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Abstract

In the description of economic models by Arrow and Hahn the existence of a market equilibrium is proved under the assumption of continuity of the excess demand function in this model. This assumption is replaced by the w-discontinuity which yields to an extension of the class of mathematical models to economies of such kind. There are studied some properties of w-discontinuous mappings, and based on them, for a new economic model the existence of a certain equilibrium is proved.

Keywords: Metric space, discontinuity, fixed point theorem, economic models, supply, demand, excess demand, market equilibrium

MSC 2000: primary 91B24, 91B42, 91B50, 26A15, secondary 47H10

1 Introduction

The classical microeconomic models have their origins mainly in the work of L. Walras [17], (1954), a wider discussion of them is presented by K. J. Arrow

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and G. Debreu [3], (1954) and also by K. J. Arrow and F. H. Hahn [4], (1971) (we make use of the last one). An extended description of the classical model can also be found in textbooks on microeconomics, for example, H. Varian [16], (1990), D. M. Kreps [12], (1990) or W. Nicholson [15], (1992). For a strictly functional-analytic approach we refer to the book of C. D. Aliprantis, D. J. Brown and O. Burkinshaw [2], (1990).

One of the basic assumptions in mathematical modelling of the standard economic model is the continuity of the excess demand function involved. There are reasons to maintain that the necessity of this assumption is caused by the methods provided by mathematics. First of all the fixed points theorems of Brouwer and Kakutani have to be mentioned, since both require the continuity of the maps. They are the main tools for establishing the existence of an equilibrium. However, the necessity of the assumption of continuity has also some economic motivation: in a neoclassical exchange economy due to the strict convexity and strict monotony of the preferences of all consumers the excess demand function is continuous (see [2], Th.1.4.4). In fact this is a different assumption about the behavior of consumers. The paper offers a possibility to substitute the continuity of the excess demand function by the w-discontinuity of this function and therefore to deal, in some extend, with unstable economies. We will examine the properties of w-discontinuous mappings and finally, under some additional conditions, we prove the existence of a generalized equilibrium. The scheme of the proof is traditional, however it is worth to mention that the Walras' Law is not supposed in this type of economy. Another assumption is needed (Assumption 3') instead. The proposal on the Walras' Law follows from the hypothesis that all consumers and all producers (or households and firms) act in a maximal rational way by taking into consideration their budget constraints. But the maximal rational way is possible only when each consumer and each producer is thoroughly familiar with the price system of all the goods. In the real situation even on the scale of one small state (for example, Latvia), this is not possible.

2 w-discontinuous mappings and their properties

A class of mappings between metric spaces which are moderately discontinuous is defined as follows. Let (X, d) and (Y, ϱ) be two metric spaces and ω

a positive number.

Definition 1 *Let* $f\colon \mathrm{dom} f \longrightarrow Y$*, where* $\mathrm{dom} f \subseteq X$*. Let* $x_0 \in \mathrm{dom} f$ *and* $w > 0$*. A positive number* δ *is called an* ω-**tolerance** *of the map* f *at the point* x_0*, if any point* $x \in \mathrm{dom} f$ *which satisfies the condition* $d(x, x_0) < \delta$ *satisfies also the inequality* $\varrho(f(x), f(x_0)) < \omega$*.*

Denote the set of all ω-tolerances of f at x_0 by $\mathcal{T}(f, x_0, \omega)$. A number $\delta > 0$ belongs to $\mathcal{T}(f, x_0, \omega)$ if

$$x \in \mathrm{dom}(f) \cap B(x_0; \delta) \Longrightarrow f(x) \in B(f(x_0); \omega),$$

where $B(x; r)$ denotes the open ball in a metric space centered at the point x with radius r.
One has immediately the following properties of ω-tolerances[1] where, in what follows, $\omega, \omega_i (i \in \{1, \ldots, k\}, k \in \mathbb{N})$ are supposed to be positive numbers and intersections of domains are assumed to be nonempty.

(i) $\omega_1 < \omega$ implies $\mathcal{T}(f, x_0, \omega_1) \subset \mathcal{T}(f, x_0, \omega)$

(ii) $0 < \delta_1 < \delta$ and $\delta \in \mathcal{T}(f, x_0, \omega)$ imply $\delta_1 \in \mathcal{T}(f, x_0, \omega)$,

(iii) Let $f_1, \ldots, f_k$ be a finite number of mappings and let x_0 belong to $\bigcap_{i=1}^{k} \mathrm{dom} f_i$.
If $\delta_i \in \mathcal{T}(f_i, x_0, \omega_i)$ $i = 1, \ldots, k$, then

$$\delta := \min_{1 \le i \le k} \delta_i \in \mathcal{T}(f_i, x_0, \omega_1 + \cdots + \omega_k) \quad i = 1, \ldots, k.$$

(iv) Let Y be a real normed space with the norm $\|\cdot\|_Y$. Consider for $f_i\colon X \longrightarrow Y$, $i = 1, \ldots, k$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ the linear combination $g = \alpha_1 f_1 + \cdots + \alpha_k f_k$. Let x_0 belong to $\bigcap_{i=1}^{k} \mathrm{dom} f_i$. Then

$$\delta_i \in \mathcal{T}(f_i, x_0, \omega_i),\ i = 1, \ldots, k \quad \text{imply} \quad \min_{1 \le i \le k} \delta_i \in \mathcal{T}(g, x_0, \sum_{i=1}^{k} |\alpha_i| \omega_i).$$

[1] We adapt some ideas which are used in [13] for continuous functions.

(v) Let be $Y=\mathbb{R}$. For $f\colon X \longrightarrow \mathbb{R}$ define the functions $|f|(x)=|f(x)|$, $f_+(x)=\max\{f(x),0\}$ and[2] $f_-(x)=\max\{-f(x),0\}$. Let x_0 belong to $\operatorname{dom} f$. Then

$$\mathcal{T}(f,x_0,\omega)\subset\mathcal{T}(g,x_0,\omega),$$

where g stands for $|f|, f_+, f_-$.

(vi) For the mappings $f_i\colon \operatorname{dom} f_i \longrightarrow \mathbb{R}$, $\operatorname{dom} f_i \subset X$, $i=1,2$ define the functions $f_1\vee f_2=\max\{f_1,f_2\}$ and $f_1\wedge f_2=\min\{f_1,f_2\}$ pointwise on $\operatorname{dom} f_1\cap\operatorname{dom} f_2$. Let x_0 belong to $\operatorname{dom} f_1\cap\operatorname{dom} f_2$. Then

$$\delta_i\in\mathcal{T}(f_i,x_0,\omega_i),\ i=1,2 \quad \text{imply} \quad \min\{\delta_1,\delta_2\}\in\mathcal{T}(g,x_0,\omega_1+\omega_2),$$

where $g=f_1\vee f_2$ or $g=f_1\wedge f_2$.

(vii) Let (Z,d_Z) be a metric space, $f\colon X\longrightarrow Y$, $g\colon Y\longrightarrow Z$ and $\operatorname{im} f\subset \operatorname{dom} g$. Define $(g\circ f)(x)=g\left(f(x)\right)$, $x\in\operatorname{dom} f$. Let x_0 belong to $\operatorname{dom} f$. If $\sigma\in\mathcal{T}(g,f(x_0),\omega)$ then

$$\mathcal{T}(f,x_0,\sigma)\subset\mathcal{T}(g\circ f,x_0,\omega).$$

We establish only the properties (iv) - (vii), since (i) and (ii) are obvious and (iii) immediately follows from (i) and (ii).

(iv). If $\delta_i\in\mathcal{T}(f_i,x_0,\omega_i)$ and $\delta=\min\{\delta_1,\ldots,\delta_k\}$ then according to (ii) for each $i=1,\ldots,k$ one has $\delta\in\mathcal{T}(f_i,x_0,\omega_i)$. If $x\in X$ and $d(x,x_0)<\delta$ then

$$\| g(x)-g(x_0)\|_Y=\| \sum_{i=1}^k \alpha_i f_i(x)-\sum_{i=1}^k \alpha_i f(x_0)\|_Y=\|\sum_{i=1}^k \alpha_i\left(f_i(x)-f_i(x_0)\right)\|_Y\le \sum_{i=1}^k |\alpha_i|\,\|f_i(x)-f_i(x_0)\|_Y\le\sum_{i=1}^k|\alpha_i|\omega_i. \tag{1}$$

(v). The property (v) for $|f|$ follows from the inequality $\big||a|-|b|\big|\le|a-b|$ for real numbers a,b. The proof of the other parts of (v) makes use of the relations $f_+=\frac{1}{2}(f+|f|)$ and $f_-=\frac{1}{2}(f-|f|)$.

(vi). For the proof use the relations $f_1\vee f_2=\frac{1}{2}(f_1+f_2+|f_1-f_2|)$ and $f_1\wedge f_2=\frac{1}{2}(f_1+f_2-|f_1-f_2|)$. If $\delta=\min\{\delta_1,\delta_2\}$ then by (iv) and (v)

[2] It is sufficient for f to belong to a normed vector lattice of functions defined on X, where $|f|$ is the modul of the element f.

$\delta \in \mathcal{T}(f_1 \pm f_2, x_0, \omega_1 + \omega_2) \subset \mathcal{T}(|f_1 - f_2|, x_0, \omega_1 + \omega_2)$, and so $\delta \in \mathcal{T}(f_1 \vee f_2, x_0, \omega_1 + \omega_2)$ and $\delta \in \mathcal{T}(f_1 \wedge f_2, x_0, \omega_1 + \omega_2)$.
(vii). If $\delta \in \mathcal{T}(f, x_0, \sigma)$ and $x \in \mathrm{dom} f$ satisfies $d(x, x_0) < \delta$ then $\varrho(f(x), f(x_0)) < \sigma$ and, since $\sigma \in \mathcal{T}(g, f(x_0), \omega)$, one has $d_Z\big(g(f(x)), g(f(x_0))\big) < \omega$, i. e. $\delta \in \mathcal{T}(g \circ f, x_0, \omega)$.

Corollary 1 *For f, g and $\alpha \neq 0$ one has*

$$\mathcal{T}(f, x_0, \tfrac{\varepsilon}{2} + w_1) \cap \mathcal{T}(g, x_0, \tfrac{\varepsilon}{2} + w_2) \subset \mathcal{T}(f + g, x_0, \varepsilon + w_1 + w_2) \quad \textit{and}$$

$$\mathcal{T}(f, x_0, \tfrac{\varepsilon}{|\alpha|} + w) \subset \mathcal{T}(\alpha f, x_0, \varepsilon + |\alpha| w).$$

Definition 2 *A mapping $f \colon X \to Y$ is said to be* **w-discontinuous at the point** *$x_0 \in X$ if for every $\varepsilon > 0$ there exists an $\varepsilon + w$-tolerance of f at the point x_0.*

The w-discontinuity of a mapping f at x_0 means that $\mathcal{T}(f, x_0, \varepsilon + w) \neq \emptyset$ for $\forall \varepsilon > 0$, i. e. for each $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $x \in X$ and $d(x, x_0) < \delta$ then $\varrho(f(x), f(x_0)) < \varepsilon + w$. Of course, the constant w may not be the best possible (smallest) one. Very often, especially in economic applications, there is known only a rough upper estimation for the "jump".
A mapping f is called **w-discontinuous in** X if it is w-discontinuous at all points of X.
The notion of w-discontinuous maps is not new. It is already found in [14] as the concept of *oscillation* or in [6] as *continuity defect*. The notion of w-discontinuity (former w-continuity) was introduced by the first author in [5].

Example 1 The usual Dirichlet function on $\mathbb{R}$ and also the generalized Dirichlet function $f : \mathbb{R}^n \to \{0, 1\}$, defined for all $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ by

$$f(x) = \begin{cases} 1, & \text{if all components } x_i \in \mathbb{Q} \\ 0, & \text{if there exists } i_0 \text{ such that } x_{i_0} \in \mathbb{R} \setminus \mathbb{Q} \end{cases},$$

are 1-discontinuous (and consequently, due to (i), for any $\omega \geq 1$ also ω-discontinuous) functions.

Example 2 The number w in the definition 2 may dramatically depend on the value of the function at the point x_0. The functions

$$f_0(x) = \mathrm{sign}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases},$$

and

$$f_1(x) = \begin{cases} f_0(x), & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}, \qquad f_2(x) = \begin{cases} f_0(x), & \text{if } x \neq 0 \\ 2, & \text{if } x = 0 \end{cases}$$

coincide in any neighborhood of 0 but f_0, f_1 are 2- and f_2 is 3-discontinuous at 0.

Proposition 1 *If f is continuous at the point x_0 and g is w-discontinuous at the point $f(x_0)$ then $g \circ f$ is w-discontinuous at x_0.*

Proof. Indeed, $\mathcal{T}(f, x_0, \varepsilon) \neq \emptyset$ and $\mathcal{T}(g, f(x_0), \varepsilon + w) \neq \emptyset$ for any $\varepsilon > 0$. If $\sigma \in \mathcal{T}(g, f(x_0), \varepsilon + w)$ then by view of (vii) each $\delta \in \mathcal{T}(f, x_0, \sigma)$ belongs to $\mathcal{T}(g \circ f, x_0, \varepsilon + w)$. That means $x \in \mathrm{dom} f$ and $d(x, x_0) < \delta$ imply $\varrho(f(x), f(x_0)) < \sigma$ and the latter in turn yields $d_Z\big(g(f(x)), g(f(x_0))\big) < \varepsilon + w$. ■

If, conversely, f is w-discontinuous at x_0 and g continuous at $f(x_0)$ then the set $\mathcal{T}(f, x_0, \sigma)$ is not empty only for sufficiently large positive $\sigma \in \mathcal{T}(g, f(x_0), \varepsilon)$. Consider for example $\mathcal{T}(f, 0, \sigma)$, $\sigma \in \mathcal{T}(g, 0, \frac{1}{2})$ with $f(x) = \mathrm{sign}(x)$, $g(x) = x$ and $\sigma = \frac{1}{4}$. In order to apply the property (vii) to the mapping $g \circ f$ the number σ has to satisfy, for example, $\sigma > w$. This, in general, leads to an additional restrictive condition on the function g.

If X, Y, V are real normed vector spaces the following properties of w-discontinuous mappings are established by adapting the methods for continuous mappings.

Proposition 2 *Let be $f_i : X \to Y$, $\alpha_i \in \mathbb{R}$, $i = 1, \ldots, k$ and $g = \alpha_1 f_1 + \cdots + \alpha_k f_k$. Suppose $w_i > 0$ and that f_i is w_i- discontinuous on the set X for each $i = 1, \ldots, k$. Then $g = \alpha_1 f_1 + \cdots + \alpha_k f_k$ is a $|\alpha_1| w_1 + \cdots + |\alpha_k| w_k$-discontinuous mapping.*

Proof. The statement is an immediate consequence of the next property, which can be obtained from (iv): If $\varepsilon > 0$ and $\sigma = \dfrac{\varepsilon}{1 + \sum_{i=1}^k |\alpha_i|}$, then

$$\delta_i \in \mathcal{T}(f_i, x_0, \sigma + w_i),\ i = 1, \ldots, k \quad \text{implies} \quad \min_{1 \le i \le k} \delta_i \in \mathcal{T}(g, x_0, \varepsilon + \sum_{i=1}^k |\alpha_i| w_i).$$

Indeed, let be ε, σ and δ_i as above and put $\delta = \min\{\delta_1, \ldots, \delta_k\}$. Then according to (ii) one has $\delta \in \mathcal{T}(f_i, x_0, \sigma + w_i)$ for each $i = 1, \ldots, k$. If $x \in X$

and $d(x,x_0)<\delta$ then by slightly altering the estimation (1) one gets

$$\| g(x)-g(x_0) \|_Y = \| \sum_{i=1}^{k} \alpha_i \left(f_i(x)-f_i(x_0)\right) \|_Y \leq \sum_{i=1}^{k} |\alpha_i| \, \|f_i(x)-f_i(x_0)\|_Y \leq$$
$$\sum_{i=1}^{k} |\alpha_i|(\sigma+w_i) < \sigma(1+\sum_{i=1}^{k}|\alpha_i|)+\sum_{i=1}^{k}|\alpha_i|w_i = \varepsilon+\sum_{i=1}^{k}|\alpha_i|w_i.$$

■

From the Definition 2, which makes sense also for $w=0$, immediately follows that the 0-discontinuous mappings are exactly the continuous ones.

Corollary 2 *Suppose that $f,g: X\to Y$, f is w'- discontinuous and g is w''-discontinuous. Then $f+g$ and $f-g$ are $w'+w''$- discontinuous mappings. In particular, if one of the mappings, f. e. g, is continuous, then $f\pm g$ are w'- discontinuous.*

Corollary 3 *If $f\colon X\to Y$ is w- discontinuous and c is a constant then $c\cdot f$ is a $|c|w$- discontinuous mapping.*

Proposition 3 *Let $f\colon \mathrm{dom} f \longrightarrow \mathbb{R}$ and $g\colon \mathrm{dom}\, g \longrightarrow \mathbb{R}$ be w'-, w''-discontinuous functions, respectively. Then the functions $f\wedge g$ and $f\vee g$ are $w'+w''$-discontinuous on $\mathrm{dom} f\cap \mathrm{dom}\, g$.*

Proof. If $\delta_1 \in \mathcal{T}(f,x_0,\omega_1)$ and $\delta_2 \in \mathcal{T}(g,x_0,\omega_i)$ then by means of (vi) $\min\{\delta_1,\delta_2\} \in \mathcal{T}(f\vee g,x_0,\omega_1+\omega_2)$. The case of $f\wedge g$ is proved in the same way. ■

Corollary 4 *If f is w- discontinuous and g is continuous then $f\vee g$ is w-discontinuous.*

In order to consider the product of mappings we need the notation of the product in a normed space.

Definition 3 ([11]) *Let X,Y,Z be real normed vector spaces. A mapping $\pi\colon X\times Y\to Z$ is called a* **product** *if it satisfies the following conditions: for all $a,b\in X$, $u,v\in Y$ and $\lambda\in\mathbb{R}$ one has*

1. $\pi((a+b,v))=\pi((a,v))+\pi((b,v))$
2. $\pi((a,u+v))=\pi((a,u))+\pi((a,v))$

3. $\pi((\lambda a, u)) = \lambda\pi((a,u)) = \pi((a, \lambda u))$

4. $\|\pi((a,u))\|_Z \le \|a\|_X \|u\|_Y$.

A simple example is given by $X = Y = \mathbb{R}^n$, $Z = \mathbb{R}$ and $\pi((x,y)) = \langle x, y\rangle$ – the scalar product in $\mathbb{R}^n$, i.e. $\langle x, y\rangle = \sum_{i=1}^{n} x_i\, y_i$.
Let V, X, Y, Z be real normed vector spaces and let $\pi \colon X \times Y \to Z$ be a product. The product of the mappings $f\colon \mathrm{dom} f \subseteq V \to X$ and $g\colon \mathrm{dom}\, g \subseteq V \to Y$ is understood pointwise, i.e.

$$(f \cdot g)(v) = \pi\big(f(v), g(v)\big), \quad \forall v \in \mathrm{dom} f \cap \mathrm{dom}\, g,$$

where $\mathrm{dom} f$, $\mathrm{dom}\, g \subseteq V$.

Proposition 4 *Suppose that* $f\colon \mathrm{dom} f \to X$ *is* w'*-discontinuous and* $g\colon \mathrm{dom}\, g \to Y$ *is* w''*-discontinuous on* $\mathrm{dom} f \cap \mathrm{dom}\, g$*. Then* $f \cdot g$ *is a* $(w'w'' + w'\|g(x_0)\|_Y + w''\|f(x_0)\|_X)$*-discontinuous mapping at every point* $x_0 \in \mathrm{dom} f \cap \mathrm{dom}\, g$.

Proof. We choose $x_0 \in V$ and put $p = w' + w'' + \|f(x_0)\|_X + \|g(x_0)\|_Y$. For any $\varepsilon > 0$ the quadratic function

$$y(t) = t^2 + pt - \varepsilon \tag{2}$$

possesses the positive root $\varepsilon' = \frac{1}{2}(\sqrt{p^2 + 4\varepsilon} - p)$. Denote by $\delta = \min\{\delta_1, \delta_2\}$, where δ_1 is an $\varepsilon' + w'$-tolerance of f and δ_2 an $\varepsilon' + w''$- tolerance of g both at the point x_0. Then $x \in V$ and $\| x - x_0 \|_V < \delta$ imply

$$\begin{aligned}
&\|\pi\big((f(x), g(x))\big) - \pi\big((f(x_0), g(x_0))\big)\|_Z = \\
&= \|\pi\big((f(x), g(x))\big) - \pi\big((f(x_0), g(x))\big) + \pi\big((f(x_0), g(x))\big) - \pi\big((f(x_0), g(x_0))\big)\|_Z \le \\
&\le \|\pi\big((f(x), g(x))\big) - \pi\big((f(x_0), g(x))\big)\|_Z + \|\pi\big((f(x_0), g(x))\big) - \pi\big((f(x_0), g(x_0))\big)\|_Z = \\
&= \|\pi\big((f(x) - f(x_0), g(x))\big)\|_Z + \|\pi\big((f(x_0), g(x) - g(x_0))\big)\|_Z \le \\
&\le \|\pi\big((f(x) - f(x_0), g(x) - g(x_0))\big)\|_Z + \|\pi\big((f(x) - f(x_0), g(x_0))\big)\|_Z + \\
&\quad \|\pi\big((f(x_0), g(x) - g(x_0))\big)\|_Z \le \\
&\le \|f(x) - f(x_0)\|_X \, \|g(x) - g(x_0)\|_Y + \|f(x) - f(x_0)\|_X \, \|g(x_0)\|_Y + \\
&\quad \|f(x_0)\|_X \, \|g(x) - g(x_0)\|_Y < \\
&< (\varepsilon' + w')(\varepsilon' + w'') + (\varepsilon' + w')\|g(x_0)\|_Y + (\varepsilon' + w'')\|f(x_0)\|_X) = \\
&= (\varepsilon')^2 + \varepsilon'(w' + w'' + \|g(x_0)\|_Y + \|f(x_0)\|_X) + w'w'' + w'\|g(x_0)\|_Y + w''\|f(x_0)\|_X.
\end{aligned}$$

Since ε' is a root of the equation (2) one has

$$\varepsilon = (\varepsilon')^2 + p\varepsilon', \quad \text{i. e.} \quad (\varepsilon')^2 + \varepsilon'(w' + w'' + \|g(x_0)\|_Y + \|f(x_0)\|_X) = \varepsilon,$$

and so

$$\|\pi((f(x), g(x))) - \pi((f(x_0), g(x_0)))\|_Z < \varepsilon + w'w'' + w'\|g(x_0)\|_Y + w''\|f(x_0)\|_X.$$

■

Corollary 5 *If $f : V \to X$ is w-discontinuous and $g : V \to Y$ is continuous then $f \cdot g$ is a $\|g(x_0)\|_Y\, w$- discontinuous mapping at every point $x_0 \in V$.*

For the division we reconcile with simplified situation, where (X, d) is again a metric space.

Proposition 5 *Let the function $f \colon X \to \mathbb{R}$ be w-discontinuous at the point x_0 and $f(x_0) \neq 0$. If there exists a neighborhood U of x_0 and a number $\alpha_0 > 0$ such that $|f(x)| \geq \alpha_0$ for all $x \in U$ then the function $\dfrac{1}{f}$ is $\dfrac{w}{\alpha_0|f(x_0)|}$-discontinuous at x_0.*

Proof. For $\varepsilon > 0$ put $\varepsilon' = \varepsilon\,\alpha_0|f(x_0)|$. By the w-discontinuity of f there exists $\delta > 0$ such that $x \in U$ and $d(x, x_0) < \delta$ implies $|f(x) - f(x_0)| < \varepsilon' + w$. Then

$$\left|\frac{1}{f(x)} - \frac{1}{f(x_0)}\right| = \frac{|f(x_0) - f(x)|}{|f(x)f(x_0)|} < \frac{\varepsilon' + w}{|f(x)||f(x_0)|} \leq$$
$$\leq \frac{\varepsilon' + w}{\alpha_0|f(x_0)|} = \frac{\varepsilon\,\alpha_0|f(x_0)| + w}{\alpha_0|f(x_0)|} = \varepsilon + \frac{w}{\alpha_0|f(x_0)|}.$$

■

As a special case we get

Corollary 6 *If $f : X \to [1, +\infty[$ is w-discontinuous then $\dfrac{1}{f}$ is a $\dfrac{w}{f(x_0)}$-discontinuous mapping for every point $x_0 \in X$.*

If the domain of definition for a continuous mapping is compact, then its range is also compact and, in particular, bounded. The boundedness of the most functions used in economic models seems to be indispenable in studying such models. The boundedness of the range is guaranteed for w-discontinuous mappings as well, however, compactness may not hold.

Example 3 Define $f\colon [0;1] \to [0;1]$ as

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if} \quad x \in \{0,1\} \\ x, & \text{if} \quad x \in (0,1). \end{cases}$$

The function f is $\frac{1}{2}$-discontinuous and its range $(0,1)$ is bounded, but not compact.

Further on we need some notations introduced by the following definitions. Let (X,d) and (Y,ϱ) be two metric spaces and w a positive number.

Definition 4 *A mapping* $f\colon \mathrm{dom} f \longrightarrow Y$, $\mathrm{dom} f \subseteq X$ *is said to be* **uniformly w-discontinuous** *if for every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that for every two points* $x, y \in \mathrm{dom} f$ *the inequality* $d(x,y) < \delta$ *implies* $\varrho(f(x), f(y)) < \varepsilon + w$.

Definition 5 *Let* A *be a subset of* X, $\mathrm{dom} f \subseteq A$ *and* $f\colon \mathrm{dom} f \longrightarrow Y$. *A mapping* $g\colon A \to Y$ *is said to be a* μ**- approximation** *(*$\mu > 0$*) of the map* f *on* $\mathrm{dom} f$ *if*

$$\varrho(f(x), g(x)) \leq \mu \quad \forall x \in \mathrm{dom} f.$$

The following theorems are proved in [5].

Theorem 1 *If* A *is a compact subset of a normed vector space* X *and* $f\colon A \longrightarrow Y$ *is* w*-discontinuous, then* f *is uniformly* $2w$*-discontinuous.*

Now let X and Y be a real normed vector spaces.

Theorem 2 *Suppose that* X *is a normed vector space,* $A \subset X$ *is compact,* $f : A \longrightarrow Y$ *is uniformly* w*-discontinuous and* w' *is an arbitrary number* $w' > w$. *Then there exists a continuous* w'*-approximation* $\overline{f}$ *for* f *in* A *such that* $\overline{f}(x) \in \mathrm{conv}(f(A))$, $x \in A$, *where* $\mathrm{conv}(f(A))$ *denotes the convex hull of the set* $f(A)$.

Now we are able to prove the next theorem.

Theorem 3 *Suppose that* $A \subset X$ *is compact and let* $f\colon A \longrightarrow X$ *be* w*-discontinuous. Then* $f(A)$ *is bounded.*

Proof. According to Theorem 1 the mapping f is uniformly $2w$-discontinuous, and by Theorem 2 for every $w' > w$ there exists a continuous $2w'$-approximation $\overline{f}$ of the mapping f on the set A. Since A is compact and $\overline{f}$ is continuous the set $\overline{f}(A)$ is compact and, consequently, bounded. Therefore, for some $r > 0$ and $x_0 \in \overline{f}(A)$ one has $\overline{f}(A) \subset B(x_0; r)$. Because $\overline{f}$ is a $2w'$-approximation of f on A there holds the inequality

$$d(\overline{f}(x), f(x)) \leq 2w', \ \forall x \in A.$$

It follows that $f(A) \subset B(x_0; r + 2w')$, i. e. $f(A)$ is bounded. ■

The following essential result is proved by O.Zaytsev in [18] and can be considered as a generalization of the Bohl-Brouwer-Schauder fixed point theorem (see [9]) for w-discontinuous mappings.

Theorem 4 *Let K be a nonempty, compact and convex subset in a normed vector space X. For every w-discontinuous mapping $f : K \longrightarrow K$ ($w > 0$) there exists a point $x^* \in K$ such that $\| x^* - f(x^*) \| \leq w$.*

3 Market equilibrium of the standard economic model

We give the description of a simple economic model $\mathcal{E}$ considered by Arrow and Hahn in [4].
Let there be n ($n \in \mathbb{N}$) different goods (commodities) on the market: services and wares, and a finite number of economic agents: households and firms, where each household can be considered as a firm, and, vice versa, each firm can be considered as a hosehold.
Let x_{hi} be the quantity of good i which is needed to the household h. If $x_{hi} < 0$ then $|x_{hi}|$ denotes the quantity of good i which is supplied by the household h. If $x_{hi} \geq 0$ then x_{hi} is the (real) demand of good i by h, including the zero demand. The summation over all households will be indicated by $x_i = \sum_h x_{hi}$ and is the total demand of good i, $i = 1, \dots, n$.
The quantity of good i that is supplied by the firm f will be denoted by y_{fi}. Again, if $y_{fi} < 0$ then $|y_{fi}|$ is the demand (input) of good i by f. If $y_{fi} \geq 0$ then y_{fi} is the supplied quantity (output) of i by f, where the zero

supply again is included. The summation over all firms will be indicated by $y_i = \sum_f y_{fi}$ and is the supply of good i, $i = 1, \dots, n$.

The initially available amount (or resources) of good i in all households will be denoted by $\overline{x_i}$. Note that $\overline{x_i}$ must be non-negative.

A market equilibrium, which is one of the most important characteristics of any economy (see e.g. [2], [4], [7], [10]), describes (in our situation) the economic situation that the total demand of each good in the economy is satisfied by its total supply. This fact is obviously expressed by saying that the difference between the total demand of each good and its total supply is less than or equal to zero. The total supply of good i is understood as the sum of the supply of the good i and the quantity of i which is already available, i. e. the total supply of the good i equals to $y_i + \overline{x_i}$. The excess demand of good i is then defined as $x_i - y_i - \overline{x_i}$, $i = 1, ..., n$.

If economic agents at the market are faced with a system of prices, i.e. with a price vector $p = (p_1, \dots, p_n)$, where p_i is the price of one unit of good i, then the quantities x_{hi}, y_{fi} and also $x_i, y_i, \overline{x_i}$ depend on p. Now we denote the excess demand of the good i by $z_i(p)$, i.e.

$$z_i(p) = x_i(p) - \Big(y_i(p) + \overline{x_i}(p)\Big).$$

If prices are involved then an equilibrium price (a price system at which an equilibrium is reached) clears the market.

Further on we frequently make use of the natural order in $\mathbb{R}^n$ introduced by the cone

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \dots, n\},$$

i. e., for two vectors $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ and write $x \leq y$ iff $x_i \leq y_i$ for all $i = 1, \dots, n$, we write $x < y$ iff $x \leq y$ and $x_{i_0} < y_{i_0}$ for at least one index i_0. The zero vector $(0, \dots, 0) \in \mathbb{R}^n$ is denoted by $\mathbf{0}$. The norm we will use in the space $\mathbb{R}^n$ is defined as

$$\|x\| = \sum_{i=1}^{n} |x_i|,\ x = (x_1, ..., x_n) \in \mathbb{R}^n.$$

This norm is equivalent to the Euclidean norm which is introduced by means of the scalar product $\langle x, y\rangle = \sum_{i=1}^{n} x_i\, y_i$. Note that in economic publications the scalar product of two vectors $x, y \in \mathbb{R}^n$ is usually written as $x\, y$.

For the standard economic model the following four assumptions[3] have to be met.

Assumption 1. Let $p = (p_1, ..., p_n)$ be n-dimensional price vector with the prices p_i for one unit the good i as components, $i = 1, 2, ..., n$. For any p let the excess demand for i be characterized by a unique number $z_i(p)$ and so the unique vector $z(p) = (z_1(p), \ldots, z_n(p))$ - the excess demand function with excess demand functions for i as components $(i = 1, 2, ..., n)$ - is well defined.

Assumption 2. $z(p) = z(\lambda p), \quad \forall p > \mathbf{0}$ and $\lambda > 0$.

Assumption 2 asserts that z is a homogeneous vector-function of degree zero. Economically this means that the value of the excess demand function does not depend on the price system if the latter is changed for all the goods simultaneously by the same portion.
From the Assumption 2 it follows that prices can be normalized (see [4],p.20) or [7],p.10). If for some price p one has $z(p) = \mathbf{0}$ then $z(\lambda p) = \mathbf{0}$ for all prices of the ray $\{\lambda p\colon \lambda > 0\}$. Therefore, further on we consider only prices from the $n-1$-dimensional simplex of R^n

$$\Delta_n = \{p = (p_1, p_2, ..., p_n) \mid p_i \geq 0 \ \text{ and } \ \sum_{i=1}^{n} p_i = 1\}.$$

We rule out the situations when all the prices are zero or some of them are negative. Note that Δ_n is a compact and convex set in the space $\mathbb{R}^n$ equipped with one of its (equivalent) norms.

Assumption 3 or Walras' Law. $p\, z(p) = 0, \quad \forall p \in \Delta_n$.

Walras' Law can be regarded as an attempt to have a model sufficiently truly reflecting rationally motivated activities of economic agents. According to Walras' Law all the firms and all the households both spend their financial resources completely ([7]).

Assumption 4. The excess demand function z is continuous on its domain of definition Δ_n.

It means that a small change of a price system will imply only a small change in the excess demand. As a consequence from continuity of z, the standard

[3]the Assumptions (F), (H), (W) and (C) in [4]

model can be used only for the description of economies with continuous excess demand functions. Sometimes they are called stable economies.
In economies such prices are important at which the excess demand for each good is nonpositive, i.e. the total supply of each good satisfies at least its total demand.

Definition 6 *A price* $p^* \in \Delta_n$ *is called an* **equilibrium** *(price) if* $z(p^*) \leq \mathbf{0}$.

If p^* is an equilibrium price then $\sum_{i=1}^{n} z_i(p^*) \leq \mathbf{0}$.

For the standard model of an economy with a finite number of goods and agents such prices always exist as is proved in the following theorem.

Theorem 5 ([4]) *If an economy* $\mathcal{E}$ *with a finite number of goods and agents satisfies the assumptions 1–4, then there exists an equilibrium in* $\mathcal{E}$.

We remark that in the case of a neoclassical exchange economy $\mathcal{E}$ (see e.g.[2]) each agent has his excess demand vector $z_h(p)$, which is uniquely defined by means of the unique maximal element of his preferences in the budget set for the price p and his initial endowment. Then the excess demand function of the economy $\mathcal{E}$ is defined as the vector $z(p) = \sum_h z_h(p)$. It satisfies also Assumptions 2,3,4 and, in addition, also some other conditions (see [2],Th.1.4.6).
This allows to prove the existence of a price p^* even such that $z(p^*) = \mathbf{0}$.

4 Economic models with discontinuous excess demand functions

If z is the excess demand function for a neoclassical exchange economy, then z is continuous on the set

$$S = \{p \in \Delta_n \mid p_i > 0, i = 1, 2, ..., n\}$$

(see [2],Th.1.4.4 and Th.1.4.6). A neoclassical exchange economy (see [2]) is characterized by a finite set of agents, where each agent i has a non-zero initial endowment ω_i and his preference relation $\succeq_i$ is continuous, strictly monotone and strictly convex (on $\mathbb{R}^n_+$) or else his preference relation $\succeq_i$ is continuous, strictly monotone and strictly convex on interior of $\mathbb{R}^n_+$, and everything in the interior is prefered to anything on the boundary and the total endowment $\omega = \sum_i \omega_i$ is strictly positive. If the preference relation $\succeq_i$ is continuous, strictly monotone and strictly convex then the corresponding utility function and the excess demand function are continuous on the set S. We will consider the situation with a discontinuous excess demand function. It is clear that in this case the properties of the preference relations differ from them in the neoclassical exchange economy.

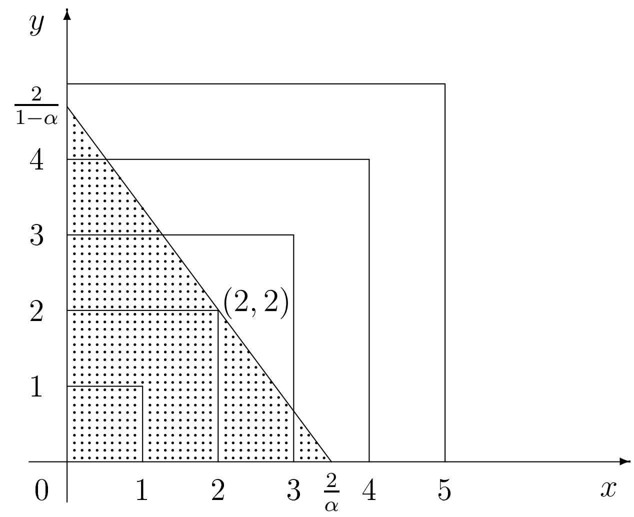

Fig.1.

For example, consider the preference relation on $\mathbb{R}^2_+$ that is represented by the utility function[4] $u(x, y) = \max\{x, y\}$ and an initial endowment $\omega = (2, 2)$. The utility function is continuous, but it is not strictly monotone (for example, $(2, 2) > (2, 1)$ but $u(2, 2) = 2 = u(2, 1)$) and it is not strictly concave, it is convex. The indifference curves for the values 1, 2, 3, 4 and 5 are illustrated in Figure 1. Let $p = (\alpha, 1 - \alpha)$ be a fixed price vector for some $0 < \alpha < 1$. We maximize the utility function u subject to the budget constraint $\alpha x + (1-\alpha)y = 2\alpha + 2(1-\alpha) = 2$. This line goes through the point $(2, 2)$ and intersects the axis in the points $(0, \frac{2}{1-\alpha})$ and $(\frac{2}{\alpha}, 0)$. From Figure 1 we see that the maximal vector of u over budget set (the dotted region in Figure 1) is the point $(0, \frac{2}{1-\alpha})$ if $\alpha > \frac{1}{2}$ and $(\frac{2}{\alpha}, 0)$ if $\alpha < \frac{1}{2}$, respectively. If

[4] i.e. $(x_1, y_1) \succeq (x_2, y_2)$ if and only if $u(x_1, y_1) \geq u(x_2, y_2)$

$\alpha = \frac{1}{2}$ then $\frac{2}{1-\alpha} = \frac{2}{\alpha}$ and therefore we have two maximizing vectors. The excess demand function in this case is

$$x(p) = x(\alpha, 1-\alpha) = \begin{cases} (0, \frac{2}{1-\alpha}), & \alpha > \frac{1}{2}, \\ \{(0,4),(4,0)\}, & \alpha = \frac{1}{2}, \\ (\frac{2}{\alpha}, 0), & \alpha < \frac{1}{2}. \end{cases}$$

In the point $(\frac{1}{2}, \frac{1}{2})$ the excess demand multifunction is discontinuous.
In [1] it is proved that in a neoclassical excange economy the condition $p_n \to p \in \partial S$ with $(p_n)_{n\in\mathbf{N}} \subset S$ implies $\lim_{n\to\infty} ||z(p_n)|| = \infty$. It is also not our case (see Theorem 3). In [1] it is shown that a utility function $u\colon X \to \mathbf{R}$ (X - topological space) representing a continuous preference relation is not necessarily continuous. If we start with an arbitrary chosen discontinuous utility function then we have no mathematical tools for finding the corresponding demand function (in the classical situation an agent maximizes the utility function with respect to the budget constraint and uses the Lagrange multiplier method for finding demand function). We note that there exist preference relations which cannot be represented by a real-valued function, for example, the lexicographic preference ordering of $\mathbf{R}^2$ (by definition $(a,b) \succeq (c,d)$ if (1) $a > c$ or (2) $a = c$ and $b > d$) (see [8], notes to chapt.4).
The above situation inspires one to consider models without explicitly given preference relations. In which cases is the excess demand function discontinuous? Consider some good i and a fixed price system p. In the case that this good is, e.g. an aeroplane or a power station, its demand $x_i(p)$ is naturally an integer. A function like $x_i(p) = \left[\frac{30000}{1+\alpha}\right]$, where $[x]$ denotes the integer part of x, provides an example.
Obviously, if the good is a piece-good (table, shoes, flower and other) then the demand for this good is an integer. Similarly, the supply of piece-goods is an integer. Therefore the demand and supply functions for piece-goods are discontinuous and consequently excess demand function too.

What can be said about the existence of an equilibrium in an economy if the excess demand function is not continuous, for example, if it is w-discontinuous? We will analyse some model of an economy with w-discontinuous excess demand functions.

For the economies under consideration we keep the two first assumptions from the standard model and change the two last as follows.

Assumption 4'. The excess demand function z is w-discontinuous on its domain of definition Δ_n.

The w-discontinuity of the excess demand function makes our model available to describe some properties of an unstable economic as well.
It is quite natural that for every price vector $p \in \Delta_n$ there exist at least one good i with the price $p_i > 0$ and such that the demand for them is satisfied, i. e. $z_i(p) \leq 0$.
If for some economy $\mathcal{E}$ with the excess demand vector $z(p)$, $p \in \Delta_n$ there holds the Walras' Law, i. e. $p\, z(p) = 0$ for any $p \in \Delta_n$, then for each $p \in \Delta_n$ the inequality

$$\gamma_p = \sum_{z_i(p) \leq 0} p_i > 0$$

is satisfied. Indeed, if for some $p = (p_1, \ldots, p_n) \in \Delta_n$ there would be $\sum\limits_{z_i(p) \leq 0} p_i = 0$, then

$$\sum_{z_i(p) \leq 0} p_i + \sum_{z_i(p) > 0} p_i = \sum_{i=1}^{n} p_i = 1$$

would imply the existence of an index i_0 such that $p_{i_0} > 0$ and $z_{i_0}(p) > \mathbf{0}$. This yields $p\, z(p) = \sum\limits_{i=1}^{n} p_i z_i(p) \geq p_{i_0} z_{i_0}(p) > 0$, a contradiction to Walras' Law.
Our next assumption requires the existence of a uniform lower bound for the sums $\sum\limits_{z_i(p) \leq 0} p_i$, for all $p \in \Delta_n$.
Assumption 3'. $\quad \gamma = \inf\limits_{p \in \Delta_n} \gamma_p > 0.$
We indicate some examples which show that Assumption (3') is independent on the Walras' Law. In each of the figures below the functions z_1 and z_2 are considered on the intervall $[p', p'']$, which is nothing than the simplex Δ_2. If we represent the vectors $p = (p_1, p_2) \in \Delta_2$ as $p = (1-t)p' + tp''$, where $t \in [0, 1]$, then $p_i = (1-t)p'_i + tp''_i$, which yields $p_1 = t$ and $p_2 = 1 - t$. For $t \in (0, 1)$ the Walras Law $p\, z(p) = p_1 z_1((p_1, p_2)) + p_2 z_2((p_1, p_2)) = 0$ now reduces to the relation $z_2(p) = -\frac{t}{1-t} z_1(p)$. For the cases $t \in \{0, 1\}$ some additional care has to be taken. We suppose that all this is true in the Figures 2 and 3, where Walras' Law is assumed to be satisfied. In the other figures it is easy to find a vector $p \in \Delta_2$ (in Figure 4, e.g. the vector p'),

where $p\,z(p) \neq 0$. In Figures 2 and 5 it is easy to see that Assumption 3' does not hold. In both cases for any $p \in \Delta_2$ we calculate $\sum_{z_i(p) \leq 0} p_i = p_2 = 1 - t$ and so $\inf_{t \in (0,1)} (1-t) = 0$. In Figures 3 and 4 Assumption 3' is satisfied with $\gamma = \min\{t_0, 1 - t_0\}$.

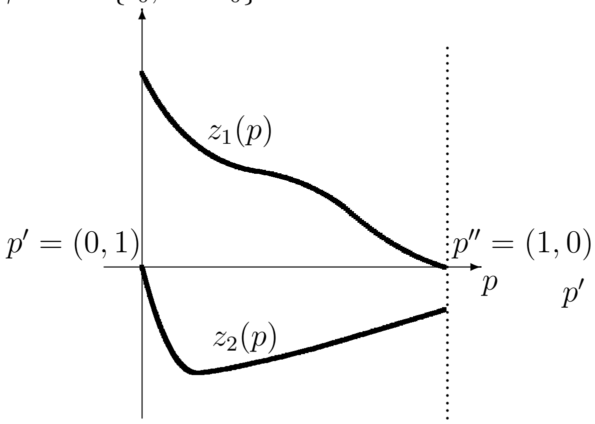

Fig.2. Walras'Law does not imply (3').

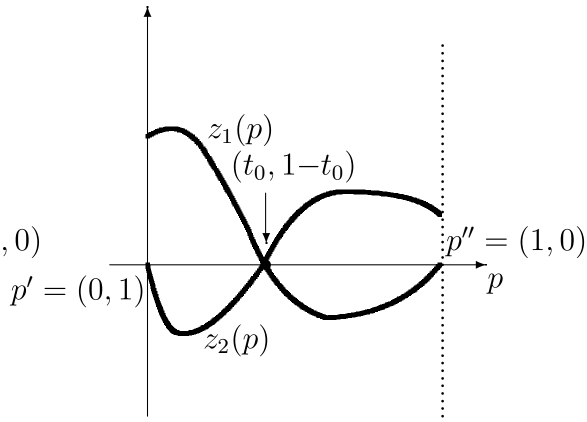

Fig.3. Walras'Law and (3') hold.

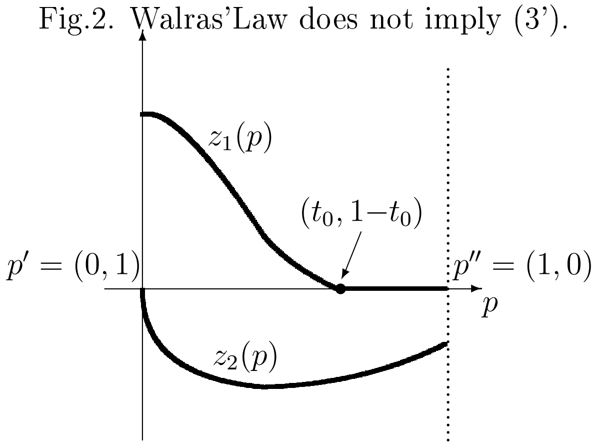

Fig.4. (3') holds but not Walras'Law.

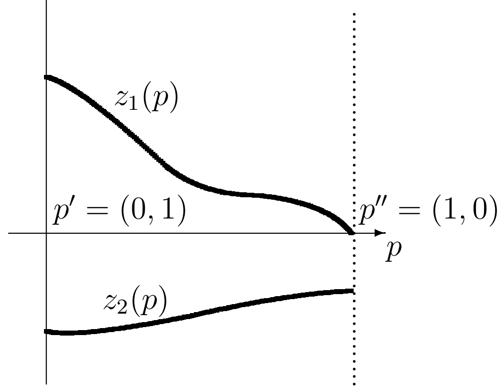

Fig.5. Neither Walras'Law nor (3').

It seems to be clear that it would be hard to find out why an equilibrium exists in our model. But it will be possible if we can estimate the unsatisfied aggregate demand. This leads to the concept of quasi- or k-equilibrium.

Definition 7 *Let k be a positive real. A price vector $p^* \in \Delta_n$ is called a* **k-equilibrium** *if it satisfies the condition*

$$\sum_{z_i(p^*)>0} z_i(p^*) \leq k.$$

The constant $k \in \mathbb{R}_+$ as a numerical value of the maximally possible unsatisfied demand for a given price $p^* \in \Delta_n$ characterizes to what state the economy differs from the market equilibrium (Definition 6).

We can prove now the following

Theorem 6 *Let $\mathcal{E}$ be an economy with n goods that satisfies the Assumptions 1,2 and the Assumption 3' with some number $\gamma > 0$. Put*

$$w_+ = w_+(n, \gamma) = \frac{1}{2n}\left(-(n+1) + \sqrt{(n+1)^2 + 8n\gamma}\right).$$

If now Assumption 4' is satisfied with $w \in [0, w_+)$, then the economy $\mathcal{E}$ possesses a k-equilibrium for each $k \geq \dfrac{nw^2 + (n+1)w}{2\gamma - nw^2 - (n+1)w}$.

Proof. For $p \in \Delta_n$ define $z_i^+(p) = \max\{0, z_i(p)\}$, $i = 1, ..., n$, $z^+(p) = (z_1^+(p), \dots, z_n^+(p))$,

$$\nu(p) = \langle p + z^+(p), e\rangle = 1 + \sum_{z_i(p)>0} z_i(p) \quad \text{and} \quad t_i(p) = \frac{p_1 + z_i^+(p)}{\nu(p)},\ i = 1, ..., n,$$

where $e = (1, \dots, 1)$ denotes the vector of $\mathbb{R}^n$ with all components equal to 1. Note that $\|e\| = n$.

Define now a map $T\colon \Delta_n \longrightarrow \Delta_n$ by $T(p) = \dfrac{p + z^+(p)}{\langle p + z^+(p), e\rangle}$. Since $0 \leq t_i(p) \leq 1$ for each i and

$$\sum_{i=1}^n t_i(p) = \frac{\sum\limits_{i=1}^n (p_i + z_i^+(p))}{\nu(p)} = \frac{1 + \sum\limits_{z_i(p)>0} z_i(p)}{\nu(p)} = 1$$

one has $T(p) : \Delta_n \longrightarrow \Delta_n$.
Now the particular maps which the map T consists of, possess the following properties:
The idendity map id on Δ_n is continuous, by Assumption 4' the map $z\colon \Delta \longrightarrow \mathbb{R}^n$ is w-discontinuous and by Corollary 4 so is z^+. By Corollary 2 the map $id + z^+$ is w-discontinuous, what by Corollary 5 implies the $w\|e\|$-discontinuity, i.e. the nw-discontinuity of $\nu(p) = \langle p + z^+(p), e\rangle$. Since $\nu\colon \Delta_n \longrightarrow [1, \infty)$ the function $\dfrac{1}{\nu}$ is $\dfrac{nw}{\nu(p)}$-discontinuous as a consequence of Corollary 6. Finally, based on Proposition 4, the map $T(p) = (p + z^+(p))\,\dfrac{1}{\nu(p)}$ is w_0-discontinuous at a every point $p \in \Delta_n$, where

$$w_0 = w_0(p) = \frac{nw^2}{\nu(p)} + \frac{w}{\nu(p)} + \frac{nw\|p + z^+(p)\|}{\nu(p)} = \frac{nw^2 + w}{\nu(p)} + nw < nw^2 + (n+1)w \tag{3}$$

and so, the map T is also $nw^2 + (n+1)w$-discontinuous on the set Δ_n.
Since Δ_n is a convex and compact subset in the normed vector space $\mathbb{R}^n$ and $T(p) : \Delta_n \longrightarrow \Delta_n$ we conclude by means of Theorem 4 that there exists a vector $p^* \in \Delta_n$ satisfying the inequality

$$\|T(p^*) - p^*\| \le nw^2 + (n+1)w.$$

Using the norm in $\mathbb{R}^n$ this yields

$$\|T(p^*) - p^*\| = \left\| \frac{p^* + z^+(p^*)}{\nu(p)} - p^* \right\| = \sum_{i=1}^n \left| \frac{p_i^* + z_i^+(p^*)}{\nu(p)} - p_i^* \right| =$$
$$= \sum_{i=1}^n \left| \frac{p_i^* + z_i^+(p^*) - p_i^* - p_i^* \sum\limits_{z_i(p^*)>0} z_i(p^*)}{\nu(p)} \right| \le nw^2 + (n+1)w.$$

Since $1 + \sum\limits_{z_i(p^*)>0} z_i(p^*) > 0$ one has

$$\sum_{i=1}^n \left| z_i^+(p^*) - p_i^* \sum_{z_i(p^*)>0} z_i(p^*) \right| \le \big(nw^2 + (n+1)w\big)\ \nu(p^*). \tag{4}$$

The left side of inequality (4) can be splitted into the two sums

$$\sum_{z_i(p^*)\le 0}\left|z_i^+(p^*)-p_i^*\sum_{z_i(p^*)>0}z_i(p^*)\right|+\sum_{z_i(p^*)>0}\left|z_i(p^*)-p_i^*\sum_{z_i(p^*)>0}z_i(p^*)\right|=$$

$$\sum_{z_i(p^*)\le 0}p_i^*\sum_{z_i(p^*)>0}z_i(p^*)+\sum_{z_i(p^*)>0}\left|z_i(p^*)-p_i^*\sum_{z_i(p^*)>0}z_i(p^*)\right|. \qquad (5)$$

Using the triangle inequality we get the estimation

$$\left|\sum_{z_i(p^*)>0}\left(z_i(p^*)-p_i^*\sum_{z_i(p^*)>0}z_i(p^*)\right)\right|\le\sum_{z_i(p^*)>0}\left|z_i(p^*)-p_i^*\sum_{z_i(p^*)>0}z_i(p^*)\right|, \quad (6)$$

and further the left hand side of (6) calculates as

$$\left|\sum_{z_i(p^*)>0}\left(z_i(p^*)-p_i^*\sum_{z_i(p^*)>0}z_i(p^*)\right)\right| = \left|\sum_{z_i(p^*)>0}z_i(p^*)\left(1-\sum_{z_i(p^*)>0}p_i^*\right)\right|=$$

$$\sum_{z_i(p^*)>0}z_i(p^*)\left(1-\sum_{z_i(p^*)>0}p_i^*\right) = \sum_{z_i(p^*)>0}z_i(p^*)\sum_{z_i(p^*)\le 0}p_i^*. \qquad (7)$$

By means of the equalities (5), (7) and the inequalities (4), (6) we obtain now

$$2\sum_{z_i(p^*)>0}z_i(p^*)\sum_{z_i(p^*)\le 0}p_i^*\le\sum_{z_i(p^*)>0}z_i(p^*)\sum_{z_i(p^*)\le 0}p_i^* + \sum_{z_i(p^*)>0}\left|z_i(p^*)-p_i^*\sum_{z_i(p^*)>0}z_i(p^*)\right|\le$$

$$\le\big(nw^2+(n+1)w\big)\;\nu(p^*).$$

It follows by means of Assumption 3'

$$2\gamma\sum_{z_i(p^*)>0}z_i(p^*)\le 2\sum_{z_i(p^*)>0}z_i(p^*)\sum_{z_i(p^*)\le 0}p_i^*\le\big(nw^2+(n+1)w\big)\;\nu(p^*).$$

Since $\nu(p^*)=1+\sum_{z_i(p^*)>0}z_i(p^*)$ the last inequality yields

$$\sum_{z_i(p^*)>0}z_i(p^*)\le\frac{nw^2+(n+1)w}{2\gamma-nw^2-(n+1)w}, \quad \text{i. e.} \quad \sum_{z_i(p^*)>0}z_i(p^*)\le k,$$

where k satisfies $k \geq \dfrac{nw^2 + (n+1)w}{2\gamma - nw^2 - (n+1)w}$.

In order to have the number $2\gamma - nw^2 - (n+1)w$ positive the value of w must belong to the interval $[0,\, w_+)$, where w_+ is the positive root of the equation $w^2 + \dfrac{n+1}{n}w - \dfrac{2\gamma}{n} = 0$. ■

Remarks.

1. Let n and $\gamma > 0$ be fixed. Then $w_+ = w_+(n, \gamma)$ is defined as indicated in the theorem. For $w \in [0, w_+)$ put

$$k_0(n, w) = \frac{nw^2 + (n+1)w}{2\gamma - nw^2 - (n+1)w}.$$

The number $k_0(n, w)$ is non-negative as was shown above. Note that a sharper estimation[5] in (3) would yield a smaller value of $k_0(n, w)$ and, therefore, would give a better result. In view of Theorem 4, however, an estimation has be obtained independently on p.

2. In Figure 6 for $n = 2$ there is shown a situation without a classic equilibrium.

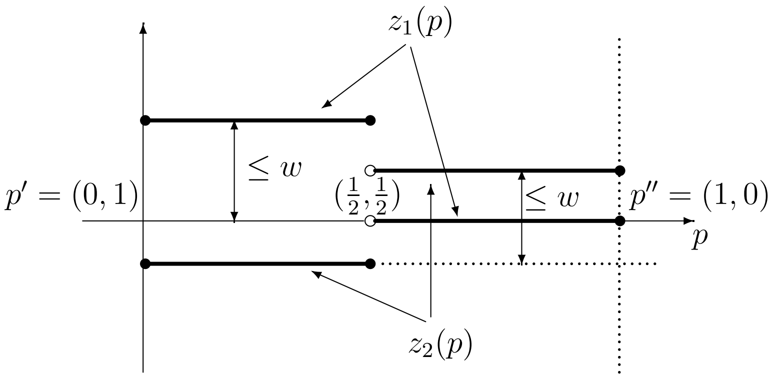

Fig.6. No classical equilibrium, but k-equilibrium exists.

It is clear that there is no $p \in \Delta_2$ which satisfies the inequality $z(p) = (z_1(p), z_2(p)) \leq \mathbf{0}$. Assumptions 1, 2, 4' are obviously fulfilled. Assumption 3' also holds. Indeed, represent $p = (p_1, p_2) \in \Delta_2$ as

$$p = (1-t)p' + tp'', \qquad t \in [0, 1],$$

[5] Our estimation is based on the rough inequality $\nu(p) > 1$.

then $t \in [0, \frac{1}{2}]$ implies $z_1(p) > 0$, $z_2(p) < 0$ and so $\gamma_p = p_2$ and $t \in (\frac{1}{2}, 1]$ implies $z_1(p) = 0$, $z_2(p) > 0$ and so $\gamma_p = p_1$. In both cases we get $\gamma_p \geq \frac{1}{2}$ which shows that the Assumption 3' holds with $\gamma = \frac{1}{2}$. Theorem 6 guarantees the existence of a k-equilibrium for $k \geq \frac{2w^2+3w}{1-2w^2-3w}$ if $w < -\frac{3}{4} + \frac{\sqrt{17}}{4}$. Note that Walras' Law is not satisfied.

3. The number $w_+(n, \gamma)$ is positive for each n and fixed $\gamma > 0$. If one takes $w = 0$ then $k_0(n, \gamma) = 0$ and with $k = 0$ there is obtained the classical case. Observe that in this case it is not necessary to use the Walras' Law for establishing a classical equilibrium.

4. Note that in the classical situation it is impossible to carry out any quantitative analysis. On the contrary, the inequalities from Theorem 6

$$w < w_+(n, \gamma) \quad \text{and} \quad k \geq k_0(n, w)$$

give a chance to analyse the behaviour of an economy for different numerical values of the parameters n, w, γ included in our model. From

$$0 \leq w_+(n, \gamma) = \frac{-(n+1) + \sqrt{(n+1)^2 + 8n\gamma}}{2n} < \frac{-(n+1) + (n+1) + \sqrt{8n\gamma}}{2n} = \sqrt{\frac{2\gamma}{n}}$$

it follows that $\lim_{n\to\infty} w_+(n, \gamma) = +0$. Since $k_0(n, 0) = 0$, the positive number k can be choosen arbitrary small. This shows that the larger the number of goods the better the chance for a classical equilibrium.

5. It is reasonable to put $k_0(n, w_+(n, \gamma)) = +\infty$. If for fixed n and γ the value w is sufficiently close to $w_+(n, \gamma)$, then k is very large. In such a case the existence of an k-equilibrium seems to be of low economic meaning.

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