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Partial differential equations

– Sobolev spaces and variational methods –

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Vorwort:

Ich habe dieses Skript zur Vorlesung Partielle Differentialgleichungen an der TU Dresden nach meinem besten Wissen und Gewissen geschrieben. Mit Sicherheit schlichen sich jedoch Druckfehler oder gar mathematische Ungenauigkeiten ein, die man beim ersten Schreiben eines Skripts nicht vermeiden kann. Möge man mir diese Fehler verzeihen.

Obwohl die Vorlesung auf Deutsch gehalten wird, habe ich mich entschieden, dieses Skript auf Englisch zu verfassen. Auf diese Weise wird eine Brücke zwischen der Vorlesung und der (meist englischsprachigen) Literatur geschlagen. Mathematik sollte jedenfalls unabhängig von der Sprache sein in der sie präsentiert wird.

Ich danke Susanne Stimpert fr die Hilfe beim Erstellen dieses Skripts. Für weitere Kommentare, die zur Verbesserungen beitragen, bin ich sehr dankbar.

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1.1 The space of test functions

For subsets Ω of $\mathbb{R}^N$, we denote by

$$C(\Omega) := \{u : \Omega \to \mathbb{C} : u \text{ is continuous}\}$$

the space of continuous complex-valued functions. Sometimes, we only consider real-valued functions, especially when order properties such as positivity or comparison are involved, but this will be clear from the context. If Ω is in addition measurable, then we denote by

$$L^p(\Omega) := \{u : \Omega \to \mathbb{C} : u \text{ is measurable, } \int_\Omega |u|^p < \infty\} \quad (p \in [1,\infty)) \text{ and}$$
$$L^\infty(\Omega) := \{u : \Omega \to \mathbb{C} : u \text{ is measurable, and } \{u > c\} \text{ is a null set for some } C \geq 0\}$$

the usual Lebesgue spaces which are equipped with the norms

$$\|u\|_{L^p} := \left(\int_\Omega |u|^p\right)^{\frac{1}{p}} \quad (p \in [1,\infty)), \text{ and}$$
$$\|u\|_{L^\infty} := \inf\{C \geq 0 : \{u > C\} \text{ is a null set}\}.$$

Strictly speaking, $L^p(\Omega)$ is a space of equivalence classes of measurable functions but it is often convenient to work with representatives instead of equivalence classes. For measurable $\Omega \subseteq \mathbb{R}^N$, we set

$$L^p_{loc}(\Omega) := \{u : \Omega \to \mathbb{C} : u \text{ is measurable and for every } K \text{ compact } \int_K |u|^p < \infty\},$$

the space of **locally p-integrable functions**. For every $p \in [1,\infty]$ one has

$$L^p(\Omega) \subseteq L^p_{loc}(\Omega) \subseteq L^1_{loc}(\Omega) \text{ and}$$
$$C(\Omega) \subseteq L^\infty_{loc}(\Omega) \subseteq L^1_{loc}(\Omega).$$

Hence, among all spaces defined above, $L^1_{loc}(\Omega)$ is the largest space.

For a function $u \in L^1_{loc}(\Omega)$ we define the **support** by

$$\operatorname{supp} u := \Omega \setminus \bigcup_{\substack{U \subseteq \Omega \text{ rel. open} \\ u=0 \text{ a.e. on } U}} U.$$

By definition, the support is relatively closed in Ω (but it is in general not closed in $\mathbb{R}^N$). The definition of the support depends on the underlying set Ω, and although the spaces $L^p([0,1])$ and $L^p((0,1))$ coincide, the support of their elements do not coincide in general; if nothing is said explicitly, the underlying set Ω is usually supposed to be open in $\mathbb{R}^N$.

We say that a function $u \in L^1_{loc}(\Omega)$ has compact support if $\operatorname{supp} u$ is compact, and we set

$$\begin{aligned} L^p_c(\Omega) &:= \{u \in L^p(\Omega) : \operatorname{supp} u \text{ is compact}\}, \\ C_c(\Omega) &:= \{u \in C(\Omega) : \operatorname{supp} u \text{ is compact}\}, \\ C^k_c(\Omega) &:= C_c(\Omega) \cap C^k(\Omega), \\ \mathrm{C}^\infty_{\mathrm{c}}(\Omega) &:= C_c(\Omega) \cap C^\infty(\Omega). \end{aligned}$$

For the definition of the latter two spaces we suppose that Ω is open. The space $\mathrm{C}^\infty_{\mathrm{c}}(\Omega)$ is sometimes also denoted by $\mathcal{D}(\Omega)$. Elements of $\mathrm{C}^\infty_{\mathrm{c}}(\Omega)$ are called **test functions**; $\mathrm{C}^\infty_{\mathrm{c}}(\Omega)$ is called the space of test functions.

Example 1.1. The function

$$u(x) := \begin{cases} e^{-\frac{1}{1-\|x\|^2}} & \text{if } \|x\|^2 = \sum_{i=1}^N x_i^2 < 1, \\ 0 & \text{else} \end{cases}$$

is a test function on $\mathbb{R}^N$ and $\operatorname{supp} u$ is the closed unit ball.

Exercise 1.2 *Show that for a function $u \in C(\Omega)$*

$$\operatorname{supp} u = \overline{\{x \in \Omega \,:\, u(x) \neq 0\}}^{\Omega},$$

where the closure is understood in the relative Euclidean topology induced on Ω.

Exercise 1.3 *Show that every $u \in L^1_{loc}(\Omega)$ is almost everywhere equal to 0 on $\Omega \setminus \operatorname{supp} u$.*

1.2 The convolution

Theorem 1.4 (Young). *Fix $p \in [1, \infty]$, $f \in L^1(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$. Then, for almost every $x \in \mathbb{R}^N$, the function*

$$y \mapsto f(x-y)g(y)$$

is integrable on $\mathbb{R}^N$, and the function $f * g : \mathbb{R}^N \to \mathbb{C}$ defined by

$$(f * g)(x) := \int_{\mathbb{R}^N} f(x-y)g(y)\, dy$$

belongs to $L^p(\mathbb{R}^N)$. Moreover,

$$\|f * g\|_{L^p} \le \|f\|_{L^1} \|g\|_{L^p}$$

Proof. *The case $p = 1$.* Let us first assume that $p = 1$. Then Tonelli's theorem yields

$$\begin{aligned}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |f(x-y)\, g(y)|\, dy\, dx &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |f(x-y)|\, dx |g(y)|\, dy \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |f(x)|\, dx |g(y)|\, dy \\ &= \|f\|_{L^1} \|g\|_{L^1} \quad < \quad \infty.\end{aligned}$$

In particular, the left-hand side is finite which is only possible if for almost every $x \in \mathbb{R}^N$, the integral $\int_{\mathbb{R}^N} |f(x-y)\, g(y)|\, dy$ is finite, that is, the function $y \mapsto f(x-y)g(y)$ is integrable. Since the left-hand side equals $\|f * g\|_{L^1}$, the above inequality also yields $\|f * g\|_{L^1} \le \|f\|_{L^1} \|g\|_{L^1}$.

The case $1 < p < \infty$. The assumption $g \in L^p(\mathbb{R}^N)$ implies $|g|^p \in L^1(\mathbb{R}^N)$, and then the first step implies that for almost every $x \in \mathbb{R}^N$, the function $y \mapsto |f(x-y)|\,|g(y)|^p$ is integrable. This, in turn, is equivalent to saying that for almost every $x \in \mathbb{R}^N$, the function $y \mapsto |f(x-y)|^{\frac{1}{p}}\, |g(y)|$ is p-integrable. On the other hand, $|f|^{\frac{1}{p'}} \in L^{p'}(\mathbb{R}^N)$, where $p' = \frac{p}{p-1}$ is the conjugate exponent. Hence, by Hölder's inequality, for almost every $x \in \mathbb{R}^N$, the function $y \mapsto |f(x-y)|\,|g(y)|$ is integrable, and

$$\begin{aligned}|f * g(x)| &\le \int_{\mathbb{R}^N} |f(x-y)|\,|g(y)|\, dy \\ &\le \left(\int_{\mathbb{R}^N} |f(x-y)|\, dy \right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^N} |f(x-y)|\,|g(y)|^p\, dy \right)^{\frac{1}{p}} \\ &= \|f\|_{L^1}^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} |f(x-y)|\,|g(y)|^p\, dy \right)^{\frac{1}{p}}.\end{aligned}$$

Raising both sides of this inequality to the p-th power and integrating over $\mathbb{R}^N$ yields, by another application of Tonelli's theorem,

$$\begin{aligned}\|f*g\|_{L^p}^p &= \int_{\mathbb{R}^N} |f*g(x)|^p\,dx \\ &\le \|f\|_{L^1}^{p-1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(x-y)||g(y)|^p\,dy\,dx \\ &= \|f\|_{L^1}^{p-1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |f(x-y)|\,dx|g(y)|^p\,dy \\ &= \|f\|_{L^1}^p\,\|g\|_{L^p}^p.\end{aligned}$$

From here follows the claim for $p \in (1,\infty)$.

The statement for $p=\infty$ is evident.

Corollary 1.5. *Fix $p \in [1,\infty]$, $f \in L^1_{loc}(\mathbb{R}^N)$ and $g \in L^p_c(\mathbb{R}^N)$ (or $f \in L^p_{loc}(\mathbb{R}^N)$ and $g \in L^1_c(\mathbb{R}^N)$). Then, for almost every $x \in \mathbb{R}^N$, the function*

$$y \mapsto f(x-y)g(y)$$

is integrable on $\mathbb{R}^N$, and the function $f*g : \mathbb{R}^N \to \mathbb{C}$ defined by

$$(f*g)(x) := \int_{\mathbb{R}^N} f(x-y)g(y)\,dy$$

belongs to $L^p_{loc}(\mathbb{R}^N)$.

We call the function $f*g$ from Theorem 1.4 and Corollary 1.5 the **convolution** of f and g. Let us discuss some properties of the convolution and its use in connection with the approximation of L^p-functions by test functions.

For subsets $A, B \subseteq \mathbb{R}^N$ we set

$$A+B := \{x+y : x \in A,\, y \in B\}.$$

Lemma 1.6 (Support of a convolution). *For appropriate functions f and g one has*

$$\operatorname{supp} f*g \subseteq \overline{\operatorname{supp} f + \operatorname{supp} g}.$$

Proof. For $x \notin \operatorname{supp} f + \operatorname{supp} g$ one has

$$(x-\operatorname{supp} f)\cap \operatorname{supp} g = \emptyset,$$

and therefore

$$(f*g)(x) = \int_{(x-\operatorname{supp} f)\cap\operatorname{supp} g} f(x-y)\,g(y)\,dy = 0.$$

In other words, $f*g=0$ everywhere on the complement of $\overline{\operatorname{supp} f + \operatorname{supp} g}$.

For every $h \in \mathbb{R}^N$ and every function $f : \mathbb{R}^N \to \mathbb{C}$ we define the **shift** $\tau_h f : \mathbb{R}^N \to \mathbb{C}$,

$$\tau_h f(x) := f(x+h) \quad (x \in \mathbb{R}^N).$$

Lemma 1.7 (Strong continuity of the shifts). *For every $p \in [1,\infty)$ and every $f \in L^p(\mathbb{R}^N)$ one has*

$$\lim_{x\to 0} \|\tau_h f - f\|_{L^p} = 0.$$

Proof. For characteristic functions $f = 1_Q$ with $Q = (a_1,b_1)\times\cdots\times(a_N,b_N)$, the claim follows from Lebesgue's dominated convergence theorem. In fact, this function is continuous at almost every point, except on the null set ∂Q. By the triangle inequality, the claim thus holds for every $f \in U$, where U is the linear span of characteristic functions of the form 1_Q. By properties of the Lebesgue measure and by properties of p-integrable functions, the space U is dense in $L^p(\mathbb{R}^N)$. Let $f \in L^p(\mathbb{R}^N)$ and $\varepsilon > 0$. Choose $g \in U$ such that $\|f - g\|_{L^p} < \varepsilon$. Then

$$\begin{aligned}
&\limsup_{h\to 0} \|\tau_h f - f\|_{L^p} \\
&\leq \limsup_{h\to 0} [\|\tau_h f - \tau_h g\|_{L^p} + \|\tau_h g - g\|_{L^p} + \|g - f\|_{L^p}] \\
&< 2\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we obtain the claim.

Theorem 1.8 (Approximation of the identity). *Let $\varphi \in L^1(\mathbb{R}^N)$ be positive and such that $\int_{\mathbb{R}^N} \varphi = 1$. Set*

$$\varphi_n(x) := n^N \varphi(nx) \quad (x \in \mathbb{R}^N, n \in \mathbb{N}),$$

so that $\varphi_n \in L^1(\mathbb{R}^N)$ is positive and $\int_{\mathbb{R}^N} \varphi_n = 1$. Then, for every $p \in [1,\infty)$ and every $f \in L^p(\mathbb{R}^N)$,

$$\lim_{n\to\infty} \|f * \varphi_n - f\|_{L^p} = 0$$

Proof. Assume first that $p = 1$, that is, $f \in L^1(\mathbb{R}^N)$. Then, by an application of Tonnelli's theorem, Lemma 1.7, and Lebesgue's dominated convergence theorem,

$$\begin{aligned}
\|f * \varphi_n - f\|_{L^1} &= \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} f(x-y)\varphi_n(y)\,dy - f(x) \right| dx \\
&= \int_{\mathbb{R}^N} \left| \int_{\mathbb{R}^N} [f(x-y) - f(x)]\,\varphi_n(y)\,dy \right| dx \\
&\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |f(x - \frac{y}{n}) - f(x)|\,dx\,\varphi(y)\,dy \\
&= \int_{\mathbb{R}^N} \|\tau_{\frac{y}{n}} f - f\|_{L^1}\,\varphi(y)\,dy \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Assume next that $p \in (1, \infty)$. Let $f \in L^1 \cap L^\infty(\mathbb{R}^N)$. Recall the interpolation inequality

$$\|f\|_{L^p} \le \|f\|_{L^1}^{\frac{1}{p}} \|f\|_{L^\infty}^{\frac{1}{p'}},$$

which is a straightforward consequence of Hölder's inequality. Applying this inequality to $f * \varphi_n - f \in L^1 \cap L^\infty(\mathbb{R}^N)$ (by Young's inequality), we obtain, by applying also the first step,

$$\begin{aligned} \|f * \varphi_n - f\|_{L^p} &\le \|f * \varphi_n - f\|_{L^1}^{\frac{1}{p}} \|f * \varphi_n - f\|_{L^\infty}^{\frac{1}{p'}} \\ &\le \|f * \varphi_n - f\|_{L^1}^{\frac{1}{p}} (2\|f\|_{L^\infty})^{\frac{1}{p'}} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Now let $f \in L^p(\mathbb{R}^N)$ be arbitrary, and let $\varepsilon > 0$. Since $L^1 \cap L^\infty(\mathbb{R}^N)$ is dense in $L^p(\Omega)$, there exists $g \in L^1 \cap L^\infty(\mathbb{R}^N)$ such that $\|f - g\|_{L^p} < \varepsilon$. Hence, by an application of Young's inequality and the preceding inequality,

$$\begin{aligned} &\limsup_{n\to\infty} \|f * \varphi_n - f\|_{L^p} \\ &\le \limsup_{n\to\infty} [\|f * \varphi_n - g * \varphi_n\|_{L^p} + \|g * \varphi_n - g\|_{L^p} + \|g - f\|_{L^p}] \\ &\le \limsup_{n\to\infty} [\|f - g\|_{L^p} \|\varphi_n\|_{L^1} + \|g - f\|_{L^p}] \\ &< 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the claim follows.

Theorem 1.9 (Smoothing effect of the convolution). *For every $f \in L^1_{loc}(\mathbb{R}^N)$ and every $g \in C^k_c(\mathbb{R}^N)$ (with $k \in \mathbb{N}_0 \cup \{\infty\}$) one has $f * g \in C^k(\mathbb{R}^N)$ and for every multi-index $\alpha \in \mathbb{N}_0^N$ of length $|\alpha| \le k$ one has*

$$\partial^\alpha (f * g) = f * (\partial^\alpha g).$$

Proof. Assume first $k = 0$. For every $x_0 \in \mathbb{R}^N$ one has, by continuity of g and by an application of Lebesgue's dominated convergence theorem,

$$\limsup_{x\to x_0} |(f * g)(x) - (f * g)(x_0)| \le \limsup_{x\to x_0} \int_{\mathbb{R}^N} |f(y)| \, |g(x - y) - g(x_0 - y)| \, dy = 0,$$

and hence $f * g$ is continuous at x_0. Since x_0 was arbitrary, we have proved that $f * g$ is continuous.

Assume next that $k = 1$. Fix $x \in \mathbb{R}^N$, $j \in \{1, \dots, N\}$, and let $e_j \in \mathbb{R}^N$ be the j-th canonical unit vector. Then, by an application of Lebesgue's dominated convergence theorem again,

$$\begin{aligned}\frac{(f*g)(x+te_j)-(f*g)(x)}{t} &= \int_{\mathbb{R}^N} f(y)\frac{g(x+te_j-y)-g(x-y)}{t}\,dy \\ &\to \int_{\mathbb{R}^N} f(y)\frac{\partial g}{\partial x_j}(x-y)\,dy \\ &= (f*\frac{\partial g}{\partial x_j})(x) \text{ as } t\to 0.\end{aligned}$$

Since $x\in\mathbb{R}^N$ was arbitrary, we have shown that the function $f*g$ is partially differentiable with respect to x_j, and the partial derivative $\frac{\partial}{\partial x_j}(f*g)$ coincides with $f*\frac{\partial g}{\partial x_j}$. Since the latter function is continuous by the first part of the proof ($k=0$), we obtain that $f*g$ is continuously partially differentiable, which is equivalent to saying that $f*g$ is continuously differentiable, that is, $f*g\in C^1(\mathbb{R}^N)$.

The case $k>1$ follows by induction.

For every subset $A\subseteq\mathbb{R}^N$ and every $\varepsilon>0$ we define

$$A^\varepsilon := \{x\in\mathbb{R}^N : \operatorname{dist}(x,A)<\varepsilon\},$$

where

$$\operatorname{dist}(x,A) := \inf\{\|x-y\| : y\in A\}.$$

Lemma 1.10 (Further examples of test functions). *For every compact subset $K\subseteq\mathbb{R}^N$ and every $\varepsilon>0$ there exists a test function $\psi\in C_c^\infty(\mathbb{R}^N)$ such that*

$$\begin{aligned}&0\le\psi(x)\le 1 \textit{ for every } x\in\mathbb{R}^N,\\ &\psi(x)=1 \textit{ for every } x\in K, \textit{ and}\\ &\psi(x)=0 \textit{ for every } x\in\mathbb{R}^N\setminus K^\varepsilon.\end{aligned}$$

Proof. Let $K\subseteq\mathbb{R}^N$ be compact and $\varepsilon>0$. Let φ be a scalar multiple of the test function from Example 1.1, so that $\varphi\ge 0$ and $\int_{\mathbb{R}^N}\varphi=1$. Put

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^N}\varphi(\frac{x}{\varepsilon}) \quad (x\in\mathbb{R}^N).$$

Then $\varphi_\varepsilon\in C_c^\infty(\mathbb{R}^N)$ is a positive test, too, $\operatorname{supp}\varphi_\varepsilon=\bar{B}(0,\varepsilon)$ (the closed ball with center 0 and radius ε) and $\int_{\mathbb{R}^N}\varphi_\varepsilon=1$. Let

$$\psi := 1_{K^\varepsilon}*\varphi_\varepsilon.$$

By the smoothing effect of the convolution (Theorem 1.9), $\psi\in C^\infty(\mathbb{R}^N)$. By Lemma 1.6 on the support of the convolution,

$$\operatorname{supp}\psi\subseteq K^\varepsilon+\bar{B}(0,\varepsilon)=K^{2\varepsilon}.$$

In particular, the support of ψ is compact, that is, $\psi \in C_c^\infty(\mathbb{R}^N)$, but this inclusion also shows that $\psi = 0$ on $\mathbb{R}^N \setminus K^{2\varepsilon}$. Since 1_{K^ε} and φ_ε are positive, their convolution is positive, too, that is, $\psi \geq 0$. Moreover, for every $x \in \mathbb{R}^N$,

$$\psi(x) = \int_{\mathbb{R}^N} 1_{K^\varepsilon}(x-y)\varphi_\varepsilon(y)\,dy \leq \int_{\mathbb{R}^N} \varphi_\varepsilon(y)\,dy = 1,$$

so that $0 \leq \psi \leq 1$ on $\mathbb{R}^N$. Finally, for every $x \in K$, the preceding inequality turns into an equality, that is, $\psi = 1$ on K. Since $\varepsilon > 0$ was arbitrary, we have proved the claim.

Theorem 1.11 (The test functions are dense in $L^p(\Omega)$). *For every open subset $\Omega \subseteq \mathbb{R}^N$ and every $p \in [1,\infty)$ the space of test functions $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.*

Proof (by truncation and regularization). Let $\Omega \subseteq \mathbb{R}^N$ be open, and let $f \in L^p(\Omega)$.

Truncation. First, we choose an increasing sequence (K_n) of compact subsets of Ω such that $\bigcup_n K_n = \Omega$. For example, the sequence given by

$$K_n := \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) \geq \frac{1}{n} \text{ and } \|x\| \leq n\}$$

will do. Next, we choose a sequence (ε_n) of positive reals such that

$$4\,\varepsilon_n < \operatorname{dist}(K_n,\partial\Omega).$$

Finally, for each n we choose a test function (cut-off function) $\psi_n \in C_c^\infty(\Omega)$ such that $0 \leq \psi_n \leq 1$ on Ω, $\psi_n = 1$ on K_n and $\psi_n = 0$ on $\Omega \setminus K_n^{\varepsilon_n}$ (see Lemma 1.10). By choice of the K_n and the ψ_n, the sequence (ψ_n) is uniformly bounded by 1 and converges pointwise to the constant function 1. In particular, by Lebesgue's dominated convergence theorem,

$$\limsup_{n\to\infty} \|f\psi_n - f\|_{L^p}^p = \limsup_{n\to\infty} \int_{\mathbb{R}^N} |f(x)\psi_n(x) - f(x)|^p\,dx = 0.$$

Regularization. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be a positive test function such that $\operatorname{supp}\varphi \subseteq \bar{B}(0,1)$ and $\int_{\mathbb{R}^N} \varphi = 1$. For example, one may take a scalar multiple of the test function from Example 1.1. For every $m \in \mathbb{N}$ we set

$$\varphi_m(x) := \varepsilon_m^{-N}\,\varphi(\frac{x}{\varepsilon_m}) \quad (x \in \mathbb{R}^N).$$

Then $\varphi_m \in C_c^\infty(\mathbb{R}^N)$ is positive, $\operatorname{supp}\varphi_m \subseteq \bar{B}(0,\frac{1}{m})$ and $\int_{\mathbb{R}^N} \varphi_m = 1$. By Theorem 1.8, for every $n \in \mathbb{N}$,

$$\lim_{m\to\infty} \|f\psi_n - (f\psi_n) * \varphi_m\|_{L^p} = 0,$$

where we have extended the function $f\psi_n$ by 0 outside Ω in order to take the convolution in $\mathbb{R}^N$. Since the sequence $(f\psi_n)$ is convergent by the first step, and since it is thus relatively compact, the above limit is uniform in $n \in \mathbb{N}$. In particular, when we put

$$f_n := (f\psi_n) * \varphi_n,$$

then

$$\lim_{n\to\infty} \|f_n - f\|_{L^p} = 0.$$

Moreover, by Theorem 1.9 on the smoothing effect of the convolution, $f_n \in C^\infty(\mathbb{R}^N)$. By Lemma 1.6 on the support of the convolution,

$$\begin{aligned}\operatorname{supp} f_n &\subseteq \operatorname{supp}(f\psi_n) + \operatorname{supp}\varphi_n \\ &\subseteq \operatorname{supp}\psi_n + \bar{B}(0,\varepsilon_n) \\ &\subseteq K_n^{\varepsilon_n} + \bar{B}(0,\varepsilon_n) \\ &\subseteq K_n^{3\varepsilon_n}.\end{aligned}$$

By the choice of ε_n, the support of f_n is thus contained in Ω, so that $f_n \in C_c^\infty(\Omega)$. Summing up, (f_n) is a desired sequence of test functions in $C_c^\infty(\Omega)$ which converges in $L^p(\Omega)$ to f. Since f was arbitrary, this implies the claim.

In the truncation part of the preceding proof, it was actually not really important that the truncation function (cut-off function) ψ_n was a test function: the characteristic function 1_{K_n} would also be fine. However, as a corollary of the above proof, which uses the slightly more complicated truncation functions, we obtain the following result.

Theorem 1.12 (Uniqueness by testing). *Let $\Omega \subseteq \mathbb{R}^N$ be open and $f \in L^1_{loc}(\Omega)$. If*

$$\int_\Omega f\varphi = 0 \text{ for every } \varphi \in C_c^\infty(\Omega),$$

then $f = 0$.

Proof. Choose (ψ_n) and (φ_n), and define (f_n) as in the proof of Theorem 1.11 above. If $f \in L^1(\Omega)$, then, by Theorem 1.11, $f_n \to f$ in $L^1(\Omega)$. On the other hand, the assumption implies $f_n = 0$ on Ω, and hence $f = 0$. If f is only locally integrable, then one may apply this reasoning to $f1_\omega$, where $\omega \subseteq \Omega$ is an open subset such that $\bar{\omega}$ is compact and contained in Ω.

1.3 Weak derivatives and Sobolev spaces

We use the following notations for the partial derivative with respect to the j-th variable in $\mathbb{R}^N$:

$$\frac{\partial}{\partial x_j} \text{ or } \partial_{x_j} \text{ or } \partial_j.$$

Given a multi-index $\alpha \in \mathbb{N}_0^N$, we use also the abbreviation

$$\partial^\alpha := \partial_{x_1}^{\alpha_1} \cdot \dots \cdot \partial_{x_N}^{\alpha_N}$$

for the partial derivatives of order $|\alpha|$.

Let $\Omega \subseteq \mathbb{R}^N$ be an open set, $f \in L^1_{loc}(\Omega)$ and $\alpha \in \mathbb{N}_0^N$. We say that $\partial^\alpha f \in L^1_{loc}(\Omega)$ if there exists $g \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega f \partial^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega g\varphi \text{ for every } \varphi \in C_c^\infty(\Omega). \tag{1.1}$$

It follows from Theorem 1.12 that the function g is uniquely determined by this identity. We write $g =: \partial^\alpha f$ and call $\partial^\alpha f$ the **weak α-th partial derivative** of f. We say that f is k **times weakly differentiable** if $\partial^\alpha f \in L^1_{loc}(\Omega)$ for all multi-indices $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq k$. By Gauß' Theorem, every k times continuously differentiable function $f : \Omega \to \mathbb{C}$ is k times weakly differentiable, and the classical partial derivatives and the weak partial derivatives coincide.

Example 1.13. Consider on $\Omega := (-1,1)$ the absolute value function $f(x) = |x|$ ($x \in (-1,1)$). Then, for every $\varphi \in C_c^\infty((-1,1))$,

$$\begin{aligned}
\int_{-1}^1 f\varphi' &= \int_{-1}^0 (-x)\varphi'(x)\,dx + \int_0^1 x\varphi'(x)\,dx \\
&= (-x)\varphi(x)\Big|_{-1}^0 - \int_{-1}^0 (-1)\varphi(x)\,dx + x\varphi(x)\Big|_0^1 - \int_0^1 \varphi(x)\,dx \\
&= -\int_{-1}^1 g\varphi,
\end{aligned}$$

where

$$g(x) := \begin{cases} -1 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0. \end{cases}$$

We have thus shown that the absolute value function f is weakly differentiable and $f' = g$. The interval $(-1,1)$ in this example may be replaced by any other open set in $\mathbb{R}$.

For every $k \in \mathbb{N}$ and every $p \in [1,\infty]$ we define the **Sobolev space**

$$\begin{aligned}
W^{k,p}(\Omega) := \{f \in L^p(\Omega) : &f \text{ is } k \text{ times weakly differentiable} \\
&\text{and } \partial^\alpha f \in L^p(\Omega) \text{ for every } \alpha \in \mathbb{N}_0^N, |\alpha| \leq k\}.
\end{aligned}$$

One easily verifies that $W^{k,p}(\Omega)$ is a vector space. We equip this space with the norm

$$\|f\|_{W^{k,p}} := \left(\sum_{\substack{\alpha\in\mathbb{N}_0^N \\ |\alpha|\le k}} \|\partial^\alpha f\|_{L^p} \right)^{\frac{1}{p}} \quad \text{if } p \in [1,\infty)$$

and

$$\|f\|_{W^{k,\infty}} := \sup_{\substack{\alpha\in\mathbb{N}_0^N \\ |\alpha|\le k}} \|\partial^\alpha f\|_{L^\infty} \text{ if } p = \infty,$$

so that $W^{k,p}(\Omega)$ is a normed space. We also define the **Sobolev spaces**

$$W_0^{k,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{W^{k,p}},$$

that is, the closure of the space of test functions in $W^{k,p}(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}}$. Finally, we set

$$H^k(\Omega) := W^{k,2}(\Omega) \text{ and}$$
$$H_0^k(\Omega) := W_0^{k,2}(\Omega).$$

Both spaces are equipped with the inner product

$$\langle f, g\rangle_{H^k} := \sum_{\substack{\alpha\in\mathbb{N}_0^N \\ |\alpha|\le k}} \langle \partial^\alpha f, \partial^\alpha g\rangle_{L^2},$$

which turns them into inner product spaces. Note that the norm induced by $\langle\cdot,\cdot\rangle_{H^k}$ coincides with the norm $\|\cdot\|_{W^{k,2}} =: \|\cdot\|_{H^k}$.

Theorem 1.14. *The Sobolev spaces $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ are Banach spaces. They are separable if $p \in [1,\infty)$, and reflexive if $p \in (1,\infty)$. The spaces $H^k(\Omega)$ and $H_0^k(\Omega)$ are separable Hilbert spaces.*

Proof. Let $A := \{\alpha \in \mathbb{N}_0^N : |\alpha| \le k\}$ be the set of all multi-indices of length less than or equal to k. Consider the cartesian product $L^p(\Omega)^A$ equipped with the norm

$$\|(f^\alpha)_{\alpha\in A}\|_{(L^p)^A} := \left(\sum_{\substack{\alpha\in\mathbb{N}_0^N \\ |\alpha|\le k}} \|f^\alpha\|_{L^p} \right)^{\frac{1}{p}} \quad \text{if } p \in [1,\infty)$$

and

$$\|(f^\alpha)_{\alpha\in A}\|_{(L^\infty)^A} := \sup_{\substack{\alpha\in\mathbb{N}_0^N \\ |\alpha|\le k}} \|f^\alpha\|_{L^\infty} \text{ if } p = \infty,$$

Then the mapping

$$j : W^{k,p}(\Omega) \to L^p(\Omega)^A, \quad f \mapsto (\partial^\alpha f)_{\alpha\in A}$$

is an isometry. Since $L^p(\Omega)$ is a Banach space, it thus suffices to show that the range of j is closed in $L^p(\Omega)^A$.

Completeness. Let (f_n) be a sequence in $W^{k,p}(\Omega)$ such that $((\partial^\alpha f_n)_{\alpha\in A})$ converges in $L^p(\Omega)^A$ to some element $(f^\alpha)_{\alpha\in A}$. Put $f := f^{(0,\dots,0)}$. By definition of $W^{k,p}(\Omega)$ and the weak partial derivatives, for every $\alpha \in A$, every $n \in \mathbb{N}$ and every $\varphi \in \mathrm{C}_c^\infty(\Omega)$,

$$\int_\Omega f_n \partial^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega \partial^\alpha f_n \varphi.$$

The convergences $f_n \to f$ and $\partial^\alpha f_n \to f^\alpha$ in $L^p(\Omega)$ thus imply that

$$\int_\Omega f \partial^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega f^\alpha \varphi \text{ for every } \alpha \in A, \varphi \in \mathrm{C}_c^\infty(\Omega).$$

Hence, by definition of the weak partial derivatives, $f \in W^{k,p}(\Omega)$ and $\partial^\alpha f = f^\alpha$. In other words, $(f^\alpha)_{\alpha\in A} = j(f)$, and we have shown that the range of j is closed. In other words, $W^{k,p}(\Omega)$ is complete.

Separability. The space $L^p(\Omega)$ being separable if $p \in [1,\infty)$, the cartesian product $L^p(\Omega)^A$ is separable, too. Since every subset of a separable metric space is separable (Exercise!), $j(W^{k,p}(\Omega)) = W^{k,p}(\Omega)$ is separable if $p \in [1,\infty)$.

Reflexivity. The space $L^p(\Omega)$ being reflexive if $p \in (1,\infty)$, the cartesian product $L^p(\Omega)^A$ is reflexive, too. Since every closed subspace of a reflexive Banach space is reflexive, $j(W^{k,p}(\Omega)) = W^{k,p}(\Omega)$ is reflexive if $p \in (1,\infty)$.

Theorem 1.15. *For every $p \in [1,\infty)$, the space $\mathrm{C}_c^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$. In other words, $W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$.*

Proof. Truncation and regularization. In fact, one may take the same sequence (f_n) as constructed in the proof of Theorem 1.11, ensuring, however, that the cut-off functions ψ_n have uniformly bounded gradient. This can easily be achieved by looking into the proof of Lemma 1.10.

Remark 1.16. For general open sets $\Omega \subseteq \mathbb{R}^N$, the spaces $W_0^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ need not coincide. For example, they do not coincide for any interval $\Omega = (a,b) \neq \mathbb{R}$ (see Theorem **??** below).

Theorem 1.17 (Friedrichs). *Let $p \in [1,\infty)$ and let $\Omega \subseteq \mathbb{R}^N$ be open. Then for every $f \in W^{1,p}(\Omega)$ there exists a sequence (f_n) in $\mathrm{C}_c^\infty(\Omega)$ such that*

$$\begin{aligned} &f_n \to f \textit{ in } L^p(\Omega) \textit{ and} \\ &\nabla f_n \to \nabla f \textit{ in } L^p(\omega) \textit{ for every } \omega \subset\subset \Omega. \end{aligned}$$

Proof. Truncation and regularization. In fact, one may take the same sequence of approximations as constructed in the proof of Theorem 1.11.

Let $\Omega \in \mathbb{R}^N$ be open. For every $f \in L^1_{loc}(\Omega)$, $\omega \subset\subset \Omega$ and $h \in \mathbb{R}^N$ with $|h| < \operatorname{dist}(\omega, \partial\Omega)$ one defines $\tau_h f$ by

$$\tau_h f(x) := f(x+h) \quad (x \in \omega).$$

Theorem 1.18. *Let $\Omega \subseteq \mathbb{R}^N$ be open, $p \in (1,\infty]$, and $f \in L^p(\Omega)$. Then the following assertions are equivalent:*

(i) $f \in W^{1,p}(\Omega)$.

(ii) *There exists a constant $C \geq 0$ such that, for every $\varphi \in C_c^\infty(\Omega)$ and every $j \in \{1,\dots,N\}$,*

$$\left| \int_\Omega f \partial_j \varphi \right| \leq C \|\varphi\|_{L^{p'}}.$$

(iii) *There exists a constant $C \geq 0$ such that, for every $\omega \subset\subset \Omega$ and every $h \in \mathbb{R}^N$ with $|h| < \operatorname{dist}(\omega, \partial\Omega)$*

$$\|f - \tau_h f\|_{L^p} \leq C|h|.$$

Moreover, if the assertions (i)–(iii) are true, then one may take $C = \|\nabla f\|_{L^p}$ in (ii) and (iii).

Proof. (i)⇒(ii) follows from the definition of weak derivative and an application of Hölder's inequality.

(ii)⇒(i) Fix $j \in \{1,\dots,N\}$. By the assumption in (ii), the linear functional $L_j : \varphi \to \int_\Omega f \partial_j \varphi$ is continuous on $C_c^\infty(\Omega)$ equipped with the norm induced from $L^{p'}(\Omega)$. Since the test functions are dense in $L^{p'}(\Omega)$ by Theorem 1.11, we may extend L_j to a bounded linear functional on $L^{p'}(\Omega)$. Recall that the dual space of $L^{p'}(\Omega)$ can be identified with $L^p(\Omega)$ (here we use $p \neq 1$), that is, there exists a function $g_j \in L^p(\Omega)$ such that, for every $\varphi \in C_c^\infty(\Omega)$

$$L_j(\varphi) = \int_\Omega f \partial_j \varphi = \int_\Omega g_j \varphi.$$

By definition of the weak derivative, this means that $\partial_j f \in L^p(\Omega)$ (and $\partial_j f = -g_j$). Since j was arbitrary, we have thus proved $f \in W^{1,p}(\Omega)$.

(i)⇒(iii) Assume first that $f \in C_c^\infty(\Omega)$. Fix $\omega \subset\subset \Omega$ and $h \in \mathbb{R}^N$ such that $|h| < \operatorname{dist}(\omega, \partial\Omega)$. Then

$$\begin{aligned} f(x+h) - f(x) &= \int_0^1 \frac{d}{dt} f(x+th)\,dt \\ &= \int_0^1 \nabla f(x+th) h\,dt, \end{aligned}$$

and therefore

$$\begin{aligned}\int_\omega |\tau_h f(x) - f(x)|^p \, dx &\le \int_\omega |h|^p \int_0^1 |\nabla f(x+th)|^p \, dt \, dx \\ &= |h|^p \int_0^1 \int_\omega |\nabla f(x+th)|^p \, dx \, dt \\ &\le |h|^p \int_0^1 \|\nabla f\|^p_{L^p(\omega')} \, dt \\ &= |h|^p \, \|\nabla f\|^p_{L^p(\omega')},\end{aligned}$$

where we have set $\omega' := \omega^{|h|} = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \omega) < |h|\}$. Note that $\omega' \subset\subset \Omega$. If $f \in W^{1,p}(\Omega)$, then, by Friedrichs' theorem (Theorem 1.17), there exists a sequence (f_n) in $C_c^\infty(\Omega)$ such that $f_n \to f$ in $L^p(\Omega)$ and $\nabla f_n \to \nabla f$ in $L^p(\omega')$. From the above inequality we thus obtain

$$\|\tau_h f - f\|_{L^p(\omega)} \le |h| \, \|\nabla f\|_{L^p(\omega')} \le |h| \, \|\nabla f\|_{L^p(\Omega)}.$$

(iii)⇒(ii) Let $\varphi \in C_c(\Omega)$ and $j \in \{1, \dots, N\}$. Choose $\omega \subset\subset \Omega$ such that $\operatorname{supp} \varphi \subseteq \omega$. Then

$$\begin{aligned}\int_\Omega f \partial_j \varphi &= \lim_{t \to 0} \int_\Omega f \frac{\tau_{te_j} \varphi - \varphi}{t} \\ &= \lim_{t \to 0} \int_\Omega \frac{\tau_{-te_j} f - f}{t} \varphi.\end{aligned}$$

By assumption, for every $t \in \mathbb{R} \setminus \{0\}$ small enough,

$$\|\tau_{-te_j} f - f\|_{L^p(\omega)} \le C|t|,$$

and hence, by Hölder's inequality,

$$\left| \int_\Omega f \partial_j \varphi \right| \le C \, \|\varphi\|_{L^{p'}}.$$

Remark 1.19. Theorem 1.18 is not true for $p = 1$. Only the implications

$$\text{(i)} \Rightarrow \text{(ii)} \Leftrightarrow \text{(iii)}$$

remain true. The space

$$\{f \in L^1(\Omega) : \exists C \ge 0 \forall \varphi \in C_c^\infty(\Omega) \forall j \in \{1, \dots, N\} \left| \int_\Omega f \partial_j \varphi \right| \le C \, \|\varphi\|_{L^\infty}\}$$

is usually denoted by $BV(\Omega)$. Elements of this space are called functions of **bounded variation**.

Theorem 1.20 (Composition rule). *Let $\Omega \subseteq \mathbb{R}^N$ be open and $p \in [1, \infty]$. Let $g \in C^1(\mathbb{R})$ satisfy $g(0) = 0$ and $\|g'\|_{L^\infty} =: M < \infty$, and let $f \in W^{1,p}(\Omega)$. Then the* *composition $g \circ f$ belongs to $W^{1,p}(\Omega)$ and*

$$\partial_j(g \circ f) = (g' \circ f) \cdot \partial_j f.$$

Proof. The assertion is true for $f \in C_c^\infty(\Omega)$ and follows for general $f \in W^{1,p}(\Omega)$ from an approximation argument using Friedrichs' theorem (Theorem 1.17). Note that for every $x \in \Omega$ and every $f \in W^{1,p}(\Omega)$,

$$|g(f(x))| = |g(f(x)) - g(0)| \leq M|f(x) - 0| = M|f(x)|,$$

and therefore $g \circ f \in L^p(\Omega)$. Moreover, $|(g' \circ f) \cdot \partial_j f| \leq M|\partial_j f|$, and hence $(g' \circ f) \cdot \partial_j f \in L^p(\Omega)$. If $f \in W^{1,p}(\Omega)$, then there exists a sequence (f_n) in $C_c^\infty(\Omega)$ such that $f_n \to f \in L^p(\Omega)$ and $\nabla f_n \to \nabla f$ in $L^p(\omega)$ for every $\omega \subset\subset \Omega$. Fix $\varphi \in C_c^\infty(\Omega)$ and $j \in \{1, \ldots, N\}$. Then, for every $n \in \mathbb{N}$,

$$\int_\Omega g \circ f_n \partial_j \varphi = -\int_\Omega (g' \circ f_n) \partial_j f_n \, \varphi.$$

Letting $n \to \infty$, we thus obtain

$$\int_\Omega g \circ f \partial_j \varphi = -\int_\Omega (g' \circ f) \partial_j f \, \varphi,$$

and from here and the above estimates follows the claim.

For real functions $f, g : \Omega \to \mathbb{R}$ we set

$$\begin{aligned} f \vee g &:= \sup\{f, g\} \quad \text{(pointwise)}, \\ f \wedge g &:= \inf\{f, g\} \quad \text{(pointwise)}, \\ f^+ &:= f \vee 0, \\ f^- &:= (-f) \vee 0, \text{ and} \\ |f| &:= f^+ + f^-. \end{aligned}$$

Corollary 1.21. *For every open subset $\Omega \subseteq \mathbb{R}^N$ and every $p \in [1, \infty]$, the real space $W^{1,p}(\Omega)$ is a vector lattice, that is, for every f, $g \in W^{1,p}(\Omega)$ one has f^+, f^-, $|f|$, $f \vee g$, $f \wedge g \in W^{1,p}(\Omega)$. Moreover,*

$$\begin{aligned} \partial_j f^+ &= \mathbf{1}_{\{f>0\}} \partial_j f, \\ \partial_j f^- &= -\mathbf{1}_{\{f<0\}} \partial_j f, \\ \partial_j |f| &= \operatorname{sgn}(f) \partial_j f, \\ \partial_j (f \vee g) &= \mathbf{1}_{\{f>g\}} \partial_j f + \mathbf{1}_{\{f \leq g\}} \partial_j g, \text{ and} \\ \partial_j (f \wedge g) &= \mathbf{1}_{\{f \leq g\}} \partial_j f + \mathbf{1}_{\{f>g\}} \partial_j g. \end{aligned}$$

Proof. For f^+ one would like to apply Theorem 1.20 with g given by

$$g(s) := \begin{cases} s & \text{if } s \geq 0, \\ 0 & \text{if } s < 0. \end{cases}$$

However, since this function is not continuously differentiable, one applies Theorem 1.20 with g_ε given by

$$g_\varepsilon(s) := \begin{cases} s - \frac{\varepsilon}{2} & \text{if } s \geq \varepsilon, \\ \frac{1}{2\varepsilon} s^2 & \text{if } 0 \leq s < \varepsilon, \\ 0 & \text{if } s < 0, \end{cases}$$

which is continuously differentiable, and then one passes to the limit $\varepsilon \to 0+$.

The other cases follow from this case by noting succesively that $f^- = (-f)^+$, $|f| = f^+ + f^-$, $f \vee g = g + (f-g)^+$ and $f \wedge g = f - (f-g)^+$.

1.4 Sobolev spaces in one dimension

In this section, let $I = (a,b)$ be an interval in $\mathbb{R}$ with $-\infty \leq a < b \leq \infty$.

Lemma 1.22. *Let $f \in L^1_{loc}(I)$ be weakly differentiable with weak derivative $\dot{f} = 0$. Then f is constant.*

Proof. Choose $\psi \in C_c^\infty(I)$ such that $\int_I \psi = 1$, and put $c := \int_I f\psi$. Then, for every $\varphi \in C_c^\infty(I)$ the function $\varphi - (\int_I \varphi)\,\psi$ has integral equal to 0 and is therefore the derivative of an other test function. By definition of the weak derivative and the assumption, we therefore obtain

$$\begin{aligned} 0 &= \int_I f(\varphi - (\int_I \varphi)\psi) \\ &= \int_I f\varphi - (\int_I \varphi)(\int_I f\psi) \\ &= \int_I f\varphi - c\int_I \varphi \\ &= \int_I (f-c)\varphi. \end{aligned}$$

By Theorem 1.12, this implies $f = c$, and therefore f is constant.

Theorem 1.23. *For every $p \in [1,\infty]$ the following assertions are true:*

a) *Every function in $f \in W^{1,p}(I)$ admits a continuous representative which extends continuously to the closed interval $\bar{I}$. If this continuous representative is denoted by f, too, then for every $s, t \in I$, $s < t$,*

$$f(t) - f(s) = \int_s^t \dot{f}(\tau)\, d\tau.$$

The continuous representative is bounded, and vanishes at $+\infty$ (if $b = +\infty$) and $-\infty$ (if $a = -\infty$) if $p < \infty$.

b) *The embedding $W^{1,p}(I) \to C_0(\bar{I})$ is continuous. More precisely,*

$$\|f\|_{L^\infty} \leq (L \wedge 1)^{-\frac{1}{p}} \|f\|_{W^{1,p}} \text{ for every } f \in W^{1,p}(I),$$

where $L := b - a \in (0, \infty]$ is the length of the interval I.

Proof. Fix $s \in I$ and define $g : I \to \mathbb{C}$ by $\int_s^t \dot{f}(\tau)\, d\tau$. It follows from Lebesgue's dominated convergence theorem that g is continuous on I and also that it admits a continuous extension to the closure of I. Moreover, for every $\varphi \in C_c^\infty(I)$,

$$\begin{aligned}
\int_I g\dot{\varphi} &= \int_a^b \int_s^t \dot{f}(\tau)\, d\tau \dot{\varphi}(t)\, dt \\
&= -\int_a^s \int_t^s \dot{f}(\tau)\, d\tau \dot{\varphi}(t)\, dt + \int_s^b \int_s^t \dot{f}(\tau)\, d\tau \dot{\varphi}(t)\, dt \\
&= -\int_a^s \int_a^\tau \dot{\varphi}(t)\, \dot{f}(\tau) d\tau + \int_s^b \int_\tau^b \dot{\varphi}(t)\, dt \dot{f}(\tau)\, d\tau \\
&= -\int_a^s \varphi(\tau)\dot{f}(\tau)\, d\tau - \int_s^b \varphi(\tau)\dot{f}(\tau)\, d\tau \\
&= -\int_I \dot{f}\varphi.
\end{aligned}$$

By definition, g is thus weakly differentiable and $\dot{g} = \dot{f}$. Hence, by Lemma 1.22, $g - f$ is a constant function. In particular, $f = g + c$ admits a continuous representative which extends continuously to the closure of I. If we denote this continuous representative by f, too, then we see from $g(s) = 0$ that $c = f(s)$, and therefore, by the definition of g,

$$f(t) - f(s) = \int_s^t \dot{f}(\tau)\, d\tau.$$

Let us show boundedness of functions in $W^{1,p}(I)$. By definition, every function $f \in W^{1,\infty}(I)$ belongs also to $L^\infty(I)$ and $\|f\|_{L^\infty} \leq \|f\|_{W^{1,\infty}}$. So consider the case when $f \in W^{1,p}(I)$ and $p < \infty$. Let $J \subseteq I$ be an interval of length equal to $L \wedge 1$. Then, by the above formula, for every $s, t \in J$, $s < t$,

$$\begin{aligned}|f(t)| &\le |f(s)| + \int_s^t |\dot f| \\ &\le |f(s)| + |t-s|^{\frac{1}{p'}} \, \|\dot f\|_{L^p(J)}.\end{aligned}$$

Integrating this inequality over $s \in J$, we obtain

$$\begin{aligned}(L\wedge 1)\,|f(t)| &\le \int_J |f(s)| + (L\wedge 1)^{1+\frac{1}{p'}} \, \|\dot f\|_{L^p(J)} \\ &\le (L\wedge 1)^{\frac{1}{p'}} \, \|f\|_{L^p(J)} + (L\wedge 1)^{1+\frac{1}{p'}} \, \|\dot f\|_{L^p(J)},\end{aligned}$$

or

$$\|f\|_{L^\infty(J)} \le (L\wedge 1)^{-\frac{1}{p}} \, \|f\|_{L^p(J)} + (L\wedge 1)^{\frac{1}{p'}} \, \|\dot f\|_{L^p(J)}.$$

From this inequality one deduces the claim in (b). It remains to show that f vanishes at $+\infty$ (if $b = +\infty$) and $-\infty$ (if $a = -\infty$) if $p < \infty$. Assume that $b = +\infty$. Then, by the preceding inequality (applied with $L = 1$), for every $s \in I$,

$$\begin{aligned}\|f\|_{W^{1,p}}^p &= \|f\|_{L^p}^p + \|\dot f\|_{L^p}^p \\ &\ge \sum_{n=0}^{\infty} [\|f\|_{L^p(s+n,s+n+1)}^p + \|\dot f\|_{L^p(s+n,s+n+1)}^p] \\ &\ge \sum_{n=0}^{\infty} \|f\|_{L^\infty(s+n,s+n+1)}^p,\end{aligned}$$

and since the left-hand side is finite, this yields

$$\lim_{t\to\infty} |f(t)| = \lim_{n\to\infty} \|f\|_{L^\infty(s+n,s+n+1)} = 0.$$

The case $a = -\infty$ is discussed similarly.

Corollary 1.24. *Let $I \subseteq \mathbb{R}$ be an open interval. A continuous function $f : I \to \mathbb{R}$ is Lipschitz continuous if and only if f is weakly differentiable and $\dot f \in L^\infty(I)$, and then the Lipschitz constant of f equals $\|\dot f\|_{L^\infty}$.*

Proof. Assume first that $\dot f \in L^\infty(I)$. Then, by Theorem 1.23, for every s, $t \in I$, $s < t$,

$$|f(t) - f(s)| \le \int_s^t |\dot f(\tau)| \, d\tau \le \|\dot f\|_{L^\infty} \, |t-s|,$$

and thus f is Lipschitz continuous with Lipschitz constant $L = \|\dot f\|_{L^\infty}$.

Conversely, assume now that f is Lipschitz continuous with Lipschitz constant L. Let $I' \subset\subset I$ be an open, bounded subinterval. Since the closure of I' is compact and contained in I, f is bounded on I'. Let $J \subset\subset I'$ be a subinterval, and let $h \in \mathbb{R}$ be such that $|h| < \operatorname{dist}(J, \partial I')$. Then

$$\|\tau_h f - f\|_{L^\infty(J)} = \sup_{t\in J} |f(t+h) - f(t)| \le L|h|.$$

By Theorem 1.18, this implies $f \in W^{1,\infty}(I')$ and $\|\dot{f}\|_{L^\infty(I')} \le L$. Since $I' \subset\subset I$ was arbitrary, we find $\|\dot{f}\|_{L^\infty(I)} \le L$.

1.5 The extension property

We say that an open subset $\Omega \subseteq \mathbb{R}^N$ has the $W^{k,p}$ **extension property** if there exists a bounded linear operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^N)$ such that, for every $f \in W^{k,p}(\Omega)$, $Ef|_\Omega = f$.

We say that the boundary of an open set $\Omega \subseteq \mathbb{R}^N$ is C^1-regular (and we write $\partial\Omega \in C^1$) if it is a C^1-manifold. The aim of this section is to prove the following result.

Theorem 1.25. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set with compact, C^1-regular boundary. Then Ω has the $W^{1,p}$ extension property for every $p \in [1,\infty]$.*

The proof of this theorem is based on two lemmas.

Lemma 1.26. *The half-space*

$$\mathbb{R}^N_+ := \{x = (x_1,\dots,x_N) \in \mathbb{R}^N : x_N > 0\}$$

has the $W^{1,p}$ extension property for every $p \in [1,\infty]$.

Proof. For every $f \in W^{1,p}(\mathbb{R}^N_+)$ we define $\tilde{f} : \mathbb{R}^N \to \mathbb{C}$ by

$$\tilde{f}(x) := \begin{cases} f(x', x_N) & \text{if } x_N > 0, \\ f(x', -x_N) & \text{if } x_N < 0, \end{cases}$$

where $x' = (x_1,\dots,x_{N-1}) \in \mathbb{R}^{N-1}$ for $x = (x_1,\dots,x_N)$. Moreover, for every $j \in \{1,\dots,N\}$ we define

$$g_j(x) := \begin{cases} (\partial_j f)(x', x_N) & \text{if } x_N > 0, \\ (-1)^{\delta_{jN}} (\partial_j f)(x', -x_N) & \text{if } x_N < 0, \end{cases}$$

where δ_{jN} is the Kronecker delta. Next, we choose a cut-off function $\eta : \mathbb{R} \to \mathbb{R}$ such that

$$\eta(s) = 1 \text{ if } |s| \ge 1,$$
$$\eta(s) = 0 \text{ if } |s| \le \frac{1}{2},$$
$$\eta(s) = \eta(-s) \text{ for every } s \in \mathbb{R},$$

and we put $\eta_\varepsilon(s) = \eta(s/\varepsilon)$ for $\varepsilon > 0$. Finally, we define $\tilde{\eta}_\varepsilon(x) := \eta_\varepsilon(x_N)$ ($x = (x', x_N) \in \mathbb{R}^N$).

Let $\varphi \in C_c^\infty(\mathbb{R}^N)$. Then the support of the test function $\varphi\,\tilde{\eta}_\varepsilon$ does not intersect the hyperplane $\{x_N = 0\}$. Therefore, by Lebesgue's dominated convergence theorem, the substitution $(x_1, \dots, x_{N-1}, x_N) \mapsto (x_1, \dots, x_{N-1}, -x_N)$, the product rule and the definition of weak derivative, for every $j \in \{1, \dots, N\}$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \tilde{f} \partial_j \varphi = \\
&= \lim_{\varepsilon\to 0+} \int_{\mathbb{R}^N} \tilde{f} \tilde{\eta}_\varepsilon \partial_j \varphi \\
&= \lim_{\varepsilon\to 0+} \left[\int_{\mathbb{R}^N_+} f(x)\tilde{\eta}_\varepsilon(x)(\partial_j\varphi)(x)\,dx + \int_{\mathbb{R}^N_+} f(x)\tilde{\eta}_\varepsilon(x)(\partial_j\varphi)(x', -x_N)\,dx \right] \\
&= \lim_{\varepsilon\to 0+} \left[\int_{\mathbb{R}^N_+} f(x)\tilde{\eta}_\varepsilon(x)\partial_j\varphi(x)\,dx + (-1)^{\delta_{jN}} \int_{\mathbb{R}^N_+} f(x)\tilde{\eta}_\varepsilon(x)\partial_j\varphi(x', -x_N)\,dx \right] \\
&= \lim_{\varepsilon\to 0+} \left[\int_{\mathbb{R}^N_+} f(x)\partial_j(\tilde{\eta}_\varepsilon(x_N)\varphi(x))\,dx + (-1)^{\delta_{jN}} \int_{\mathbb{R}^N_+} f(x)\partial_j(\tilde{\eta}_\varepsilon(x)\varphi(x', -x_N))\,dx \right.\\
&\quad \left. - \int_{\mathbb{R}^N_+} f(x)(\partial_j\tilde{\eta}_\varepsilon)(x)\varphi(x)\,dx - (-1)^{\delta_{jN}} \int_{\mathbb{R}^N_+} f(x)(\partial_j\tilde{\eta}_\varepsilon)(x)\varphi(x', -x_N)\,dx \right] \\
&= \lim_{\varepsilon\to 0+} \left[- \int_{\mathbb{R}^N_+} (\partial_j f)(x)\tilde{\eta}_\varepsilon(x)\varphi(x)\,dx - (-1)^{\delta_{jN}} \int_{\mathbb{R}^N_+} (\partial_j f)(x)\tilde{\eta}_\varepsilon(x)\varphi(x', -x_N)\,dx \right.\\
&\quad \left. - \int_{\mathbb{R}^N_+} f(x)(\partial_j\tilde{\eta}_\varepsilon)(x)\varphi(x)\,dx - (-1)^{\delta_{jN}} \int_{\mathbb{R}^N_+} f(x)(\partial_j\tilde{\eta}_\varepsilon)(x)\varphi(x', -x_N)\,dx \right] \\
&= - \int_{\mathbb{R}^N_+} \partial_j f(x)\varphi(x)\,dx - (-1)^{\delta_{jN}} \int_{\mathbb{R}^N_+} \partial_j f(x)\varphi(x', -x_N)\,dx \\
&\quad - \lim_{\varepsilon\to 0+} \left[\int_{\mathbb{R}^N_+} f(x)(\partial_j\tilde{\eta}_\varepsilon)(x)\varphi(x)\,dx + (-1)^{\delta_{jN}} \int_{\mathbb{R}^N_+} f(x)(\partial_j\tilde{\eta}_\varepsilon)(x)\varphi(x', -x_N)\,dx \right].
\end{aligned}$$

If $j \neq N$, then $\partial_j \tilde{\eta}_\varepsilon = 0$, and thus the function under the lim-sign on the right-hand side is equal to 0. If $j = N$, then, for $R \geq 0$ large enough (so that $\operatorname{supp}\varphi \subseteq B(0, R)$),

$$\begin{aligned}
&\limsup_{\varepsilon\to 0+}\left|\int_{\mathbb{R}^N_+} f(x)(\partial_N\tilde{\eta}_\varepsilon)(x)\varphi(x)\,dx+(-1)^{\delta_{jN}}\int_{\mathbb{R}^N_+} f(x)(\partial_N\tilde{\eta}_\varepsilon)(x)\varphi(x',-x_N)\,dx\right|\\
&\le \limsup_{\varepsilon\to 0+}\int_{\{\frac{\varepsilon}{2}<x_N<\varepsilon,\ |x'|\le R\}} |f(x)|\,|\eta'_\varepsilon(x_N)|\,|\varphi(x',x_N)-\varphi(x',-x_N)|\,dx\\
&\le \limsup_{\varepsilon\to 0+}\int_{\{\frac{\varepsilon}{2}<x_N<\varepsilon,\ |x'|\le R\}} |f(x)|\,\frac{\|\eta'\|_{L^\infty}}{\varepsilon}\,\|\nabla\varphi\|_\infty\,2\varepsilon\,dx\\
&=0.
\end{aligned}$$

Hence, the limit on the right-hand side of the display above equals 0 for every $j\in\{1,\dots,N\}$. We have thus proved

$$\int_{\mathbb{R}^N}\tilde{f}\partial_j\varphi=-\int_{\mathbb{R}^N} g_j\varphi$$

for every $j\in\{1,\dots,N\}$ and every $\varphi\in C_c^\infty(\mathbb{R}^N)$. In other words, $\tilde{f}\in W^{1,p}(\mathbb{R}^N)$ and $\partial_j\tilde{f}=g_j$ for every $j\in\{1,\dots,N\}$. By definition, $\tilde{f}$ is an extension of f, and $\|\tilde{f}\|_{W^{1,p}(\mathbb{R}^N)}=2\|f\|_{W^{1,p}(\mathbb{R}^N_+)}$. Thus $E:W^{1,p}(\mathbb{R}^N_+)\to W^{1,p}(\mathbb{R}^N)$, $f\mapsto\tilde{f}$ is a desired extension operator.

Lemma 1.27 (Partition of unity). *Let $K\subseteq\mathbb{R}^N$ be a compact set, and let $(U_i)_{1\le i\le n}$ be a finite open covering of K. Then there exists a finite family $(\varphi_i)_{0\le i\le 1}$ in $C^\infty(\mathbb{R}^N)$ such that*

$$\begin{aligned}
&0\le\varphi_i\le 1 \text{ for every } 0\le i\le n,\\
&\sum_{i=0}^n\varphi_i=1 \text{ on } \mathbb{R}^N,\\
&\operatorname{supp}\varphi_i\subseteq U_i \text{ for every } 1\le i\le n, \text{ and}\\
&\operatorname{supp}\varphi_0\subseteq\mathbb{R}^N\setminus K.
\end{aligned}$$

We call the family $(\varphi_i)_{0\le i\le n}$ a **partition of unity subordinate to the covering** $(U_i)_{1\le i\le n}$.

Proof. Choose $\delta>0$, and define, for every $i\in\{1,\dots,n\}$,

$$V_i:=\{x\in U_i:\operatorname{dist}(x,\partial U_i)>\delta\}.$$

Then V_i is open and a subset of U_i. A simple compactness argument shows that for $\delta>0$ small enough, $(V_i)_{1\le i\le n}$ is a finite covering of K, too. Then we define

$$A_1 := \bar{V}_1,$$
$$A_i := \bar{V}_i \setminus \bigcup_{j=1}^{i-1} A_j \text{ for every } 2 \le i \le n, \text{ and}$$
$$A_0 := \mathbb{R}^N \setminus \bigcup_{j=1}^{n} A_j.$$

Then $A_i^{\frac{\delta}{2}} \subseteq U_i$ for every $i \in \{1, \dots, n\}$, the A_i are mutually disjoint, and $\bigcup_{i=0}^n A_i = \mathbb{R}^N$. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be a positive test function such that $\int_{\mathbb{R}^N} \varphi = 1$ and $\operatorname{supp} \varphi \subseteq B(0, \frac{\delta}{4})$, and put $\varphi_i := 1_{A_i} * \varphi$ $(0 \le i \le n)$. Then $\varphi_i \in C^\infty(\mathbb{R}^N)$ by the smoothin effect of the convolution (Theorem 1.9), $\operatorname{supp} \varphi_i \subseteq U_i$ by construction, and

$$\sum_{i=0}^n \varphi_i = (\sum_{i=0}^n 1_{A_i}) * \varphi = 1_{\bigcup_{i=0}^n A_i} * \varphi = 1 * \varphi = 1.$$

Proof (of Theorem 1.25). Let $((U_i, \psi_i))_{1 \le i \le n}$ be a finite atlas of $\partial\Omega$, where U_i are open sets and $\psi_i : U_i \to \mathbb{R}^N$ are C^1-diffeomorphisms from U_i onto $V_i := \psi_i(U_i)$ such that $\psi_i(U_i \cap \Omega) \subseteq \mathbb{R}^N_+$, $\psi_i(U_i \cap \partial\Omega) \subseteq \{x \in \mathbb{R}^N : x_N = 0\}$ and $\psi_i(U_i \cap \bar{\Omega}^c) \subseteq \{x \in \mathbb{R}^N : x_N < 0\}$. We can always find such a finite atlas by definition of a C^1-manifold and since $\partial\Omega$ is compact. Moreover, without loss of generality, we may assume that the sets V_i are symmetric with respect to the hyperplane $\{x_N = 0\}$.

Let $(\varphi_i)_{0 \le i \le n}$ be a partition of unity subordinate to the covering $(U_i)_{1 \le i \le n}$. Given $f \in W^{1,p}(\Omega)$, we put $f_i := (f\varphi_i)|_\Omega$ $(0 \le i \le 1)$. Then $f_i \in W^{1,p}(\Omega)$ and

$$\operatorname{supp} f_i \subseteq \operatorname{supp} \varphi_i \cap \Omega \subseteq U_i \cap \Omega.$$

The function f_0 has compact support in Ω and may thus be extended by 0 to $\mathbb{R}^N$. We denote this extension by $\tilde{f}_0$ and note that $\tilde{f}_0 \in W^{1,p}(\mathbb{R}^N)$ and $\|\tilde{f}_0\|_{W^{1,p}(\mathbb{R}^N)} = \|f_0\|_{W^{1,p}(\Omega)}$.

For $1 \le i \le N$, the functions $f_i \circ \psi_i^{-1}$ belong to $W^{1,p}(\mathbb{R}^N_+)$ and have support in $V_i \cap \mathbb{R}^N_+$. They may thus be extended to functions in $W^{1,p}(\mathbb{R}^N)$ by reflection (see Lemma 1.26 and its proof). We denote these extensions by $\widetilde{f_i \circ \psi_i^{-1}}$ and note that these extensions have support in V_i (here we use that the V_i are symmetric with respect to the hyperplane $\{x_N = 0\}$). Now put $\tilde{f}_i := \widetilde{f_i \circ \psi_i^{-1}} \circ \psi_i$. Then $\tilde{f}_i \in W^{1,p}(\mathbb{R}^N)$ is an extension of f_i. Now, it suffices to define $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ by

$$Ef := \sum_{i=0}^n \tilde{f}_i$$

and to verify that E is a desired, bounded extension operator.

1.6 The Sobolev embedding theorems

In the following, given two Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subseteq Y$ and if the identity map $i : X \to Y$, $x \mapsto x$ is continuous.

Theorem 1.28 (Sobolev-Gagliardo-Nirenberg). *Fix $p \in [1, N)$ and define $p^* := \frac{Np}{N-p}$. Then every $f \in W^{1,p}(\mathbb{R}^N)$ belongs to $L^{p^*}(\mathbb{R}^N)$ and*

$$\|f\|_{L^{p^*}} \leq C\|\nabla f\|_{L^p}$$

In particular, $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$.

For the proof of this theorem, we need the following technical lemma.

Lemma 1.29. *Let $N \geq 2$ and $f_1, \ldots, f_N \in L^{N-1}(\mathbb{R}^{N-1})$. For every $1 \leq i \leq N$ and every $x \in \mathbb{R}^N$ we put $\tilde{x}_i := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$. Then the function*

$$f(x) := \prod_{i=1}^{N} f_i(\tilde{x}_i)$$

belongs to $L^1(\mathbb{R}^N)$ and

$$\|f\|_{L^1(\mathbb{R}^N)} \leq \prod_{i=1}^{N} \|f_i\|_{L^{N-1}(\mathbb{R}^{N-1})}.$$

Proof (incomplete). The assertion of this lemma is straightfoward to verify for $N = 2$. Now suppose that the assertion is true for some N, and let us prove that it is true for $N+1$, too. In the case $N+1$, fix x_{N+1}. We have, by Hölder's inequality,

$$\int_{\mathbb{R}^N} |f(x_1, \ldots, x_{N+1})|\, dx_1 \ldots dx_N \leq \|f_{N+1}\|_{L^N(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} \prod_{i=1}^{N} |f_i(\tilde{x}_i)|^{\frac{N}{N-1}}\, dx_1 \ldots dx_N \right)^{\frac{N-1}{N}}.$$

By assumption, $f_i \in L^N(\mathbb{R}^N)$, that is,

$$\int_{\mathbb{R}^N} |f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{N+1})|^N\, dx_1 \ldots dx_{i-1}\, dx_{i+1} \ldots dx_{N+1} < \infty.$$

In particular, for almost every $x_{N+1} \in \mathbb{R}$,

$$\int_{\mathbb{R}^{N-1}} |f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{N+1})|^N\, dx_1 \ldots dx_{i-1}\, dx_{i+1} \ldots dx_N < \infty \text{ for every } 1 \leq i \leq N.$$

Hence, by induction hypothesis, for almost every $x_{N+1} \in \mathbb{R}$,

$$\int_{\mathbb{R}^N} \prod_{i=1}^{N} |f_i(\tilde{x}_i)|^{\frac{N}{N-1}} \, dx_1 \dots \, dx_N \le \prod_{i=1}^{N} \|f_i\|_{L^N}^{\frac{1}{N-1}}.$$

or, equivalently, $|f_i|^{\frac{N}{N-1}} \in L^{N-1}(\mathbb{R}^N)$.

Proof (of Theorem 1.28). Assume first that $p = 1$ and $f \in C_c^\infty(\mathbb{R}^N)$. Then

$$|f(x)| = \left| \int_{-\infty}^{x_1} \partial_1 f(\xi, x_2, \dots, x_N) \, d\xi \right| \le \int_{-\infty}^{\infty} |\partial_1 f(\xi, x_2, \dots, x_N)| \, d\xi.$$

Similarly,

$$|f(x)| \le \int_{-\infty}^{\infty} |\partial_i f(x_1, \dots, x_{i-1}, \xi, x_{i+1}, \dots, x_N)| \, d\xi =: f_i(\tilde{x}_i),$$

where $\tilde{x}_i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_N)$. Hence,

$$|f(x)|^N \le \prod_{i=1}^{N} f_i(\tilde{x}_i).$$

Note that $f_i^{\frac{1}{N-1}} \in L^{N-1}(\mathbb{R}^{N-1})$ and

$$\begin{aligned} \|f_i^{\frac{1}{N-1}}\|_{L^{N-1}(\mathbb{R}^{N-1})} &= \left(\int_{\mathbb{R}^{N-1}} f_i(\tilde{x}_i) \, d\tilde{x}_i \right)^{\frac{1}{N-1}} \\ &= \left(\int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} |\partial_i f(x_1, \dots, x_{i-1}, \xi, x_{i+1}, \dots, x_N)| \, d\xi \, d\tilde{x}_i \right)^{\frac{1}{N-1}} \\ &= \|\partial_i f\|_{L^1(\mathbb{R}^N)}^{\frac{1}{N-1}}. \end{aligned}$$

As a consequence, by Lemma 1.29,

$$\int_{\mathbb{R}^N} |f(x)|^{\frac{N}{N-1}} \, dx \le= \prod_{i=1}^{N} \|\partial_i f\|_{L^1(\mathbb{R}^N)}^{\frac{1}{N-1}},$$

or

$$\|f\|_{L^{\frac{N}{N-1}}} \le \prod_{i=1}^{N} \|\partial_i f\|_{L^1(\mathbb{R}^N)}^{\frac{1}{N}}.$$

Applying this formula to $|f|^{t-1} f$ with $t \ge 1$, we obtain

$$\begin{aligned} \|f\|^t_{L^{\frac{tN}{N-1}}} &\le \prod_{i=1}^{N} \|t|f|^{t-1}\partial_i f\|^{\frac{1}{N}}_{L^1(\mathbb{R}^N)} \\ &\le t \prod_{i=1}^{N} \||f|^{t-1}\|^{\frac{1}{N}}_{L^{\frac{p}{p-1}}} \|\partial_i f\|^{\frac{1}{N}}_{L^p(\mathbb{R}^N)} \\ &= t\|f\|^{t-1}_{L^{\frac{(t-1)p}{p-1}}} \prod_{i=1}^{N} \|\partial_i f\|^{\frac{1}{N}}_{L^p(\mathbb{R}^N)} \end{aligned} \tag{1.2}$$

Choose t such that $\frac{tN}{N-1} = \frac{(t-1)p}{p-1}$, that is, $t = \frac{N-1}{N}p^*$. Then

$$\|f\|_{L^{p^*}} \le C \prod_{i=1}^{N} \|\partial_i f\|^{\frac{1}{N}}_{L^p(\mathbb{R}^N)}$$

for some constant C which depends only on t, and from here follows immediately

$$\|f\|_{L^{p^*}} \le C\|\nabla f\|_{L^p(\mathbb{R}^N)}.$$

The claim for general $f \in W^{1,p}(\mathbb{R}^N)$ follows from this inequality and an approximation argument using Theorem 1.15.

Corollary 1.30. *For every $p \in [1,N)$ and every $q \in [p,p^*]$ one has*

$$W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N).$$

Proof. By definition of the Sobolev spaces, $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, and by Theorem 1.28, $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$. The interpolation inequality (a straightforward application of Hölder's inequality) yields

$$L^p \cap L^{p*}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \text{ for every } q \in [p,p^*],$$

and putting the embeddings together yields the claim.

Corollary 1.31 (The limit case $p = N$). *For every $q \in [N,\infty)$ one has*

$$W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N).$$

Proof. Let $f \in C_c^\infty(\mathbb{R}^N)$. We apply (1.2) with $p = N$ and obtain

$$\|f\|^t_{L^{\frac{tN}{N-1}}} \le t\|f\|^{t-1}_{L^{\frac{(t-1)N}{N-1}}} \|\nabla f\|_{L^N(\mathbb{R}^N)}.$$

By Young's inequality,

$$\|f\|_{L^{\frac{tN}{N-1}}} \le C[\|f\|_{L^{\frac{(t-1)N}{N-1}}} + \|\nabla f\|_{L^N(\mathbb{R}^N)}]. \tag{1.3}$$

Choosing $t = N$ in this inequality, we obtain

$$\|f\|_{L^{\frac{N^2}{N-1}}} \le C\|f\|_{W^{1,N}},$$

and thus, by interpolation,

$$\|f\|_{L^q} \le C\|f\|_{W^{1,N}} \text{ for every } q \in [N, \frac{N^2}{N-1}]$$

Iterating this argument by choosing now $t = N+1$, $t = N+2$, ... in (1.3), we obtain

$$\|f\|_{L^q} \le C\|f\|_{W^{1,N}} \text{ for every } q \in [N, \infty).$$

This inequality holds, by an approximation argument, finally for all $f \in W^{1,N}(\mathbb{R}^N)$.

Theorem 1.32 (Morrey). *For every* $p \in (N, \infty)$,

$$W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N).$$

More precisely, for $\alpha := 1 - \frac{N}{p}$ *there exists a constant* $C = C(p,N) \ge 0$ *such that for every* $f \in W^{1,p}(\mathbb{R}^N)$, x, $y \in \mathbb{R}^N$,

$$|f(x) - f(y)| \le C|x-y|^\alpha \|\nabla f\|_{L^p}.$$

Proof. Let Q be cube with sides parallel to the coordinate axes and of length $r > 0$, such that $0 \in Q$. For every $x \in Q$ one has

$$\begin{aligned} |f(x) - f(0)| &= \left| \int_0^1 \frac{d}{dt} f(tx)\, dt \right| \\ &\le \int_0^1 |\nabla f(tx) \cdot x|\, dt \\ &\le r \int_0^1 |\nabla f(tx)|\, dt. \end{aligned}$$

Put $\bar{f} := \frac{1}{|Q|} \int_Q f$. By integrating the above inequality over $x \in Q$, and by using Hölder's inequality, we obtain

$$\begin{aligned} |\bar{f} - f(0)| &\le \frac{r}{|Q|} \int_Q \int_0^1 |\nabla f(tx)|\, dt\, dx \\ &= \frac{1}{r^{N-1}} \int_0^1 \int_Q |\nabla f(tx)|\, dx\, dt \\ &= \frac{1}{r^{N-1}} \int_0^1 \int_{tQ} |\nabla f(x)|\, dx\, \frac{dt}{t^N} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{r^{N-1}} \int_0^1 \left(\int_{tQ} |\nabla f|^p \right)^{\frac{1}{p}} |tQ|^{\frac{p-1}{p}} \, \frac{dt}{t^N} \\
&\leq r^{1-\frac{N}{p}} \, \|\nabla f\|_{L^p(Q)} \int_0^1 t^{-\frac{N}{p}} \, dt \\
&= \frac{r^{1-\frac{N}{p}}}{1-\frac{N}{p}} \, \|\nabla f\|_{L^p(Q)}.
\end{aligned}
$$

Clearly, this inequality holds also with $x \in \mathbb{R}^N$ instead of 0 (and with a cube Q containing x). Hence, for every cube Q with sides parallel to the coordinate axes and of length $r > 0$, and for every x, $y \in Q$,

$$
|f(x) - f(y)| \leq \frac{2r^{1-\frac{N}{p}}}{1-\frac{N}{p}} \, \|\nabla f\|_{L^p(Q)}.
$$

Since we may choose Q minimal (so that, for example, $r = 2|x-y|$), and by an approximation argument (Theorem 1.15), we obtain the Hölder continuity of $f \in W^{1,p}(\mathbb{R}^N)$. Moreover, applying again the above inequality with 0 replaced by $x \in \mathbb{R}^N$ and with a cube of side length $r = 1$, we obtain

$$
\begin{aligned}
|f(x)| &\leq |\bar{f}| + |\bar{f} - f(x)| \\
&\leq \|f\|_{L^p(Q)} + \frac{1}{1-\frac{N}{p}} \, \|\nabla f\|_{L^p(Q)} \\
&\leq C \, \|f\|_{W^{1,p}(\mathbb{R}^N)},
\end{aligned}
$$

and from here and an approximation argument, we obtain the boundedness of $f \in W^{1,p}(\mathbb{R}^N)$.

Summarizing the results of this section, we are in the position to state the first version of the Sobolev embedding theorem. For an open set $\Omega \subseteq \mathbb{R}^N$, $m \in \mathbb{N}_0$ and $\theta \in [0,1]$ we set

$$
C^{0,\theta}(\Omega) := \{ f \in C(\Omega) : \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\theta} < \infty,
$$

and

$$
C^{m,\alpha}(\Omega) := \{ f \in C^m(\Omega) : \partial^\alpha f \in C^{0,\theta}(\Omega) \text{ for all } \alpha \in \mathbb{N}_0^N, |\alpha| = m \}.
$$

Theorem 1.33 (Sobolev embedding theorem). *Let $k \in \mathbb{N}$ and $p \in [1, \infty)$. Then the following assertions are true:*

a) *If $kp < N$, then*

$$W^{k,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \textit{ for every } q \in [p, \frac{Np}{N-kp}].$$

b) *If $kp = N$, then*

$$W^{k,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \textit{ for every } q \in [p, \infty).$$

c) *If $kp > N$, then*

$$W^{k,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N) \cap C^{m,\alpha}(\mathbb{R}^N),$$

where

$$m := \begin{cases} [k - \frac{N}{p}] & \textit{if } k - \frac{N}{p} \notin \mathbb{N}, \\ k - \frac{N}{p} - 1 & \textit{if } k - \frac{N}{p} \notin \mathbb{N}, \end{cases}$$

and $\theta = k - \frac{N}{p} - m$.

Proof. For $k = 1$, the assertions (a)-(c) follow respectively from Corollary 1.30, Corollary 1.31 and Theorem 1.32. For $k \geq 2$, the assertions follow by an iteration (or induction) argument.

Corollary 1.34 (Sobolev embedding theorem). *Let $k \in \mathbb{N}$, $p \in [1, \infty)$ and let $\Omega \subseteq \mathbb{R}^N$ be an open set which has the $W^{1,q}$-extension property for every $q \in [1, \infty)$ (for example, Ω has compact, C^1 regular boundary $\partial\Omega$). Then the following assertions are true:*

a) *If $kp < N$, then*

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega) \textit{ for every } q \in [p, \frac{Np}{N-kp}].$$

b) *If $kp = N$, then*

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega) \textit{ for every } q \in [p, \infty).$$

c) *If $kp > N$, then*

$$W^{k,p}(\Omega) \hookrightarrow L^\infty(\Omega) \cap C^{m,\alpha}(\Omega),$$

where

$$m := \begin{cases} [k - \frac{N}{p}] & \textit{if } k - \frac{N}{p} \notin \mathbb{N}, \\ k - \frac{N}{p} - 1 & \textit{if } k - \frac{N}{p} \notin \mathbb{N}, \end{cases}$$

and $\theta = k - \frac{N}{p} - m$.

These assertions (a)-(c) remain true for arbitrary open $\Omega \subseteq \mathbb{R}^N$ if the spaces $W^{k,p}(\Omega)$ are replaced by $W_0^{k,p}(\Omega)$.

Proof. For $k = 1$, the assertions follow from the assumption that Ω has the $W^{1,q}$-extension property for every $q \in [1, \infty)$ and from the Sobolev embedding theorem for the $W^{k,p}(\mathbb{R}^N)$-spaces (Theorem 1.33). For $k \geq 2$, the assertions follow by an iteration (or induction) argument.

If $\Omega \subseteq \mathbb{R}^N$ is an arbitrary open set, one may argue similarly by noting that functions in $W_0^{k,p}(\Omega)$ may be extended by 0 outside Ω to functions in $W^{k,p}(\mathbb{R}^N)$. In fact, for test functions in $C_c^\infty(\Omega)$ this is obvious, and for general functions in $W_0^{k,p}(\Omega)$ this follows by an approximation argument using the definition of $W_0^{k,p}(\mathbb{R}^N)$.

Lemma 1.35. *For every open, bounded subset $\Omega \subseteq \mathbb{R}^N$, every $p \in [1,\infty]$ and every $\varphi \in L^1(\mathbb{R}^N)$ the operator*

$$\begin{aligned} T_\varphi : L^p(\mathbb{R}^N) &\to L^p(\Omega), \\ f &\mapsto (f * \varphi)|_\Omega, \end{aligned}$$

is compact.

Proof. For $\varphi \in C_c(\mathbb{R}^N)$ the assertions follows from the theorem of Arzelà-Ascoli. For general $\varphi \in L^1(\mathbb{R}^N)$ the assertion follows by approximation of φ, Young's inequality (Theorem 1.4) and the fact that the compact operators form a closed subspace of the space of bounded linear operators. In fact, if $\varphi \in L^1(\mathbb{R}^N)$, then we find a sequence (φ_n) in $C_c^\infty(\mathbb{R}^N)$ such that $\lim_{n\to\infty} \|\varphi - \varphi_n\|_{L^1} = 0$ (Theorem 1.11). As a consequence

$$\begin{aligned} \|T_\varphi - T_{\varphi_n}\| &= \sup_{\|f\|_{L^p(\Omega)} \leq 1} \|T_\varphi f - T_{\varphi_n} f\|_{L^p(\Omega)} \\ &\leq \sup_{\|f\|_{L^p(\Omega)} \leq 1} \|f * \varphi - f * \varphi_n\|_{L^p(\mathbb{R}^N)} \\ &\leq \sup_{\|f\|_{L^p(\Omega)} \leq 1} \|f\|_{L^p(\mathbb{R}^N)} \|\varphi - \varphi_n\|_{L^1(\mathbb{R}^N)} \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Since the T_{φ_n} are compact and converge in operator norm to T_φ, T_φ is compact, too.

Lemma 1.36. *For every $f \in C_c^\infty(\Omega)$ one has*

$$f(0) = \frac{-1}{\sigma_{N-1}} \int_{\mathbb{R}^N} \nabla f(y) \frac{y}{|y|^N} \, dy.$$

Proof.

Theorem 1.37 (Rellich-Kondrachev). *Let $k \in \mathbb{N}$, $p \in [1,\infty)$ and let $\Omega \subseteq \mathbb{R}^N$ be a bounded, open set. Then the following assertions are true:*

a) *If $kp < N$, then*

$$W_0^{k,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega) \textit{ for every } q \in [p, \frac{Np}{N-kp}).$$

b) *If $kp = N$, then*

$$W_0^{k,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega) \textit{ for every } q \in [p, \infty).$$

c) *If $kp > N$, then*

$$W_0^{k,p}(\Omega) \overset{c}{\hookrightarrow} L^\infty(\Omega) \cap C^{m,\theta'}(\bar{\Omega}),$$

where

$$m := \begin{cases} [k - \frac{N}{p}] & \textit{if } k - \frac{N}{p} \notin \mathbb{N}, \\ k - \frac{N}{p} - 1 & \textit{if } k - \frac{N}{p} \notin \mathbb{N}. \end{cases}$$

and $\theta' \in (0, k - \frac{N}{p} - m)$.

These assertions (a)-(c) remain true for bounded, open sets $\Omega \subseteq \mathbb{R}^N$ which have the $W^{1,q}$-extension property for every $q \in [1, \infty)$, and then the spaces $W_0^{k,p}(\Omega)$ may be replaced by $W^{k,p}(\Omega)$.

Proof. First assume that $k = 1$. Choose $R > 0$ such that $\Omega \subseteq B(0, R)$ and define $\varphi \in L^1(\mathbb{R}^N)$ by

$$\varphi(x) := \frac{1}{\sigma_{N-1}} \frac{x}{|x|^N} 1_{B(0,2R)}(x) \quad (x \in \mathbb{R}^N).$$

Let $f \in C_c^\infty(\Omega) \subseteq C_c^\infty(\mathbb{R}^N)$. By Lemma 1.36 (applied to the function $f(x - \cdot)$), for every $x \in \Omega$,

$$\begin{aligned} f(x) &= \frac{1}{\sigma_{N-1}} \int_{\mathbb{R}^N} \nabla f(x-y) \frac{y}{|y|} \, dy \\ &= \frac{1}{\sigma_{N-1}} \int_{B(0,2R)} \nabla f(x-y) \frac{y}{|y|} \, dy \\ &= \nabla f * \varphi(x). \end{aligned}$$

Hence, the continuous operator

$$\begin{aligned} W_0^{1,p}(\Omega) &\to L^p(\Omega; \mathbb{R}^N) \to L^p(\Omega), \\ f &\mapsto \quad \nabla f \qquad \mapsto (\nabla f * \varphi)|_\Omega, \end{aligned}$$

coincides with the identity on the space $C_c^\infty(\Omega)$. This space being dense in $W_0^{1,p}(\Omega)$ (by definition), the above operator thus is the canonical embedding of $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$. However, the second operator is compact by Lemma 1.35, and thus the embedding itself is compact.

Next, assume $p < N$, and let $q \in [p, p^*)$, where $p^* = \frac{Np}{N-p}$. Then $\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{p^*}$ for some $\theta \in (0,1]$. Let (f_n) be a bounded sequence in $W^{1,p}(\Omega)$. Then, by the preceding step, there exists a subsequence (denoted again by (f_n)) which converges in $L^p(\Omega)$ to some element in f. Moreover, by the Sobolev embedding theorem (Corollary 1.34 (a)), the sequence (f_n) is bounded in $L^{p*}(\Omega)$. Hence, by the interpolation inequality, and since $\theta > 0$,

$$\|f_n - f\|_{L^q} \le \|f_n - f\|_{L^p}^{\theta} \, \|f_n - f\|_{L^{p^*}}^{1-\theta} \to 0,$$

that is, (f_n) converges also in $L^q(\Omega)$. Since the bounded sequence (f_n) was arbitrary, we have proved that the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact. The case $p = N$ is treated similarly, and in the case $p > N$ one uses the case (c) of the Sobolev embedding theorem (Corollary 1.34) and Arzelà-Ascoli.

The case $k \ge 2$ follows by induction.

If $\Omega \subseteq \mathbb{R}^N$ is bounded, open and has the $W^{1,p}(\Omega)$-extension property, and if we replace the spaces $W_0^{k,p}(\Omega)$ by $W^{k,p}(\Omega)$, then one may argue as follows. Again, we consider first the case $k = 1$. We choose a test function $\psi \in C_c^\infty(\mathbb{R}^N)$ such that $\psi = 1$ on Ω. Moreover, we choose $R > 0$ such that $\operatorname{supp} \psi \subseteq B(0,R)$. Let $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N)$ be any extension operator, and consider then the extension operator $\tilde{E} : W^{1,p}(\Omega) \to W_0^{1,p}(B(0,R))$ given by $\tilde{E}f := (\psi \cdot Ef)|_\Omega$. Using now the first step (with $W_0^{1,p}(\Omega)$ replaced by $W_0^{1,p}(B(0,R))$), we obtain the claim for $k = 1$. The case $k \ge 2$ follows again by induction.

Remark 1.38. Note that the range of possible exponents q in Theorem 1.37 (a) differs from the range of possible exponents in Corollary 1.34 (a): the limit case $q = p^*$ is excluded. In fact, the embedding

$$W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$$

is continuous, but in general not compact.

1.7 The space $W_0^{1,p}(\Omega)$ and the Poincaré inequality

Theorem 1.39 (Poincaré inequality). *Let $\Omega \subseteq \mathbb{R}^N$ be contained in a strip of the form*

$$S := \mathbb{R}^{N-1} \times (a,b),$$

and let $p \in [1,\infty]$. Then

$$\frac{1}{(b-a)^p} \|u\|_{L^p}^p \le \|\nabla u\|_{L^p}^p \text{ for every } u \in W_0^{1,p}(\Omega).$$

Proof. Assume first that $u \in C_c^\infty(\Omega)$. We extend u by 0 to $\mathbb{R}^N$ and denote this extension by u, too. Then, for every $x = (\bar{x}, x_N) \in \Omega$ ($\bar{x} \in \mathbb{R}^{N-1}$, $x_N \in \mathbb{R}$),

$$\begin{aligned} u(x) &= u(\bar{x}, x_N) - u(\bar{x}, a) \\ &= \int_a^{x_N} \partial_N u(\bar{x}, \xi)\, d\xi. \end{aligned}$$

As a consequence, if $p < \infty$, then

$$\begin{aligned} \int_\Omega |u(x)|^p\, dx &= \int_{\mathbb{R}^{N-1}} \int_a^b |u(\bar{x}, x_N)|^p\, dx_N\, d\bar{x} \\ &\le \int_{\mathbb{R}^{N-1}} \int_a^b \left(\int_a^{x_N} |\partial_N u(\bar{x}, \xi)|\, d\xi \right)^p dx_N\, d\bar{x} \\ &\le \int_{\mathbb{R}^{N-1}} \int_a^b (b-a)^{p-1} \int_a^b |\partial_N u(\bar{x}, \xi)|^p\, d\xi\, dx_N\, d\bar{x} \\ &= \int_{\mathbb{R}^{N-1}} (b-a)^p \int_a^b |\partial_N u(\bar{x}, \xi)|^p\, d\xi\, d\bar{x} \\ &= (b-a)^p \int_\Omega |\partial_N u(x)|^p\, dx \\ &\le (b-a)^p \int_\Omega |\nabla u|^p\, dx. \end{aligned}$$

This inequality and an approximation argument yield the claim.

Chapter 2
Elliptic equations

2.1 The Lax-Milgram lemma

Let V be a Hilbert space. A sesquilinear form $a : V \times V \to \mathbb{C}$ is **coercive** if there exists $\eta > 0$ such that

$$\operatorname{Re} a(u,u) \geq \eta \, \|u\|_V^2 \text{ for every } u \in V.$$

If H is a second Hilbert space such that $V \hookrightarrow H$ with dense, continuous embedding, then the form a is called H-**elliptic** (or shortly **elliptic** if the Hilbert space H is clear from the context) if there exist $\eta > 0$ and $\omega \in \mathbb{R}$ such that

$$\operatorname{Re} a(u,u) + \omega \, \|u\|_H^2 \geq \eta \, \|u\|_V^2 \text{ for every } u \in V,$$

Clearly, a coercive sesquilinear form is H-elliptic for every Hilbert space H into which V is continuously and densely embedded.

Lemma 2.1 (Lax-Milgram). *Let $a : V \times V \to \mathbb{C}$ be a continuous, coercive, sesquilinear form on a Hilbert space V. Then, for every continuous, antilinear $f : V \to \mathbb{C}$ there exists a unique element $u \in V$ such that*

$$a(u,v) = \langle f, v \rangle_{V',V} \text{ for all } v \in V.$$

If V is a real Hilbert space, then "sesquilinear" and "antilinear" should be replaced by "bilinear" and "linear", respectively. Moreover, if V is real and if a is in addition symmetric, then the unique element $u \in V$ above is characterised by the equality

$$\frac{1}{2} a(u,u) - \langle f, u \rangle_{V',V} = \min_{v \in V} \frac{1}{2} a(v,v) - \langle f, v \rangle_{V',V} \tag{2.1}$$

Proof. For every $u \in V$ we denote by Au the continuous, antilinear form $V \to \mathbb{C}$, $v \mapsto a(u,v)$. We thus obtain a continuous, linear operator $A : V \to V'$ (where V' is in this proof the space of all continuous, antilinear forms on V). We have to show that A is bijective.

The operator A is injective and has closed range. By coercivity of a, for every $v \in V$,

$$\begin{aligned}\eta \|v\|_V^2 &\le \operatorname{Re} a(v,v)\\ &\le |a(v,v)|\\ &= |\langle Av, v\rangle_{V',V}|\\ &\le \|Av\|_{V'} \|v\|_V,\end{aligned}$$

which implies

$$\eta \|v\|_V \le \|Av\|_{V'} \text{ for every } v \in V.$$

As a consequence of this inequality, A is injective and has closed range.

The operator A is surjective. Since A has closed range, it suffices to show that A has dense range in V'. If the range of A is not dense in V', then, by the Hahn-Banach theorem and by reflexivity of V, there exists $u \in V \setminus \{0\}$ such that

$$\langle Av, u\rangle_{V',V} = a(v,u) = 0 \text{ for every } v \in V.$$

Choosing $v = u$ in this equality, we obtain a contradiction to coercivity of a. Hence, the range of A is dense in V' which, together with the preceding step, yields that A is surjective.

Now let us assume that V is a real Hilbert space and that a is symmetric. Let $u \in V$ be such that $a(u,v) = \langle f, v\rangle_{V',V}$ for every $v \in V$. Then, for every $v \in V$,

$$\begin{aligned}\frac{1}{2}a(u+v,u+v) - \langle f, u+v\rangle_{V',V} &= \frac{1}{2}a(u,u) + a(u,v) + \frac{1}{2}a(v,v) - \langle f,u\rangle_{V',V} - \langle f,v\rangle_{V',V}\\ &= \frac{1}{2}a(u,u) + \frac{1}{2}a(v,v) - \langle f,u\rangle_{V',V}\\ &\ge \frac{1}{2}a(u,u) - \langle f,u\rangle_{V',V},\end{aligned}$$

which yields (2.1).

Conversely, assume that (2.1) holds. Then, for every $v \in V$,

$$\frac{1}{2}a(u,u) - \langle f,u\rangle_{V',V} \le \frac{1}{2}a(u+v,u+v) - \langle f,u+v\rangle_{V',V},$$

which implies

$$0 \le a(u,v) + \frac{1}{2}a(v,v) - \langle f,v\rangle_{V',V} \text{ for every } v \in V.$$

Replacing v by tv with $t > 0$, diving the resulting inequality by t, and letting $t \to 0+$, we deduce

$$0 \le a(u,v) - \langle f,v\rangle_{V',V} \text{ for every } v \in V.$$

By noting that this inequality holds both for v and $-v$, we conclude that

$$0 = a(u,v) - \langle f,v\rangle_{V',V} \text{ for every } v \in V.$$

Proof (Second proof of Lemma 2.1 under the additional assumption that V is separable). Existence. Let (V_n) be an increasing sequence of finite dimensional subspaces of V such that $\bigcup_n V_n$ is dense in V. For the existence of such a sequence, we use separability of V; for example, if (w_k) is a dense sequence in V, then one may choose $V_n := \operatorname{span}\{w_k : 1 \le k \le n\}$.

For each n we consider the finite dimensional, linear problem of finding $u_n \in V_n$ such that

$$a(u_n,v) = f(v) \text{ for every } v \in V_n. \tag{2.2}$$

In order to translate this into a linear problem in $\mathbb{C}^m$ ($m = \dim V_n$), one may choose a basis $(b_i)_{1\le i\le m}$ of V_n, define the coefficients $\beta_i := f(b_i) \in \mathbb{C}$ and $\alpha_{ij} := a(b_i,b_j) \in \mathbb{C}$ ($1 \le i,\ j \le m$), and write $u_n = \sum_i \xi_i b_i$ with $\xi_i \in \mathbb{C}$. Then the problem (2.2) is equivalent to the problem

$$\sum_{i=1}^m \alpha_{ij}\xi_i = \beta_j \text{ for every } 1 \le j \le m.$$

If $u_n \in V_n$ is a solution of (**??**), then the coercivity of a yields

$$\begin{aligned} \|f\|_{V'}\|u_n\|_V &\ge \operatorname{Re} f(u_n) \\ &= \operatorname{Re} a(u_n,u_n) \\ &= \eta\,\|u_n\|_V^2, \end{aligned}$$

or, equivalently,

$$\|u_n\|_V \le \frac{1}{\eta}\,\|f\|_{V'}. \tag{2.3}$$

This inequality shows first injectivity of the underlying linear operator, which together with the fact that V_n is finite dimensional implies bijectivity of the underlying linear operator. Hence, the problem (2.2) admits a unique solution $u_n \in V_n$. However, inequality 2.3 also implies that the resulting sequence (u_n) is bounded in V. Since V is reflexive, there exists a subsequence of (u_n) (which we denote again by (u_n)) which converges weakly to some element $u \in V$. In particular, for the continuous, linear functionals $a(\cdot,v) : V \to \mathbb{C}$ ($v \in V$) one has $a(u_n,v) \to a(u,v)$ ($n \to \infty$). However, for all $v \in V_k$ and all $n \ge k$ one has $a(u_n,v) = f(v)$. Hence

$$a(u,v) = f(v) \text{ for all } v \in V_n \text{ and all} n.$$

Since $\bigcup_n V_n$ is dense in V and since a and f are continuous, we obtain

$$a(u,v) = f(v) \text{ for all } v \in V.$$

Uniqueness. Uniqueness of a solution $u \in V$ of the above equation follows again from coercivity of a, similarly as in the finite dimensional case.

2.2 The Laplace operator

In this section we consider the problem

$$\begin{aligned} \lambda u - \Delta u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{2.4}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open set, $\lambda \in \mathbb{R}$, $f : \Omega \to \mathbb{C}$ is a given function, $u : \Omega \to \mathbb{C}$ is the unknown function, and

$$\Delta := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$$

is the **Laplace operator**. While the first line in (2.4) is a partial differential equation in which the unknown function u and its partial derivatives (here, the second, not mixed partial derivatives) appear, the second line in (2.4) is a boundary condition. It is called (homogeneous) Dirichlet boundary condition.

Note that if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a **classical solution** of (2.4), that is, u satisfies (2.4) in the usual sense (using classical partial derivatives), then we may multiply the first line in (2.4) by the complex conjugate of a test function $\varphi \in C_c^\infty(\Omega)$ and integrate over Ω. An integration by parts then yields

$$\begin{aligned} \int_\Omega f\bar{\varphi} &= \lambda \int_\Omega u\bar{\varphi} - \int_\Omega \Delta u\bar{\varphi} \\ &= \lambda \int_\Omega u\bar{\varphi} - \int_\Omega \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}\bar{\varphi} \\ &= \lambda \int_\Omega u\bar{\varphi} - \int_\Omega \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}\overline{\frac{\partial \varphi}{\partial x_i}} \\ &= \lambda \int_\Omega u\bar{\varphi} - \int_\Omega \nabla u\overline{\nabla\varphi}. \end{aligned}$$

Given $f \in L^2(\Omega)$, we now call a function $u \in H_0^1(\Omega)$ a **weak solution** of (2.4) if

$$\lambda \int_\Omega u\bar{\varphi} - \int_\Omega \nabla u\overline{\nabla\varphi} = \int_\Omega f\bar{\varphi} \text{ for every } \varphi \in H_0^1(\Omega). \tag{2.5}$$

Theorem 2.2 (The Laplace operator with Dirichlet boundary conditions). *Let $\lambda_1 := \lambda_1(\Omega)$ be the Poincaré constant of the set Ω, that is, the optimal (largest) constant $\lambda \geq 0$ such that*

$$\lambda \int_\Omega |u|^2 \leq \int_\Omega |\nabla u|^2 \text{ for every } u \in H_0^1(\Omega).$$

Then, for every $\lambda > -\lambda_1$ and every $f \in L^2(\Omega)$ the problem (2.4) *admits a unique weak solution $u \in H_0^1(\Omega)$. For this weak solution we have the estimate*

$$\|u\|_{L^2} \leq \frac{1}{\lambda + \lambda_1} \|f\|_{L^2}.$$

Proof. Consider on the Hilbert space $H_0^1(\Omega)$ the sesquilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ given by

$$a(u,v) := \lambda \int_\Omega u\bar{v} - \int_\Omega \nabla u \overline{\nabla v} \quad (u, v \in H_0^1(\Omega)).$$

Then a is continuous since for every u, $v \in H_0^1(\Omega)$, by the Cauchy-Schwarz inequality,

$$\begin{aligned} |a(u,v)| |\lambda| \, \|u\|_{L^2} \|v\|_{L^2} + \|\nabla u\|_{L^2} \|v\|_{L^2} \\ (1 + |\lambda|) \, \|u\|_{H_0^1} \|v\|_{H_0^1}. \end{aligned}$$

We show that a is also coercive. Choose first $\varepsilon > 0$ such that $\lambda + \lambda_1 > \varepsilon$, and choose next $\mu \in (0,1]$ such that $\varepsilon - \mu\lambda_1 > 0$. Then, for every $u \in H_0^1(\Omega)$, by the definition of λ_1,

$$\begin{aligned} a(u,u) &= \lambda \int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \\ &= (\lambda + \lambda_1 - \varepsilon) \int_\Omega |u|^2 - (\lambda_1 - \varepsilon) \int_\Omega |u|^2 \\ &\quad + (1 - \mu) \int_\Omega |\nabla u|^2 + \mu \int_\Omega |\nabla u|^2 \\ &\geq (\lambda + \lambda_1 - \varepsilon) \int_\Omega |u|^2 + (\varepsilon - \mu\lambda_1) \int_\Omega |u|^2 + \mu \int_\Omega |\nabla u|^2 \\ &\geq \eta \|u\|_{H_0^1}^2, \end{aligned}$$

where $\eta = \min\{\lambda + \lambda_1 - \varepsilon, \mu\} > 0$. Hence, a is coercive.

Consider next the mapping $\ell : H_0^1(\Omega) \to \mathbb{C}, v \mapsto \int_\Omega f\bar{v}$, which is well defined and continuous by the Cauchy-Schwarz inequality, and antilinear. Existence and uniqueness of a weak solution of (2.4) thus follows from the Lax-Milgram lemma (Lemma 2.1) applied to a and ℓ. For the estimate, note that we have, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\|f\|_{L^2}\,\|u\|_{L^2} &\geq \int_\Omega fu \\ &= \lambda \int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \\ &\geq (\lambda + \lambda_1) \int_\Omega |u|^2 \\ &= (\lambda+\lambda_1)\,\|u\|_{L^2}^2 .\end{aligned}$$

Theorem 2.3 (The limit case $\lambda = -\lambda_1$). *Assume that $\Omega \subseteq \mathbb{R}^N$ is open and bounded. Then the problem*

$$\begin{aligned} -\lambda_1 u - \Delta u &= 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{2.6}$$

admits a weak solution $u \in H_0^1(\Omega)$ which is not $= 0$.

Proof. By definition of λ_1,

$$\lambda_1 = \inf_{\substack{u \in H_0^1(\Omega) \\ u \neq 0}} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} = \inf_{\substack{u \in H_0^1(\Omega) \\ \|u\|_{L^2}=1}} \int_\Omega |\nabla u|^2 .$$

By definition of the infimum, there exists a sequence (u_n) in $H_0^1(\Omega)$ such that

$$\begin{aligned} \int_\Omega |\nabla u_n|^2 &\to \lambda_1 \text{ as } n \to \infty, \text{ and} \\ \|u_n\|_{L^2} &= 1 \text{ for every } n. \end{aligned}$$

This sequence is thus bounded in $H_0^1(\Omega)$. By the theorem of Rellich-Kondrachev (Theorem 1.37), there exists a subsequence of (u_n) (which we denote again by (u_n)) and $u \in L^2(\Omega)$ such that

$$u_n \to u \text{ in } L^2(\Omega).$$

Moreover, since $H_0^1(\Omega)$ is reflexive, there exists a further subsequence (again denoted by (u_n)) which converges weakly in $H_0^1(\Omega)$ to some element $v \in H_0^1(\Omega)$. Since weak convergence in $H_0^1(\Omega)$ implies weak convergence in $L^2(\Omega)$, since strong convergence implies weak convergence, and since weak limits are unique, we obtain $u = v \in H_0^1(\Omega)$. By the geometric version of the Hahn-Banach theorem, and since the function $v \mapsto \int_\Omega |\nabla v|^2$ is convex and continuous,

$$\int_\Omega |\nabla u|^2 \le \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2.$$

Moreover, by the strong convergence in $L^2(\Omega)$,

$$\|u\|_{L^2} = \lim_{n\to\infty} \|u_n\|_{L^2} = 1.$$

In particular, $u \neq 0$. The preceding two (in-)equalities, the definition of λ_1, and the choice of the sequence (u_n) imply

$$\begin{aligned} \lambda_1 &= \lambda_1 \int_\Omega |u|^2 \\ &\le \int_\Omega |\nabla u|^2 \\ &\le \liminf_{n\to\infty} \int_\Omega |\nabla u_n|^2 \\ &= \lambda_1 . \end{aligned}$$

In particular, the inequality signs in this chain of inequalities can be replaced by equality signs. This means that

$$\int_\Omega |\nabla u|^2 - \lambda_1 \int_\Omega |u|^2 = 0.$$

In other words, u is a global minimizer of the (positive) function $v \mapsto \int_\Omega |\nabla v|^2 - \lambda_1 \int_\Omega |v|^2$. Proceeding now like in the second step of the proof of the lemma of Lax-Milgram (Lemma 2.1), we deduce from this

$$\int_\Omega \nabla u \overline{\nabla v} - \lambda_1 \int_\Omega u\bar{v} = 0 \text{ for every } v \in H_0^1(\Omega).$$

In other words, u is a weak solution of (2.6).

Theorem 2.4 (The case $N = 1$). *Let $\Omega = (a,b)$ be an interval in $\mathbb{R}$ ($-\infty \le a < b \le \infty$). Then every weak solution $u \in H_0^1(a,b)$ of* (2.4) *($\lambda \in \mathbb{R}$, $f \in L^2(a,b)$) belongs to $H^2(a,b) \cap H_0^1(a,b)$ and $\lambda u - u'' = f$. Moreover, u belongs to $C_0(a,b)$ which means in particular that u admits a continuous extension to a and b (if they are finite) and $u(a) = u(b) = 0$.*

Proof. Since u is a weak solution,

$$\int_a^b f\bar{v} = \lambda \int_a^b u\bar{v} + \int_a^b u'\bar{v}' \text{ for every } H_0^1(a,b).$$

In particular, since $C_c^\infty((a,b)) \subseteq H_0^1(a,b)$,

$$\int_a^b u'\varphi' = -\int_a^b (\lambda u - f)\varphi \text{ for every } \varphi \in C_c^\infty((a,b)).$$

By definition of the weak derivative, and since $u - f \in L^2(a,b)$, this equality implies

$$u' \in H^1(a,b) \text{ and } (u')' = \lambda u - f.$$

In other words,

$$u \in H^2(a,b) \text{ and } \lambda u - u'' = f.$$

The fact that u admits a continuous extension to the closure of the interval (a,b) follows from properties of the Sobolev spaces on intervals (see Theorem). The fact that u vanishes in a and b (if they are finite) follows from Theorem .

Now let us consider the problem

$$\begin{aligned} \lambda u - \Delta u &= f \text{ in } \Omega, \\ \partial_\nu u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{2.7}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open set with C^1-regular boundary $\partial\Omega$, ν is the outer normal vector, $\partial_\nu u = \nabla u \cdot \nu$ is the outer normal derivative, $\lambda \in \mathbb{R}$, $f : \Omega \to \mathbb{C}$ is a given function, $u : \Omega \to \mathbb{C}$ is the unknown function. The boundary condition in (2.7) is called (homogeneous) *Neumann boundary condition*.

We call a function $u \in H^1(\Omega)$ a *weak solution* of problem (2.7) if

$$\lambda \int_\Omega u\bar{\varphi} - \int_\Omega \nabla u \overline{\nabla\varphi} = \int_\Omega f\bar{\varphi} \text{ for every } \varphi \in H^1(\Omega). \tag{2.8}$$

Theorem 2.5 (The Laplace operator with Neumann boundary conditions). *For every $\lambda > 0$ and every $f \in L^2(\Omega)$ the problem (2.7) admits a unique weak solution $u \in H^1(\Omega)$. For this weak solution we have the estimate*

$$\|u\|_{L^2} \le \frac{1}{\lambda} \|f\|_{L^2}.$$

Proof. Consider on the Hilbert space $H^1(\Omega)$ the sesquilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ given by

$$a(u,v) := \lambda \int_\Omega u\bar{v} + \int_\Omega \nabla u \overline{\nabla v} \quad (u,v \in H^1(\Omega)).$$

Then a is continuous since for every u, $v \in H^1(\Omega)$, by the Cauchy-Schwarz inequality,

$$\begin{aligned} |a(u,v)| &\le \lambda \|u\|_{L^2} \|v\|_{L^2} + \|\nabla u\|_{L^2} \|v\|_{L^2} \\ &\le (1+\lambda) \|u\|_{H^1} \|v\|_{H^1}. \end{aligned}$$

We show that a is also coercive. In fact, for every $u \in H_0^1(\Omega)$,

$$\begin{aligned} a(u,u) &= \lambda \int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \\ &\geq \eta \|u\|_{H_0^1}^2, \end{aligned}$$

where $\eta = \min\{\lambda, 1\} > 0$. Hence, a is coercive.

Consider next the mapping $\ell : H^1(\Omega) \to \mathbb{C}, v \mapsto \int_\Omega f\bar{v}$, which is well defined and continuous by the Cauchy-Schwarz inequality, and antilinear. Existence and uniqueness of a weak solution of (2.7) thus follows from the Lax-Milgram lemma (Lemma 2.1) applied to a and ℓ. For the estimate, note that we have, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \|f\|_{L^2} \|u\|_{L^2} &\geq \int_\Omega fu \\ &= \lambda \int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \\ &\geq \lambda \int_\Omega |u|^2 \\ &= \lambda \|u\|_{L^2}^2. \end{aligned}$$

Theorem 2.6 (Neumann boundary conditions in the case $N = 1$). *Let $\Omega = (a,b)$ be an interval in $\mathbb{R}$ ($-\infty \leq a < b \leq \infty$). Then every weak solution $u \in H^1(a,b)$ of* (2.7) *($\lambda \in \mathbb{R}$, $f \in L^2(a,b)$) belongs to $H^2(a,b)$ and $\lambda u - u'' = f$. Moreover, u' admits a continuous extension to a and b (if they are finite) and $u'(a) = u'(b) = 0$.*

Proof. Since u is a weak solution,

$$\int_a^b f\bar{v} = \lambda \int_a^b u\bar{v} + \int_a^b u'\bar{v}' \text{ for every } H^1(a,b).$$

In particular, since $C_c^\infty((a,b)) \subseteq H^1(a,b)$,

$$\int_a^b u'\varphi' = -\int_a^b (\lambda u - f)\varphi \text{ for every } \varphi \in C_c^\infty((a,b)).$$

By definition of the weak derivative, and since $u - f \in L^2(a,b)$, this equality implies

$$u' \in H^1(a,b) \text{ and } (u')' = \lambda u - f.$$

In other words,

$$u \in H^2(a,b) \text{ and } \lambda u - u'' = f.$$

The fact that u' admits a continuous extension to the closure of the interval (a,b) follows from properties of the Sobolev spaces on intervals (see Theorem

). Assume, for simplicity, that both a and b are finite. For every $\varphi \in H^1(a,b)$, we have, by an integration by parts,

$$\begin{aligned}\int_a^b f\bar\varphi &= \lambda \int_a^b u\bar\varphi + \int_a^b u'\bar\varphi' \\ &= \lambda \int_a^b u\bar\varphi + \big[u'\bar\varphi\big]_a^b - \int_a^b u''\bar\varphi \\ &= \int_a^b (\lambda u - u'')\bar\varphi + u'(b)\bar\varphi(b) - u'(a)\bar\varphi(b) \\ &= \int_a^b f\bar\varphi + u'(b)\bar\varphi(b) - u'(a)\bar\varphi(b).\end{aligned}$$

In other words,

$$u'(b)\bar\varphi(b) - u'(a)\bar\varphi(b) = 0 \text{ for every } \varphi \in H^1(a,b).$$

Choosing now successively $\varphi(x) := \frac{x-a}{b-a}$ and $\varphi(x) = \frac{b-x}{b-a}$, we obtain

$$u'(b) = u'(a) = 0.$$

2.3 General elliptic operators in divergence form and inhomogeneous Dirichlet boundary conditions

Consider the elliptic operator L which is formally given by

$$Lu = \sum_{i,j=1}^N \partial_i(a_{ij}(x)\partial_j u) + \sum_{i=1}^N [\partial_i(b_i(x)u) + c_i(x)\partial_i u] + d(x)u,$$

where

$$a_{ij}, b_i, c_i, d \in L^\infty(\Omega) \quad (1 \le i, j \le N),$$

and $\Omega \subseteq \mathbb{R}^N$ is an open set. We assume that the coefficients a_{ij} are **uniformly elliptic** in the sense that there exists $\eta > 0$ such that

$$\sum_{i,j=1}^N a_{ij}(x)\xi_j\overline{\xi_i} \ge \eta|\xi|^2 \text{ for all } \xi \in \mathbb{C}^N, x \in \Omega. \tag{2.9}$$

There are no further conditions on the lower order conditions b_i, c_i and d, and in fact, the boundedness condition on these coefficients may be relaxed a little bit. All coefficients may be complex valued. We then consider the problem

$$\begin{aligned} \lambda u - Lu &= f + \operatorname{div} g \text{ in } \Omega, \\ u &= h \text{ on } \partial\Omega, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} &f, g_1, \dots, g_N \in L^2(\Omega) \text{ and} \\ &h \in H^1(\Omega). \end{aligned}$$

We say that a function $u \in H^1(\Omega)$ is a **weak solution** of the problem

$$\lambda u - Lu = f + \operatorname{div} g \text{ in } \Omega, \tag{2.11}$$

if, for all $\varphi \in C_c^\infty(\Omega)$,

$$\begin{aligned} &\lambda \int_\Omega u\bar{\varphi} + \sum_{i,j=1}^N \int_\Omega a_{ij}(x)\partial_j u \overline{\partial_i \varphi} + \\ &+ \sum_{i=1}^N \int_\Omega [-b_i(x) u \overline{\partial_i \varphi} + c_i(x)\partial_i u \bar{\varphi} + \int_\Omega d(x) u \bar{\varphi} \\ &= \int_\Omega f\bar{\varphi} + \sum_{i=1}^N \int_\Omega g_i \overline{\partial_i \varphi}. \end{aligned} \tag{2.12}$$

Note that by an approximation argument, if the above equality holds for all test functions $\varphi \in C_c^\infty(\Omega)$, then it holds for all $\varphi \in H_0^1(\Omega)$, and vice versa. Next, we say that the inhomogeneous Dirichlet boundary condition

$$u = h \text{ on } \partial\Omega \tag{2.13}$$

is satisfied, if

$$u - h \in H_0^1(\Omega). \tag{2.14}$$

Accordingly, $u \in H^1(\Omega)$ is a weak solution of (2.10) if it satisfies both (2.12) and (2.14).

Theorem 2.7. *Assume that Ω, a_{ij}, b_i, c_i, d, f, g_i and h are as above. Then there exists a real number $\hat{\lambda}$ such that for all $\lambda \in \mathbb{C}$ with* $\operatorname{Re}\lambda > \hat{\lambda}$ *the problem* (2.10) *admits a unique weak solution $u \in H^1(\Omega)$.*

Proof. Assume first that $h = 0$. Then we define the sesquilinear form $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ by

$$a(u,v) = \lambda \int_\Omega u\bar{v} + \sum_{i,j=1}^N \int_\Omega a_{ij}(x)\partial_j u \overline{\partial_i v} + \\ + \sum_{i=1}^N \int_\Omega [-b_i(x)u\overline{\partial_i v} + c_i(x)\partial_i u\bar{v} + \int_\Omega d(x)u\bar{v}.$$

Then a is continuous since for every u, $v \in H_0^1(\Omega)$, by the Cauchy-Schwarz inequality,

$$|a(u,v)| \le |\lambda| \, \|u\|_{L^2}\|v\|_{L^2} + \sum_{i,j=1}^N \|a_{ij}\|_{L^\infty} \|\partial_j u\|_{L^2}\|\partial_i v\|_{L^2} + \\ + \sum_{i=1}^N [\|b_i\|_{L^\infty} \|u\|_{L^2}\|\partial_i v\|_{L^2} + \|c_i\|_{L^\infty} \|\partial_i u\|_{L^2}\|v\|_{L^2}] + \|d\|_{L^\infty} \|u\|_{L^2}\|v\|_{L^2} \\ \le C \|u\|_{H^1}\|v\|_{H^1},$$

where $C \ge 0$ is for example the sum of $|\lambda|$ and the L^∞-norms of the coefficients a_{ij}, b_i, c_i, d.

We show that a is also coercive whenever

$$\operatorname{Re}\lambda > \hat{\lambda} := \|d\|_{L^\infty} + \frac{1}{2\eta} \sum_{i=1}^N (\|b_i\|_{L^\infty}^2 + \|c_i\|_{L^\infty}^2).$$

In fact, for every such $\lambda \in \mathbb{C}$ and every $u \in H_0^1(\Omega)$, by the uniform ellipticity condition, by the Cauchy-Schwarz inequality and by Young's inequality,

$$\operatorname{Re} a(u,u) = \operatorname{Re}\lambda \int_\Omega |u|^2 + \operatorname{Re} \sum_{i,j=1}^N \int_\Omega a_{ij}\partial_j u \overline{\partial_i u} + \\ + \operatorname{Re} \sum_{i=1}^N \int_\Omega [-b_i(x)u\overline{\partial_i u} + c_i(x)\partial_i u\bar{u} + \int_\Omega \operatorname{Re} d(x)|u|^2 \\ \ge (\operatorname{Re}\lambda - \|d\|_{L^\infty}) \|u\|_{L^2}^2 + \eta \|\nabla u\|_{L^2}^2 - \\ - \left(\sum_{i=1}^N (\|b_i\|_{L^\infty}^2 + \|c_i\|_{L^\infty}^2) \right)^{\frac{1}{2}} \|u\|_{L^2}\|\nabla u\|_{L^2} \\ \ge (\operatorname{Re}\lambda - \hat{\lambda})\|u\|_{L^2}^2 + \frac{\eta}{2}\|\nabla u\|_{L^2}^2 \\ \ge \tilde{\eta}\|u\|_{H_0^1}^2,$$

where $\tilde{\eta} = \min\{\operatorname{Re}\lambda - \hat{\lambda}, \frac{\eta}{2}\} > 0$. Hence, a is coercive.

Consider next the mapping $\ell : H^1(\Omega) \to \mathbb{C}$ given by

$$\ell(v) = \int_\Omega f\bar{v} - \sum_{i=1}^N \int_\Omega g_i \overline{\partial_i v},$$

which is well defined and continuous by the Cauchy-Schwarz inequality, and antilinear. Existence and uniqueness of a weak solution of (2.10) thus follows from the Lax-Milgram lemma (Lemma 2.1) applied to a and ℓ.

Let now $h \in H^1(\Omega)$ be arbitrary, and define

$$\hat{f} := f + \lambda h + \sum_{i=1}^N c_i \partial_i h + dh \in L^2(\Omega) \text{ and}$$

$$\hat{g}_i := g_i + \sum_{j=1}^N a_{ij}\partial_j h + b_i h \in L^2(\Omega) \quad (1 \le i \le N).$$

Then one easily verifies that $u \in H^1(\Omega)$ is a weak solution of (2.10) if and only if $w = u - h \in H_0^1(\Omega)$ is a weak solution of

$$\begin{aligned} \lambda w - Lw &= \hat{f} + \operatorname{div}\hat{g} \text{ in } \Omega, \\ w &= 0 \text{ on } \partial\Omega, \end{aligned}$$

and from this equivalence and the first step one obtains existence and uniqueness of a weak solution of (2.10).

2.4 The comparison and maximum principles

Theorem 2.8 (Comparison principle I). *Let $\Omega \subseteq \mathbb{R}^N$ be open, $f \in L^2(\Omega)$ and $\lambda > -\lambda_1(\Omega)$. Let $u \in H_0^1(\Omega)$ be the unique weak solution of* (2.4). *If $f \ge 0$, then $u \ge 0$, and similarly if $f \le 0$, then $u \le 0$.*

Proof. Let f, λ and u be as in the assumption and assume that $f \le 0$. Taking $\varphi = u^+ \in H_0^1(\Omega)$ as a test function in the definition of a weak solution, we obtain

$$\begin{aligned} 0 &\ge \int_\Omega f u^+ \\ &= \lambda \int_\Omega u u^+ + \int_\Omega \nabla u \nabla u^+ \\ &= \lambda \int_\Omega (u^+)^2 + \int_\Omega |\nabla u^+|^2 \\ &\ge (\lambda + \lambda_1) \int_\Omega (u^+)^2. \end{aligned}$$

Since $\lambda + \lambda_1 > 0$, this inequality implies $u^+ = 0$ which means that $u \le 0$. The case $f \ge 0$ is proved similarly, or follows from this case by multiplying the equation (2.4) by -1.

We consider now again the problem (2.10) with a general elliptic operator in divergence form and inhomogeneous boundary conditions, but we assume that the coefficients a_{ij}, b_i, c_i and d, as well as the functions f, g_i and h are real valued. Repeating the existence and uniqueness proof (proof of Theorem 2.7) in the *real* Hilbert space $H^1(\Omega)$, we see that the unique weak solution $u \in H^1(\Omega)$ is real valued, too.

We say that $u \in H^1(\Omega)$ is a **subsolution** (resp. **supersolution**) of (2.11) and we write

$$\lambda u - Lu \le f + \operatorname{div} g \text{ in } \Omega \quad (\text{resp. } \ge) \tag{2.15}$$

if, for every *positive* test function $\varphi \in C_c^\infty(\Omega)$,

$$\begin{aligned} &\lambda \int_\Omega u\varphi + \sum_{i,j=1}^N \int_\Omega a_{ij}(x)\partial_j u \partial_i \varphi + \\ &+ \sum_{i=1}^N \int_\Omega [-b_i(x) u \partial_i \varphi + c_i(x)\partial_i u \varphi + \int_\Omega d(x) u\varphi \\ &\le \int_\Omega f\varphi + \sum_{i=1}^N \int_\Omega g_i \partial_i \varphi \quad (\text{resp. } \ge). \end{aligned} \tag{2.16}$$

Moreover, given a constant $k \in \mathbb{R}$, we write

$$u \le k \text{ on } \partial\Omega$$

if

$$(u-k)^+ \in H_0^1(\Omega),$$

and we define

$$\begin{aligned} \sup_{\partial\Omega} u &:= \{k \in \mathbb{R} : u \le k \text{ on } \partial\Omega\} \\ &= \{k \in \mathbb{R} : (u-k)^+ \in H_0^1(\Omega)\}, \text{ and} \\ \inf_{\partial\Omega} u &:= -\sup_{\partial\Omega}(-u) \\ &= \sup\{k \in \mathbb{R} : (u+k)^- \in H_0^1(\Omega)\}. \end{aligned}$$

Theorem 2.9 (Comparison principle II). *Let $\Omega \subseteq \mathbb{R}^N$ be open, $f \in L^2(\Omega)$ and $\lambda > -\lambda_1(\Omega)$. Let $u \in H^1(\Omega)$ be such that*

$$\begin{aligned} \lambda u - \Delta u &\le 0 \textit{ in } \Omega, \\ u &\le 0 \textit{ on } \partial\Omega. \end{aligned}$$

Then $u \geq 0$.

Proof. Let λ and u be as in the assumption. Taking $\varphi = u^+ \in H_0^1(\Omega)$ as a test function in the definition of a weak solution, we obtain

$$\begin{aligned} 0 &\geq \lambda \int_\Omega u u^+ + \int_\Omega \nabla u \nabla u^+ \\ &= \lambda \int_\Omega (u^+)^2 + \int_\Omega |\nabla u^+|^2 \\ &\geq (\lambda + \lambda_1) \int_\Omega (u^+)^2. \end{aligned}$$

Since $\lambda + \lambda_1 > 0$, this inequality implies $u^+ = 0$ which means that $u \leq 0$.

Theorem 2.10. *Let $\Omega \subseteq \mathbb{R}^N$ be open, $\lambda > 0$ and $f \in L^2 \cap L^\infty(\Omega)$. Let $u \in H_0^1(\Omega)$ be the unique weak solution of* (2.4)*. Then $u \in L^\infty(\Omega)$ and*

$$\|u\|_{L^\infty} \leq \frac{\|f\|_{L^\infty}}{\lambda}.$$

Proof. Let $k := \frac{\|f\|_{L^\infty}}{\lambda}$. Since $u \in H_0^1(\Omega)$ is a weak solution of (2.4),

$$\lambda \int_\Omega u\varphi + \int_\Omega \nabla u \nabla \varphi = \int_\Omega f\varphi \text{ for every } \varphi \in H_0^1(\Omega).$$

Taking $\varpi = (u-k)^+ \in H_0^1(\Omega)$ as a test function (here we use $\lambda > 0$!), we obtain the equality

$$\lambda \int_\Omega (u-k)(u-k)^+ + \int_\Omega \nabla u \nabla (u-k)^+ = \int_\Omega (f - \lambda k)(u-k)^+,$$

and hence

$$\lambda \int_\Omega ((u-k)^+)^2 + \int_\Omega |\nabla (u-k)^+|^2 = \int_\Omega (f - \|f\|_{L^\infty})(u-k)^+ \leq 0.$$

This inequality implies $(u-k)^+ = 0$ which is only possible if $u \leq k = \frac{\|f\|_{L^\infty}}{\lambda}$. Similarly, one proves $u \geq -\frac{\|f\|_{L^\infty}}{\lambda}$. Taking both inequalities together yields the claim.

For measurable functions $f : \Omega \to \mathbb{R}$ we denote by $\sup_\Omega f$ the **essential supremum**, that is,

$$\sup_\Omega f := \inf\{k \in \mathbb{R} : f \leq k \text{ almost everywhere}\},$$

and accordingly for $\inf_\Omega f$.

Theorem 2.11. *Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded, $\lambda > -\lambda_1$ and $f \in L^2(\Omega)$. Let $u \in H^1(\Omega)$ be such that*

$$\lambda u - \Delta u \le f \text{ in } \Omega.$$

Then

$$u \le \sup_{\partial\Omega} u + k(\lambda)(\sup_{\Omega} f - \lambda \sup_{\partial\Omega} u)^+,$$

where $k(\lambda) := \|w\|_{L^\infty}$ and $w \in H_0^1(\Omega)$ is the unique weak solution of

$$\begin{aligned} \lambda w - \Delta w &= 1 \text{ in } \Omega, \\ w &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{2.17}$$

Similarly, if

$$\lambda u - \Delta u \ge f \text{ in } \Omega,$$

then

$$u \ge \inf_{\partial\Omega} u - k(\lambda)(\lambda \inf_{\partial\Omega} u - \inf_{\Omega} f)^+.$$

Proof. Set $m := \sup_{\partial\Omega} u$ and $M := \sup_{\Omega} f - \lambda \sup_{\partial\Omega} u$. Then the assumption and the definition of m and M imply

$$\begin{aligned} \lambda(u-m) + \Delta(u-m) \le f - \lambda m \le M \text{ in } \Omega, \\ u - m \le 0 \text{ on } \partial\Omega. \end{aligned}$$

If $M \le 0$, then Theorem 2.9 implies $u - m \le 0$, so that $u \le \sup_{\partial\Omega} u$. If $M \ge 0$, then

$$\begin{aligned} \lambda(u-m-Mw) + \Delta(u-m-Mw) \le 0 \text{ in } \Omega, \\ u - m - Mw \le 0 \text{ on } \partial\Omega, \end{aligned}$$

where w is the unique solution of (2.17). Now Theorem 2.9 implies $u - m - Mw \le 0$, so that $u \le \sup_{\partial\Omega} u + \|w\|_\infty M$. Hence, in both cases we obtain the required estimate. The case with $\lambda u - \Delta u \ge f$ follows from the first case by multiplying this inequality with -1 and by replacing u by $-u$.

A function $u \in H^1(\Omega)$ is **harmonic** if $-\Delta u = 0$ in Ω.

Corollary 2.12 (Weak maximum principle). *Let $\Omega \subseteq \mathbb{R}^N$ be open and bounded. Then, for every harmonic function $u \in H^1(\Omega)$,*

$$\inf_{\partial\Omega} u \le u \le \sup_{\partial\Omega} u.$$

Proof. Apply Theorem 2.11 with $\lambda = 0$ and $f = 0$.

2.5 Regularity of weak solutions of elliptic equations

Theorem 2.13 (Elliptic equations in $\mathbb{R}^N$). *Let the coefficients $a_{ij} \in W^{1,\infty}(\mathbb{R}^N)$ be uniformly elliptic, $f \in L^2(\mathbb{R}^N)$, and let $u \in H^1(\mathbb{R}^N)$ be a weak solution of*

$$-\sum_{i,j=1}^{N} \partial_j (a_{ij}(x)\partial_i u) = f \text{ in } \mathbb{R}^N.$$

Then $u \in H^2(\mathbb{R}^N)$ and

$$\|\partial_i \partial_j u\|_{L^2} \leq \|f\|_{L^2} \text{ for every } 1 \leq i, j \leq N.$$

Proof. We prove the statement in the case when $a_{ij} = \delta_{ij}$, which corresponds to the Laplace operator. The argument for coefficients in $W^{1,\infty}(\mathbb{R}^N)$ is very similar. For every $h \in \mathbb{R}^N$, $h \neq 0$ and every function $u : \mathbb{R}^N \to \mathbb{C}$ we define $D_h u : \mathbb{R}^N \to \mathbb{C}$ by

$$D_h u(x) = \frac{u(x+h) - u(x)}{|h|} \quad (x \in \mathbb{R}^N).$$

Recall from Theorem 1.18 that a function $u \in L^2(\mathbb{R}^N)$ belongs to $H^1(\mathbb{R}^N)$ if and only if there exists a constant $C \geq 0$ such that $\|D_h u\|_{L^2} \leq C$ for every $h \in \mathbb{R}^N$, $h \neq 0$, and then one may choose $C = \|\nabla u\|_{L^2}$. We shall apply this characterisation to the partial derivatives / the gradient of u. Since $u \in H^1(\mathbb{R}^N)$ is a weak solution of $-\Delta u = f$ in $\mathbb{R}^N$, for every $\varphi \in H^1(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi = \int_{\mathbb{R}^N} f \varphi.$$

Inserting $\varphi = D_{-h}(D_h u)$ into this equality yields

$$\begin{aligned} \int_{\mathbb{R}^N} f D_{-h}(D_h u) &= \int_{\mathbb{R}^N} \nabla u \, \nabla D_{-h}(D_h u) \\ &= \int_{\mathbb{R}^N} |\nabla D_h u|^2 \\ &= \int_{\mathbb{R}^N} |D_h(\nabla u)|^2. \end{aligned}$$

As a consequence, by the Cauchy-Schwarz inequality and the characterisation from Theorem 1.18,

$$\|D_h(\nabla u)\|_{L^2}^2 \leq \|f\|_{L^2} \, \|D_{-h}(D_h u)\|_{L^2} \leq \|f\|_{L^2} \, \|D_h(\nabla u)\|_{L^2},$$

or

$$\|D_h(\nabla u)\|_{L^2} \leq \|f\|_{L^2}.$$

From here and Theorem 1.18 follows the claim.

Corollary 2.14 (Elliptic equations on domains - inner regularity). *Let $\Omega \subseteq \mathbb{R}^N$ be open. Let the coefficients $a_{ij} \in W^{1,\infty}(\Omega)$ be uniformly elliptic, $f \in L^2(\Omega)$, and let $u \in H^1(\Omega)$ be a weak solution of*

$$-\sum_{i,j=1}^{N} \partial_j(a_{ij}(x)\partial_i u) = f \text{ in } \Omega.$$

Then $u \in H^2_{loc}(\Omega)$, and for every compact $K \subseteq \Omega$ there exists a constant $C_K \geq 0$ such that

$$\|\partial_i \partial_j u\|_{L^2(K)} \leq C_K \|f\|_{L^2(\Omega)} \text{ for every } 1 \leq i,j \leq N.$$

Proof.

Corollary 2.15 (Elliptic equations in the half-space; Dirichlet boundary conditions). *Let the coefficients $a_{ij} \in W^{1,\infty}(\mathbb{R}^N_+)$ be uniformly elliptic, $f \in L^2(\mathbb{R}^N_+)$, and let $u \in H^1_0(\mathbb{R}^N_+)$ be a weak solution of*

$$-\sum_{i,j=1}^{N} \partial_j(a_{ij}(x)\partial_i u) = f \text{ in } \mathbb{R}^N.$$

Then $u \in H^2(\mathbb{R}^N)$ and

$$\|\partial_i \partial_j u\|_{L^2} \leq \|f\|_{L^2} \text{ for every } 1 \leq i,j \leq N.$$

Theorem 2.16 (Elliptic equations in domains - regularity up to the boundary; Dirichlet boundary conditions). *Let $\Omega \subseteq \mathbb{R}^N$ be open with bounded, C^2-regular boundary $\partial\Omega$. Let the coefficients $a_{ij} \in W^{1,\infty}(\Omega)$ be uniformly elliptic, $f \in L^2(\Omega)$, and let $u \in H^1_0(\Omega)$ be a weak solution of*

$$-\sum_{i,j=1}^{N} \partial_j(a_{ij}(x)\partial_i u) = f \text{ in } \Omega.$$

Then $u \in H^2(\Omega)$, and there exists a constant $C \geq 0$ depending only on the boundary $\partial\Omega$ and the coefficients a_{ij} such that

$$\|u\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

Chapter 3
Evolution equations

3.1 Wellposedness results for abstract diffusion equations, wave equations and Schrödinger equations

Let V and H be complex Hilbert spaces such that $V \subseteq H$ with dense and continuous embedding. Let $a : V \times V \to \mathbb{C}$ be an elliptic, bounded, sesquilinear form. Associated with this form is an operator $A : H \supseteq \operatorname{dom} A \to H$ given by

$$\begin{aligned} \operatorname{dom} A &:= \{u \in V : \exists f \in H \forall v \in V : a(u,v) = \langle f, v \rangle_H\}, \\ Au &:= f. \end{aligned}$$

If the form a is in addition symmetric, the following theorem asserts that A is "essentially" a multiplication operator on an abstract L^2-space. It is a very general form of the theorem which states that every hermitian matrix is diagonalisable over an orthonormal basis of eigenvectors.

Theorem 3.1 (Spectral theorem for symmetric, elliptic forms). *Let V, H, a and A be as above. Assume in addition that a is symmetric. Then there exists a measure space $(B, \mathcal{B}, \mu)$, a (real) measurable function $m : B \to \mathbb{R}$ which is bounded from below, and a unitary operator $U : H \to L^2(B, \mathrm{d}\mu)$ such that, given the multiplication operator $M : L^2(B, \mathrm{d}\mu) \supseteq \operatorname{dom} M \to L^2(B, \mathrm{d}\mu)$ with*

$$\begin{aligned} \operatorname{dom} M &:= \{f \in L^2(B, \mathrm{d}\mu) : mf \in L^2(B, \mathrm{d}\mu)\} \\ &= L^2(B, (1+m^2)\mathrm{d}\mu), \\ Mf &:= mf, \end{aligned}$$

one has $U(\operatorname{dom} A) = \operatorname{dom} M$, $U(V) = L^2(B, (1+m^2)^{\frac{1}{2}}\mathrm{d}\mu)$ and the diagram

$$\begin{array}{ccc} \operatorname{dom} A & \xrightarrow{A} & H \\ \big\downarrow{\scriptstyle U} & & \big\uparrow{\scriptstyle U^*=U^{-1}} \\ \operatorname{dom} M & \xrightarrow{M} & L^2(B, \mathrm{d}\mu) \end{array}$$

commutes.

We shall not prove this result here. For a proof, see for instance [Reed and Simon (1980)].

Theorem 3.2 (Wellposedness of abstract diffusion equations). *Let V, H, a and A be as above. Assume in addition that a is symmetric. Then for every $u_0 \in H$ there exists a unique function $u \in C(\mathbb{R}_+; H) \cap C^\infty((0,\infty); H)$ such that $u(t) \in \operatorname{dom} A$ for every $t > 0$ and*

$$\dot{u}(t) + Au(t) = 0 \text{ in } (0,\infty), \quad u(0) = u_0. \tag{3.1}$$

Proof.

Theorem 3.3 (Wellposedness of abstract wave equations). *Let V, H, a and A be as above. Assume in addition that a is symmetric. Then for every $u_0 \in \operatorname{dom} A$, $u_1 \in V$ there exists a unique function $u \in C^2(\mathbb{R}_+; H) \cap C^1(\mathbb{R}_+; V)$ such that $u(t) \in \operatorname{dom} A$ for every $t \geq 0$ and*

$$\ddot{u}(t) + Au(t) = 0 \text{ in } (0,\infty), \quad u(0) = u_0, \, \dot{u}(0) = u_1. \tag{3.2}$$

Moreover, one has energy conservation in the sense that for every $t \geq 0$

$$\|\dot{u}(t)\|_H^2 + a(u(t), u(t)) = \|u_1\|_H^2 + a(u_0, u_0).$$

Theorem 3.4 (Wellposedness of abstract Schrödinger equations). *Let V, H, a and A be as above. Assume in addition that a is symmetric. Then for every $u_0 \in \operatorname{dom} A$ there exists a unique function $u \in C^1(\mathbb{R}_+; H)$ such that $u(t) \in \operatorname{dom} A$ for every $t \geq 0$ and*

$$\dot{u}(t) + iAu(t) = 0 \text{ in } (0,\infty), \quad u(0) = u_0. \tag{3.3}$$

Moreover, one has energy conservation in the sense that for every $t \geq 0$

$$\|u(t)\|_H^2 = \|u_0\|_H^2.$$

3.2 The comparison principle for diffusion equations

Let $\Omega \subseteq \mathbb{R}^N$ be open, $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous and $u_0 \in L^2(\Omega)$ real-valued. Let a solution u of

$$\partial_t u - \Delta u - f(u) = 0 \text{ in } (0,T)\times\Omega,$$
$$u(0,\cdot) = u_0 \text{ in } \Omega,$$

be a function $u \in C([0,T];L^2(\Omega)) \cap C^1(]0,T[;L^2(\Omega)) \cap C(]0,T[;H^1(\Omega))$, such that for all $\phi \in \mathbb{C}_c^\infty(\Omega)$ and all $t \in]0,T]$

$$\int_\Omega \partial_t u(t,x)\phi(x)dx + \int_\Omega \nabla u(t,x)\nabla\phi(x)dx + \int_\Omega f(u(t,x))\phi(x)dx = 0$$

and $u(0) = u_0$. Via approxiation one gets

$$\int_0^\tau \int_\Omega \partial_t u(t,x)\phi(t,x)dxdt + \int_0^\tau \int_\Omega \nabla u(t,x)\nabla\phi(t,x)dxdt + \int_0^\tau \int_\Omega f(u(t,x))\phi(t,x)dxdt = 0$$

for all $\phi \in C([0,T],H_0^1(\Omega))$ with $\phi(0) = 0$ and all $\tau \in]0,T]$.

Notation: $Q_T := (0,T)\times\Omega$.

The **parabolic boundary** of Q_T is the set

$$\Gamma_T = (\{0\}\times\bar{\Omega}) \cup ([0,T]\times\partial\Omega).$$

We denote

$$edge\partial_t u - \Delta u + f(u) \le \partial_t v - \Delta v + f(v)$$

in Q_T if $u,v \in C([0,T];L^2(\Omega)) \cap C^1(]0,T[;L^2(\Omega)) \cap C(]0,T[;H^1(\Omega))$ and

$$\int_0^\tau \int_\Omega \partial_t u\phi + \int_0^\tau \int_\Omega \nabla u\nabla\phi + \int_0^\tau \int_\Omega f(u)\phi \le \int_0^\tau \int_\Omega \partial_t v\phi + \int_0^\tau \int_\Omega \nabla v\nabla\phi + \int_0^\tau \int_\Omega f(v)\phi$$

for all $\phi \in C([0,T];H_0^1(\Omega))$, $\phi \ge 0$, $\phi(0) = 0$ and all $\tau \in]0,T]$.

We denote $u \le v$ in Γ_T if

$$(u-v)^+ \in C([0,T];L^2(\Omega)) \cap C(]0,T[;H_0^1(\Omega))$$
$$(u-v)^+(0) = 0.$$

Theorem 3.5 (Comparison principle). *Let*

$$\partial_t u - \Delta u + f(u) \le \partial_t v - \Delta v + f(v) \textit{ in } Q_T$$
$$u \le v \textit{ on } \Gamma_T.$$

Then $u \le v$ in Q_T.

Corollary 3.6. *Let $f(0) = 0$ and u a solution of*

$$\begin{aligned} \partial_t u - \Delta u + f(u) = 0 \; &in \; Q_T \\ u = 0 \; &on \; (0,T) \times \partial\Omega \\ u(0,\cdot) = u_0 \; &in \; \Omega. \end{aligned}$$

If $u_0 \geq 0$ in Ω, then $u \geq 0$ in Q_T.

Proof. Choose $v = 0$ in Q_T as comparison.

Proof (Comparison principle). As required for all $0 < \tau' \leq \tau \leq T$ holds

$$\begin{aligned} \int_{\tau'}^{\tau}\int_\Omega (\partial_t u - \partial_t v)(u-v)^+ + \int_{\tau'}^{\tau}\int_\Omega (\nabla u - \nabla v)\nabla(u-v)^+ &\leq - \int_{\tau'}^{\tau}\int_\Omega (f(u) - f(v))(u-v)^+ \\ &\leq L \int_{\tau'}^{\tau}\int_\Omega |u-v|(u-v)^+ \\ &\leq L \int_{\tau'}^{\tau}\int_\Omega ((u-v)^+)^2 \end{aligned}$$

with Lipschitz constant L. Set

$$a(t) = \int_\Omega ((u-v)^+(t,x))^2 dx,$$

then $a \in C([0,T])$ and with the inequality we get

$$a(\tau) - a(\tau') \leq 2L \int_{\tau'}^{\tau} a(t) dt$$

due to

$$\begin{aligned} \int_{\tau'}^{\tau}\int_\Omega \partial_t(u-v)(u-v)^+ &= \frac{1}{2}\int_{\tau'}^{\tau}\frac{d}{dt}\int_\Omega ((u-v)^+)^2 = \frac{1}{2}(a(\tau) - a(\tau')), \\ \int_{\tau'}^{\tau}\int_\Omega \nabla(u-v)\nabla(u-v)^+ &= \int_{\tau'}^{\tau}\int_\Omega |\nabla(u-v)^+|^2 = 0 \end{aligned}$$

For $t' \to 0$ and with $a(0) = 0$ we get

$$a(\tau) \leq 2L \int_0^T a(t) dt \quad \forall \tau \in [0,T].$$

Then (with Gronwall's Lemma)

$$a \equiv 0 \quad \Longrightarrow \quad (u-v)^+ = 0 \text{ in } Q_T,$$

i.e. $u \leq v$ in Q_T.

Chapter 4
Distributions

4.1 The topology in $\mathcal{D}(\Omega)$

Let $\Omega \subseteq \mathbb{R}^N$ be open. In this chapter we write $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$ to denote the space of test functions in Ω.

In order to equip the space of test functions with a topology, choose a sequence of bounded, open sets $(\Omega_k)_{k\in\mathbb{N}}$ such that $\bar{\Omega}_k \subseteq \Omega_{k+1}$ and $\bigcup_{k\in\mathbb{N}} \Omega_k = \Omega$, for example

$$\Omega_k := \left\{ x \in \Omega \,:\, \operatorname{dist}(x, \Omega^c) \geq \frac{1}{k}, |x| \leq k \right\}, \quad k \in \mathbb{N}.$$

Let $E_k := \left\{ \phi \in \mathcal{D}(\Omega) \,:\, \operatorname{supp} \phi \subseteq \bar{\Omega}_k \right\}$. Every space E_k is equipped with the countable family $(p_\alpha)_{\alpha\in\mathbb{N}_0^N}$ of seminorms given by

$$p_\alpha(\phi) := |\partial^\alpha \phi|_\infty,$$

and from here one can construct a metric d_k as following:

$$d_k(\phi, \psi) := \sum_{\alpha\in\mathbb{N}_0^N} c_\alpha \left(p_\alpha(\phi - \psi) \wedge 1 \right), \quad (\phi, \psi \in E_k),$$

where the $c_\alpha > 0$ are such that $\sum_{\alpha\in\mathbb{N}_0^N} c_\alpha < \infty$.

Remark 4.1. For every $k \in \mathbb{N}$ the space (E_k, d_k) is a Fréchet space, that is, a complete, metric vector space (proof: exercise).

Let the topology τ on $\mathcal{D}(\Omega)$ be the finest topology on $\mathcal{D}(\Omega)$, such that every mapping

$$\begin{aligned} g_k : (E_k, d_k) &\to (\mathcal{D}(\Omega), \tau) \\ \phi &\mapsto \phi \end{aligned}$$

is continuous (final/inductive topology). Together with this topology $\mathcal{D}(\Omega)$ becomes a topological vector space, which means that $\mathcal{D}(\Omega)$ is not only a vector space and a topological space, but also that the addition and the multiplication,

$$+ : \mathcal{D}(\Omega) \times \mathcal{D}(\Omega) \to \mathcal{D}(\Omega) \text{ and}$$
$$\cdot : \mathbb{K} \times \mathcal{D}(\Omega) \to \mathcal{D}(\Omega),$$

are continuous.

Exercise 4.2 *A sequence $(\phi_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\Omega)$ converges to $\phi \in \mathcal{D}(\Omega)$ with respect to τ if and only if there exists a $k \in \mathbb{N}$ such that $\phi_n, \phi \in E_k$, that is, $\operatorname{supp}\phi, \operatorname{supp}\phi_n \subseteq \bar{\Omega}_k$, and $d_k(\phi_n, \phi) \to 0$ for $k \to \infty$.*

Exercise 4.3 *The topology τ does not depend on the choice of the sequence $(\Omega_k)_{k\in\mathbb{N}}$.*

4.2 Distributions

We denote by

$$\mathcal{D}(\Omega)' := \{T : \mathcal{D}(\Omega) \to \mathbb{K} : T \text{ is linear and continuous}\}$$

the dual space of $(D(\Omega), \tau)$. The elements of $\mathcal{D}(\Omega)'$ are called **distributions on** Ω, $\mathcal{D}(\Omega)'$ is called **space of distributions on** Ω. We equip $\mathcal{D}(\Omega)'$ with the **weak-* topology** τ', which is the coarsest topology such that all mappings

$$(\mathcal{D}(\Omega)', \tau') \to \mathbb{K}$$
$$T \mapsto T(\phi),$$

with $\phi \in \mathcal{D}(\Omega)$ are continuous. In the following, given $\phi \in \mathcal{D}(\Omega)$ and $T \in \mathcal{D}(\Omega)'$, we shall also use the notation

$$\langle T, \phi \rangle := T(\phi).$$

A sequence $(T_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\Omega)'$ converges to $T \in \mathcal{D}(\Omega)'$ with respect to the topology τ' if and only if $T_n(\phi) \to T(\phi)$ for all $\phi \in \mathcal{D}(\Omega)$.

Examples 4.4 (Distributions).

a) Let $f \in L^1_{loc}(\Omega)$ $(\supseteq L^p(\Omega), C(\Omega))$. Then

$$T_f : \mathcal{D}(\Omega) \to \mathbb{K},$$
$$\phi \mapsto \langle T_f, \phi \rangle := \int_\Omega f \cdot \phi,$$

is continuous and linear, that is, a distribution. By uniqueness by testing (Theorem 1.12),

$$T_f = 0 \quad \Longrightarrow \quad f = 0.$$

Thus all locally integrable functions are distributions and the mapping

$$\begin{aligned} L^1_{loc}(\Omega) &\to \mathcal{D}(\Omega)' \\ f &\mapsto T_f \end{aligned}$$

is linear and injective, and so $L^1_{loc}(\Omega) \subseteq \mathcal{D}(\Omega)'$. This embedding is even continuous, if one equips $L^1_{loc}(\Omega)$ with the usual Fréchet topology as follows: let $(\Omega_k)_{k\in\mathbb{N}}$ be a sequence of open, bounded subsets of Ω, such that

$$\bar{\Omega}_k \subseteq \Omega_{k+1}, \quad \bigcup_{k\in\mathbb{N}} \Omega_k = \Omega.$$

Define for every $k \in \mathbb{N}$ the seminorm p_k by

$$p_k(f) := \int_{\Omega_k} |f| \qquad (f \in L^1_{loc}(\Omega)),$$

and then the metric

$$d(f,g) = \sum_{k\in\mathbb{N}} 2^{-k}(p_k(f-g) \wedge 1).$$

Then d is a complete metric on $L^1_{loc}(\Omega)$.

b) Let

$$M^b_{loc}(\Omega) := \{\mu : \mathcal{B}(\Omega) \to \mathbb{R} : \ \mu \text{ is a signed Radon measure}\}$$

be the space of all signed Radon measures μ, i.e. $\mu = \mu_1 - \mu_2$, with μ_1, μ_2 positive Radon measures, i.e. positive regular Borel measures with the property that

$$\mu_i(K) < \infty \qquad \forall i \in \{1,2\} \quad \forall K \subseteq \Omega \text{ compact}.$$

For all $\mu \in M^b_{loc}(\Omega)$

$$\begin{aligned} T_\mu : \mathcal{D}(\Omega) &\to \mathbb{R} \\ \phi &\mapsto \langle T_\mu, \phi \rangle := \int_\Omega \phi \, d\mu \end{aligned}$$

is linear and continuous, that is, a distribution.

For example, point evaluation

$$T_{\delta_{x_0}} : \mathcal{D}(\Omega) \to \mathbb{R}$$
$$\phi \mapsto \langle T_{\delta_{x_0}}, \phi\rangle := \qquad \phi(x_0) = \int_\Omega \phi d\delta_{x_0}$$

is a distribution for every $x_0 \in \Omega$. It is called the Dirac distribution in the point x_0.

c) For all $\alpha \in \mathbb{N}_0^N$, $\Omega \subseteq \mathbb{R}^N$, $x_0 \in \Omega$

$$T : \mathcal{D}(\Omega) \to \mathbb{R}$$
$$\phi \mapsto \langle T, \phi\rangle := D^\alpha \phi(x_0)$$

is linear and continuous, that is, a distribution.

We call $m \in \mathbb{N}_0$ the **order of a distribution** $T \in \mathcal{D}(\Omega)'$, if for all compact subsets $K \subseteq \Omega$ there exists a constant $c_K \geq 0$, such that for all $\phi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}\phi \subseteq K$ the estimate

$$|\langle T, \phi\rangle| \leq c_K \cdot \|\phi\|_{C^m} \quad \text{with}$$
$$\|\phi\|_{C^m} := \sum_{\alpha\in\mathbb{N}_0^N, |\alpha|\leq m} \|D^\alpha\phi\|_{L^\infty}$$

holds.

Remark 4.5. a) Distributions of the form T_f or T_μ with $f \in L^1_{loc}(\Omega)$ or $\mu \in M^b_{loc}(\Omega)$ are of order 0. Indeed, for all $f \in L^1_{loc}(\Omega)$, all compact $K \subseteq \Omega$ and all $\phi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}(\phi) \subseteq K$ one has

$$|\langle T_f, \phi\rangle| = |\int_\Omega f\phi| \leq \int_K |f \cdot g| \leq \left(\int_K |f|\right) \|\phi\|_{C^\infty}$$

b) We now equip $C^m_c(\Omega)$ with a topology in a similar way as we did with $C^\infty_c(\Omega)$. Let $(\Omega_k)_{k\in\mathbb{N}}$ be a sequence of open, bounded subsets of Ω such that $\bar{\Omega}_k \subseteq \Omega_{k+1}$ and $\bigcup_k \Omega_k$. Then we set

$$E_k := \{\phi \in C^m_c(\Omega) : \operatorname{supp}\phi \subseteq \bar{\Omega}_k\}$$

and equip this space with the norm

$$\|\phi\|_{C^m} := \sum_{\alpha\in\mathbb{N}_0^N, |\alpha|\leq m} \|D^\alpha\phi\|_{L^\infty}.$$

Then let τ_m be the finest topology on $C^m_c(\Omega)$, such that every mapping

$$\begin{aligned}(E_k, \|\cdot\|_{C^m}) &\to (C_c^m(\Omega), \tau_m) \\ \phi &\mapsto \phi\end{aligned}$$

is continuous. A distribution $T \in \mathcal{D}(\Omega)'$ is of order $m \in \mathbb{N}_0$, if and only if T extends to a linear, continuous mapping $C_c^m(\Omega) \to \mathbb{K}$.

Theorem 4.6 (Riesz-Markov).

$$\{T \in \mathcal{D}(\Omega)' : T \textit{ is of order } 0\} = M_{loc}^b(\Omega) = C_c(\Omega)'$$

(without proof)

Theorem 4.7. *Let $T \in \mathcal{D}(\Omega)'$ be positive, that is, $\langle T, \phi\rangle \geq 0$ for all $\phi \in \mathcal{D}(\Omega)$ with $\phi \geq 0$. Then T is of order 0, that is, T is a positive Radon measure.*

Let $T \in \mathcal{D}(\Omega)'$. Define

$$O_T := \bigcup \{U \subseteq \Omega : U \text{ is open and } \forall \phi \in \mathcal{D}(\Omega) \text{ with } \operatorname{supp}\phi \subseteq U \text{ one has } \langle T, \phi\rangle = 0\}.$$

We call $\operatorname{supp} T := \Omega \setminus O_T$ the **support** of T.

Example 4.8. If

$$\langle T, \phi\rangle = D^\alpha \phi(x_0)$$

then

$$\operatorname{supp} T = \{x_0\}.$$

For all $T \in \mathcal{D}(\Omega)'$, $\alpha \in \mathbb{N}_0^N$ we define $D^\alpha T \in \mathcal{D}(\Omega)'$ by

$$\langle D^\alpha T, \phi\rangle := (-1)^{|\alpha|}\langle T, D^\alpha \phi\rangle \qquad (\phi \in \mathcal{D}(\Omega)).$$

The distribution $D^\alpha T$ is called **α-th partial derivative** of T.

Theorem 4.9 (Consistency of the distributional partial derivatives). *For all $f \in W_{loc}^{1,1}(\Omega) = \{g \in L_{loc}^1(\Omega) : \partial_i g \in L_{loc}^1(\Omega)\, \forall 1 \leq i \leq N\}$ and all $1 \leq i \leq N$ one has*

$$\partial_i T_f = T_{\partial_i f}.$$

Thus, the partial derivative ∂_i in the distributional sense coincides with the weak partial derivative ∂_i of functions in the Sobolev space $W_{loc}^{1,1}(\Omega)$. In particular, it coincides with the classical partial derivate on $C^1(\Omega)$.

Proof. By definition of ∂_i on $\mathcal{D}(\Omega)'$, using the definitions of T_f and the weak derivatives we get

$$\begin{aligned}\langle \partial_i T_f, \phi\rangle &= -\langle T_f, \partial_i \phi\rangle \\ &= -\int_\Omega f \partial_i \phi \\ &= \int_\Omega \partial_i f \phi \\ &= \langle T_{\partial_i f}, \phi\rangle \quad \forall \phi \in \mathcal{D}(\Omega).\end{aligned}$$

4.3 The product and the convolution

We define the **product** $T \cdot \psi \in \mathcal{D}(\Omega)'$ of a distribution $T \in \mathcal{D}(\Omega)'$ and a test function $\psi \in \mathcal{D}(\Omega)$ by

$$\langle T \cdot \psi, \phi\rangle := \langle T, \psi \cdot \phi\rangle \quad (\phi \in \mathcal{D}(\Omega)).$$

Lemma 4.10 (Consistency of the product). *If $f \in L^1_{loc}(\Omega)$ and $\psi \in \mathcal{D}(\Omega)$ then*

$$T_f \cdot \psi = T_{f\psi}.$$

Proof. For every $f \in L^1_{loc}(\Omega)$ and ψ, $\phi \in \mathcal{D}(\Omega)$ one has

$$\langle T_f \cdot \psi, \phi\rangle = \langle T_f, \psi \cdot \phi\rangle = \int_\Omega f \cdot \psi \cdot \phi = \langle T_{f \cdot \psi}, \phi\rangle.$$

Lemma 4.11. *For every $T \in \mathcal{D}(\Omega)'$ and every $\psi \in \mathcal{D}(\Omega)$ one has*

$$\operatorname{supp} T \cdot \psi \subseteq \operatorname{supp} T \cap \operatorname{supp} \psi \quad (\subseteq \operatorname{supp} \psi)$$

For a distribution $T \in \mathcal{D}(\Omega)'$ and a test function $\phi \in \mathcal{D}(\mathbb{R}^N)$ we define the **convolution** $T * \phi$ by setting

$$T * \phi(x) := \langle T, \phi(x - \cdot)\rangle$$

for all $x \in \Omega$, such that $\phi(x - \cdot) \in \mathcal{D}(\Omega)$, so $x - \operatorname{supp} \phi \subseteq \Omega$.

One observes that the set $\{x \in \Omega : \phi(x - \cdot) \in \mathcal{D}(\Omega)\} = U_\phi$ is an open subset with respect to Ω.

Lemma 4.12. *If $T \in \mathcal{D}(\Omega)'$ and $\phi \in \mathcal{D}(\Omega)$ then*

$$T * \phi \in C^\infty(U_\phi).$$

If $\operatorname{supp} T + \operatorname{supp} \phi \subseteq \Omega$ and $\operatorname{supp} T$ are compact, and if we extend $T * \phi$ by 0, then $T * \phi \in \mathcal{D}(\Omega)$. Moreover,

$$D^\alpha (T * \phi) = T * D^\alpha \phi = D^\alpha T * \phi.$$

Theorem 4.13. *The space $\mathcal{D}(\Omega)$ is sequentially dense in $\mathcal{D}(\Omega)'$ with respect to the weak-* topology τ', that is, for all $\tau \in \mathcal{D}(\Omega)$ there exists a sequence $(\phi_n)_{n\in\mathbb{N}} \in \mathcal{D}(\Omega)$, such that*

$$T_{\phi_n} \to T \quad \text{in } \mathcal{D}(\Omega)'.$$

Proof. Truncation and regularization.

Let $(\phi_n)_{n\in\mathbb{N}}$ be an approximation of the identity, that is, $\phi_1 \in \mathcal{D}(\mathbb{R}^N)$, $\phi_1 \geq 0$, $\int_{\mathbb{R}^N} \phi_1 = 1$ and

$$\phi_n x) := n^N \phi_1(n \cdot x) \qquad (n \in \mathbb{N}, x \in \mathbb{R}^N).$$

Then $\phi_n \geq 0$ and $\int_{\mathbb{R}^N} \phi_n \geq 1$ for every n. Moreover, for all $\psi \in \mathcal{D}(\mathbb{R}^N)$

$$\langle T_{\phi_n}, \psi \rangle = \int_{\mathbb{R}^N} \phi_n(x)\psi(x)dx \to \psi(0) = \langle T_{\delta_0}, \psi \rangle, \quad (n \to \infty),$$

that is, the approximation of the identity (ϕ_n) converges to the Dirac distribution T_{δ_0} in $\mathcal{D}(\mathbb{R}^N)'$.

Remark 4.14. For all $\phi \in \mathcal{D}(\mathbb{R}^N)$ one has

$$T_{\delta_0} * \phi(x) = \langle T_{\delta_0}, \phi(x - \cdot) \rangle = \phi(x),$$

or

$$T_{\delta_0} * \phi = \phi.$$

4.4 Tempered distributions

We define the space

$$\mathcal{S}(\mathbb{R}^N) := \{f \in C^\infty(\mathbb{R}^N) : \forall \alpha, \beta \in \mathbb{N}_0^N : \int_{\mathbb{R}^N} |x^\beta \partial^\alpha f(x)|^2 \, \mathrm{d}x < \infty\}.$$

Elements of $\mathcal{S}(\mathbb{R}^N)$ are called the **rapidly decreasing functions** or **Schwartz (test) functions**. Clearly, the space of (classical) test functions $C_c^\infty(\mathbb{R}^N) = \mathcal{D}(\mathbb{R}^N)$ is a subspace of $\mathcal{S}(\mathbb{R}^N)$, but it is a proper subspace since the function $f(x) = e^{-x^2}$ is an example of a Schwartz test function which does not have compact support.

It is an exercise to show that

$$\mathcal{S}(\mathbb{R}^N) = \{f \in C^\infty(\mathbb{R}^N) : \forall \alpha, \beta \in \mathbb{N}_0^N : \int_{\mathbb{R}^N} |x^\beta \partial^\alpha f(x)| \, \mathrm{d}x < \infty\}$$
$$= \{f \in C^\infty(\mathbb{R}^N) : \forall \alpha, \beta \in \mathbb{N}_0^N : \sup_{x \in \mathbb{R}^N} |x^\beta \partial^\alpha f(x)| < \infty\}.$$

The space $\mathcal{S}(\mathbb{R}^N)$ is equipped with the topology induced by the countable family of seminorms $(\|\cdot\|_{\alpha,\beta})_{\alpha,\beta \in \mathbb{N}_0^N}$, where

$$\|f\|_{\alpha,\beta} := \left(\int_{\mathbb{R}^N} |x^\beta \partial^\alpha f(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

This countable family of seminorms induces in a natural way a metric d given by

$$d(f,g) := \sum_{\alpha,\beta \in \mathbb{N}_0^N} c_{\alpha,\beta} \frac{\|f-g\|_{\alpha,\beta}}{1 + \|f-g\|_{\alpha,\beta}},$$

where the coefficients $c_{\alpha,\beta} > 0$ are fixed such that $\sum_{\alpha,\beta \in \mathbb{N}_0^N} c_{\alpha,\beta} < \infty$. We have

$$f_n \to f \text{ in } \mathcal{S}(\mathbb{R}^N) \quad \Leftrightarrow \quad \forall \alpha, \beta \in \mathbb{N}_0^N : \|f_n - f\|_{\alpha,\beta} \to 0$$
$$\Leftrightarrow \quad d(f_n, f) \to 0,$$

and the space $\mathcal{S}(\mathbb{R}^N)$ is complete. In other words, the countable family of seminorms turns $\mathcal{S}(\mathbb{R}^N)$ into a Fréchet space.

From the definition of the space $\mathcal{S}(\mathbb{R}^N)$ we immediately obtain the following lemma which is, however, worth of being stated separately.

Lemma 4.15. *For every $f \in \mathcal{S}(\mathbb{R}^N)$ and every polynomial $p : \mathbb{C}^N \to \mathbb{C}$ the product pf and the (sum of) partial derivatives $p(\partial)f$ belong again to $\mathcal{S}(\mathbb{R}^N)$. In other words, the mappings*

$$f \mapsto pf \quad \textit{and}$$
$$f \mapsto p(\partial)f$$

leave the space $\mathcal{S}(\mathbb{R}^N)$ invariant.

Elements of the dual space

$$\mathcal{S}(\mathbb{R}^N)' := \{T : \mathcal{S}(\mathbb{R}^N) \to \mathbb{C} : T \text{ is linear and continuous}\}$$

are called **tempered distributions**. Since $\mathcal{D}(\mathbb{R}^N) \subseteq \mathcal{S}(\mathbb{R}^N)$ with dense and continuous embedding, we obtain the inclusion

$$\mathcal{D}(\mathbb{R}^N) \subseteq \mathcal{S}(\mathbb{R}^N) \subseteq \mathcal{S}(\mathbb{R}^N)' \subseteq \mathcal{D}(\mathbb{R}^N)'.$$

In particular, every tempered distribution is a distribution. For every multi-index $\alpha \in \mathbb{N}_0^N$ and every tempered distribution $T \in \mathcal{S}(\mathbb{R}^N)'$ we define the

partial derivative $\partial^\alpha T \in \mathcal{S}(\mathbb{R}^N)'$ and the product $x^\alpha T \in \mathcal{S}(\mathbb{R}^N)'$ by

$$\langle \partial^\alpha T, \varphi\rangle := (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi\rangle \quad \text{and}$$
$$\langle x^\alpha T, \varphi\rangle := \langle T, x^\alpha \varphi\rangle.$$

These linear operations are consistent with the classical partial derivatives and the product, respectively, on the space $\mathcal{S}(\mathbb{R}^N)$.

4.5 The Fourier transform

4.5.1 The Fourier transform on $L^1(\mathbb{R}^N)$

For every $f \in L^1(\mathbb{R}^N)$ we define the **Fourier transform** $\mathcal{F} f$ and the **adjoint Fourier transform** $\bar{\mathcal{F}} f$ by

$$\mathcal{F} f(x) := \int_{\mathbb{R}^N} e^{-ixy} f(y) \, \mathrm{d}y \quad \text{and}$$
$$\bar{\mathcal{F}} f(x) := \int_{\mathbb{R}^N} e^{ixy} f(y) \, \mathrm{d}y \quad (x \in \mathbb{R}^N).$$

The integrals are absolutely convergent, and we have the trivial estimates

$$|\mathcal{F} f(x)|, |\bar{\mathcal{F}} f(x)| \leq \|f\|_{L^1} \text{ for every } x \in \mathbb{R}^N.$$

In particular, the functions $\mathcal{F} f$ and $\bar{\mathcal{F}} f$ are bounded.

Theorem 4.16 (Riemann-Lebesgue). *For every $f \in L^1(\mathbb{R}^N)$ one has $\mathcal{F} f, \bar{\mathcal{F}} f \in C_0(\mathbb{R}^N)$.*

Proof. The fact that the Fourier transform $\mathcal{F} f$ is continuous follows easily from Lebesgue's dominated convergence theorem. Next, for every $x \in \mathbb{R}^N$, $x \neq 0$,

$$\mathcal{F} f(x) = \frac{1}{2} \int_{\mathbb{R}^N} (e^{-ixy} - e^{-ixy} e^{i\pi \frac{x \cdot x}{|x|^2}}) f(y) \, \mathrm{d}y$$
$$= \frac{1}{2} \int_{\mathbb{R}^N} e^{ixy} (f(y) - f(y + \frac{\pi x}{|x|^2})) \, \mathrm{d}y.$$

Since the shift group on $L^1(\mathbb{R}^N)$ is strongly continuous by Lemma 1.7, we thus obtain

$$|\mathcal{F} f(x)| \leq \frac{1}{2} \int_{\mathbb{R}^N} |f(y) - f(y + \frac{\pi x}{|x|^2})| \, \mathrm{d}y \to 0 \text{ as } |x| \to \infty.$$

The arguments for the adjoint Fourier transform are similar.

Corollary 4.17. *The Fourier transform $\mathcal{F}$ and the adjoint Fourier transform are bounded, linear operators from $L^1(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$.*

We need the following basic lemma in order to prove the inversion formula for the Fourier transform.

Lemma 4.18 (Féjer kernel). *One has, for $a > 0$,*

$$\int_{\mathbb{R}} \frac{\sin^2 ax}{x^2} \, \mathrm{d}x = a\pi.$$

Proof. We define

$$f(\lambda) := \int_0^\infty e^{-\lambda x} \frac{\sin^2 ax}{x^2} \, \mathrm{d}x \quad (\lambda \in (0,\infty)).$$

Then $f \in C^\infty((0,\infty))$ and

$$\lim_{\lambda \to 0+} f(\lambda) = \int_0^\infty \frac{\sin^2 ax}{x^2} \, \mathrm{d}x = \frac{1}{2} \int_{\mathbb{R}} \frac{\sin^2 ax}{x^2} \, \mathrm{d}x, \text{ and}$$
$$\lim_{\lambda \to \infty} f(\lambda) = 0.$$

A simple computation shows

$$\begin{aligned} f'(\lambda) &= -\int_0^\infty e^{-\lambda x} \frac{\sin^2 ax}{x} \, \mathrm{d}x, \text{ and} \\ f''(\lambda) &= \int_0^\infty e^{-\lambda x} \sin^2 ax \, \mathrm{d}x \\ &= \int_0^\infty e^{-\lambda x} \left(\frac{e^{iax} - e^{-iax}}{2i} \right)^2 \mathrm{d}x \\ &= -\frac{1}{4} \left(\frac{1}{\lambda - 2ia} - \frac{2}{\lambda} + \frac{1}{\lambda + 2ia} \right) \\ &= \frac{1}{4} \left(\frac{2}{\lambda} - \frac{2\lambda}{\lambda^2 + 4a^2} \right). \end{aligned}$$

As a consequence,

$$f'(\lambda) = \frac{1}{4} \log \frac{\lambda^2}{\lambda^2 + 4a^2}.$$

In order to integrate this function, we make the ansatz

$$f(\lambda) = \frac{1}{4} \left(\lambda \log \frac{\lambda^2}{\lambda^2 + 4a^2} + g(\lambda) \right),$$

which leads to the equation

$$g'(\lambda) = -\frac{8a^2}{\lambda^2 + 4a^2},$$

that is,

$$g(\lambda) = -4a \arctan\frac{\lambda}{2a} + C$$

Together with the condition $\lim_{\lambda\to\infty} f(\lambda) = 0$ we thus find

$$f(\lambda) = \frac{1}{4}\left(\lambda \log\frac{\lambda^2}{\lambda^2 + 4a^2} + 4a(\frac{\pi}{2} - \arctan\frac{\lambda}{2a})\right).$$

This yields

$$\lim_{\lambda\to 0} f(\lambda) = a\frac{\pi}{2},$$

which implies the claim.

Before stating the following theorem we define for every $r \in \mathbb{R}^N$ with $r_k \geq 0$ the set

$$Q_r := \bigtimes_{k=1}^{N} [-r_k, r_k].$$

Theorem 4.19 (Inversion formula for the Fourier transform I). *Let $f \in L^1(\mathbb{R}^N)$. For every $R > 0$ we put*

$$g_R(x) := \frac{1}{(2\pi R)^N}\int_{[0,R]^N}\int_{Q_r} e^{ixy}\mathcal{F}f(y)\,\mathrm{d}y\,\mathrm{d}r \quad (x \in \mathbb{R}^N).$$

Then $g_R \in L^1(\mathbb{R}^N)$ and

$$\lim_{R\to\infty} \|g_R - f\|_{L^1} = 0.$$

Proof. For every $R > 0$ and every $x \in \mathbb{R}^N$ we compute, using Fubini's theorem,

$$\begin{aligned}
&\frac{1}{(2\pi R)^N}\int_{[0,R]^N}\int_{Q_r} e^{ixy}\mathcal{F}f(y)\,\mathrm{d}y\,\mathrm{d}r \\
&= \frac{1}{(2\pi R)^N}\int_{[0,R]^N}\int_{\mathbb{R}^N}\int_{Q_r} e^{iy(x-z)}\,\mathrm{d}y f(z)\,\mathrm{d}z\,\mathrm{d}r \\
&= \frac{1}{(\pi R)^N}\int_{\mathbb{R}^N}\int_{[0,R]^N}\prod_{k=1}^{N}\frac{\sin(r_k(x_k - z_k))}{x_k - z_k}\,\mathrm{d}r f(z)\,\mathrm{d}z \\
&= \int_{\mathbb{R}^N}\prod_{k=1}^{N}\frac{\sin^2(\frac{R}{2}(x_k - z_k))}{\frac{R}{2}\pi(x_k - z_k)^2} f(z)\,\mathrm{d}z \\
&= \int_{\mathbb{R}} k_R(x - z) f(z)\,\mathrm{d}z \\
&= k_R * f(x)
\end{aligned}$$

where

$$k_R(x) := \prod_{k=1}^{N} \frac{\sin^2(\frac{R}{2}x_k)}{\frac{R}{2}\pi x_k^2} \quad (x \in \mathbb{R}^N)$$

is the **Féjer kernel**. Note that

$$k_R \in L^1(\mathbb{R}^N),$$
$$k_R \geq 0,$$
$$k_R(x) = R^N k_2(Rx) \text{ for every } x \in \mathbb{R}^N \text{ and}$$
$$\int_{\mathbb{R}^N} k_R(x)\,\mathrm{d}x = 1 \quad \text{(Lemma 4.18)}$$

for every $R > 0$. Hence, $(k_R)_{R\nearrow\infty}$ is an approximate identity, and the claim follows from Young's inequality and Theorem 1.8.

Corollary 4.20 (Inversion formula for the Fourier transform II). *Let $f \in L^1(\mathbb{R}^N)$ be such that $\mathcal{F} f \in L^1(\mathbb{R}^N)$. Then $\bar{\mathcal{F}} f \in L^1(\mathbb{R}^N)$ and*

$$f = \frac{1}{(2\pi)^N}\bar{\mathcal{F}}(\mathcal{F} f) \text{ and}$$
$$f = \frac{1}{(2\pi)^N}\mathcal{F}(\bar{\mathcal{F}} f).$$

Proof. Since

$$\bar{\mathcal{F}} f(x) = \int_{\mathbb{R}^N} e^{ixy} f(y)\,\mathrm{d}y = \mathcal{F} f(-x) \text{ for every } x \in \mathbb{R}^N,$$

we immediately obtain $\bar{\mathcal{F}} f \in L^1(\mathbb{R}^N)$.

Now let g_R be defined as in the preceding theorem. For every $R > 0$ and every $x \in \mathbb{R}^N$ we then have

$$\begin{aligned} g_R(x) &- \frac{1}{(2\pi)^N}\bar{\mathcal{F}}(\mathcal{F} f)(x) = \\ &= \frac{1}{(2\pi)^N}\left[\frac{1}{R^N}\int_{[0,R]^N}\int_{Q_r} e^{ixy}\mathcal{F} f(y)\,\mathrm{d}y\,\mathrm{d}r - \int_{\mathbb{R}^N} e^{ixy}\mathcal{F} f(y)\, udy\right] \\ &= \frac{1}{(2\pi)^N}\frac{1}{R^N}\int_{[0,R]^N}\int_{Q_r^c} e^{ixy}\mathcal{F} f(y)\,\mathrm{d}y\,\mathrm{d}r, \end{aligned}$$

and hence, for every $L > 0$

$$
\begin{aligned}
&\limsup_{R\to\infty} \left| g_R(x) - \frac{1}{(2\pi)^N} \bar{\mathcal{F}}(\mathcal{F} f)(x) \right| \leq \\
&\leq \frac{1}{(2\pi)^N} \Big[\limsup_{R\to\infty} \left| \frac{1}{R^N} \int_{[L,R]^N} \int_{Q_r^c} e^{ixy} \mathcal{F} f(y) \,\mathrm{d}y \,\mathrm{d}r \right| \\
&\quad + \limsup_{R\to\infty} \left| \frac{1}{R^N} \int_{[0,R]^N \setminus [L,R]^N} \int_{Q_r^c} e^{ixy} \mathcal{F} f(y) \,\mathrm{d}y \,\mathrm{d}r \right| \Big] \\
&\leq \frac{1}{(2\pi)^N} \Big[\limsup_{R\to\infty} \frac{(R-L)^N}{R^N} \int_{([-L,L]^N)^c} |\mathcal{F} f(y)| \,\mathrm{d}y + \limsup_{R\to\infty} \frac{NLR^{N-1}}{R^N} \|\mathcal{F} f\|_{L^1} \Big] \\
&\leq \frac{1}{(2\pi)^N} \int_{([-L,L]^N)^c} |\mathcal{F} f(y)| \,\mathrm{d}y
\end{aligned}
$$

Since $L > 0$ was arbitrary, and since

$$
\lim_{L\to\infty} \int_{([-L,L]^N)^c} |\mathcal{F} f(y)| \,\mathrm{d}y = 0,
$$

we thus obtain

$$
\lim_{R\to\infty} g_R(x) = \frac{1}{(2\pi)^N} \bar{\mathcal{F}}(\mathcal{F} f)(x) \text{ for every } x \in \mathbb{R}^N.
$$

Combining this with the first inversion formula, we obtain the first identity. The second identity is proved similarly.

Corollary 4.21. *The Fourier transforms* $\mathcal{F}, \bar{\mathcal{F}} : L^1(\mathbb{R}^N) \to \mathrm{C}_0(\mathbb{R}^N)$ *are injective.*

Remark 4.22. The Fourier transform $\mathcal{F}$ on L^1 is not surjective onto C_0.

Lemma 4.23 (Fourier transform and convolution). *For every* $f, g \in L^1(\mathbb{R}^N)$ *one has*

$$
\begin{aligned}
\mathcal{F}(f * g) &= \mathcal{F} f \mathcal{F} g \text{ and} \\
\bar{\mathcal{F}}(f * g) &= \bar{\mathcal{F}} f \bar{\mathcal{F}} g.
\end{aligned}
$$

Proof. For every $x \in \mathbb{R}^N$ we compute, using Fubini's theorem,

$$\begin{aligned}\mathcal{F}(f*g)(x) &= \int_{\mathbb{R}^N} e^{-ixy} f*g(y)\,\mathrm{d}y\\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-ixy} f(y-z)g(z)\,\mathrm{d}y\,\mathrm{d}z\\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-ix(y+z)} f(y)\,\mathrm{d}y\,g(z)\,\mathrm{d}z\\ &= \int_{\mathbb{R}^N} e^{-ixy} f(y)\,\mathrm{d}y \int_{\mathbb{R}^N} e^{-ixz} g(z)\,\mathrm{d}z\\ &= \mathcal{F}f(x)\mathcal{F}g(x).\end{aligned}$$

The second identity is proved similarly.

4.5.2 The Fourier transform on $\mathcal{S}(\mathbb{R}^N)$

Lemma 4.24. *For every $f \in \mathcal{S}(\mathbb{R}^N)$ and every polynomial $p : \mathbb{C}^N \to \mathbb{C}$ one has $\mathcal{F}f$, $\bar{\mathcal{F}}f \in \mathrm{C}^\infty(\mathbb{R}^N)$, and*

$$\begin{aligned}\mathcal{F}(p(\partial)f) &= p(i\cdot)\mathcal{F}f,\\ \mathcal{F}(p(-i\cdot)f) &= p(\partial)\mathcal{F}f,\\ \bar{\mathcal{F}}(p(\partial)f) &= p(-i\cdot)\bar{\mathcal{F}}f, \text{ and}\\ \bar{\mathcal{F}}(p(i\cdot)f) &= p(\partial)\bar{\mathcal{F}}f.\end{aligned}$$

Proof. Let $f \in \mathcal{S}(\mathbb{R}^N)$ and $k \in \{1,\ldots,N\}$. Then

$$\begin{aligned}\mathcal{F}(-i\cdot_k f)(x) &= -\int_{\mathbb{R}^N} e^{-ixy} iy_k f(y)\,\mathrm{d}y\\ &= \int_{\mathbb{R}^N} \frac{\partial}{\partial x_k}(e^{-ixy}) f(y)\,\mathrm{d}y\\ &= \partial_k \int_{\mathbb{R}^N} e^{-ixy} f(y)\,\mathrm{d}y\\ &= \partial_k \mathcal{F}f(x).\end{aligned}$$

Moreover, by an integration by parts,

$$\begin{aligned}
\mathcal{F}(\partial_k f)(x) &= \int_{\mathbb{R}^N} e^{-ixy} \frac{\partial}{\partial y_k} f(y) \, \mathrm{d}y \\
&= -\int_{\mathbb{R}^N} \frac{\partial}{\partial y_k} (e^{-ixy}) f(y) \, \mathrm{d}y \\
&= -ix_k \int_{\mathbb{R}^N} e^{-ixy} f(y) \, \mathrm{d}y \\
&= -ix_k \mathcal{F} f(x).
\end{aligned}$$

The first two equalities follow from these two identities and by induction. The proofs for the adjoint Fourier transform $\bar{\mathcal{F}}$ are similar.

Theorem 4.25. *For every $f \in \mathcal{S}(\mathbb{R}^N)$ one has $\mathcal{F} f, \bar{\mathcal{F}} f \in \mathcal{S}(\mathbb{R}^N)$ and the Fourier transforms $\mathcal{F}, \bar{\mathcal{F}} : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N; X)$ are linear, continuous isomorphisms.*

Proof. The statement essentially follows from the preceding Lemma 4.24.

4.5.3 The Fourier transform on L^2

Theorem 4.26 (Parseval's identity).

a) *For every $T \in L^1(\mathbb{R}^N; \mathcal{L}(X, Y))$ and every $f \in L^1(\mathbb{R}^N; X)$ one has*

$$\int_{\mathbb{R}^N} \mathcal{F} T(x) f(x) \, \mathrm{d}x = \int_{\mathbb{R}^N} T(x) \mathcal{F} f(x) \, \mathrm{d}x.$$

b) *For every $f, g \in L^1(\mathbb{R}^N)$ one has*

$$\int_{\mathbb{R}^N} \mathcal{F} f(x) \bar{g}(x) \, \mathrm{d}x = \int_{\mathbb{R}^N} f(x) \overline{\bar{\mathcal{F}} g(x)} \, \mathrm{d}x.$$

c) *For every $f, g \in L^1(\mathbb{R}^N)$ such that $\mathcal{F} f, \mathcal{F} g \in L^1(\mathbb{R}^N)$ one has*

$$\int_{\mathbb{R}^N} f(x) \bar{g}(x) \, \mathrm{d}x = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \mathcal{F} f(x) \overline{\mathcal{F} g(x)} \, \mathrm{d}x.$$

Similar identities hold if we replace everywhere $\mathcal{F}$ by $\bar{\mathcal{F}}$ and vice versa.

Proof. (a) We calculate, using Fubini's theorem,

$$\begin{aligned}
\int_{\mathbb{R}^N} \mathcal{F} T(x) f(x) \, \mathrm{d}x &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{-ixy} T(y) \, \mathrm{d}y \, f(x) \, \mathrm{d}x \\
&= \int_{\mathbb{R}^N} T(y) \int_{\mathbb{R}^N} e^{-ixy} f(x) \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{\mathbb{R}^N} T(y) \mathcal{F} f(y) \, \mathrm{d}y.
\end{aligned}$$

(b) is proved in a similar way and (c) follows from (b) by using the Inversion Formula II (Corollary 4.20).

Theorem 4.27 (Plancherel). *The Fourier transforms* $\mathcal{F}, \bar{\mathcal{F}} : C_c^\infty(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ *extend uniquely to bounded, linear operators on* $L^2(\mathbb{R}^N)$*. The operators* $\frac{1}{\sqrt{2\pi}^N}\mathcal{F}$, $\frac{1}{\sqrt{2\pi}^N}\bar{\mathcal{F}} : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ *are unitary and*

$$(\frac{1}{\sqrt{2\pi}^N}\mathcal{F})^* = \frac{1}{\sqrt{2\pi}^N}\bar{\mathcal{F}}.$$

Proof. From Parseval's identity (Theorem 4.26 (c)) we obtain, that for every $f, g \in C_c^\infty(\mathbb{R}^N)$

$$\langle \mathcal{F} f, \mathcal{F} g\rangle_{L^2} = (2\pi)^N \langle f, g\rangle_{L^2},$$

and in particular,

$$\|\mathcal{F} f\|_{L^2}^2 = (2\pi)^N \|f\|_{L^2}^2.$$

As a consequence, since $C_c^\infty(\mathbb{R}^N)$ is dense in $L^2(\mathbb{R}^N)$, $\mathcal{F}$ extends in a unique way to a bounded, linear operator on $L^2(\mathbb{R}^N)$. Moreover, we see from the above equality that $\frac{1}{\sqrt{2\pi}^N}\mathcal{F}$ is isometric. As a consequence, this operator is injective and has closed range. However, from the inversion formula we see that $C_c^\infty(\mathbb{R}^N)$ is contained in the range. Hence, $\frac{1}{\sqrt{2\pi}^N}\mathcal{F}$ is surjective, and thus unitary.

The arguments for $\bar{\mathcal{F}}$ are similar.

4.5.4 The Fourier transform on $\mathcal{S}(\mathbb{R}^N)'$

4.6 The theorem of Malgrange-Ehrenpreis

In this section we state and prove the theorem of Malgrange-Ehrenpreis. The proof given here follows the lines of the proof of H. König and may be found, for example, in [Walter (1994)].

Let $p : \mathbb{C}^N \to \mathbb{C}$ be a complex polynomial of degree $m \in \mathbb{N}$,

$$p(z) = \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| \le m}} a_\alpha z^\alpha \quad (z \in \mathbb{C}^N)$$

with fixed coefficients $a_\alpha \in \mathbb{C}$. Denote by $\dot{p}$ the main part of this polynomial, that is,

$$\dot{p}(z) := \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| = m}} a_\alpha z^\alpha \quad (z \in \mathbb{C}^N)$$

with $\dot{p} \neq 0$.

We call a tempered distribution $T \in \mathcal{S}(\mathbb{R}^N)'$ a **fundamental solution** for the partial differential operator with constant coefficients

$$p(\partial) = \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| \leq m}} a_\alpha \partial^\alpha$$

(which in principle acts on $\mathcal{D}(\Omega)'$ for every open $\Omega \subseteq \mathbb{R}^N$) if

$$p(\partial)T = \delta_0,$$

where δ_0 is the Dirac distribution in 0. Given a fundamental solution for the differential operator $p(\partial)$, one may solve the partial differential equation

$$p(\partial)u = f \text{ in } \mathbb{R}^N$$

for given right-hand side $f \in \mathcal{D}(\mathbb{R}^N)$ by putting $u := T * f \in C^\infty(\mathbb{R}^N)$. In fact, for this function u one has

$$p(\partial)u = p(\partial)(T * f) = (p(\partial)T) * f = \delta_0 * f = f.$$

Theorem 4.28 (Malgrange-Ehrenpreis). *For every polynomial $p : \mathbb{C}^N \to \mathbb{C}$ the differential operator $p(\partial)$ admits a fundamental solution.*

Let us note some observations which are useful for the proof of this theorem.

Observation 4.29. For every $a = (a_1, \dots, a_N) \in \mathbb{C}^N$ and every $n \in \mathbb{N}$ one has

$$\left(\sum_{j=1}^N a_j \right)^n = \sum_{|\alpha|=n} \frac{n!}{\alpha!} a^\alpha$$

and therefore

$$\left(1 + \sum_{j=1}^N a_j \right)^n = \sum_{|\alpha| \leq n} \frac{n!}{(n - |\alpha|)!\alpha!} a^\alpha.$$

If $z = (z_1, \dots, z_n) \in \mathbb{C}^N$ and $a_j = z_j \bar{z}_j = |z_j|^2$, then one has

$$(1 + |z|^2)^n = \sum_{|\alpha| \leq n} \frac{n!}{(n - |\alpha|)!\alpha!} z^\alpha \bar{z}^\alpha.$$

Observation 4.30. Let Γ be the unit circle in $\mathbb{C}$, $\Gamma^N \subseteq \mathbb{C}^N$ the N-fold cartesian product. We write

$$\int_\Gamma f(z)\mathrm{d}\tau(z) := \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\,\mathrm{d}\theta.$$

Note that for every $c \in \Gamma$,

$$\int_\Gamma f(cz)\mathrm{d}\tau(z) = \int_\Gamma f(z)\mathrm{d}\tau(z).$$

Accordingly, we define for $f : \Gamma^N \to \mathbb{C}$ the iterated integral $\int_{\Gamma^N} f(z)\,\mathrm{d}\tau_N(z)$. For every pair of multi-indices $\alpha, \beta \in \mathbb{N}_0^N$ one has

$$\int_{\Gamma^N} z^\alpha \bar{z}^\beta\,\mathrm{d}\tau_N(z) = \delta_{\alpha\beta} = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ 1 & \text{if } \alpha = \beta. \end{cases}$$

Let us define the **trace** of the main part of the polynomial p by

$$s(p) = \sum_{|\alpha|=m} |a_\alpha|^2 > 0.$$

One has

$$s(p) = \int_{\Gamma^N} |\dot{p}(z)|^2 \mathrm{d}\tau_N(z),$$

because

$$\begin{aligned} |\dot{p}(z)|^2 &= \dot{p}(z)\,\overline{\dot{p}(z)} \\ &= \sum_{|\alpha|=m}\sum_{|\beta|=m} a_\alpha \overline{a_\beta}\, z^\alpha \bar{z}^\beta. \end{aligned}$$

Observation 4.31. Let $g : \mathbb{C} \to \mathbb{C}$ be holomorphic and let $q : \mathbb{C} \to \mathbb{C}$ be a polynomial, $q(z) = \sum_{k=0}^m c_k z^k$ with $c_m \neq 0$. Then Cauchy's theorem implies

$$g(0)\,\overline{c_m} = \int_\Gamma g(z)\,\overline{q(z)}\,z^m\,\mathrm{d}\tau(z).$$

Indeed, note that for $z \in \Gamma$ one has $\bar{z} = z^{-1}$ and therefore

$$\overline{q(z)}\,z^m = \sum_{k=0}^m \overline{c_k} z^{m-k}.$$

Observation 4.32. Let $f : \mathbb{C}^N \to \mathbb{C}$ be holomorphic. Then, for every $z \in \mathbb{C}^N$,

$$f(z)\,s(p) = \int_{\Gamma^N} f(z+s)\,\overline{p(z+s)}\,\dot{p}(s)\,\mathrm{d}\tau_N(s).$$

In fact, if one fixes $z, s \in \mathbb{C}^N$, and if one sets

$$g(t) := f(z+ts) \quad (t \in \mathbb{C}), \text{ and}$$
$$q(t) := p(z+ts) = \dot{p}(s)t^m + \sum_{j=0}^{m-1} A_j(z,s)\, t^j,$$

where the A_j are independent of t, then $g(0) = f(z)$, $c_m = \dot{p}(s)$, and the preceding observation yields

$$\begin{aligned} f(z)\,|\dot{p}(s)|^2 &= f(z)\overline{\dot{p}(s)}\dot{p}(s) \\ &= \int_\Gamma f(z+ts)\,\overline{p(z+ts)}\,t^m\,\dot{p}(s)\,\mathrm{d}\tau(t) \\ &= \int_\Gamma f(z+ts)\,\overline{p(z+ts)}\,\dot{p}(ts)\,\mathrm{d}\tau(t). \end{aligned}$$

Integrating both sides over $s \in \Gamma^N$ yields

$$\begin{aligned} f(z)\,s(p) &= \int_\Gamma \int_{\Gamma^N} f(z+ts)\overline{p(z+ts)}\dot{p}(ts)\,\mathrm{d}\tau_N(s)\,\mathrm{d}\tau(t) \\ &= \int_{\Gamma^N} f(z+s)\overline{p(z+s)}\dot{p}(s)\,\mathrm{d}\tau_N(s). \end{aligned}$$

Proof (of Theorem 4.28). Let us define the function $\langle\cdot\rangle : \mathbb{C} \to \mathbb{C}$ by

$$\langle t \rangle := \begin{cases} \frac{\bar{t}}{t} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Let $u \in \mathcal{D}(\mathbb{R}^N)$ be a test function. Then its Fourier transform $\mathcal{F}u : \mathbb{R}^N \to \mathbb{C}$ extends to a holomorphic function $\mathbb{C}^N \to \mathbb{C}$ which we also denote by $\mathcal{F}u$. The inversion formula for the Fourier transform yields

$$u(0) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \mathcal{F}u(x)\,\mathrm{d}x,$$

and therefore

$$\begin{aligned}(2\pi)^N s(p)u(0) &= \int_{\mathbb{R}^N} s(p)\mathcal{F}u(x)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N}\int_{\Gamma^N} \mathcal{F}u(x+s)\overline{p(x+s)}\,\dot{p}(s)\,\mathrm{d}\tau_N(s)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N}\int_{\Gamma^N} p(x+s)\mathcal{F}u(x+s)\,\langle p(x+s)\rangle\,\dot{p}(s)\,\mathrm{d}\tau_N(s)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N}\int_{\Gamma^N} \mathcal{F}(p(-i\partial)u)(x+s)\,\langle p(x+s)\rangle\,\dot{p}(s)\,\mathrm{d}\tau_N(s)\,\mathrm{d}x.\end{aligned}$$

Setting $v = p(-i\partial)u$, we have

$$\begin{aligned}(1+|z|^2)^n\mathcal{F}v(z) &= \sum_{|\alpha|\le n} c_\alpha\langle z^\alpha\rangle z^{2\alpha}\mathcal{F}v(z)\\ &= \sum_{|\alpha|\le n} c_\alpha\langle z^\alpha\rangle\mathcal{F}(\partial^{2\alpha}v)(z)(-1)^{|\alpha|},\end{aligned}$$

where $c_\alpha := \frac{n!}{(n-|\alpha|)!\alpha!}$. Hence,

$$(2\pi)^N s(p)u(0) = \int_{\mathbb{R}^N}\int_{\Gamma^N}\sum_{|\alpha|\le n}(-1)^{|\alpha|}c_\alpha\,\frac{\langle (x+s)^\alpha p(x+s)\rangle}{(1+|x+s|^2)^n}\,\dot{p}(s)\mathcal{F}(\partial^{2\alpha}v)(x+s)\,\mathrm{d}\tau_N(s)\,\mathrm{d}x.$$

We take n large enough, so that $\frac{N}{2} < n$. Then the function I_α given by

$$I_\alpha(x,s) := (-1)^{|\alpha|}c_\alpha\,\frac{\langle (x+s)^\alpha p(x+s)\rangle}{(1+|x+s|^2)^n}\,\dot{p}(s)$$

is integrable over $\mathbb{R}^N\times\Gamma^N$. By interchanging a sum and some integrals, we obtain

$$\begin{aligned}(2\pi)^N s(p)u(0) &= \int_{\mathbb{R}^N}\int_{\Gamma^N}\sum_{|\alpha|\le n} I_\alpha(x,s)\mathcal{F}(\partial^{2\alpha}v)(x+s)\,\mathrm{d}\tau_N(s)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N}\int_{\Gamma^N}\sum_{|\alpha|\le n} I_\alpha(x,s)\int_{\mathbb{R}^N} e^{-i(x+s)y}\,\partial^{2\alpha}v(y)\,\mathrm{d}y\,\mathrm{d}\tau_N(s)\,\mathrm{d}x\\ &= \int_{\mathbb{R}^N}\left[\sum_{|\alpha|\le n}\int_{\mathbb{R}^N}\int_{\Gamma^N} I_\alpha(x,s)e^{-i(x+s)y}\,\mathrm{d}\tau_N(s)\,\mathrm{d}x\right]\partial^{2\alpha}v(y)\,\mathrm{d}y.\end{aligned}$$

Setting

$$K_p(y) := \sum_{|\alpha|\le n}\int_{\mathbb{R}^N}\int_{\Gamma^N} I_\alpha(x,s)e^{-i(x+s)y}\,\mathrm{d}\tau_N(s)\,\mathrm{d}x \quad (y\in\mathbb{R}^N),$$

we have $K_p \in L^1_{loc}(\mathbb{R}^N)$. Now it suffices to put

$$T := \frac{1}{s(p)\,(2\pi)^N} \sum_{|\alpha| \le m} \partial^{2\alpha} K_p,$$

and one obtains

$$\langle p(i\partial)T, u\rangle = \langle T, p(-i\partial)u\rangle = \langle T, v\rangle = u(0),$$

or, in different notation,

$$p(i\partial)T = \delta_0.$$

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