmes dynamiques dissipatifs et applications*, Masson, Paris, 1990.
11. E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc., Providence, R.I., 1957.
12. P. C. Kunstmann and L. Weis, *Maximal L^p regularity for parabolic equations, Fourier multiplier theorems and H^∞ functional calculus*, Levico Lectures, Proceedings of the Autumn School on Evolution Equations and Semigroups (M. Iannelli, R. Nagel, S. Piazzera eds.), vol. 69, Springer Verlag, Heidelberg, Berlin, 2004, pp. 65–320.
13. J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.
14. E. M. Ouhabaz, *Analysis of Heat Equations on Domains*, London Mathematical Society Monographs, vol. 30, Princeton University Press, Princeton, 2004.
15. H. Triebel, *Theory of Function Spaces*, Birkhäuser, Basel, 1983.