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Quelques méthodes de résolution pour les équations non-linéaires

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CHAPTER 1

Introduction

1. * Examples of nonlinear problems

1.1. Roots of polynomials. Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial.

Problem: Prove existence of a root of p, that is, prove that the equation

$$p(z) = 0$$

admits a solution. If possible, try to find an explicit formula for a solution, or try to locate a solution.

The same questions may be asked for polynomials $p : \mathbb{C}^n \to \mathbb{C}^n$.

1.2. Ordinary differential equations. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function, and let $x_0 \in \mathbb{R}^n$. Prove existence (and uniqueness) of a local solution of the ordinary differential equation with initial value

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0.$$

1.3. Optimization problems. Let $j : \mathbb{R} \to \mathbb{R}$ be a convex function and define the cost functional J on the space $C([0, 1])$ by

$$J(u) = \int_0^1 j(u(s))\, ds, \quad u \in C([0, 1]).$$

Prove that the cost functional J admits a global (or local) minimum.

1.4. Nonlinear diffusion. Let $\Omega \subset \mathbb{R}^n$ be a an open set. Let $u : [0, T] \times \Omega \to \mathbb{R}$ be some function depending on a time variable $t \in [0, T]$ and a space variable $x \in \Omega$. For example, this function may in the applications be an energy density, a population density, or an image.

In the following, we think of u being an energy density. If $O \subset \Omega$ is a small volume (with smooth boundary ∂O), then

$$\int_O u(t, x)\, dx$$

is the total energy in the volume O at time t. The total energy in O can only change if there is an energy transport through the boundary, or if there is an energy source within O. According to Fourier's law, an energy transport is only possible in the opposite direction of the gradient ∇u; recall that the gradient ∇u points into the direction in which u increases most, in particular, into the direction in which there is a higher energy density, and energy transport is directed to regions with lower energy density.

Hence,

$$\frac{\partial}{\partial t}\int_O u\,dx = \int_{\partial O} a(|\nabla u|)\frac{\nabla u}{|\nabla u|} n\,d\sigma,$$

where $a : \mathbb{R}_+ \to \mathbb{R}_+$ is some given function (the diffusion coefficient function), the integral over the boundary ∂O is taken with respect to the surface measure and $n = n(x)$ is the outer normal in a point $x \in \partial O$.

By changing the order of differentiation and integration on the left-hand side, and by applying the divergence theorem to the integral on the right-hand side, we obtain

$$\int_O \frac{\partial u}{\partial t}\,dx = \int_O \operatorname{div}\big(a(|\nabla u|)\frac{\nabla u}{|\nabla u|}\big)\,dx.$$

Since this last inequality holds for every arbitrary volume $O \subset \Omega$, we obtain that the energy density u satisfies the following partial differential equation:

$$\frac{\partial u}{\partial t} - \operatorname{div}\big(a(|\nabla u|)\frac{\nabla u}{|\nabla u|}\big) = 0. \tag{1.1}$$

This is a quite general example of a diffusion equation which appears in heat conduction, population dynamics, geometric flows, image analysis, . . . , depending on the choice for the diffusion coefficient a.

For example, if we choose $a(s) = s$, then

$$\operatorname{div}\big(a(|\nabla u|)\frac{\nabla u}{|\nabla u|}\big) = \operatorname{div}\nabla u =: \Delta u$$

is the *Laplace operator*, and the equation (1.1) is the linear diffusion equation

$$\frac{\partial u}{\partial t} - \Delta u = 0.$$

If the diffusion coefficient is nonlinear but homogeneous, for example if $a(s) = s^{p-1}$ for some $p \geq 1$, then

$$\operatorname{div}\big(a(|\nabla u|)\frac{\nabla u}{|\nabla u|}\big) = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) =: \Delta_p u$$

is the p-Laplace operator, and the equation (1.1) becomes the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} - \Delta_p u = 0$$

involving the p-Laplace operator. Note that the 2-Laplace operator is just the Laplace operator defined before. This equation will serve as a model problem for nonlinear diffusion.

In the applications, other diffusion coefficients appear. For example, the function $a(s) = \frac{s}{\sqrt{1+s^2}}$ leads to the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\big) = 0$$

which is related to the mean curvature flow of surfaces, and only slightly different diffusion coefficients are also used in image analysis.

1.5. Nonlinear elliptic problems. Instead of the time dependent problems from the previous section, we may also consider the stationary (time-independent) problems

$$-\Delta_p u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \partial\Omega,$$

or, more generally,

$$-\operatorname{div}\big(a(|\nabla u|)\frac{\nabla u}{|\nabla u|}\big) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \partial\Omega.$$

Problem: Prove that for every f in a certain class of functions there exists a unique solution u.

Before solving this problem, one actually has to define the notion of *solution*; for example, one has to say in which class of functions a solution should live, and in which sense it solves the PDEs above.

2. The Sobolev space $W^{1,p}(\Omega)$

Let $\Omega \subset \mathbb{R}^n$ be an open set. For every function $u \in C^1(\Omega)$ we define its *support* by

$$\operatorname{supp} u := \overline{\{x \in \Omega : u(x) \neq 0\}};$$

the closure is to be taken in $\mathbb{R}^n$. Then we define the space of all compactly supported C^1 functions:

$$C^1_c(\Omega) := \{u \in C^1(\Omega) : \operatorname{supp} u \text{ is compact and contained in } \Omega\}.$$

For every $1 \le p < \infty$ we define the *Sobolev space*

$$W^{1,p}(\Omega) := \{u \in L^p(\Omega) : \forall 1 \le i \le n\, \exists v_i \in L^p(\Omega)\, \forall \varphi \in C^1(\Omega) \\ \int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega v_i \varphi\}.$$

We note that the elements v_i are uniquely determined, if they exist, and we write $\frac{\partial u}{\partial x_i} := v_i$. We call $\frac{\partial u}{\partial x_i}$ the *weak partial derivative* of u with respect to x_i.

We equip the space $W^{1,p}(\Omega)$ with the norm

$$\|u\|_{W^{1,p}} := (\|u\|^p_{L^p} + \sum_{i=1}^n \|\frac{\partial u}{\partial x_i}\|^p_{L^p})^{\frac{1}{p}}.$$

Then the space $W^{1,p}(\Omega)$ is a Banach space.

We further define

$$W_0^{1,p}(\Omega) := \overline{C_c^1(\Omega)}^{\|\cdot\|_{W^{1,p}}}.$$

Whenever X is a Banach space, we denote by X' its dual space, which is the space

$$X' := \{x' : X \to \mathbb{R} : x' \text{ is linear and continuous}\}.$$

It is equipped with the norm

$$\|x'\|_{X'} := \sup_{\|x\|_X \leq 1} |x'(x)|.$$

Instead of $x'(x)$ we will also write $\langle x', x\rangle_{X',X}$.

The dual space of $W_0^{1,p}(\Omega)$ is denoted by $W^{-1,p'}(\Omega)$ with $p' = \frac{p}{p-1}$, that is

$$W_0^{1,p}(\Omega)' =: W^{-1,p'}(\Omega).$$

For every $u \in L^{p'}(\Omega)$ and every $1 \leq i \leq n$ we define the *weak partial derivative* $\frac{\partial u}{\partial x_i}$ as an element in $W^{-1,p'}(\Omega)$ by

$$\langle \frac{\partial u}{\partial x_i}, v\rangle_{W^{-1,p'},W^{1,p}} := -\int_\Omega u \frac{\partial v}{\partial x_i}\, dx.$$

Lemma 2.1. *For every $1 \leq p < \infty$, the operators*

$$\begin{aligned} \frac{\partial}{\partial x_i} : W^{1,p}(\Omega) &\to L^p(\Omega), \\ u &\mapsto \frac{\partial u}{\partial x_i}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial x_i} : L^{p'}(\Omega) &\to W^{-1,p'}(\Omega), \\ u &\mapsto \frac{\partial u}{\partial x_i} \end{aligned}$$

are linear and continuous.

Proof. The two operators are clearly linear. For the first operator, one has

$$\|\frac{\partial u}{\partial x_i}\|_{L^p} \leq \|u\|_{W^{1,p}},$$

by the definition of the norm in $W^{1,p}$. For the second operator, one calculates, using Hölder's inequality,

$$\begin{aligned} \|\frac{\partial u}{\partial x_i}\|_{W^{-1,p'}} &= \sup_{\|v\|_{W_0^{1,p}}\leq 1} |\langle \frac{\partial u}{\partial x_i}, v\rangle_{W^{-1,p'},W_0^{1,p}}| \\ &= \sup_{\|v\|_{W_0^{1,p}}\leq 1} |\int u \frac{\partial v}{\partial x_i}| \\ &\leq \sup_{\|v\|_{W_0^{1,p}}\leq 1} \|u\|_{L^{p'}} \, \|\frac{\partial v}{\partial x_i}\|_{L^p} \\ &\leq \|u\|_{L^{p'}}. \end{aligned}$$

Hence, both operators are continuous. □

The following lemma is an immediate consequence of the preceding lemma.

Lemma 2.2. *For every $1 \leq p < \infty$, the operators*

$$\begin{aligned} \operatorname{div} : W^{1,p}(\Omega)^n &\to L^p(\Omega), \\ u = (u_i) &\mapsto \sum_i \frac{\partial u_i}{\partial x_i}, \end{aligned}$$

and

$$\begin{aligned} \operatorname{div} : L^{p'}(\Omega)^n &\to W^{-1,p'}(\Omega), \\ u = (u_i) &\mapsto \sum_i \frac{\partial u_i}{\partial x_i} \end{aligned}$$

are linear and continuous.

3. * The p-Laplace operator

Let $\Omega \subset \mathbb{R}^n$ be an open set. The p-Laplace operator ($p \geq 1$) is the partial differential operator which to every function $u : \Omega \to \mathbb{R}$ assigns the function

$$\Delta_p u(x) := \operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)), \quad x \in \Omega.$$

We simply write Δ instead of Δ_2 and call the 2-Laplace operator simply Laplace operator.

In the following, we will realize the p-Laplace operator as an abstract operator between two Banach spaces and use functional analytic methods in order to solve elliptic and parabolic PDEs involving the p-Laplace operator. We will see that several abstract methods will apply.

DEFINITION 3.1 (p-Laplace operator). Let $1 \le p < \infty$, and let $\Omega \subset \mathbb{R}^n$ be an open set. We define the *Dirichlet p-Laplace operator* on Ω to be the operator

$$\begin{aligned} \Delta_p^\Omega : W_0^{1,p}(\Omega) &\to W^{-1,p'}(\Omega), \\ u &\mapsto \Delta_p^\Omega u := \operatorname{div}(|\nabla u|^{p-2}\nabla u). \end{aligned}$$

LEMMA 3.2. *The Dirichlet p-Laplace operator is well defined and continuous. Moreover, there exist constants $C \ge 0$, $\eta > 0$ such that for every $u \in W_0^{1,p}(\Omega)$*

$$\|\Delta_p^\Omega u\|_{W^{-1,p'}} \le C\, \|u\|_{W^{1,p}}^{p-1}$$

and

$$-\langle \Delta_p^\Omega u, u\rangle_{W^{-1,p'},W_0^{1,p}} \ge \eta\, \|\nabla u\|_{L^p}^p.$$

PROOF. The operator

$$\begin{aligned} \operatorname{div} : L^{p'}(\Omega)^n &\to W^{-1,p'}(\Omega), \\ u = (u_i) &\mapsto \operatorname{div} u := \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} \end{aligned}$$

is linear and continuous by Lemma 2.2, and Δ_p^Ω is the composition of the operator

$$\begin{aligned} D : W_0^{1,p}(\Omega) &\to L^{p'}(\Omega)^n, \\ u &\mapsto |\nabla u|^{p-2}\nabla u, \end{aligned}$$

and the operator div. We show that the operator D is well defined and continuous.

First of all, for every $u \in W_0^{1,p}(\Omega)$

$$\int_\Omega |Du|^{p'} = \int_\Omega |\nabla u|^{(p-1)p'} = \int_\Omega |\nabla u|^p < \infty,$$

which implies that D is well defined. So it remains to show that D is continuous.

Let $(u_n) \subset W_0^{1,p}(\Omega)$ be converging to some $u \in W_0^{1,p}(\Omega)$. Then $\nabla u_n \to \nabla u$ in $L^p(\Omega)^n$. For every convergent sequence in L^p, we find a subsequence which converges almost everywhere and which is dominated by some function in L^p, that is, after passing to a subsequence (!) which we denote again by (u_n), we have $\nabla u_n \to \nabla u$ almost everywhere and $|\nabla u_n| \le g$ for some $g \in L^p(\Omega)$ and all n. Hence, $|\nabla u_n|^{p-2}\nabla u_n \to |\nabla u|^{p-2}\nabla u$ almost everywhere, and $|\nabla u_n|^{p-1} \le g^{p-1} \in L^{\frac{p}{p-1}}(\Omega) = L^{p'}(\Omega)$ for every n. By Lebesgue's dominated convergence theorem, this implies $|\nabla u_n|^{p-2}\nabla u_n \to |\nabla u|^{p-2}\nabla u$ in $L^{p'}(\Omega)$.

We have thus shown that for every convergent sequence $(u_n) \subset W^{1,p}(\Omega)$, $u_n \to u$, we find a subsequence (again denoted by (u_n)) such that $Du_n \to Du$ in $L^{p'}(\Omega)$. This implies that D is continuous, as the following short argument by contradiction shows. Assume that D is not continous. Then there exists a convergent sequence $(u_n) \subset W^{1,p}(\Omega)$, $u_n \to u$, such that (Du_n) does not converge to Du in $L^p(\Omega)$. The property that (Du_n) does not converge to Du means that there exists a subsequence of (u_n) (which we denote again by (u_n)) and some $\varepsilon > 0$ such that

$$\|Du_n - Du\|_{L^{p'}} \ge \varepsilon \text{ for every } n.$$

But the subsequence (u_n) is still convergent to u, and by what has been said before, there exists again a subsequence (again denoted by (u_n)) such that $Du_n \to Du$ in $L^{p'}(\Omega)$, a contradition to the estimate above. Hence, the assumption that D is not continuous must be false, and therefore D is continuous.

It remains to show the two estimates. First of all,

$$\begin{aligned} \|\Delta_p^\Omega u\|_{W^{-1,p'}} &= \sup_{\|v\|_{W_0^{1,p}}\leq 1} |\langle \Delta_p^\Omega u, v\rangle_{W^{-1,p'},W_0^{1,p}}| \\ &= \sup_{\|v\|_{W_0^{1,p}}\leq 1} \Big| \int_\Omega |\nabla u|^{p-2}\nabla u \,\nabla v\Big| \\ &\leq \sup_{\|v\|_{W_0^{1,p}}\leq 1} \|\nabla u\|_{L^p}^{p-1} \, \|\nabla v\|_{L^p} \\ &\leq \|\nabla u\|_{L^p}^{p-1} \\ &\leq \|u\|_{W^{1,p}}^{p-1}. \end{aligned}$$

Secondly,

$$-\langle \Delta_p^\Omega u, u\rangle_{W^{-1,p'},W_0^{1,p}} = \int_\Omega |\nabla u|^p,$$

and the claim is completely proved. □

CHAPTER 2

Minimization of convex functions

In the following, X denotes a Banach space with norm $\|\cdot\|$. The space

$$X' := \{x' : X \to \mathbb{K} : x' \text{ is linear and bounded}\}$$

is the *dual space* of X, that is, the space of all linear and bounded functionals on X. The dual space X' is a Banach space for the norm

$$\|x'\| := \sup_{\substack{x \in X \\ \|x\| \leq 1}} |x'(x)|. \tag{0.1}$$

1. Reflexive Banach spaces

The following theorem, one version of the Hahn-Banach theorem, is standard in any functional analysis course and it will not be proved here.

THEOREM 1.1 (Hahn-Banach; extension of bounded functionals). *Let X be a normed space and $U \subset X$ a linear subspace. Then for every bounded linear $u' : U \to \mathbb{K}$ there exists a bounded linear extension $x' : X \to \mathbb{K}$ (i.e. $x'|_U = u'$) such that $\|x'\| = \|u'\|$.*

COROLLARY 1.2. *If X is a normed space, then for every $x \in X \setminus \{0\}$ there exists $x' \in X'$ such that*

$$\|x'\| = 1 \text{ and } x'(x) = \|x\|.$$

PROOF. By the Hahn-Banach theorem (Theorem 1.1), there exists an extension $x' \in X'$ of the functional $u' : \operatorname{span}\{x\} \to \mathbb{K}$ defined by $u'(\lambda x) = \lambda\|x\|$ such that $\|x'\| = \|u'\| = 1$. □

COROLLARY 1.3. *If X is a normed space, then for every $x \in X$*

$$\|x\| = \sup_{\substack{x' \in X' \\ \|x'\| \leq 1}} |x'(x)|. \tag{1.1}$$

PROOF. For every $x' \in X'$ with $\|x'\| \leq 1$ one has

$$|x'(x)| \leq \|x'\| \, \|x\| \leq \|x\|,$$

which proves one of the required inequalities. The other inequality follows from Corollary 1.2. □

REMARK 1.4. The equality (1.1) should be compared to the definition (0.1) of the norm of an element $x' \in X'$.

From now on, it will be convenient to use the following notation. Given a normed space X and elements $x \in X$, $x' \in X'$, we write

$$\langle x', x\rangle := \langle x', x\rangle_{X'\times X} := x'(x).$$

For the bracket $\langle\cdot,\cdot\rangle$, we note the following properties. The function

$$\begin{aligned} \langle\cdot,\cdot\rangle : X'\times X &\to \mathbb{K}, \\ (x', x) &\mapsto \langle x', x\rangle = x'(x) \end{aligned}$$

is bilinear and for every $x' \in X'$, $x \in X$,

$$|\langle x', x\rangle| \leq \|x'\|\,\|x\|.$$

The bracket $\langle\cdot,\cdot\rangle$ thus appeals to the notion of the scalar product on inner product spaces, and the last inequality appeals to the Cauchy-Schwarz inequality, but note, however, that the bracket is *not* a scalar product since it is defined on a pair of two different spaces. Moreover, even if $X = H$ is a complex Hilbert space, then the bracket differs from the scalar product in that it is bilinear instead of sesquilinear.

Corollary 1.5. *Let X be a normed space, $U \subset X$ a closed linear subspace and $x \in X \setminus U$. Then there exists $x' \in X'$ such that*

$$\langle x', x\rangle \neq 0 \textit{ and } \langle x', u\rangle = 0 \textit{ for every } u \in U.$$

Proof. Let $\pi : X \to X/U$ be the quotient map ($\pi(x) = x + U$). Since $x \notin U$, we have $\pi(x) \neq 0$. By Corollary 1.2, there exists $\varphi \in (X/U)'$ such that $\langle \varphi, \pi(x)\rangle \neq 0$. Then $x' := \varphi \circ \pi \in X'$ is a desired functional we are looking for. □

Corollary 1.6. *If X is a normed space such that X' is separable, then X is separable, too.*

Proof. Let $D' = \{x_n' : n \in \mathbb{N}\}$ be a dense subset of the unit sphere of X'. For every $n \in \mathbb{N}$ we choose an element $x_n \in X$ such that $\|x_n\| \leq 1$ and $|\langle x_n', x_n\rangle| \geq \frac{1}{2}$. We claim that $D := \operatorname{span}\{x_n : n \in \mathbb{N}\}$ is dense in X. If this was not true, i.e. if $\bar{D} \neq X$, then, by Corollary 1.5, we find an element $x' \in X' \setminus \{0\}$ such that $x'(x_n) = 0$ for every $n \in \mathbb{N}$. We may without loss of generality assume that $\|x'\| = 1$. Since D' is dense in the unit sphere of X', we find $n_0 \in \mathbb{N}$ such that $\|x' - x_{n_0}'\| \leq \frac{1}{4}$. But then

$$\frac{1}{2} \leq |\langle x_{n_0}', x_{n_0}\rangle| = |\langle x_{n_0}' - x', x_{n_0}\rangle| \leq \|x_{n_0}' - x'\|\,\|x_{n_0}\| \leq \frac{1}{4},$$

which is a contradiction. Hence, $\bar{D} = X$ and X is separable. □

Given a normed space X, we call

$$X'' := (X')'$$

the *bidual* of X.

Lemma 1.7. *Let X be a normed space. Then the mapping*

$$\begin{aligned} J : X &\to X'', \\ x &\mapsto (x' \mapsto \langle x', x\rangle), \end{aligned}$$

is well defined and isometric.

Proof. The linearity of $x' \mapsto \langle x', x \rangle$ is clear, and from the inequality

$$|Jx(x')| = |\langle x', x \rangle| \leq \|x'\| \, \|x\|,$$

follows that $Jx \in X''$ (i.e. J is well defined) and $\|Jx\| \leq \|x\|$. The fact that J is isometric follows from Corollary 1.2. □

Definition 1.8. A Banach space X is called *reflexive* if the isometry J from Lemma 1.7 is surjective, i.e. if $JX = X''$. In other words: a normed space X is reflexive if for every $x'' \in X''$ there exists $x \in X$ such that

$$\langle x'', x' \rangle = \langle x', x \rangle \text{ for all } x' \in X'.$$

Remark 1.9. It may happen that the spaces X and X'' are isomorphic without X being reflexive (the example of such a Banach space is however quite involved). We emphasize that reflexivity means that the special operator J is an isomorphism.

Lemma 1.10. *Every Hilbert space is reflexive.*

Proof. By the Theorem of Riesz-Fréchet, we may identify H with its dual H' and thus also H with its bidual H''. The identification is done via the scalar product. It should be noted, however, that for complex Hilbert spaces, the identification of H with its dual H' is only antilinear, but after the second identification (H' with H'') it turns out that the identification of H with H'' is linear.

It is finally an exercise to show that this identification of H with H'' coincides with the mapping J from Lemma 1.7. □

Lemma 1.11. *Every finite dimensional Banach space is reflexive.*

Proof. It suffices to remark that if X is finite dimensional, then

$$\dim X = \dim X' = \dim X'' < \infty.$$

Surjectivity of the mapping J (which is always injective) thus follows from linear algebra. □

Theorem 1.12. *The space $L^p(\Omega)$ is reflexive if $1 < p < \infty$ ($(\Omega, \mathcal{A}, \mu)$ being an arbitrary measure space).*

Lemma 1.13. *The spaces l^1, $L^1(\Omega)$ ($\Omega \subset \mathbb{R}^N$) and $C([0,1])$ are not reflexive.*

Proof. For every $t \in [0,1]$, let $\delta_t \in C([0,1])'$ be defined by

$$\langle \delta_t, f \rangle := f(t), \quad f \in C([0,1]).$$

Then $\|\delta_t\| = 1$ and whenever $t \neq s$, then

$$\|\delta_t - \delta_s\| = 2.$$

In particular, the uncountably many balls $B(\delta_t, \frac{1}{2})$ ($t \in [0,1]$) are mutually disjoint so that $C([0,1])'$ is not separable.

Now, if $C([0,1])$ were reflexive, then $C([0,1])'' = C([0,1])$ would be separable (since $C([0,1])$ is separable), and then, by Corollary 1.6, $C([0,1])'$ would be separable; a contradiction to what has been said before. This proves that $C([0,1])$ is not reflexive.

The cases of l^1 and $L^1(\Omega)$ are proved similarly. They are separable Banach spaces with nonseparable dual. □

Theorem 1.14. *Every closed subspace of a reflexive Banach space is reflexive.*

Proof. Let X be a reflexive Banach space, and let $U \subset X$ be a closed subspace. Let $u'' \in U''$. Then the mapping $x'' : X' \to \mathbb{K}$ defined by

$$\langle x'', x' \rangle = \langle u'', x'|_U \rangle, \quad x' \in X',$$

is linear and bounded, i.e. $x'' \in X''$. By reflexivity of X, there exists $x \in X$ such that

$$\langle x', x \rangle = \langle u'', x'|_U \rangle, \quad x' \in X'. \tag{1.2}$$

Assume that $x \notin U$. Then, by Corollary 1.3, there exists $x' \in X'$ such that $x'|_U = 0$ and $\langle x', x \rangle \neq 0$; a contradiction to the last equality. Hence, $x \in U$. We need to show that

$$\langle u'', u' \rangle = \langle u', x \rangle, \forall u' \in U'. \tag{1.3}$$

However, if $u' \in U'$, then, by Hahn-Banach we can choose an extension $x' \in X'$, i.e. $x'|_U = u'$. The equation (1.3) thus follows from (1.2). □

Corollary 1.15. *The Sobolev spaces $W^{k,p}(\Omega)$ ($\Omega \subset \mathbb{R}^N$ open) are reflexive if $1 < p < \infty$, $k \in \mathbb{N}$.*

Proof. For example, for $k = 1$, the operator

$$\begin{aligned} T : W^{1,p}(\Omega) &\to L^p(\Omega)^{1+N}, \\ u &\mapsto (u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}), \end{aligned}$$

is isometric, so that we may consider $W^{1,p}(\Omega)$ as a closed subspace of $L^p(\Omega)^{1+N}$ which is reflexive by Theorem 1.12. The claim follows from Theorem 1.14. □

Corollary 1.16. *A Banach space is reflexive if and only if its dual is reflexive.*

Proof. Assume that the Banach space X is reflexive. Let $x''' \in X'''$ (the tridual!). Then the mapping $x' : X \to \mathbb{K}$ defined by

$$\langle x', x \rangle := \langle x''', J_X(x) \rangle, \quad x \in X,$$

is linear and bounded, i.e. $x' \in X'$ (here J_X denotes the isometry $X \to X''$). Let $x'' \in X''$ be arbitrary. Since X is reflexive, there exists $x \in X$ such that $J_X x = x''$. Hence,

$$\langle x''', x'' \rangle = \langle x''', J_X x \rangle = \langle x', x \rangle = \langle x'', x' \rangle,$$

which proves that $J_{X'} x' = x'''$, i.e. the isometry $J_{X'} : X' \to X'''$ is surjective. Hence, X' is reflexive.

On the other hand, assume that X' is reflexive. Then X'' is reflexive by the preceeding argument, and therefore X (considered as a closed subspace of X'' via the isometry J) is reflexive by Theorem 1.14. □

Definition 1.17. Let X be a normed space. We say that a sequence $(x_n) \subset X$ *converges weakly* to some $x \in X$ if

$$\lim_{n\to\infty} \langle x', x_n\rangle = \langle x', x\rangle \text{ for every } x' \in X'.$$

Notations: if (x_n) converges weakly to x, then we write $x_n \rightharpoonup x$, $w - \lim_{n\to\infty} x_n = x$, $x_n \to x$ in $\sigma(X, X')$, or $x_n \to x$ weakly.

Theorem 1.18. *In a reflexive Banach space every bounded sequence admits a weakly convergent subsequence.*

Proof. Let (x_n) be a bounded sequence in a reflexive Banach space X. We first assume that X is separable. Then X'' is separable by reflexivity, and X' is separable by Corollary 1.6. Let $(x'_m) \subset X'$ be a dense sequence.

Since $(\langle x'_1, x_n\rangle)$ is bounded by the boundedness of (x_n), there exists a subsequence $(x_{\varphi_1(n)})$ of (x_n) ($\varphi_1 : \mathbb{N} \to \mathbb{N}$ is increasing, unbounded) such that

$$\lim_{n\to\infty} \langle x'_1, x_{\varphi_1(n)}\rangle \text{ exists.}$$

Similarly, there exists a subsequence $(x_{\varphi_2(n)})$ of $(x_{\varphi_1(n)})$ such that

$$\lim_{n\to\infty} \langle x'_2, x_{\varphi_2(n)}\rangle \text{ exists.}$$

Note that for this subsequence, we also have that

$$\lim_{n\to\infty} \langle x'_1, x_{\varphi_2(n)}\rangle \text{ exists.}$$

Iterating this argument, we find a subsequence $(x_{\varphi_3(n)})$ of $(x_{\varphi_2(n)})$ and finally for every $m \in \mathbb{N}$, $m \geq 2$, a subsequence $(x_{\varphi_m(n)})$ of $(x_{\varphi_{m-1}(n)})$ such that

$$\lim_{n\to\infty} \langle x'_j, x_{\varphi_m(n)}\rangle \text{ exists for every } 1 \leq j \leq m.$$

Let $(y_n) := (x_{\varphi_n(n)})$ be the 'diagonal sequence'. Then (y_n) is a subsequence of (x_n) and

$$\lim_{n\to\infty} \langle x'_m, y_n\rangle \text{ exists for every } m \in \mathbb{N}.$$

Let $x' \in X'$ be arbitrary, and let $\varepsilon > 0$. Since $\{x'_m : m \in \mathbb{N}\}$ is dense in X', there exists $m \in \mathbb{N}$ such that

$$\|x' - x'_m\| \leq \varepsilon.$$

Then there exists $n_0 \in \mathbb{N}$ such that for every $\mu, \nu \geq n_0$

$$|\langle x'_m, y_\mu - y_\nu\rangle| \leq \varepsilon.$$

Hence, for every $\mu, \nu \geq n_0$,

$$\begin{aligned} |\langle x', y_\mu - y_\nu\rangle| &\leq |\langle x' - x'_m, y_\mu - y_\nu\rangle| + |\langle x'_m, y_\mu - y_\nu\rangle| \\ &\leq \varepsilon(2M + 1), \end{aligned}$$

where $M = \sup_n \|y_n\| < \infty$ is independent of ε, μ and ν. As a consequence,

$$\langle x'', x' \rangle := \lim_{n\to\infty} \langle x', y_n \rangle \text{ exists for every } x' \in X',$$

and x'' is a bounded linear functional on X'.

Since X is reflexive, there exists $x \in X$ such that $Jx = x''$. For this x, we have by definition of J

$$\lim_{n\to\infty} \langle x', y_n \rangle = \langle x', x \rangle \text{ exists for every } x' \in X',$$

i.e. (y_n) converges weakly to x.

If X is not separable as we first assumed, then one may replace X by $\tilde{X} := \overline{\operatorname{span}}\{x_n : n \in \mathbb{N}\}$ which is separable. By the above, there exists $x \in \tilde{X}$ and a subsequence of (x_n) (which we denote again by (x_n)) such that for every $\tilde{x}' \in \tilde{X}'$,

$$\lim_{n\to\infty} \langle \tilde{x}', x_n \rangle = \langle \tilde{x}', x \rangle,$$

i.e. (x_n) converges weakly in $\tilde{X}$. If $x' \in X'$, then $x'|_{\tilde{X}} \in \tilde{X}'$, and it follows easily that the sequence (x_n) also converges weakly in X to the element x. □

2. Main theorem

We start by stating a second version of the Hahn-Banach theorem. We will not prove this theorem. We only recall that a subset K of a Banach space X is *convex* if for every $x, y \in K$ and every $t \in [0, 1]$ one has $tx + (1 - t)y \in K$.

Theorem 2.1 (Hahn-Banach; separation of convex sets). *Let X be a Banach space, $K \subset X$ a closed, nonempty, convex subset, and $x_0 \in X \setminus K$. Then there exists $x' \in X'$ and $\varepsilon > 0$ such that*

$$\operatorname{Re} \langle x', x \rangle + \varepsilon \le \operatorname{Re} \langle x', x_0 \rangle, \quad x \in K.$$

Corollary 2.2. *Let X be a Banach space and $K \subset X$ a closed, convex subset (closed for the norm topology). If $(x_n) \subset K$ converges weakly to some $x \in X$, then $x \in K$.*

Proof. Assume the contrary, i.e. $x \notin K$. By the Hahn-Banach theorem (Theorem 2.1), there exist $x' \in X'$ and $\varepsilon > 0$ such that

$$\operatorname{Re} \langle x', x_n \rangle + \varepsilon \le \operatorname{Re} \langle x', x \rangle \text{ for every } n \in \mathbb{N},$$

a contradiction to the assumption that $x_n \rightharpoonup x$. □

A function $f : K \to \mathbb{R} \cup \{+\infty\}$ on a convex subset K of a Banach space X is called *convex* if for every $x, y \in K$, and every $t \in [0, 1]$,

$$f(tx + (1 - t)y) \le t f(x) + (1 - t) f(y). \tag{2.1}$$

Let $K \subset X$ be an arbitrary subset of a Banach space. A function $f : K \to \mathbb{R} \cup \{+\infty\}$ is called *lower semicontinuous* if for every sequence $(x_n) \subset K$ and every $x \in K$ one has

$$x = \lim_{n\to\infty} x_n \quad \Rightarrow \quad f(x) \le \liminf_{n\to\infty} f(x_n).$$

LEMMA 2.3. *A function* $f : K \to \mathbb{R} \cup \{+\infty\}$ *is lower semicontinuous if and only if for every* $c \in \mathbb{R}$ *the set* $\{x \in K : f(x) \le x\}$ *is closed in* K*.*

PROOF. Assume first that f is lower semicontinuous. Let $c \in \mathbb{R}$ and let $K_c := \{x \in K : f(x) \le c\}$. Let $(x_n) \subset K_c$ be a convergent sequence such that $x = \lim_{n\to\infty} x_n \in K$. Then, by lower semicontinuity,

$$f(x) \le \liminf_{n\to\infty} f(x_n) \le c,$$

so that $x \in K_c$. Hence, K_c is closed in K.

Assume now that $K_c := \{x \in K : f(x) \le c\}$ is closed for every $c \in \mathbb{R}$. Let $(x_n) \subset K$ be a convergent sequence such that $x = \lim_{n\to\infty} x_n \in K$. We have to show that $f(x) \le \liminf_{n\to\infty} f(x_n) =: c$. If this inequality was not true then there exists $\varepsilon > 0$ such that

$$f(x) \ge \liminf_{n\to\infty} f(x_n) + \varepsilon = c + \varepsilon.$$

In addition, there exists a subsequence (x_{n_k}) of (x_n) such that $\lim_{k\to\infty} f(x_{n_k}) = c$. This means that $x_{n_k} \in K_{c+\frac{\varepsilon}{2}}$ for all k large enough. Since $x_{n_k} \to x$ and since $K_{c+\frac{\varepsilon}{2}}$ is closed in K, this implies $x \in K_{c+\frac{\varepsilon}{2}}$, or, equivalently,

$$f(x) \le c + \frac{\varepsilon}{2},$$

which is a contradiction to the above inequality. Hence, we have shown that f is lower semicontinuous. □

COROLLARY 2.4. *Let* X *be a Banach space,* $K \subset X$ *a closed, convex subset, and* $f : K \to \mathbb{R} \cup \{+\infty\}$ *a lower semicontinuous, convex function. If* $(x_n) \subset K$ *converges weakly to* $x \in K$*, then*

$$f(x) \le \liminf_{n\to\infty} f(x_n).$$

PROOF. For every $c \in \mathbb{R}$, the set $K_c := \{x \in K : f(x) \le c\}$ is closed (by lower semicontinuity of f and by Lemma 2.3) and convex (by convexity of f). After extracting a subsequence, if necessary, we may assume that $c := \liminf_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(x_n)$. Then for every $\varepsilon > 0$ the sequence (x_n) is eventually in $K_{c+\varepsilon}$, i.e. except for finitely many x_n, the sequence (x_n) lies in $K_{c+\varepsilon}$. Hence, by Corollary 2.2, $x \in K_{c+\varepsilon}$, which means that $f(x) \le c + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the claim follows. □

THEOREM 2.5. *Let* X *be a reflexive Banach space,* $K \subset X$ *a closed, convex, nonempty subset, and* $f : K \to \mathbb{R} \cup \{+\infty\}$ *a lower semicontinuous, convex function such that*

$$\lim_{\substack{\|x\|\to\infty \\ x\in K}} f(x) = +\infty \qquad \textit{(weak coercivity).}$$

Then there exists $x_0 \in K$ *such that*

$$f(x_0) = \inf\{f(x) : x \in K\} > -\infty.$$

PROOF. Let $(x_n) \subset K$ be such that $\lim_{n\to\infty} f(x_n) = \inf\{f(x) : x \in K\}$. By the coercivity assumption on f, the sequence (x_n) is bounded. Since X is reflexive, there exists a weakly convergent subsequence (Theorem 1.18); we denote by x_0 the limit. By Corollary 2.2, $x_0 \in K$. By Corollary 2.4,

$$f(x_0) \le \lim_{n\to\infty} f(x_n) = \inf\{f(x) : x \in K\}.$$

The claim is proved. □

3. * A nonlinear elliptic problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $p \ge 2$. Let $f : \Omega \to \mathbb{R}$ be some function in $L^2(\Omega)$. We consider the nonlinear elliptic boundary value problem

$$\begin{cases} -\Delta_p u(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

We call a function $u \in W_0^{1,p}(\Omega)$ a *weak solution* of this problem if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi = \int_\Omega f\varphi \quad \text{for every } \varphi \in C_c^1(\Omega). \tag{3.2}$$

Note that $u \in W_0^{1,p}(\Omega)$ is a weak solution of (3.1) if and only if $-\Delta_p^\Omega u = f$, where Δ_p^Ω is the p-Laplace operator defined in Chapter 1, Section 3.

In the following, we will give an other characterization and we will see that $u \in W_0^{1,p}(\Omega)$ is a weak solution of (3.1) if and only if u is a critical point of some real valued *energy function*. Part of this energy function is introduced in the following lemma.

LEMMA 3.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set, $1 \le p < \infty$, and define*

$$\begin{aligned} E_0 : W_0^{1,p}(\Omega) &\to \mathbb{R}, \\ u &\mapsto E_0(u) := \frac{1}{p}\int_\Omega |\nabla u|^p. \end{aligned}$$

Then the function E_0 is convex, of class C^1, and for every $\varphi \in W_0^{1,p}(\Omega)$ one has

$$E_0'(u)\varphi = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi.$$

In other words, if $p \ge 2$ and if $\Omega \subset \mathbb{R}^n$ is bounded, then $E_0'(u) = -\Delta_p^\Omega u$.

PROOF. We consider the function

$$\begin{aligned} |\cdot| : W_0^{1,p}(\Omega) &\to \mathbb{R}, \\ u &\mapsto |u| := \Big(\int_\Omega |\nabla u|^p\Big)^{\frac{1}{p}}, \end{aligned}$$

which is a semi-norm on $W_0^{1,p}(\Omega)$. This means that it satisfies all the properties of a norm except the implication $|u| = 0 \Rightarrow u = 0$ which is not true in general.

In particular, for every $u, v \in W_0^{1,p}(\Omega)$, the triangle inequality

$$|u + v| \le |u| + |v|$$

is true, and this implies the triangle inequality from above

$$|u - v| \ge \big||u| - |v|\big|.$$

This triangle inequality from above implies, that if $u_n \to u$ in $W_0^{1,p}(\Omega)$, then

$$0 \leftarrow \|u_n - u\|_{W^{1,p}} \ge |u_n - u| \ge \big||u_n| - |u|\big|,$$

and hence the application $|\cdot|$ is continuous. Moreover, for every $u, v \in W_0^{1,p}(\Omega)$ and every $t \in [0,1]$ the triangle ineqzality implies

$$|tu + (1-t)v| \le t|u| + (1-t)|v|,$$

so that $|\cdot|$ is also convex.

Since also the function $\mathbb{R}_+ \to \mathbb{R}$, $s \mapsto \frac{1}{p}s^p$ is continuous and convex, and since E_0 is the composition of $|\cdot|$ with this latter function, we obtain that E_0 is continuous and convex.

Next, we note that for every $u \in W_0^{1,p}(\Omega)$ the operator

$$\begin{aligned} T_u : W_0^{1,p}(\Omega) &\to \mathbb{R}, \\ h &\mapsto Th = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla h \end{aligned}$$

is well defined, linear and continuous. Moreover, one can show that for every $u \in W_0^{1,p}(\Omega)$

$$\lim_{\|h\|_{W_0^{1,p}} \to 0} \frac{E_0(u+h) - E_0(u) - T_u h}{\|h\|_{W_0^{1,p}}} = 0.$$

In fact, this equality is a consequence of the differentiability of the function $\mathbb{R}^n \to \mathbb{R}$, $x \to |x|^p$, where now $|\cdot|$ denotes the euclidean norm, and several convergence theorems from measure and integration theory; we omit the detailed proof. This last equality implies, by definition, that the function E_0 is differentiable and $E_0'(u) = T_u$, that is,

$$E_0'(u)\varphi = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi \quad \text{for every } u, \varphi \in W_0^{1,p}(\Omega).$$

Hence, if $p \ge 2$ and if $\Omega \subset \mathbb{R}^n$ is bounded, then, for every $u \in W_0^{1,p}(\Omega)$ one has $E_0'(u) = -\Delta_p^\Omega u$, or simply $E_0' = -\Delta_p^\Omega$. Since, by Lemma 3.2 (Chapter 1), the operator Δ_p^Ω is continuous, we obtain that the function E_0 is C^1 in this case. In the general case, that is, for $1 \le p < \infty$ and $\Omega \subset \mathbb{R}^n$ open, the continuity of E_0' is proved as in Lemma 3.2 (Chapter 1). □

In order to prove the main result in this section, we recall the Poincaré inequality.

THEOREM 3.2 (Poincaré inequality). *Let* $\Omega \subset \mathbb{R}^n$ *be a* bounded *domain, and let* $1 \le p < \infty$. *Then there exists a constant* $C \ge 0$ *such that*

$$\int_\Omega |u|^p \le C^p \int_\Omega |\nabla u|^p \quad \textit{for every } u \in W_0^{1,p}(\Omega).$$

We note that the Poincaré inequality implies that

$$\|u\| := \left(\int_\Omega |\nabla u|^p \right)^{\frac{1}{p}}$$

defines an equivalent norm on $W_0^{1,p}(\Omega)$ if $\Omega \subset \mathbb{R}^n$ is bounded. Clearly,

$$\|u\| \le \|u\|_{W_0^{1,p}} \quad \text{for every } u \in W_0^{1,p},$$

by the definition of the norm in $W^{1,p}$. On the other hand,

$$\begin{aligned} \|u\|_{W_0^{1,p}} &\le C\,(\|u\|_{L^p} + \|\nabla u\|_{L^p}) \\ &\le C\,\|\nabla u\|_{L^p} = C\,\|u\|, \end{aligned}$$

by the Poincaré inequality.

THEOREM 3.3. *Let* $\Omega \subset \mathbb{R}^n$ *be bounded and open, and let* $p \ge 2$. *Then for every* $f \in L^2(\Omega)$ *there exists a unique weak solution* $u \in W_0^{1,p}(\Omega)$ *of the problem* (3.1).

PROOF. Let $f \in L^2(\Omega)$, and define the function

$$\begin{aligned} E : W_0^{1,p}(\Omega) &\to \mathbb{R}, \\ u &\mapsto E(u) = \frac{1}{p} \int_\Omega |\nabla u|^p - \int_\Omega f u. \end{aligned}$$

We claim that this function is convex, of class C^1, and for every $\varphi \in W_0^{1,p}(\Omega)$ one has

$$E'(u)\varphi = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi - \int_\Omega f \varphi.$$

In fact, note that E is the sum of the function E_0 from Lemma 3.1 and a continuous, linear function. Since every continuous, linear function is convex and of class C^1, by Lemma 3.1, the function E is convex and of class C^1. The formula above follows also from Lemma 3.1, and from derivating a linear function.

As a consequence, a function $u \in W_0^{1,p}(\Omega)$ is a weak solution of (3.1) if and only if u is a *critical point* of E, that is, if and only if $E'(u) = 0$!

Existence: By the Poincaré inequality, and by the Cauchy-Schwarz inequality,

$$\begin{aligned} E(u) &\ge \frac{1}{2p} \int_\Omega |\nabla u|^p + \frac{1}{2pC^p} \int_\Omega |u|^p - \|f\|_2 \|u\|_2 \\ &\ge \eta \|u\|_{W_0^{1,p}}^p - \|f\|_2 \|u\|_{W_0^{1,p}} \\ &= \|u\|_{W_0^{1,p}} (\eta \|u\|_{W_0^{1,p}}^{p-1} - \|f\|_2). \end{aligned}$$

Since $p > 1$, this implies

$$\lim_{\|u\|_{W_0^{1,p}}\to\infty} E(u) = \infty,$$

that is, E is weakly coercive. Since the space $W_0^{1,p}(\Omega)$ is reflexive, by Theorem 2.5 about the minimization of convex functions, there exists $u \in W_0^{1,p}(\Omega)$ such that

$$u = \inf_{W_0^{1,p}} E,$$

that is, u is a global minimum. Since every local (or global) minimum of E is a critical point of E, we have thus proved existence of a weak solution of (3.1).

Uniqueness: Assume that $v \in W_0^{1,p}(\Omega)$ is a second weak solution. Then $E'(v) = E'(u) = 0$. Since E is in addition convex, we obtain that the function $f : [0,1] \to \mathbb{R}$, $f(t) = E(tu + (1-t)u)$ is convex and $f'(0) = f'(1) = 0$. Hence, $f'(t) = 0$ for every $t \in [0,1]$ (the derivative of a convex function is increasing), so that f is constant. Hence, v is also a global minimum of E. If $u \neq v$, then the strict convexity of E (!!) implies

$$E(\frac{u+v}{2}) < \frac{E(u)+E(v)}{2} = \inf E,$$

which is a contradiction. Hence, we must have $u = v$. □

4. * The von Neumann minimax theorem

In the following theorem, we call a function $f : K \to \mathbb{R}$ on a convex subset K of a Banach space X *concave* if $-f$ is convex, or, equivalently, if for every $x, y \in K$ and every $t \in [0,1]$,

$$f(tx + (1-t)y) \geq t\, f(x) + (1-t)\, f(y). \tag{4.1}$$

A function $f : K \to \mathbb{R}$ is called *strictly convex* (resp. *strictly concave*) if for every x, $y \in K$, $x \neq y$, $f(x) = f(y)$ the inequality in (2.1) (resp. (4.1)) is strict for $t \in (0,1)$.

Theorem 4.1 (von Neumann). *Let K and L be two closed, bounded, nonempty, convex subsets of reflexive Banach spaces X and Y, respectively. Let $f : K \times L \to \mathbb{R}$ be a continuous function such that*

$$x \mapsto f(x,y) \text{ is strictly convex for every } y \in L, \text{ and}$$
$$y \mapsto f(x,y) \text{ is concave for every } x \in K.$$

Then there exists $(\bar{x}, \bar{y}) \in K \times L$ such that

$$f(\bar{x}, y) \leq f(\bar{x}, \bar{y}) \leq f(x, \bar{y}) \text{ for every } x \in K,\ y \in L. \tag{4.2}$$

Remark 4.2. A point $(\bar{x}, \bar{y}) \in K \times L$ satisfying (4.2) is called a *saddle point* of f.

A saddle point is a point of *equilibrium* in a two-person zero-sum game in the following sense: If the player controlling the strategy x modifies his strategy when the second player plays $\bar{y}$, he increases his loss; hence, it is his interest to play $\bar{x}$. Similarly, if the player controlling the strategy y modifies his strategy when the first player plays $\bar{x}$, he diminishes his gain; thus it is in his interest to play $\bar{y}$. This property of equilibrium of saddle points justifies their use as a (reasonable) solution in a two-person zero-sum game ([**3**]).

Proof. Define the function $F : L \to \mathbb{R}$ by $F(y) := \inf_{x\in K} f(x,y)$ $(y \in L)$. By Theorem 2.5, for every $y \in L$ there exists $x \in K$ such that $F(y) = f(x,y)$. By strict convexity, this element x is uniquely determined. We denote $x := \Phi(y)$ and thus obtain

$$F(y) = \inf_{x\in K} f(x,y) = f(\Phi(y),y), \quad y \in L. \tag{4.3}$$

By concavity of the function $y \mapsto f(x,y)$ and by the definition of F, for every y_1, $y_2 \in L$ and every $t \in [0,1]$,

$$\begin{aligned} F(ty_1 + (1-t)y_2) &= f(\Phi(ty_1 + (1-t)y_2), ty_1 + (1-t)y_2) \\ &\geq t\, f(\Phi(ty_1 + (1-t)y_2), y_1) + (1-t)\, f(\Phi(ty_1 + (1-t)y_2), y_2) \\ &\geq t\, F(y_1) + (1-t)\, F(y_2), \end{aligned}$$

so that F is concave. Moreover, F is upper semicontinuous: let $(y_n) \subset L$ be convergent to $y \in L$. For every $x \in K$ and every $n \in \mathbb{N}$ one has $F(y_n) \leq f(x,y_n)$, and taking the limes superior on both sides, we obtain, by continuity of f,

$$\limsup_{n\to\infty} F(y_n) \leq \limsup_{n\to\infty} f(x,y_n) = f(x,y).$$

Since $x \in K$ was arbitrary, this inequality implies $\limsup_{n\to\infty} F(y_n) \leq F(y)$, i.e. F is upper semicontinuous.

By Theorem 2.5 (applied to $-F$), there exists $\bar{y} \in L$ such that

$$f(\Phi(\bar{y}), \bar{y}) = F(\bar{y}) = \sup_{y\in L} F(y).$$

We put $\bar{x} = \Phi(\bar{y})$ and show that $(\bar{x}, \bar{y})$ is a saddle point. Clearly, for every $x \in K$,

$$f(\bar{x}, \bar{y}) \leq f(x, \bar{y}). \tag{4.4}$$

Therefore it remains to show that for every $y \in L$,

$$f(\bar{x}, \bar{y}) \geq f(\bar{x}, y). \tag{4.5}$$

Let $y \in L$ be arbitrary and put $y_n := (1 - \frac{1}{n})\bar{y} + \frac{1}{n}y$ and $x_n = \Phi(y_n)$. Then, by concavity,

$$\begin{aligned} F(\bar{y}) \geq F(y_n) &= f(x_n, y_n) \\ &\geq (1 - \frac{1}{n}) f(x_n, \bar{y}) + \frac{1}{n} f(x_n, y) \\ &\geq (1 - \frac{1}{n}) F(\bar{y}) + \frac{1}{n} f(x_n, y), \end{aligned}$$

or

$$F(\bar{y}) \geq f(x_n, y) \text{ for every } n \in \mathbb{N}.$$

Since K is bounded and closed, the sequence $(x_n) \subset K$ has a weakly convergent subsequence which converges to some element $x_0 \in K$ (Theorem 1.18 and Corollary 2.2). By the preceeding inequality and Corollary 2.4,

$$F(\bar{y}) \geq f(x_0, y).$$

This is just the remaining inequality (4.5) if we can prove that $x_0 = \bar{x}$. By concavity, for every $x \in K$ and every $n \in \mathbb{N}$,

$$\begin{aligned} f(x, y_n) &\geq f(x_n, y_n) \\ &\geq (1 - \frac{1}{n}) f(x_n, \bar{y}) + \frac{1}{n} f(x_n, y) \\ &\geq (1 - \frac{1}{n}) f(x_n, \bar{y}) + \frac{1}{n} F(y). \end{aligned}$$

Letting $n \to \infty$ in this inequality and using Corollary 2.4 again, we obtain that for every $x \in K$,

$$f(x, \bar{y}) \geq f(x_0, \bar{y}).$$

Hence, $x_0 = \Phi(\bar{y}) = \bar{x}$ and the theorem is proved. □

5. * The brachistochrone problem

The following problem was asked by Johann Bernoulli in 1696:

> For given two points A and B in a vertical plane, find a curve connecting A and B which is optimal among all other such curves in the following sense. The point P of unit mass which starts from A with zero velocity and moves along this curve only due to the gravitational force will reach the point B in a minimal time.

Without loss of generality, we may assume that in the xy-plane we have $A = (0, a)$ and $B = (b, 0)$ for some a, $b > 0$. We will look for a curve connecting A and B and which is in addition a graph of a continuously differentiable function $y : [0, b] \to \mathbb{R}$ satisfying $y(0) = a$ and $y(b) = 0$.

The principle of conservation of energy implies that

$$\frac{1}{2} v(t)^2 = g\, y(x(t)),$$

where v is the velocity of the point P, g is the gravitational constant and $x(t)$ is the x-coordinate of the point P at time t (and $y(x(t))$ is the height of the point P). Note that

$$v(t) = \sqrt{1 + y'(x(t))^2}\, \dot{x}(t),$$

and therefore

$$\dot{x}(t) = \frac{dx}{dt}(t) = \sqrt{\frac{2gy(x(t))}{1 + y'(x(t))^2}}.$$

Hence, the time T at which the point P reaches the point B is given by

$$T = \int_0^T dt = \int_0^b \sqrt{\frac{1 + y'(x)^2}{2gy(x)}}\, dx.$$

The problem is therefore to minimize the functional T given by

$$T(y) = \int_0^b \sqrt{\frac{1 + y'(x)^2}{2gy(x)}}\, dx,$$

where y varies in the convex set

$$K := \{y \in W^{1,p}(0, b) : y(0) = a \text{ and } y(b) = 0\}$$

and $p \geq 1$ is to be fixed. It is easy to check that the functional T is convex and that for every $p \geq 1$ the set K is closed in $W^{1,p}(0, 1)$. However, the space $W^{1,p}(0, 1)$ is reflexive only if $p > 1$. On the other hand, the functional T is coercive only if $p = 1$.

Hence, we can *not* apply the main theorem of this section on minimization of convex functionals (Theorem 2.5), unless we replace the set K by a bounded convex subset which is likely to contain the global minimum of T!

CHAPTER 3

Iterative methods

1. * Newton's method

THEOREM 1.1 (Newton's method). *Let X and Y be two Banach spaces, $U \subset X$ an open set. Let $f \in C^1(U; Y)$ and assume that there exists $\bar{x} \in U$ such that (i) $f(\bar{x}) = 0$ and (ii) $f'(\bar{x}) \in \mathcal{L}(X, Y)$ is an isomorphism. Then for every $L \in (0, 1)$ there exists a neighbourhood $V \subset U$ of $\bar{x}$ such that for every $x_0 \in V$ the operator $f'(x_0)$ is an isomorphism, the sequence (x_n) defined iteratively by*

$$x_{n+1} = x_n - f'(x_n)^{-1} f(x_n), \quad n \geq 0, \tag{1.1}$$

remains in V and $\|x_n - \bar{x}\| \leq L^n \|x_0 - \bar{x}\|$ for every $n \in \mathbb{N}$. In particular, $\lim_{n\to\infty} x_n = \bar{x}$.

REMARK 1.2. The iteration given by (1.1) is called *Newton iteration*.

PROOF OF THEOREM 1.1. By continuity, there exists a neighbourhood $\tilde{V} \subset U$ of $\bar{x}$ such that $f'(x)$ is an isomorphism for all $x \in \tilde{V}$. It will be useful to define the auxiliary function $\varphi : \tilde{V} \to X$ by

$$\varphi(x) := x - f'(x)^{-1} f(x), \quad x \in \tilde{V}.$$

Since $f(\bar{x}) = 0$, we find that for every $x \in \tilde{V}$

$$\begin{aligned} \varphi(x) - \varphi(\bar{x}) &= x - f'(x)^{-1}(f(x) - f(\bar{x})) - \bar{x} \\ &= x - \bar{x} - f'(x)^{-1}(f'(\bar{x})(x - \bar{x}) + o(x - \bar{x})), \end{aligned}$$

so that by the continuity of $f'(\cdot)^{-1}$

$$\lim_{x\to\bar{x}} \frac{\|\varphi(x) - \varphi(\bar{x})\|}{\|x - \bar{x}\|} = 0.$$

In particular, for every $L \in (0, 1)$ there exists $r > 0$ such that $V := B(\bar{x}, r) \subset \tilde{V} \subset U$ and such that for every $x \in V$

$$\|\varphi(x) - \bar{x}\| = \|\varphi(x) - \varphi(\bar{x})\| \leq L \|x - \bar{x}\|.$$

This implies that for every $x_0 \in V$ one has $\varphi(x_0) \in V$ and if we define iteratively $x_{n+1} = \varphi(x_n) = \varphi^{n+1}(x_0)$, then

$$\|x_n - \bar{x}\| \leq L^n \|x_0 - \bar{x}\| \to 0 \text{ as } n \to \infty.$$

□

2. Local inverse theorem and implicit function theorem

Let X and Y be two Banach spaces and let U be an open subset of X. The following are two classical theorems in differential calculus.

THEOREM 2.1 (Local inverse theorem). *Let $f : U \to Y$ be continuously differentiable and $\bar{x} \in U$ such that $f(\bar{x}) : X \to Y$ is an isomorphism, that is, bounded, bijective and the inverse is also bounded. Then there exist neighbourhoods $V \subset U$ of $\bar{x}$ and $W \subset Y$ of $f(\bar{x})$ such that $f : V \to W$ is a C^1 diffeomorphism, that is f is continuously differentiable, bijective and the inverse $f^{-1} : W \to V$ is continuously differentiable.*

THEOREM 2.2 (Implicit function theorem). *Assume that $X = X_1 \times X_2$ for two Banach spaces, and let $f : X \supset U \to Y$ be continuously differentiable and $\bar{x} = (\bar{x}_1, \bar{x}_2) \in U$ such that $\frac{\partial f}{\partial x_2}(\bar{x}) : X_2 \to Y$ is an isomorphism. Then there exist neighbourhoods $U_1 \subset X_1$ of $\bar{x}_1$ and $U_2 \subset X_2$ of $\bar{x}_2$, $U_1 \times U_2 \subset U$, and a continuously differentiable function $g : U_1 \to U_2$ such that*

$$\{x \in U_1 \times U_2 : f(x) = f(\bar{x})\} = \{(x_1, g(x_1)) : x_1 \in U_1\}.$$

For the proof of the local inverse theorem, we need the following lemma.

LEMMA 2.3. *Let $f : U \to Y$ be continuously differentiable such that $f : U \to f(U)$ is a homeomorphism, that is, continuous, bijective and with continuous inverse. Then f is a C^1 diffeomorphism if and only if for every $x \in U$ the derivative $f'(x) : X \to Y$ is an isomorphism.*

PROOF. Assume first that f is a C^1 diffeomorphism. When we differentiate the identities $x = f^{-1}(f(x))$ and $y = f(f^{-1}(y))$, which are true for every $x \in U$ and every $y \in f(U)$, then we find

$$\begin{aligned} I_X &= (f^{-1})'(f(x))f'(x) \quad \text{for every } x \in U \text{ and} \\ I_Y &= f'(f^{-1}(y))(f^{-1})'(y) \\ &= f'(x)(f^{-1})'(f(x)) \quad \text{for every } x = f^{-1}(y) \in U. \end{aligned}$$

As a consequence, $f'(x)$ is an isomorphism for every $x \in U$.

For the converse, assume that $f'(x)$ is an isomorphism for every $x \in U$. For every $x_1, x_2 \in U$ one has, by differentiability,

$$f(x_2) = f(x_1) + f'(x_1)(x_2 - x_1) + o(x_2 - x_1),$$

where o depends on x_1 and $\lim_{x_2 \to x_1} \frac{o(x_2 - x_1)}{\|x_2 - x_1\|} = 0$. We have $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$ if we put $y_i := f(x_i)$. Hence, the above identity becomes

$$y_2 = y_1 + f'(f^{-1}(y_1))(f^{-1}(y_2) - f^{-1}(y_1)) + o(f^{-1}(y_2) - f^{-1}(y_1)).$$

To this identity, we apply the inverse operator $(f'(f^{-1}(y_1)))^{-1}$ and we obtain

$$f^{-1}(y_2) = f^{-1}(y_1) + (f'(f^{-1}(y_1)))^{-1}(y_2 - y_1) - (f'(f^{-1}(y_1)))^{-1}o(f^{-1}(y_2) - f^{-1}(y_1)).$$

Since f^{-1} is continuous, the last term on the right-hand side of the last equality is sublinear. Hence, f^{-1} is differentiable and

$$(f^{-1})'(y_1) = (f'(f^{-1}(y_1)))^{-1}.$$

From this identity (using that f^{-1} and f' are continuous) we obtain that f^{-1} is continuously differentiable. The claim is proved. □

Proof of the local inverse theorem. Consider the function

$$\begin{aligned} g : U &\to X, \\ x &\mapsto f'(\bar{x})^{-1} f(x). \end{aligned}$$

It suffices to show that $g : V \to W$ is a C^1 diffeomorphism for appropriate neighbourhoods V of $\bar{x}$ and W of $g(\bar{x})$.

Consider also the function

$$\begin{aligned} \varphi : U &\to X, \\ x &\mapsto x - g(x). \end{aligned}$$

This function φ is continuously differentiable and $\varphi'(x) = I - f'(\bar{x})^{-1} f'(x)$ for every $x \in U$. In particular, $\varphi'(\bar{x}) = 0$. By continuity of φ', there exists $r > 0$ and $L < 1$ such that $\|\varphi'(x)\| \le L$ for every $x \in \bar{B}(\bar{x}, r) \subset U$. Hence,

$$\|\varphi(x_1) - \varphi(x_2)\| \le L\,\|x_1 - x_2\| \quad \text{for every } x_1,\ x_2 \in \bar{B}(\bar{x}, r).$$

By the definition of φ, this implies

$$\begin{aligned} \|g(x_1) - g(x_2)\| &= \|x_1 - x_2 - (\varphi(x_1) - \varphi(x_2))\| \\ &\ge \|x_1 - x_2\| - L\,\|x_1 - x_2\| \\ &= (1 - L)\,\|x_1 - x_2\|. \end{aligned} \tag{2.1}$$

We claim that for every $y \in \bar{B}(g(\bar{x}), (1-L)r)$ there exists a unique $x \in \bar{B}(\bar{x}, r)$ such that $g(x) = y$.

The uniqueness follows from (2.1).

In order to prove existence, let $x_0 = \bar{x}$, and then define recursively $x_{n+1} = y + \varphi(x_n) = y + x_n - f'(\bar{x})^{-1} f(x_n)$ for every $n \ge 0$. Then

$$\begin{aligned} \|x_n - \bar{x}\| &= \Big\|\sum_{k=0}^{n-1} x_{k+1} - x_k\Big\| \\ &\le \|x_1 - x_0\| + \sum_{k=1}^{n-1} \|\varphi(x_k) - \varphi(x_{k-1})\| \\ &\le \sum_{k=0}^{n-1} L^k\,\|x_1 - x_0\| \\ &= \frac{1 - L^n}{1 - L}\,\|y - g(\bar{x})\| \\ &\le (1 - L^n)\,r \quad \le \quad r, \end{aligned}$$

which implies $x_n \in \bar{B}(\bar{x}, r)$ for every $n \geq 0$. Similarly, for every $n \geq m \geq 0$,

$$\|x_n - x_m\| \leq \sum_{k=m}^{n-1} L^k \|y - g(\bar{x})\|,$$

so that the sequence (x_n) is a Cauchy sequence in $\bar{B}(\bar{x}, r)$. Since $\bar{B}(\bar{x}, r)$ is complete, there exists $\lim_{n\to\infty} x_n =: x \in \bar{B}(\bar{x}, r)$. By continuity,

$$x = y + \varphi(x) = y + x - g(x),$$

or

$$g(x) = y.$$

This proves the above claim, that is, g is locally invertible. It remains to show that g^{-1} is continuous (then g is a homeomorphism, and therefore a C^1 diffeomorphism by Lemma 2.3). Contiunity of the inverse function, however, is a direct consequence of (2.1) (which even implies Lipschitz continuity). □

Remark 2.4. The iteration formula

$$x_{n+1} = y + x_n - f'(\bar{x})^{-1} f(x_n)$$

used in the proof of the local inverse theorem in order to find a solution of $g(x) = f'(\bar{x})^{-1} f(x) = y$ should be compared to the Newton iteration

$$x_{n+1} = y + x_n - f'(x_n)^{-1} f(x_n).$$

Proof of the implicit function theorem. Consider the function

$$\begin{aligned} F : U &\to X_1 \times Y, \\ (x_1, x_2) &\mapsto (x_1, f(x_1, x_2)). \end{aligned}$$

Then F is continuously differentiable and

$$F'(\bar{x})(h_1, h_2) = (h_1, \frac{\partial f}{\partial x_1}(\bar{x})h_1 + \frac{\partial f}{\partial x_2}(\bar{x})h_2).$$

In particular, by the assumption, $F'(\bar{x})$ is locally invertible with inverse

$$F'(\bar{x})^{-1}(y_1, y_2) = (y_1, (\frac{\partial f}{\partial x_2}(\bar{x}))^{-1}(y_2 - \frac{\partial f}{\partial x_1}(\bar{x})y_1)).$$

By the local inverse theorem (Theorem 2.1), there exists a neighbourhood U_1 of $\bar{x}_1$, a neighbourhood U_2 of $\bar{x}_2$ and a neighbourhood V of $(\bar{x}_1, f(\bar{x})) = F(\bar{x})$ such that $F : U_1 \times U_2 \to V$ is a C^1 diffeomorphism. The inverse is of the form

$$F^{-1}(y_1, y_2) = (y_1, h_2(y_1, y_2)),$$

where h_2 is a function such that $f(y_1, h_2(y_1, y_2)) = y_2$. Let

$$\tilde{U}_1 := \{x_1 \in U_1 : (x_1, f(\bar{x})) \in V\}.$$

Then $\tilde{U}_1$ is open by continuity of the function $x_1 \mapsto (x_1, f(\bar{x}))$, and $\bar{x}_1 \in \tilde{U}_1$. We restrict F to $\tilde{U}_1 \times U_2$, and we define

$$\begin{aligned} g : \tilde{U}_1 &\to X_2, \\ x_1 &\mapsto g(x_1) = F^{-1}(x_1, f(\bar{x}))_2, \end{aligned} \tag{2.2}$$

where $F^{-1}(\cdot)_2$ denotes the second component of $F^{-1}(\cdot)$. Then g is continuously differentiable, $g(\tilde{U}_1) \subset U_2$ and g satisfies the required property of the implicit function. □

Lemma 2.5 (Higher regularity of the local inverse). *Let $f \in C^k(U;Y)$ for some $k \geq 1$ and assum that $f : U \to f(U)$ is a C^1 diffeomorphism. Then f is a C^k diffeomorphism, that is, f^{-1} is k times continuously differentiable.*

Proof. For every $y \in f(U)$ we have

$$(f^{-1})'(y) = f'(f^{-1}(y))^{-1}.$$

The proof therefore follows by induction on k. □

Lemma 2.6 (Higher regularity of the implicit function). *If, in the implicit function theorem (Theorem 2.2), the function f is k times continuously differentiable, then the implicit function g is also k times continuously differentiable.*

Proof. This follows from the previous lemma (Lemma 2.5) and the definition of the implicit function in the proof of the implicit function theorem. □

3. * Parameter dependence of solutions of ordinary differential equations

Let P and X be two Banach spaces and let $f \in C^k(P \times X; X)$. Consider the ordinary differential equation

$$\dot{x}(t) = f(p, x(t)), \quad x(0) = 0, \tag{3.1}$$

where p is a parameter. Fix a parameter $p_0 \in P$, let $I_0 \subset \mathbb{R}$ be a compact intervall such that $0 \in I_0$, and let a solution $x_0 \in C^1(I_0; X)$ be a solution of the above problem for the parameter $p = p_0$.

Theorem 3.1. *Then there exists a neighbourhood $U_0 \subset P$ of p_0 and a k times continuously differentiable function $g : U_0 \to C^1(I_0; X)$ such that for every $p \in U_0$ the function $x_p = g(p)$ is the unique solution of* (3.1) *for the parameter p. All solutions of* (3.1) *in a neighbourhood of* (p_0, x_0) *are of this form.*

Proof. Let $C^1_0(I_0; X) = \{x \in C^1(I_0; X) : x(0) = 0\}$ be equipped with the norm $\|x\|_{C^1} = \|x\|_\infty + \|\dot{x}\|_\infty$, so that C^1_0 is a Banach space. Consider the function

$$\begin{aligned} F : P \times C^1_0(I_0; X) &\to C(I_0; X), \\ (p, x) &\mapsto \dot{x} - f(p, x). \end{aligned}$$

Then, by definition of F, $F(p_0, x_0) = 0$. Moreover, the function F is k times continuously differentiable and $\frac{\partial F}{\partial x}(p_0, x_0)$ is an isomorphism from $C^1_0(I_0; X)$ onto $C(I_0; X)$ (!!).

By the implicit function theorem (Theorem 2.2), there exists a neighbourhood U_0 of p_0 and k times continuously differentiable function $g : U_0 \to C^1_0(I_0; X)$ (we use also Lemma 2.6) such that for every $p \in U_0$ one has $F(p, g(p)) = 0$, that is, $g(p)$ is the solution of (3.1) for the parameter p, and it also follows from the implicit function theorem, that every solution of (3.1) is of this form. □

4. * A bifurcation theorem and ordinary differential equations

We follow [**8**, Section 4.3].

THEOREM 4.1 (Crandall-Rabinowitz). *Let X and Y be two Banach spaces, $U \subset \mathbb{R} \times X$ be an open set, let $f \in C^2(U; Y)$ and $(\bar\lambda, \bar x) \in U$. Assume that*
(i) $f(\lambda, \bar x) = 0$ for all λ in a neighbourhood of $\bar\lambda$,
(ii) $\dim \operatorname{Ker} \frac{\partial f}{\partial x}(\bar\lambda, \bar x) = \operatorname{codim} \operatorname{Rg} \frac{\partial f}{\partial x}(\bar\lambda, \bar x) = 1$*, and*
(iii) if $x_0 \in \operatorname{Ker} \frac{\partial f}{\partial x}(\bar\lambda, \bar x) \setminus \{0\}$, then $\frac{\partial^2 f}{\partial\lambda\partial x}(\bar\lambda, \bar x)(1, x_0) \notin \operatorname{Rg} \frac{\partial f}{\partial x}(\bar\lambda, \bar x)$.

Denote by X_1 the topological complement of $\operatorname{Ker} \frac{\partial f}{\partial x}(\bar\lambda, \bar x)$ in X.

Then there exists a continuously differentiable curve $(\lambda, x) : (-\delta, \delta) \to \mathbb{R} \times X_1$ such that

$$(\lambda(0), x(0)) = (\bar\lambda, \bar x) \quad \text{and} \quad f(\lambda(t), tx_0 + tx(t)) = 0 \quad \text{for every } t \in (-\delta, \delta).$$

Moreover, there is a neighbourhood $V \subset U$ of $(\bar\lambda, \bar x)$ such that

$$f(\lambda, x) = 0 \quad \text{for } (\lambda, x) \in V$$

if and only if

$$\text{either} \quad x = 0 \quad \text{or} \quad \lambda = \lambda(t),\ x = tx_0 + tx(t).$$

PROOF. For simplicity, we assume that $(\bar\lambda, \bar x) = (0, 0)$. Fix

$$x_0 \in \operatorname{Ker} \frac{\partial f}{\partial x}(\bar\lambda, \bar x), \qquad x_0 \neq 0,$$

and consider the function $F : \mathbb{R} \times \mathbb{R} \times X_1 \to Y$ which is given by

$$F(t, \lambda, x) = \begin{cases} \frac{1}{t} f(\lambda, t(x_0 + x_1)) & \text{for } t \neq 0, \\ \frac{\partial f}{\partial x}(\lambda, 0)(x_0 + x_1) & \text{for } t = 0. \end{cases}$$

Then

$$F(0, 0, 0) = 0$$

and the operator

$$\begin{aligned} \mathbb{R} \times X_1 &\to Y, \\ (\lambda, x_1) &\mapsto \frac{\partial F}{\partial \lambda}(0, 0, 0)\lambda + \frac{\partial F}{\partial x}(0, 0, 0)x_1 \end{aligned}$$

is an isomorphism by assumptions (ii) and (iii). The claim follows from the implicit function theorem (Theorem 2.2). □

EXAMPLE 4.2. We study the periodic boundary value problem

$$\begin{cases} \ddot x(t) + \lambda x(t) + g(\lambda, t, x(t), \dot x(t)) = 0, & t \in [0, 2\pi], \\ x(0) = x(2\pi), \\ \dot x(0) = \dot x(2\pi). \end{cases} \tag{4.1}$$

The function $g : \mathbb{R} \times [0, 2\pi] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g = g(\lambda, t, x, p)$ satisfies the following assumptions:
(i) g is k times continuously differentiable for some $k \geq 2$, and 2π-periodic with

respect to t,
(ii) $g(\lambda, t, 0, 0) = 0$, and
(iii) $\frac{\partial g}{\partial x}(\lambda, t, 0, 0) = \frac{\partial g}{\partial p}(\lambda, t, 0, 0) = 0$.

We will study the above problem near the point $\lambda = 0$ which is a simple eigenvalue of the associated eigenvalue problem

$$\begin{cases} \ddot{x}(t) + \lambda x(t) = 0, \quad t \in [0, 2\pi], \\ x(0) = x(2\pi), \\ \dot{x}(0) = \dot{x}(2\pi). \end{cases} \tag{4.2}$$

Let

$$\begin{aligned} X &= \{x \in C^2([0, 2\pi]) : x(0) = x(2\pi), \dot{x}(0) = \dot{x}(1) \text{ and } \ddot{x}(0) = \ddot{x}(2\pi)\} \quad \text{and} \\ Y &= \{y \in C([0, 2\pi]) : y(0) = y(2\pi)\}. \end{aligned}$$

The spaces X and Y are Banach spaces when they are equipped with the norms

$$\begin{aligned} ||x||_X &= ||x||_\infty + ||\dot{x}||_\infty + ||\ddot{x}||_\infty \quad \text{and} \\ ||y||_Y &= ||y||_\infty, \end{aligned}$$

respectively. Let us define $f : \mathbb{R} \times X \to Y$ by

$$f(\lambda, x) = \ddot{x} + \lambda x + g(\lambda, \cdot, x, \dot{x}).$$

It follows from (i) that f is well-defined and k times continuously differentiable. Moreover, by hypothesis (iii), we have

$$\frac{\partial f}{\partial x}(\lambda, 0)w = \ddot{w} + \lambda w,$$

which implies

$$\dim \operatorname{Ker} \frac{\partial f}{\partial x}(0, 0) = 1;$$

in fact, the only functions lying in X and satisfying $\ddot{x} = 0$ are the constant functions.

Next, let $y \in \operatorname{Rg} \frac{\partial f}{\partial x}(0, 0)$. Then there exists a function $x \in X$ such that $\ddot{x} = y$. Integrating this equality over the interval $[0, 2\pi]$ implies

$$\int_0^{2\pi} y = \int_0^{2\pi} \ddot{x} = \dot{x}(1) - \dot{x}(0) = 0,$$

so that

$$\operatorname{Rg} \frac{\partial f}{\partial x}(0, 0) \subset \{y \in Y : \int_0^{2\pi} y = 0\}.$$

On the other hand, let $y \in Y$ be such that $\int_0^{2\pi} y = 0$. Define

$$x(t) := \int_0^t (t - s)y(s)\, ds - t \int_0^{2\pi} (2\pi - s)y(s)\, ds.$$

Then $x \in X$ and $\ddot{x} = y$. We have therefore proved the equality

$$\operatorname{Rg} \frac{\partial f}{\partial x}(0, 0) = \{y \in Y : \int_0^{2\pi} y = 0\}.$$

From this we deduce

$$\operatorname{codim} \operatorname{Rg} \frac{\partial f}{\partial x}(0,0) = 1.$$

Note that Ker $\frac{\partial f}{\partial x}(0,0)$ is the space of constant functions and that a topological complement is given by

$$X_1 = \{x \in X : \int_0^{2\pi} x(t)\, dt\}.$$

Since

$$\frac{\partial^2 f}{\partial \lambda\, \partial x}(0,0)1 = 1 \quad \text{and} \quad 1 \notin \operatorname{Rg} \frac{\partial f}{\partial x}(0,0),$$

the condition (iii) of Theorem 4.1 is satisfied. It follows from the Crandall-Rabinowitz theorem (Theorem 4.1) that $\lambda = 0$ is a point of bifurcation of (4.1).

In particular, the point $(0,0) \in \mathbb{R} \times X$ belongs to the branch of trivial solutions $(\lambda, 0)$, but also to the branch

$$\Gamma = \{(\lambda(s), s + sx(s)) : s \in (-\delta, \delta)\}$$

where $(\lambda, x) : (-\delta, \delta) \to \mathbb{R} \times X$ is a curve satisfying

$$x(0) = 0, \ \frac{d}{ds}x(0) = 0, \quad \lambda(0) = 0.$$

Hence, for any $s \in (-\delta, \delta)$, $s \neq 0$, the nontrivial solution $s + sx(s)$ (sum of the constant function s and the perturbation $sx(s)$) belongs to X_1.

CHAPTER 4

Monotone operators

1. Monotone operators

Definition 1.1. Let v be a real Banach space, and let V' be its dual space. An operator $A : V \to V'$ is *monotone* if for every $u, v \in V$ one has

$$\langle Au - Av, u - v\rangle_{V',V} \geq 0.$$

Example 1.2. Let $\Omega \subset \mathbb{R}^n$ be open and bounded. For every $p \geq 2$ and every $1 \leq i \leq n$, the linear operator

$$\begin{aligned} B_i : W_0^{1,p}(\Omega) &\to W^{-1,p'}(\Omega), \\ u &\mapsto \frac{\partial u}{\partial x_i}, \end{aligned}$$

is monotone. In fact, for every $u \in W_0^{1,p}(\Omega)$, by an integration by parts,

$$\begin{aligned} \langle B_i u, u\rangle &= \int_\Omega \frac{\partial u}{\partial x_i}\, u \\ &= -\int_\Omega u\, \frac{\partial u}{\partial x_i} \\ &= -\langle B_i u, u\rangle, \end{aligned}$$

so that

$$\langle B_i u, u\rangle = 0.$$

By linearity, B_i is hence monotone.

Example 1.3. The negative p-Laplace operator $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is monotone. In fact, for every $u, v \in W_0^{1,p}(\Omega)$,

$$\begin{aligned} -\langle \Delta_p u - \Delta_p v, u - v\rangle_{W^{-1,p'},W_0^{1,p}} &= \int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)(\nabla u - \nabla v) \\ &\geq \int_\Omega (|\nabla u|^p + |\nabla v|^p - |\nabla u|^{p-1}\,|\nabla v| - |\nabla u|\,|\nabla v|^{p-1}) \\ &= \int_\Omega (|\nabla u|^{p-1} - |\nabla v|^{p-1})\,(|\nabla u| - |\nabla v|) \\ &\geq 0. \end{aligned}$$

The fact, that $-\Delta_p$ is a monotone operator, can also be deduced from the following simple lemma.

LEMMA 1.4. *Let $\varphi : V \to \mathbb{R}$ be a continuously differentiable, convex function. Then $\varphi' : V \to V'$ is monotone.*

PROOF. For every $u, v \in V$, the function $t \mapsto \varphi(tu + (1-t)v)$ is convex which means that its derivative is increasing. In particular,

$$\frac{d}{dt}\varphi(tu + (1-t)v)|_{t=1} \geq \frac{d}{dt}\varphi(tu + (1-t)v)|_{t=0},$$

which means

$$\langle \varphi'(u), u - v \rangle \geq \langle \varphi'(v), u - v \rangle.$$

Hence, φ' is monotone. □

Since the negative p-Laplace operator $-\Delta_p$ is the derivative of the continuously differentiable and convex function $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by $\varphi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p$, the preceding lemma provides another proof of the monotonicity of $-\Delta_p$.

DEFINITION 1.5. Let V be a Banach space. An operator $A : V \to V'$ is
(i) *hemi-continuous* if for every $u, v, w \in V$ the function $t \mapsto \langle A(u + tv), w \rangle$ is continous,
(ii) *bounded* if it maps bounded sets into bounded sets, and
(iii) *pseudo-monotone* if A is bounded and if

$$\left.\begin{array}{l} u_n \rightharpoonup u \text{ in } V \text{ and} \\ \limsup_{n\to\infty} \langle Au_n, u_n - u \rangle \leq 0 \end{array}\right\} \quad \Rightarrow \quad \liminf_{n\to\infty} \langle Au_n, u_n - v \rangle \geq \langle Au, u - v \rangle.$$

LEMMA 1.6. *Let V be a Banach space and $A : V \to V'$ be an operator. Consider the following properties:*
(i) A is monotone, bounded and hemicontinuous,
(ii) A is pseudo-monotone,
(iii) A satisfies

$$\left.\begin{array}{l} u_n \rightharpoonup u \text{ in } V, \\ Au_n \rightharpoonup \chi \text{ in } V' \text{ and} \\ \limsup_{n\to\infty} \langle Au_n, u_n \rangle \leq \langle \chi, u \rangle \end{array}\right\} \quad \Rightarrow \quad Au = \chi.$$

Then (i)⇒(ii)⇒(iii).

PROOF. (i)⇒(ii) Assume that A is monotone, bounded and hemicontinuous, and let $(u_n) \subset V$ be a sequence satisfying

$$u_n \rightharpoonup u \text{ in } V \text{ and } \limsup_{n\to\infty} \langle Au_n, u_n - u \rangle \leq 0.$$

By monotonicity of A, we have

$$\langle Au_n, u_n - u \rangle \geq \langle Au, u_n - u \rangle.$$

The weak convergence of (u_n) implies

$$\liminf_{n\to\infty} \langle Au_n, u_n - u \rangle \geq 0.$$

Together with the assumption above, this implies

$$\lim_{n\to\infty} \langle Au_n, u_n - u\rangle = 0. \tag{1.1}$$

Let $v \in V$ and define $w := (1-\lambda)u + \lambda v$ ($\lambda \in (0,1)$). By monotonicity,

$$\langle Au_n - Aw, u_n - w\rangle \geq 0.$$

Together with the definition of w, this implies

$$\begin{aligned} \langle Au_n, u_n - w\rangle &= \langle Au_n, u_n - u\rangle + \langle Au_n, u - w\rangle \\ &= \langle Au_n, u_n - u\rangle + \lambda\,\langle Au_n, u - v\rangle \\ &\geq \langle Aw, u_n - w\rangle \\ &= \langle Aw, u_n - u\rangle + \lambda\,\langle Aw, u - v\rangle. \end{aligned}$$

Since $u_n \rightharpoonup u$ and by (1.1), we obtain

$$\liminf_{n\to\infty} \langle Au_n, u - v\rangle \geq \langle A((1-\lambda)u + \lambda v), u - v\rangle.$$

Letting $\lambda \searrow 0$ and using the hemi-continuity of A, we finally obtain

$$\liminf_{n\to\infty} \langle Au_n, u - v\rangle \geq \langle Au, u - v\rangle.$$

Hence, A is pseudo-monotone.

(ii)⇒(iii) Assume that A is pseudo-monotone, and let $(u_n) \subset V$ be a sequence such that $u_n \rightharpoonup u$, $Au_n \rightharpoonup \chi$ and $\limsup_{n\to\infty}\langle Au_n, u_n\rangle \leq \langle \chi, u\rangle$. Then

$$\limsup_{n\to\infty} \langle Au_n, u_n - u\rangle \leq 0$$

which together with the pseudo-monotonicity implies

$$\liminf_{n\to\infty} \langle Au_n, u_n - v\rangle \geq \langle Au, u - v\rangle \quad \text{for every } v \in V.$$

Together with the assumption above, this implies

$$\langle \chi, u\rangle - \langle \chi, v\rangle \geq \langle Au, u - v\rangle,$$

or

$$\langle \chi - Au, u - v\rangle \geq 0 \quad \text{for every } v \in V.$$

This is equivalent to

$$\langle \chi - Au, v\rangle \geq 0 \quad \text{for every } v \in V,$$

which in turn implies (the inequality is true for v and $-v$)

$$\langle \chi - Au, v\rangle = 0 \quad \text{for every } v \in V.$$

Hence, $Au = \chi$. □

Corollary 1.7. *Let V be a reflexive Banach space, and let $A : V \to V'$ be a monotone, bounded, hemicontinuous operator. Then*

$$u_n \to u \text{ in } V \quad \Rightarrow \quad Au_n \rightharpoonup Au \text{ in } V'.$$

Proof. Assume that $u_n \to u$ in V. Since A is bounded, the sequence (Au_n) is bounded in V'. Since V is reflexive, and after passing to a subsequence, there exists $\chi \in V'$ such that $Au_n \rightharpoonup \chi$ in V'.

Moreover,

$$\begin{aligned} \langle Au_n, u_n \rangle &= \langle Au_n, u_n - u \rangle + \langle Au_n, u \rangle \\ &\leq \|Au_n\| \, \|u_n - u\| + \langle Au_n, u \rangle. \end{aligned}$$

Hence,

$$\limsup_{n\to\infty} \langle Au_n, u_n \rangle \leq \langle \chi, u \rangle.$$

By Lemma 1.6 (implication (i)⇒(iii)), we obtain $Au = \chi$. □

2. Surjectivity of monotone operators

In this section we give a sufficient condition for the surjectivity of a monotone operator $A : V \to V'$. Before, however, we recall Brouwer's fixed point theorem, without proof.

Theorem 2.1 (Brouwer's fixed point theorem). *Let $C \subset \mathbb{R}^n$ be a nonempty, compact, convex set, and let $f : C \to C$ be a continuous function. Then f has a fixed point, that is, there exists $x \in C$ such that $f(x) = x$.*

Corollary 2.2. *Let $f \in C(\mathbb{R}^n; \mathbb{R}^n)$. Assume that there exists $\varrho > 0$ such that $\langle f(x), x \rangle_{\mathbb{R}^n} \geq 0$ whenever $\|x\| = \varrho$. Then there exists $x \in \mathbb{R}^n$ such that $\|x\| \leq \varrho$ and $f(x) = 0$.*

Proof. Assume, on the contrary, that $f(x) \neq 0$ whenever $\|x\| \leq \varrho$, and let $C := \bar{B}(0, \varrho)$. Then the function $g : C \to C$ given by $g(x) = -\varrho \frac{f(x)}{\|f(x)\|}$ is well defined and continuous. By Brouwer's fixed point theorem, there exists $x \in C$ such that $x = g(x) = -\varrho \frac{f(x)}{\|f(x)\|}$. Since $\|g(x)\| = \varrho$, this implies $\|x\| = \varrho$. Therefore,

$$\varrho^2 = \langle x, x \rangle = -\varrho \langle \frac{f(x)}{\|f(x)\|}, x \rangle \leq 0,$$

using also the assumption on f. This is a contradiction to $\varrho > 0$, and therefore, there exists $x \in C$ such that $f(x) = 0$. □

Theorem 2.3. *Let V be a separable, reflexive Banach space. Let $A : V \to V'$ be a monotone, bounded, hemicontinuous operator and assume that A is also* coercive, *that is,*

$$\lim_{\|v\|\to\infty} \frac{\langle Av, v \rangle}{\|v\|} = \infty.$$

Then A is surjective, that is, for every $f \in V'$ there exists $u \in V$ such that $Au = f$.

Proof. Let $f \in V'$. We have to solve the equation $Au = f$.

Let (w_m) be a total sequence, that is, a sequence such that span $\{w_m : m\}$ is dense in V; the existence of such a sequence is guaranteed by the assumption that V is separable.

Let $V_m :=$ span $\{w_k : 1 \leq k \leq m\}$.

We first prove that for every m there exists $u_m \in V_m$ such that

$$\langle Au_m, w_k\rangle = \langle f, w_k\rangle \quad \text{for every } 1 \le k \le m. \tag{2.1}$$

For every $u \in V_m$ we restrict the linear functional $Au \in V' = \mathcal{L}(V,\mathbb{R})$ to the closed subspace V_m, and we thus obtain a linear functional on V_m. In other words, we define an operator $A_m : V_m \to V_m'$ by

$$\langle A_m u, w\rangle_{V_m',V_m} := \langle Au, w\rangle_{V',V}.$$

By coercivity, there exists $\varrho > 0$ such that for every $u \in V$, $\|u\| \ge \varrho$,

$$\begin{aligned} \langle A_m u - f, u\rangle_{V_m',V_m} &= \langle Au - f, u\rangle_{V',V} \\ &\ge \langle Au, u\rangle - \|f\|\,\|u\| \\ &= \|u\|\,\Big(\frac{\langle Au, u\rangle}{\|u\|} - \|f\|\Big) \\ &\ge 0. \end{aligned}$$

The operator A_m inherits the properties of A, that is, A_m is monotone, bounded, hemicontinuous. By Corollary 1.7, it therefore maps convergent sequences in V_m into weakly convergent sequences in V_m'; more precisely, if $u_n \to u$ in V_m, then $Au_m \rightharpoonup Au$ in V_m'. However, the space V_m' being finite dimensional, weak convergence and norm convergence coincide, and hence A_m is continuous.

By the continuity of A_m, by the above inequality, and by Corollary 2.2, there exists $u_m \in V_m$ such that $A_m u_m - f = 0$. In other words, for every $w \in V_m$,

$$\langle Au_m - f, w\rangle_{V',V} = \langle A_m u_m - f, w\rangle_{V_m',V_m} = 0,$$

so that we have proved (2.1).

By the preceding equality, for every m,

$$\langle Au_m, u_m\rangle = \langle f, u_m\rangle \le \|f\|\,\|u_m\|.$$

Therefore, the sequence $(\frac{\langle Au_m, u_m\rangle}{\|u_m\|})$ is bounded in V. By coercivity of A, this implies that the sequence (u_m) is bounded in V. Since A is bounded, also the sequence (Au_m) is bounded. Since V and V' are reflexive, and after passing to a subsequence, there exists $u \in V$, $\chi \in V'$ such that

$$u_m \rightharpoonup u \text{ in } V \quad \text{and} \quad Au_m \rightharpoonup \chi \text{ in } V'.$$

For every k we have

$$\langle \chi, w_k\rangle = \lim_{m\to\infty} \langle Au_m, w_k\rangle = \langle f, w_k\rangle.$$

Since the sequence (w_k) is total in V, this implies $\chi = f$. Moreover,

$$\begin{aligned} \limsup_{m\to\infty} \langle Au_m, u_m\rangle &= \limsup_{m\to\infty} \langle f, u_m\rangle \\ &= \lim_{m\to\infty} \langle f, u_m\rangle \\ &= \langle f, u\rangle. \end{aligned}$$

By Lemma 1.6 (implication (i)⇒(iii)), $Au = f$. □

Lemma 2.4. *Let $A : V \to V'$ be monotone and assume that one of the following conditions holds:*
(i) A is strictly monotone, *that is*

$$\langle Au - Av, u - v\rangle > 0 \quad \text{for every } u,\ v \in V,\ u \neq v,$$

(ii) A is hemicontinuous, V is strictly convex, and $Au = Av$ implies $\|u\| = \|v\|$.
Then A is injective.

Proof. (i) If $Au = Av$, then $\langle Au - Av, u - v\rangle = 0$, and therefore $u = v$ by strict monotonicity.
(ii) We first prove that for every $f \in V'$

$$Au = f \quad \Leftrightarrow \quad \forall v \in V : \langle Av - f, v - u\rangle \geq 0. \tag{2.2}$$

In fact, if $Au = f$, then $\langle Av - f, v - u\rangle \geq 0$ by monotonicity of A. For the converse implication, let $w \in V$, $\lambda \geq 0$ and put $v = u + \lambda w$. Then

$$\langle A(u + \lambda w) - f, \lambda w\rangle \geq 0,$$

or

$$\langle A(u + \lambda w) - f, w\rangle \geq 0.$$

Letting $\lambda \searrow 0$ and using that A is hemicontinuous, we obtain

$$\langle Au - f, w\rangle \geq 0.$$

Replacing w by $-w$, we obtain $\langle Au - f, w\rangle = 0$, and since $w \in V$ is arbitrary, $Au = f$. Hence we have proved (2.2).

Let $S := \{u \in V : Au = f\}$ be the set of all solutions of the equation $Au = f$. For every $v \in V$, the set $S_v := \{u \in V : \langle Av - f, v - u\rangle \geq 0\}$ is convex, and by (2.2), $S = \bigcap_{v\in V} S_v$ is therefore convex, too. By assumption, $S \subset \{\|u\| = \varrho\}$ for some $\varrho \geq 0$. Since V is strictly convex, the set S is therefore reduced to at most one point. As a consequence, A is injective. □

3. * A nonlinear elliptic problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $p \geq 2$. Let $b \in \mathbb{R}^n$, and let $f : \Omega \to \mathbb{R}$ be some function in $L^2(\Omega)$. We consider the nonlinear elliptic boundary value problem

$$\begin{cases} -\Delta_p u(x) + b \cdot \nabla u(x) = f(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

We call a function $u \in W_0^{1,p}(\Omega)$ a *weak solution* of this problem if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi + \sum_{i=1}^n \int_\Omega b_i \frac{\partial u}{\partial x_i}\varphi = \int_\Omega f\varphi \quad \text{for every } \varphi \in C_c^1(\Omega). \tag{3.2}$$

Note that $u \in W_0^{1,p}(\Omega)$ is a weak solution of (3.1) if and only if $-\Delta_p^\Omega u + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} = f$, where Δ_p^Ω is the p-Laplace operator defined in Chapter 1, Section 3.

Theorem 3.1. *For every* $f \in L^2(\Omega)$ *there exists a unique weak solution* $u \in W_0^{1,p}(\Omega)$ *of the problem* (3.1).

For the proof, we first prove a general result.

Lemma 3.2. *Let* $b_i \in C(\mathbb{R};\mathbb{R})$ *be a function satisfying the growth condition*

$$|b_i(s)| \leq C\,(1+|s|)^{p-2} \quad \text{for some } C \geq 0 \text{ and all } s \in \mathbb{R}. \tag{3.3}$$

Then the operator

$$\begin{aligned} B_i : W_0^{1,p}(\Omega) &\to W^{-1,p'}(\Omega), \\ u &\mapsto b_i(u)\frac{\partial u}{\partial x_i}, \end{aligned}$$

is well defined, bounded and hemicontinuous. If b_i *is constant, then* B_i *is in addition monotone.*

Proof. Let $u, v \in W_0^{1,p}(\Omega)$. Then, by the growth estimate (3.3) and by Hölder's inequality,

$$\begin{aligned} \int_\Omega |B_i(u)v| &= \int_\Omega |b_i(u)\,\frac{\partial u}{\partial x_i}\,v| \\ &\leq C\int_\Omega (1+|u|)^{p-2}\,|\frac{\partial u}{\partial x_i}\,v| \\ &\leq C\,(\int_\Omega (1+|u|)^{\frac{p(p-2)}{p-1}}\,|v|^{\frac{p}{p-1}})^{\frac{p-1}{p}}\,\|\frac{\partial u}{\partial x_i}\|_p \\ &\leq C\,(\int_\Omega (1+|u|)^p)^{\frac{p-2}{p}}\,\|v\|_p\,\|\frac{\partial u}{\partial x_i}\|_p \\ &< \infty, \end{aligned}$$

so that B_i is well-defined. From this estimate we obtain in addition for every $u \in W_0^{1,p}(\Omega)$

$$\begin{aligned} \|B_i(u)\|_{W^{-1,p'}} &= \sup_{\|v\|_{W_0^{1,p}}\leq 1} \Big|\int_\Omega B_i(u)v\Big| \\ &\leq \sup_{\|v\|_{W_0^{1,p}}\leq 1} \|1+|u|\|_p^{p-2}\,\|v\|_p\,\|\frac{\partial u}{\partial x_i}\|_p \\ &\leq (C+\|u\|_p)^{p-2}\,\|u\|_{W_0^{1,p}}, \end{aligned}$$

so that B_i is bounded.

Next, let $u, v, w \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} |\langle B_i(u+tv)-B_i(u), w\rangle| &\leq \int_\Omega |b_i(u+tv)-b_i(u)|\,|\frac{\partial u}{\partial x_i}|\,|w| + \\ &\quad + t\int_\Omega |b_i(u+tv)|\,|\frac{\partial v}{\partial x_i}|\,|w| \\ &\to 0 \quad \text{as } t \to 0, \end{aligned}$$