On Matrices Satisfying a Maximum Principle with Respect to a Cone

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A matrix equation $\mathbf{A}x = y$ is considered in the space $\mathbb{R}^n$ that is ordered by a cone K. In case $K = \mathbb{R}^n_+$ it is known that a matrix $\mathbf{A}$ is said to satisfy the maximum principle if $\mathbf{A}$ is invertible and for each $y \in \mathbb{R}^n_+ \setminus \{0\}$ the solution x belongs to $\mathbb{R}^n_+$ and is such that $x_i = \max_{1\leq k\leq n} x_k$ and $y_i > 0$ for some i. This concept is generalized to finitely generated and to circular cones $K \subset \mathbb{R}^n$. This is achieved by "evaluating" x and y with the aid of elements of a given base for the polar cone K° of K. The maximum principle is characterized geometrically by means of the behavior under $\mathbf{A}^{-1}$ of convex boundary parts of a base for K. A weighted maximum principle is investigated and an infinite dimensional example is indicated.

1 Introduction

We consider the equation

$$\mathbf{A}x = y, \quad y \in K, \tag{1}$$

where $\mathbf{A}$ is an invertible (n,n)-matrix and K a cone [1] in $\mathbb{R}^n$. Roughly speaking, we ask how certain order properties of y induce properties of the solution x of (1).

In [4] and [5] matrices $\mathbf{A}$ are considered for which $\max_{1\leq i\leq n} x_i$ is attained at a component x_i of $x = (x_1, \dots, x_n)^T$ which corresponds to a positive component y_i of the given right side $y \in K$, where $K = \mathbb{R}^n_+$ and $x \in K$. Matrices with this property are said to satisfy the maximum principle. In [6] results

[1] The precise meaning of the concepts used in this paper will be explained in the next section.

of [4] and [5] are generalized to matrices satisfying the following condition: For a given vector $\gamma = (\gamma_1, \ldots, \gamma_n)^T$, with $\gamma_i > 0$ for each i, the maximum of $(\gamma_1 x_1, \ldots, \gamma_n x_n)^T$, where x again is the solution of (1), is attained at components corresponding to nonzero components of y. A matrix possessing the last property is said to satisfy the weighted maximum principle with respect to γ. The investigation in the mentioned papers and especially the necessary and sufficient condition in [6] make essential use of the natural order introduced in $\mathbb{R}^n$ by the cone $\mathbb{R}^n_+$.

In the present paper, which also extends some results of the Diploma thesis of the first author ([2]), we replace $\mathbb{R}^n_+$ by a more general cone K. In particular, we shall consider arbitrary finitely generated cones (equivalently, polyhedral cones) and certain circular cones. The main idea is to characterize properties of y and x in (1) by means of elements of a base F (which is assumed to exist) for the polar cone K°. More precisely, if F is a fixed base for K°, then for $x, y \in K$ we denote

$$
\begin{aligned}
&F^+(y) = \{z \in F : \langle y, z\rangle > 0\} \quad \text{and} \\
&F^{\max}(x) = \{z \in F : \langle x, z\rangle = \alpha(x)\}, \text{where } \ \alpha(x) = \sup\{\langle x, z\rangle : z \in F\}.
\end{aligned}
$$

Notice that if F is compact, then for each $x \in K$, $\alpha(x)$ is finite and $F^{\max}(x)$ is nonempty.

An (n, n)-matrix $\mathbf{A}$ — considered as a linear operator in the ordered space $(\mathbb{R}^n, K)$ — is said to be *inverse monotone in* $(\mathbb{R}^n, K)$ if $\mathbf{A}$ is invertible and $y \in K$ implies $\mathbf{A}^{-1}y \in K$. Further, an (n, n)-matrix $\mathbf{A}$ is said to *satisfy the maximum principle with respect to the base* F if

1. $\mathbf{A}$ is inverse monotone in $(\mathbb{R}^n, K)$ and
2. $F^{\max}(\mathbf{A}^{-1}y) \cap F^+(y) \neq \emptyset$ for each $y \in K \setminus \{0\}$.

Roughly speaking, $\mathbf{A}$ satisfies the maximum principle if for each $y \in K$, the (unique) solution $x = \mathbf{A}^{-1}y$ of the equation $\mathbf{A}x = y$ belongs to K and is such that there exists at least one element in the base F that both “maximally evaluates” x and “positively evaluates” y.

Notice that for each $y \in \operatorname{int}(K)$ (provided such a y exists) there holds $F^+(y) = F$. Thus if F is compact, then $F^{\max}(\mathbf{A}^{-1}y) \cap F^+(y) \neq \emptyset$ for these y. Hence condition 2 is actually a condition on boundary points y of K.

If $K = \mathbb{R}^n_+$ and the convex hull of the coordinate vectors $e^1, \ldots, e^n$ is chosen as base F for $K^\circ = \mathbb{R}^n_+$, then condition 2 is equivalent to

$$\max_{1 \le i \le n} x_i = \max_{i \in N^+(y)} x_i,$$

where $N^+(y) = \{i \in \{1, \dots, n\} : y_i > 0\}$. Hence the maximum principle in the sense of [4] and [5] turns out to be a special case of the above. The weighted maximum principle introduced in [6] is also covered by the approach studied in the present paper, see Section 5.

2 Preliminaries

We consider the Euclidean space $\mathbb{R}^n$ with scalar product $\langle \cdot, \cdot \rangle$ and associated norm $\|x\| = \sqrt{\langle x, x \rangle}$, where $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$. If S is a subset of $\mathbb{R}^n$, we write $\mathrm{int}(S)$ for the interior of S. Real numbers are denoted by small greek letters.

A *wedge* in $\mathbb{R}^n$ is a nonempty subset K of $\mathbb{R}^n$ such that

$$x, y \in K \quad \text{and} \quad \alpha, \beta \geq 0 \quad \text{implies} \quad \alpha x + \beta y \in K.$$

A *cone* is a wedge K satisfying $K \cap (-K) = \{0\}$. A wedge K is said to be *reproducing* if $K - K = \mathbb{R}^n$. The *polar* of a wedge K is defined by

$$K^\circ = \{z \in \mathbb{R}^n : \ \langle x, z \rangle \geq 0 \text{ for each } x \in K\}.$$

A *base* for a wedge K is a convex subset D of K such that for each $x \in K \setminus \{0\}$ there exists a unique number $\alpha > 0$ such that $\frac{1}{\alpha} x \in D$. Notice that the polar of a wedge is a wedge, a wedge with a base is a cone, and a base never contains 0.

For a nonempty subset S of $\mathbb{R}^n$,

$$\mathrm{c}(S) = \{x \in \mathbb{R}^n : \ x = \sum_{i=1}^{m} \lambda_i x^i, \ \lambda_i \geq 0, \ x^i \in S, \ i = 1, ..., m\}$$

denotes the *positive-linear hull* (which is always a wedge), and

$$\mathrm{co}(S) = \{x \in \mathbb{R}^n : \ x = \sum_{i=1}^{m} \lambda_i x^i, \ \lambda_i \geq 0, \ \sum_{i=1}^{m} \lambda_i = 1, \ x^i \in S, \ i = 1, ..., m\}$$

denotes the *convex hull* of S. In these representations, m denotes a positive integer depending on x. If S is a nonempty finite set, say $S = \{x^1, ..., x^r\}$, then the positive-linear hull of S, briefly written $\mathrm{c}\{x^1, ..., x^r\}$, is said to be a *finitely generated wedge*. A *finitely generated cone* is a finitely generated wedge that is a cone. A *polyhedral wedge* is the intersection of a finite number of closed homogeneous halfspaces, and so is a closed set. A *polyhedral cone* is a polyhedral wedge that is a cone. The convex hull of $\{x^1, ..., x^r\}$, written $\mathrm{co}\{x^1, ..., x^r\}$, is said to be a *polytope*. A polytope is a compact set with a finite number of extreme points, which are also called the *vertices* of the polytope. A *face* of a polytope P is a proper convex subset Φ of P such that

$$x, y \in P, \quad x \neq y, \quad (x, y) \cap \Phi \neq \emptyset \quad \text{implies} \quad [x, y] \subset \Phi.$$

Here (x, y) and $[x, y]$ denote the open and closed segment joining the points x and y, respectively.

Some basic properties of wedges and cones are crucial for our purpose. Most of them are well known and some even hold in infinite dimensional normed spaces. For convenient reference we gather these facts in the following theorem.

Theorem 1 *Let K be a wedge in $\mathbb{R}^n$.*
(a) *$K^{\circ\circ}$ is the closure of K.*
(b) *The following statements are equivalent:*
 (b_1) *K is reproducing.*
 (b_2) $\mathrm{int}(K) \neq \emptyset$.
 (b_3) *K° is a cone.*
(c) *If K is closed, then* $\mathrm{int}(K)$ *is nonempty if and only if K° has a compact base.*
(d) *If K is closed, then for each $z \in \mathbb{R}^n$ the following statements are equivalent:*
 (d_1) $z \in \mathrm{int}(K^\circ)$.
 (d_2) *$\langle x, z\rangle > 0$ for each $x \in K \setminus \{0\}$.*
 (d_3) *$\{y \in K : \langle y, z\rangle = 1\}$ is a base for K.*
(e) *If K is a closed cone, then K has a compact base.*
(f) *If $K = \mathrm{c}(S)$, where S is a compact convex subset of $\mathbb{R}^n$ such that $0 \notin S$, then K is a closed cone.*
(g) *If K is a closed reproducing cone, then* $\mathrm{int}(K) \cap \mathrm{int}(K^\circ) \neq \emptyset$.
(h) *K is a finitely generated wedge if and only if K is a polyhedral wedge. Hence any finitely generated wedge is closed.*
(i) *If K is a finitely generated wedge, then so is K°.*

Proof. (a): See [1], 3.1.7. (bipolar theorem).

(b): See [7], Theorem IV.1.1. and Theorem IV.5.1.

(c): See [1], 3.8.6.

(d_1) $\Rightarrow$ (d_2): Assume $\langle x, z\rangle = 0$ for some $x \in K \setminus \{0\}$. It follows from (d_1) that for each $y \in \mathbb{R}^n$ there exists $\delta > 0$ such that $z + \beta y \in K^\circ$ whenever

$\beta \in (-\delta, \delta)$. This implies for each $\beta \in (-\delta, \delta)$

$$0 \leq \langle x, z + \beta y \rangle = \langle x, z \rangle + \beta \langle x, y \rangle = \beta \langle x, y \rangle.$$

We thus obtain $\langle x, y \rangle = 0$ for arbitrary $y \in \mathbb{R}^n$, which is impossible since $x \neq 0$. Hence (d_2) holds.

(d_2) $\Rightarrow$ (d_3): Obvious.

(d_3) $\Rightarrow$ (d_1): Since $A = \{x \in K : \ \|x\| = 1\}$ is compact, $\alpha = \min_{x \in A} \langle x, z \rangle$ exists and, by (d_3), is positive. Now let $y \in \mathbb{R}^n$ be such that $\|y - z\| \leq \alpha$. Then for each $x \in K$, it follows that

$$\langle x, y \rangle = \langle x, y - z \rangle + \langle x, z \rangle \geq -\|x\|\alpha + \alpha\|x\| = 0.$$

Hence $y \in K^\circ$. Thus (d_1) holds.

(e): Since $(K^\circ)^\circ$, which by (a) equals K, is a cone, it follows by applying (b) to K° that $\mathrm{int}(K^\circ) \neq \emptyset$. This in turn implies by (c) that K has a compact base. (Notice that K° is always closed.)

(f): K is a cone: Let $x, -x \in K$. Then

$$x = \sum_{i=1}^{p} \lambda_i x^i, \quad -x = \sum_{j=1}^{q} \mu_j y^j,$$

where

$$\lambda_i, \mu_j \geq 0, \quad x^i, y^j \in S \quad \text{for} \quad i = 1, \ldots, p, \quad j = 1, \ldots, q.$$

Let $\alpha = \sum_{i=1}^{p} \lambda_i + \sum_{j=1}^{q} \mu_j$ and suppose $\alpha > 0$. Then

$$0 = \frac{1}{\alpha}(x - x) = \sum_{i=1}^{p} \frac{\lambda_i}{\alpha} x^i + \sum_{j=1}^{q} \frac{\mu_j}{\alpha} y^j \in S,$$

contradicting the hypothesis. Hence $\alpha = 0$ and so, in particular, $\lambda_i = 0$ for each i, which implies $x = 0$.
K is closed: Let $(x^k)_{k \in \mathbb{N}}$ denote a sequence in K converging to $x \in \mathbb{R}^n$. Since S is convex, we can write, for each $k \in \mathbb{N}$,

$$x^k = \lambda_k y^k, \quad \text{where } \lambda_k \geq 0, \ y^k \in S.$$

We may assume that $x \neq 0$ and that $\lambda_k > 0$ for each $k \in \mathbb{N}$. According to the assumptions on S, there exist $\varrho > 0$ and $\sigma > 0$ such that, for each $k \in \mathbb{N}$, we have $\varrho \leq \|y^k\| \leq \sigma$ and so

$$0 < \frac{1}{\sigma}\inf\|x^k\| \leq \lambda_k \leq \frac{1}{\varrho}\sup\|x^k\| < +\infty.$$

Hence $(\lambda_k)_{k\in\mathbb{N}}$ contains a subsequence $(\lambda_{k_i})_{i\in\mathbb{N}}$ converging to some $\lambda > 0$. We thus obtain

$$y^{k_i} = \frac{1}{\lambda_{k_i}}x^{k_i} \ \rightarrow \ \frac{1}{\lambda}x.$$

Since S is closed, we have $\frac{1}{\lambda}x \in S$ and so $x \in \mathrm{c}(S) = K$.

(g): According to the hypothesis, K and K° are closed cones (cf. (b)) and so, by virtue of (e) have compact bases D and F, respectively. Set $S = \mathrm{co}(D \cup F)$. Assume $0 \in S$. Since D and F are convex, we may write

$$0 = \lambda_1 x^1 + \lambda_2 x^2, \quad \text{where } \lambda_1, \lambda_2 \geq 0,\ \lambda_1 + \lambda_2 = 1,\ x^1 \in D,\ x^2 \in F.$$

Since $0 \notin D \cup F$, it follows immediately that $\lambda_1 > 0$ and $\lambda_2 > 0$. Therefore

$$\langle x^1, x^2\rangle = -\frac{\lambda_1}{\lambda_2}\langle x^1, x^1\rangle = -\frac{\lambda_1}{\lambda_2}\|x^1\|^2 < 0,$$

contradicting the fact that $x^1 \in K$ and $x^2 \in K^\circ$. Hence $0 \notin S$. Moreover, S is compact and convex. According to (f), the wedge $K_1 = \mathrm{c}(S)$ is thus a closed cone. It now follows from (e) again that $K_1 (= K_1^{\circ\circ})$ has a compact base. This in turn implies, according to (c), that $\mathrm{int}(K_1^\circ) \neq \emptyset$. The assertion then follows from

$$\mathrm{int}(K_1^\circ) \subset \mathrm{int}(K) \cap \mathrm{int}(K^\circ).$$

(h), (i): See [3], 2.8. ■

3 Finitely generated cones

In this section we shall characterize matrices satisfying the maximum principle under the assumption that the cone K in $\mathbb{R}^n$ is reproducing and finitely generated. Hence K is also closed (Theorem 1(h)) and there exists an element

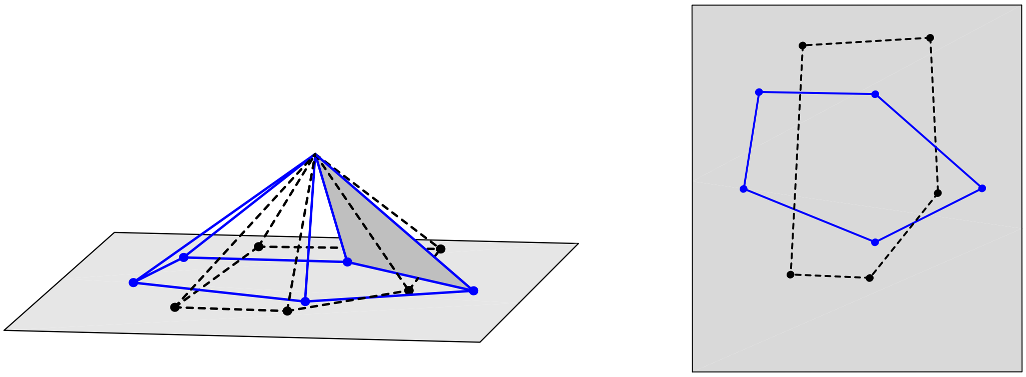

Figure 1. The cones K and K° and their bases

$u^0 \in \text{int}(K) \cap \text{int}(K^\circ)$ (Theorem 1(g)). In what follows we keep such a u^0 fixed. Obviously we may assume $\|u^0\| = 1$. Using the hyperplane

$$E = \{x \in \mathbb{R}^n : \langle x, u^0 \rangle = 1\}$$

we define

$$D = E \cap K \qquad \text{and} \qquad F = E \cap K^\circ.$$

By Theorem 1(d), D and F are bases for K and K°, respectively (see Figure 1)[2]. We have the following representations:

$$K = \text{c}\{x^1, \ldots, x^m\}, \;\; D = \text{co}\{x^1, \ldots, x^m\}, \;\; \text{where } x^1, ..., x^m \in E, \tag{2}$$

$$K^\circ = \text{c}\{z^1, \ldots, z^r\}, \;\; F = \text{co}\{z^1, \ldots, z^r\}, \;\; \text{where } z^1, ..., z^r \in E. \tag{3}$$

In fact, since K is finitely generated, there exist $x^1, ..., x^m \in \mathbb{R}^n \setminus \{0\}$ such that $K = \text{c}\{x^1, ..., x^m\}$. For each $i = 1, ..., m$, we have $\langle x^i, u^0 \rangle > 0$ (Theorem 1(d)). Hence we may assume that $\langle x^i, u^0 \rangle = 1$ and so $x^i \in E$. It is then immediate that $D = \text{co}\{x^1, ..., x^m\}$. The same argument applies to K°.
We call $x^1, ..., x^m$ *generating vectors* of K (with respect to u^0) if (2) holds and none of the x^i is a convex combination of the others. This implies that the x^i are the vertices of D. Analogous remarks apply to z^j.

Convention: Henceforth, it is assumed that the representations (2) and (3) are given, where $x^1, ..., x^m$ and $z^1, ..., z^r$ are generating vectors of K and K°, respectively.

[2] We express our gratitude to Mrs. Karla Nestler for her patience in preparing the figures according to our intentions.

For $x \in K$ we define the following index sets:

$$\begin{aligned} J(x) \ &= \{j \in \{1, \dots, r\} : \quad \langle x, z^j \rangle = \max_{k=1,\dots,r} \langle x, z^k \rangle\}, \\ J_0(x) &= \{j \in \{1, \dots, r\} : \quad \langle x, z^j \rangle = 0\}. \end{aligned}$$

We refer to the definition of $\alpha(x)$, $F^{\max}(x)$ and $F^+(y)$ given in the Introduction.

Lemma 2 *For each $x \in K$ there holds*

$$F^{\max}(x) = \mathrm{co}\{z^j : j \in J(x)\}.$$

Proof. Since the linear optimization problem

$$\text{maximize } \langle x, z \rangle \quad \text{subject to } z \in F$$

has a solution at a vertex of F and so at some z^j, it follows that

$$\alpha(x) = \langle x, z^j \rangle \quad \text{for each } j \in J(x)$$

and consequently, since $F^{\max}(x)$ is convex, $\mathrm{co}\{z^j : j \in J(x)\} \subset F^{\max}(x)$.

Conversely, if $z \in F^{\max}(x)$, then there exist an index set $J \subset \{1, \dots, r\}$ and a representation

$$z = \sum_{j \in J} \lambda_j z^j, \quad \text{where } \lambda_j > 0 \text{ for each } j \in J \text{ and } \sum_{j \in J} \lambda_j = 1.$$

The assumption that $\langle x, z^{j_0} \rangle < \alpha(x)$ for some $j_0 \in J$ leads to the contradiction

$$\begin{aligned} \alpha(x) = \langle x, z \rangle &= \lambda_{j_0} \langle x, z^{j_0} \rangle + \sum_{j \in J, j \neq j_0} \lambda_j \langle x, z^j \rangle \\ &< \alpha(x) \left(\lambda_{j_0} + \sum_{j \in J, j \neq j_0} \lambda_j \right) = \alpha(x). \end{aligned}$$

Hence $\langle x, z^j \rangle = \alpha(x)$ for each $j \in J$ and so $z \in \mathrm{co}\{z^j : j \in J(x)\}$. ■

Lemma 3 *Let $y \in K$.*
(a) *The following statements are equivalent:*
 (a_1) $F^+(y) = F$.
 (a_2) $J_0(y) = \emptyset$.

(a$_3$) $y \in \mathrm{int}(K)$.

(b) *If y is a boundary point of K, then*

$$F^+(y) = F \setminus \mathrm{co}(\{z^j : j \in J_0(y)\}).$$

Proof. Using $F = \mathrm{co}\{z^1, ..., z^r\}$ yields for each $z \in F$ that

$$\langle y, z\rangle = 0 \quad \text{if and only if} \quad z \in \mathrm{co}(\{z^j : j \in J_0(y)\}). \tag{4}$$

From this the equivalence of (a$_1$) and (a$_2$) follows immediately.

The equivalence of (a$_1$) and (a$_3$) follows by applying Theorem 1(d) to K° (notice that K is closed by Theorem 1(h) and so $K^{\circ\circ} = K$ by Theorem 1(a)).

Assertion (b) is a consequence of (a) and (4) again. ■

Lemma 4 *Let $x, y \in K$, $y \neq 0$. If the set $F^{\max}(x) \cap F^+(y)$ is nonempty, then it contains some z^j, where $j \in J(x)$.*

Proof. Let $z \in F^{\max}(x) \cap F^+(y)$. Then by Lemma 2, $z = \sum_{j\in J(x)} \lambda_j z^j$, where $\lambda_j \geq 0$ for each $j \in J(x)$ and $\sum_{j\in J(x)} \lambda_j = 1$. We thus obtain $0 < \langle y, z\rangle = \sum_{j\in J(x)} \lambda_j \langle y, z^j\rangle$ and so $\langle y, z^j\rangle > 0$ for at least one $j \in J(x)$. ■

For each $j = 1, ..., r$ we set

$$I_0(z^j) = \{\, i \in \{1, ..., m\} : \ \langle x^i, z^j\rangle = 0\}.$$

Notice that for each $j = 1, ..., r$, the index set $I_0(z^j)$ is nonempty since z^j is a vertex of F and so is a boundary point of K°. Indeed, apply Lemma 3 to $z^j, K^\circ, I_0(z^j)$ instead of $y, K, J_0(y)$, respectively.

Lemma 5 *Let Φ be a face of D and let $I \subset \{1, ..., m\}$ be such that $\Phi = \mathrm{co}(\{x^i : \ i \in I\})$. Then there exists $j_0 \in \{1, ..., r\}$ satisfying $I \subset I_0(z^{j_0})$.*

Proof. Since $\Phi \neq D$, the elements of Φ belong to the relative boundary of D. Denote by $|I|$ the cardinal number of I. Then for $b = \frac{1}{|I|}\sum_{i\in I} x^i \in \Phi$ we have $J_0(b) \neq \emptyset$ by Lemma 3(b) applied to K°. If now $j_0 \in J_0(b)$, then

$$0 = \langle b, z^{j_0}\rangle = \frac{1}{|I|}\sum_{i\in I} \langle x^i, z^{j_0}\rangle,$$

hence $\langle x^i, z^{j_0}\rangle = 0$ for each $i \in I$. ■

For each $j = 1, \dots, r$ we further denote

$$D_j = \{x \in D: \ x = u^0 - \sum_{i \in I_0(z^j)} \alpha_i (x^i - u^0) \text{ for some } \alpha_i \geq 0; \ i \in I_0(z^j)\}. \quad (5)$$

The following theorem provides a decomposition of the base D and characterizes those z^j that belong to $F^{\max}(x)$ for a given $x \in D$.

Theorem 6 *The following statements hold.*
(a) $\bigcup_{j=1}^r D_j = D$.
(b) *For each $x \in D$ and each $j \in \{1, ..., r\}$, the element z^j belongs to $F^{\max}(x)$ if and only if $x \in D_j$.*

Proof. (a): We have to show that $D \subset \bigcup_{j=1}^r D_j$. This will be done in two steps.

Notice first that since u^0 is a relatively interior point of D, there exists a positive number ε such that the set

$$B_E(u^0, \varepsilon) = \{y \in E: \ \|y - u^0\| < \varepsilon\}$$

is contained in D.

1. We show that $B_E(u^0, \varepsilon) \subset \bigcup_{j=1}^r D_j$. Let $x \in B_E(u^0, \varepsilon), \ x \neq u^0$, be given. Then $x' = 2u^0 - x$ is also in $B_E(u^0, \varepsilon)$ and so in D. Since D is a polytope, there exists an x'' on the relative boundary of D such that $x' = \lambda x'' + (1-\lambda)u^0$ with some $\lambda \in (0,1)$. Let Φ be a face of D containing x'' and let j_0 be an index as in Lemma 5. We then have $x'' = \sum_{i \in I_0(z^{j_0})} \gamma_i x^i$ with $\gamma_i \geq 0, \ \sum_{i \in I_0(z^{j_0})} \gamma_i = 1$, and it follows that

$$x' = u^0 + \sum_{i \in I_0(z^{j_0})} \gamma'_i (x^i - u^0), \quad \text{where} \quad \gamma'_i = \lambda \gamma_i.$$

Therefore $x \in D_{j_0}$ follows from

$$x = 2u^0 - x' = u^0 - \sum_{i \in I_0(z^{j_0})} \gamma'_i (x^i - u^0).$$

2. Now let x be an arbitrary element of $D \setminus \{u^0\}$. Then

$$\tilde{x} = u^0 + \frac{\varepsilon}{2\|x - u^0\|}(x - u^0)$$

belongs to $B_E(u^0, \varepsilon)$ and thus, according to the first step, has a representation of the form $\tilde{x} = u^0 - \sum_{i \in I_0(z^{j_0})} \alpha_i (x^i - u^0)$ with some $j_0 \in \{1, ..., r\}$ and $\alpha_i \geq 0$

for $i \in I_0(z^{j_0})$. Consequently we obtain

$$x = u^0 - \sum_{i \in I_0(z^{j_0})} \alpha_i'(x^i - u^0) \quad \text{with} \quad \alpha_i' = \frac{2\|x - u^0\|}{\varepsilon}\alpha_i$$

and so $x \in D_{j_0}$.

(b): We fix $x \in D$ and $j \in \{1, ..., r\}$. Assume first that $x \in D_j$. Then with some $\alpha_i \geq 0$, we have $x = u^0 - \sum_{i \in I_0(z^j)} \alpha_i(x^i - u^0)$. Taking into account that $\langle u^0, z^j \rangle = 1$ and $\langle x^i, z^j \rangle = 0$ for each $i \in I_0(z^j)$, we further get

$$\langle x, z^j \rangle = 1 + \sum_{i \in I_0(z^j)} \alpha_i. \tag{6}$$

For each $k \in \{1, ..., r\}$ we similarly obtain

$$\langle x, z^k \rangle = 1 + \sum_{i \in I_0(z^j)} \alpha_i \; - \sum_{i \in I_0(z^j)} \alpha_i \langle x^i, z^k \rangle \; \leq \; 1 + \sum_{i \in I_0(z^j)} \alpha_i. \tag{7}$$

We can thus conclude that $j \in J(x)$ and so, by Lemma 2, $z^j \in F^{\max}(x)$.

Conversely, suppose now $x \notin D_j$. According to (a) we have $x \in D_{j_0}$ for some $j_0 \neq j$ and so, by virtue of (6) with j replaced by j_0,

$$\langle x, z^{j_0} \rangle = 1 + \sum_{i \in I_0(z^{j_0})} \alpha_i.$$

Since $x \notin D_j$, there exists an $i_0 \in I_0(z^{j_0}) \setminus I_0(z^j)$ such that $\alpha_{i_0} > 0$. Consequently, with k replaced by j in equation (7),

$$\begin{aligned} \langle x, z^j \rangle &= 1 + \sum_{i \in I_0(z^{j_0})} \alpha_i - \sum_{i \in I_0(z^{j_0})} \alpha_i \langle x^i, z^j \rangle \\ &\leq 1 + \sum_{i \in I_0(z^{j_0})} \alpha_i - \alpha_{i_0} \langle x^{i_0}, z^j \rangle \; < \; 1 + \sum_{i \in I_0(z^{j_0})} \alpha_i, \end{aligned}$$

which implies $z^j \notin F^{\max}(x)$. ■

The decomposition of the base D is shown in Figure 2.

For a nonempty index set $I \subset \{1, ..., m\}$ we denote

$$J_+(I) = \{1, ..., r\} \setminus \bigcap_{i \in I} J_0(x^i) = \left\{ j \in \{1, \ldots, r\} : \exists i \in I, \langle z^j, x^i \rangle > 0 \right\}.$$

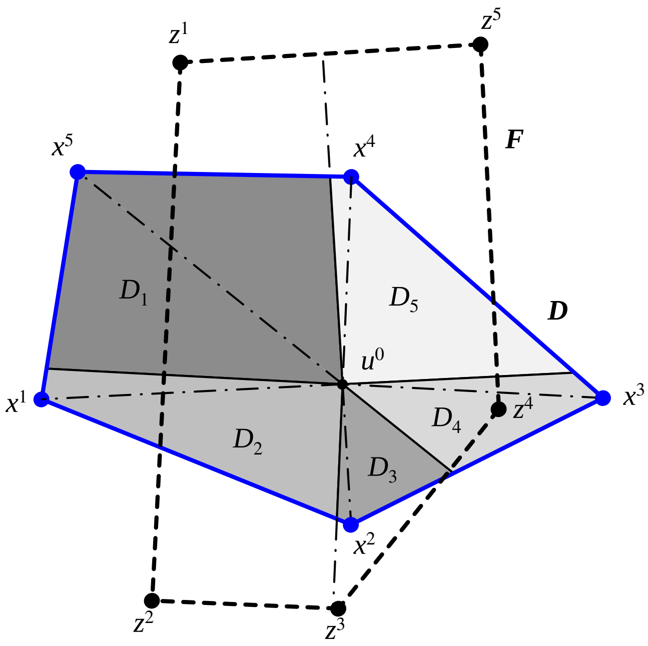

Figure 2. Decomposition of the base D

Observe that $J_+(I)$ is always nonempty. Now we can formulate a condition on a matrix $\mathbf{A}$ that will turn out to characterize the maximum principle property. Let $\mathbf{A}$ be an invertible matrix. We say that $\mathbf{A}^{-1}$ *satisfies condition* (G^*) *with respect to the base* D if for each nonempty subset I of $\{1,\dots,m\}$ such that $\mathrm{co}\{x^i : i \in I\}$ is a face of D there holds the inclusion

$$(G^*) \qquad \mathbf{A}^{-1}\left(\mathrm{co}\{x^i : i \in I\}\right) \subset \bigcup_{j\in J_+(I)} \mathrm{c}(D_j).$$

Theorem 7 *For every invertible* (n,n)*-matrix* $\mathbf{A}$ *the following statements are equivalent:*
(a) $\mathbf{A}$ *satisfies the maximum principle with respect to* F.
(b) $\mathbf{A}^{-1}$ *satisfies condition* (G^*) *with respect to* D.

Proof. (a) $\Rightarrow$(b): Suppose that $\mathbf{A}$ satisfies the maximum principle with respect to F. Let $I \subset \{1,...,m\}$ be such that $\Phi = \mathrm{co}\{x^i : i \in I\}$ is a face of D. Let $y \in \Phi$, i. e. $y = \sum_{i\in I}\beta_i x^i$, where $\beta_i \geq 0$, $\sum_{i\in I}\beta_i = 1$. Since $\mathbf{A}$ is inverse monotone, it follows that $x = \mathbf{A}^{-1}y \in K$. We have to show that $x \in \bigcup_{j\in J_+(I)}\mathrm{c}(D_j)$. Assume the contrary. According to Theorem 6 this means that $z^j \notin F^{\max}(x)$ for each $j \in J_+(I)$. On the other hand, if $j \in \{1,...,r\}$ is not in $J_+(I)$, then for each $i \in I$ we have $\langle x^i, z^j\rangle = 0$, which implies $\langle y, z^j\rangle = \sum_{i\in I}\beta_i\langle x^i, z^j\rangle = 0$ and so $z^j \notin F^+(y)$. We have thus shown that $z^j \notin F^{\max}(\mathbf{A}^{-1}y)\cap F^+(y)$ for each $j \in \{1,...,r\}$. But this implies by Lemma 4 that $F^{\max}(\mathbf{A}^{-1}y)\cap F^+(y)$ is empty, which contradicts (a).

(b) $\Rightarrow$(a): Suppose that $\mathbf{A}^{-1}$ satisfies condition (G^*) with respect to D. Notice first that $\mathbf{A}$ is inverse monotone in $(\mathbb{R}^n, K)$. Now let $y \in K\setminus\{0\}$ be given.

Since

$$F^{\max}(\mathbf{A}^{-1}(\lambda y)) \cap F^{+}(\lambda y) = F^{\max}(\mathbf{A}^{-1} y) \cap F^{+}(y) \tag{8}$$

we may, without loss of generality, assume that $y \in D$. If $y \in \operatorname{int}(K)$, then according to Lemma 2 and Lemma 3

$$F^{\max}(x) \cap F^{+}(y) = \operatorname{co}(\{z^j : j \in J(x)\}) \neq \emptyset.$$

Hence we may further assume that y is a boundary point of K. Denote by I the smallest nonempty subset of $\{1, \ldots, m\}$ such that $\operatorname{co}\{x^i : i \in I\}$ is a face of D that contains y. Then there exists a representation of y as

$$y = \sum_{i \in I} \beta_i x^i, \quad \beta_i > 0 \quad \text{for} \quad i \in I \quad \text{and} \quad \sum_{i \in I} \beta_i = 1. \tag{9}$$

According to (G^*) there exist an index $j_0 \in J_+(I)$ and a number $\lambda > 0$ such that $x = \mathbf{A}^{-1} y \in \lambda D_{j_0}$ (notice that D_{j_0} is convex). Theorem 6(b) and (8) therefore imply $z^{j_0} \in F^{\max}(\lambda^{-1} x) = F^{\max}(x)$. Moreover, by definition of $J_+(I)$ we have $\langle x^{i_0}, z^{j_0} \rangle > 0$ for some $i_0 \in I$. Since the corresponding β_{i_0} is also positive, (9) implies $\langle y, z^{j_0} \rangle > 0$. Hence $z^{j_0} \in F^{\max}(x) \cap F^{+}(y)$. ■

Remark
The proof (cf. (8)) shows that the theorem remains true if $\mathbf{A}$ is only assumed to be a positively homogeneous operator in $(\mathbb{R}^n, K)$.

Example
Consider $\mathbb{R}^3$ with the cone $K = \mathrm{c}\{x^1, \ldots, x^5\}$, where

$$x^1 = \sqrt{3}\left(\tfrac{3}{4},\ \tfrac{1}{4},\ 0\right)^T, \quad x^2 = \sqrt{3}\left(\tfrac{1}{4},\ \tfrac{3}{4},\ 0\right)^T, \quad x^3 = \sqrt{3}\left(0,\ \tfrac{1}{2},\ \tfrac{1}{2}\right)^T,$$

$$x^4 = \sqrt{3}\left(0,\ \tfrac{1}{4},\ \tfrac{3}{4}\right)^T, \quad x^5 = \sqrt{3}\left(\tfrac{3}{4},\ 0,\ \tfrac{1}{4}\right)^T.$$

Then we have $K^\circ = \mathrm{c}\{z^1, \ldots, z^5\}$, where

$$z^1 = \sqrt{3}\,(0,\ 0,\ 1)^T, \qquad z^2 = \sqrt{3}\left(1,\ -\tfrac{1}{3},\ \tfrac{1}{3}\right)^T, \quad z^3 = \sqrt{3}\,(1,\ 0,\ 0)^T,$$

$$z^4 = \sqrt{3}\left(\tfrac{1}{7},\ \tfrac{9}{7},\ -\tfrac{3}{7}\right)^T, \quad z^5 = \sqrt{3}\left(-\tfrac{1}{5},\ \tfrac{3}{5},\ \tfrac{3}{5}\right)^T.$$

The x^i and the z^j $(i, j = 1, \ldots, 5)$ turn out to be generating vectors of K and K° respectively on the plane $E = \{x \in \mathbb{R}^3 \ : \ \langle x, u^0 \rangle = 1\}$, where

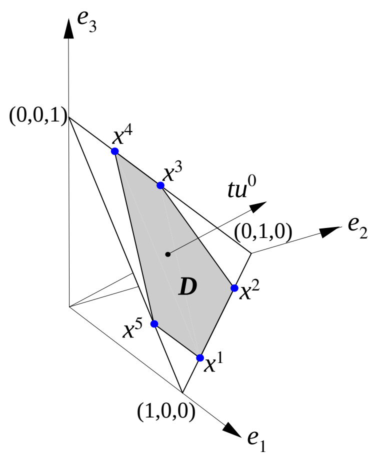

Figure 3. The base D

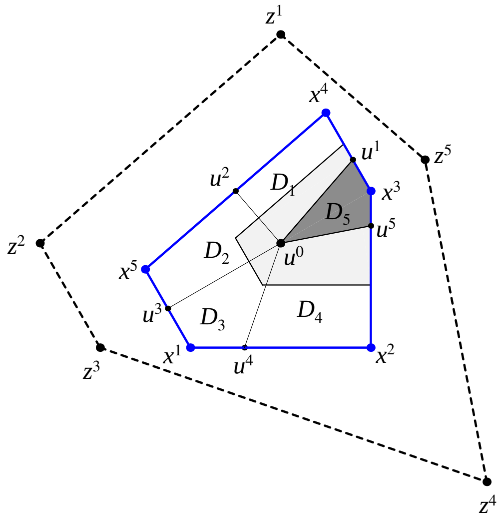

Figure 4: D and F on the plane E

$u^0 = \sqrt{3}\,(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$ (cf. Figure 3). The corresponding bases D and F are the convex hulls of the respective generating vectors. For each $i = 1, \ldots, 5$ let $\alpha_i \geq 0$ be such that $u^i = u^0 - \alpha_i(x^i - u^0)$ is a relative boundary point of D. Then according to (5) each D_j is the convex hull of u^0, u^i for $i \in I_0(z^j)$ and some x^k. For example $D_5 = \mathrm{co}\{u^0, u^1, x^3, u^5\}$ (see Figure 4).

For each $\lambda \in \mathbb{R} \setminus \{1\}$ we consider the following matrix and its inverse:

$$\mathbf{A}_\lambda = \frac{1}{\lambda - 1}\begin{pmatrix} -1 & 0 & 0 \\ \frac{\lambda}{2} & \frac{\lambda}{2} - 1 & \frac{\lambda}{2} \\ \frac{\lambda}{2} & \frac{\lambda}{2} & \frac{\lambda}{2} - 1 \end{pmatrix}, \ \mathbf{A}_\lambda^{-1} = \begin{pmatrix} 1 - \lambda & 0 & 0 \\ \frac{\lambda}{2} & 1 - \frac{\lambda}{2} & \frac{\lambda}{2} \\ \frac{\lambda}{2} & \frac{\lambda}{2} & 1 - \frac{\lambda}{2} \end{pmatrix}.$$

From $\mathbf{A}_\lambda^{-1}x = x + \lambda(x^3 - x)$ for each $x \in E$ we conclude that $\mathbf{A}_\lambda$ is inverse monotone in $(\mathbb{R}^3, K)$ if and only if $\lambda \in [0, 1)$. For these λ we now investigate condition (G^*).

Let $I = \{1, 5\}$. Then $\mathrm{co}(\{x^i \ : \ i \in I\}) = [x^1, x^5]$ is a face of D and $J_+(I) = \{1, \ldots, 4\}$. If λ is such that (G^*) holds, then

$$\mathbf{A}_\lambda^{-1}([x^1, x^5]) \subset \bigcup_{i=1}^{4} \mathrm{c}(D_j).$$

In view of the decomposition of D into the sets D_j, this inclusion is violated if and only if there exists $\tilde{x} \in [x^1, x^5]$ such that

$$\mathbf{A}_\lambda^{-1}\tilde{x} = u^0 - \alpha_1(x^1 - u^0) - \alpha_5(x^5 - u^0), \quad \text{where} \quad \alpha_1 > 0, \ \alpha_5 > 0.$$

Denote the set of these $\tilde{x}$ by X_λ. A straightforward calculation results in the following tabular:

λ	X_λ
$0 \le \lambda \le \frac{5}{9}$	$\emptyset$
$\frac{5}{9} < \lambda \le \frac{5}{7}$	$(\tilde{x}^1, \tilde{x}^5)$, where $\tilde{x}^1 = \frac{5-7\lambda}{5(1-\lambda)} x^1 + \frac{2\lambda}{5(1-\lambda)} x^5,\ \tilde{x}^5 = \frac{2\lambda}{5(1-\lambda)} x^1 + \frac{5-7\lambda}{5(1-\lambda)} x^5$
$\frac{5}{7} < \lambda < 1$	$[x^1, x^5]$

Consequently, $0 \le \lambda \le \frac{5}{9}$ is a necessary condition for $\mathbf{A}_\lambda^{-1}$ to satisfy condition (G^*) with respect to D. An analogous inspection of the other faces of D shows that this condition is also sufficient.

According to Theorem 7 we conclude that $\mathbf{A}_\lambda$ satisfies the maximum principle with respect to F if and only if $\lambda \in [0, \frac{5}{9}]$.

4 Circular cones

Now we study the maximum principle with respect to a class of cones that are not finitely generated.

A subset $K \subset \mathbb{R}^n$ is called a *circular cone* if there exist an element $u^0 \in \mathbb{R}^n$, $||u^0|| = 1$, and a number $\beta \in (0,1)$ such that

$$K = \{x \in \mathbb{R}^n : \beta||x|| \le \langle x, u^0 \rangle\}. \tag{10}$$

Each circular cone is a cone. Consider the continuous function $g(x) = \beta\|x\| - \langle x, u^0 \rangle$, $x \in \mathbb{R}^n$, to see that K is closed and that $u^0 \in \mathrm{int}(K)$. Furthermore, Theorem 1(d) yields that

$$D = E \cap K, \quad \text{where} \quad E = \{x \in \mathbb{R}^n : \ \langle x, u^0 \rangle = 1\},$$

is a compact base for K. It is easy to verify that another representation of D is

$$D = \left\{ x \in \mathbb{R}^n : \langle x, u^0 \rangle = 1, \ \ ||x - u^0|| \le \varrho \right\}, \text{ where } \varrho = \frac{\sqrt{1-\beta^2}}{\beta}.$$

(This justifies the notion circular cone.) The following lemma describes the polar K° of K.

Lemma 8 *The polar K° of K is a cicular cone having the base*

$$F = \left\{ z \in \mathbb{R}^n : \langle z, u^0 \rangle = 1, \quad ||z - u^0|| \leq \frac{1}{\varrho} \right\}. \tag{11}$$

Proof. Since $u^0 \in \mathrm{int}(K) = \mathrm{int}(K^{\circ\circ})$, Theorem 1(d) implies that $F = E \cap K^\circ$ is a base for K°. It remains to show that for each $z \in E$ we have $z \in K^\circ$ if and only if $||z - u^0|| \leq \frac{1}{\varrho}$.

Take a fixed $z \in E$. The Cauchy-Schwarz inequality yields

$$\begin{aligned} \langle x, z \rangle &= \langle x - u^0, z - u^0 \rangle + 1 \geq -||x - u^0|| \cdot ||z - u^0|| + 1 \\ &\geq -\varrho||z - u^0|| + 1 \qquad \text{for each } x \in D. \end{aligned}$$

Hence $||z - u^0|| \leq \frac{1}{\varrho}$ is sufficient for $z \in K^\circ$.
On the other hand, the element $x^0 = u^0 - \frac{\varrho}{||z-u^0||}(z - u^0)$ belongs to D and $\langle x^0, z \rangle = -\varrho||z - u^0|| + 1$. To guarantee the relation $\langle x^0, z \rangle \geq 0$, the condition $||z - u^0|| \leq \frac{1}{\varrho}$ is necessary. ■

Next we shall characterize the sets $F^{\max}(x)$ and $F^+(y)$.

Lemma 9 *For each $x, y \in D$, the following statements hold:*

(a) $$F^{\max}(x) = \begin{cases} F & \text{if } x = u^0, \\ \left\{ u^0 + \frac{1}{\varrho||x-u^0||}(x - u^0) \right\} & \text{if } x \neq u^0. \end{cases}$$

(b) $$F^+(y) = \begin{cases} F & \text{if } ||y - u^0|| < \varrho, \\ F \setminus \left\{ u^0 - \frac{1}{\varrho^2}(y - u^0) \right\} & \text{if } ||y - u^0|| = \varrho. \end{cases}$$

Proof. (a): We assume that $x \neq u^0$, the proof for $x = u^0$ being elementary. For each $z \in F$, we estimate

$$\langle x, z \rangle = \langle x - u^0, z - u^0 \rangle + 1 \leq ||x - u^0|| \cdot ||z - u^0|| + 1 \leq \frac{||x - u^0||}{\varrho} + 1,$$

and the upper bound is attained by $z^0 = u^0 + \frac{1}{\varrho||x-u^0||}(x - u^0)$, which belongs to F and so to $F^{\max}(x)$.

Assume there is a $z' \in F^{\max}(x)$ such that $z' \neq z^0$. Then $z'' = (z^0 + z')/2$ also belongs to $F^{\max}(x)$ because $F^{\max}(x)$ is convex. Since $||z'' - u^0|| < \frac{1}{\varrho}$, the linear function $\langle x, \cdot \rangle$ attains its maximum at an interior point of the feasible domain F and consequently is constant there. But this contradicts the fact that $u^0, z^0 \in F$, $\langle x, u^0 \rangle = 1$, $\langle x, z^0 \rangle > 1$. Thus we have $F^{\max}(x) = \{z^0\}$.

(b): Let $y \in D$ be such that $||y - u^0|| = \varrho$. Then $z^0 = u^0 - \frac{1}{\varrho^2}(y - u^0)$ belongs to F and $\langle y, z^0 \rangle = 0$, i. e. $z^0 \notin F^+(y)$. Assuming there exists $z' \in F$ such that $z' \neq z^0$ and $\langle y, z' \rangle = 0$, we conclude by similar arguments as in the proof of (a) that $F^+(y) = \emptyset$, which contradicts $\langle y, u^0 \rangle = 1$.

If $||y - u^0|| < \varrho$, then $x \in \text{int}(K)$, and the conclusion follows. ■

Theorem 10 *Let* $\mathbf{A}$ *be an invertible* (n, n)*-matrix. Then the following statements are equivalent:*

(a) $\mathbf{A}$ *satisfies the maximum principle with respect to the base* F *given in relation* (11).

(b) *For each* $y \in D$ *with* $||y - u^0|| = \varrho$*, there exist* $y' \in D$*,* $\lambda > 0$ *and* $\alpha \in [0, 1]$ *such that*

$$\mathbf{A}^{-1}y = \lambda[u^0 + \alpha(y' - u^0)], \text{ where } y' - u^0 \neq -(y - u^0). \tag{12}$$

Proof. (a) $\Rightarrow$ (b): Let $y \in D$ with $||y - u^0|| = \varrho$. There exist $u \in D$ and $\lambda > 0$ such that $u = \frac{1}{\lambda}\mathbf{A}^{-1}y$.

If $u = u^0$, then (12) is satisfied with $\alpha = 0$ and $y' = y$.

If $u \neq u^0$, then $\mathbf{A}^{-1}y = \lambda[u^0 + (u - u^0)] = \lambda[u^0 + \alpha(y' - u^0)]$, where $\alpha = \frac{||u-u^0||}{\varrho} \in (0, 1]$ and $y' = u^0 + \frac{\varrho}{||u-u^0||}(u - u^0) \in D$, $||y' - u^0|| = \varrho$. It remains to show that $y' - u^0 \neq -(y - u^0)$. Lemma 9 yields

$$\begin{aligned} F^+(y) &= F \setminus \{u^0 - \frac{1}{\varrho^2}(y - u^0)\} \quad \text{and} \\ F^{\max}(\mathbf{A}^{-1}y) &= F^{\max}(u^0 + \alpha(y' - u^0)) = \{u^0 + \frac{1}{\varrho^2}(y' - u^0)\}. \end{aligned}$$

With $\mathbf{A}$ satisfying the maximum principle we conclude $u^0 - \frac{1}{\varrho^2}(y - u^0) \neq u^0 + \frac{1}{\varrho^2}(y' - u^0)$.

(b) $\Rightarrow$ (a): For each $y \in D$ with $||y - u^0|| = \varrho$ we have $\mathbf{A}^{-1}y \in K$; since $\mathbf{A}$ is linear, the inverse monotony of $\mathbf{A}$ follows.

Let $y \in D$, $||y - u^0|| < \varrho$. Then, by Lemma 9(b) we have $F^+(y) = F$. Since $F^{\max}(\mathbf{A}^{-1}y) \neq \emptyset$ always holds, we get

$$F^{\max}(\mathbf{A}^{-1}y) \cap F^+(y) = F^{\max}(\mathbf{A}^{-1}y) \cap F \neq \emptyset.$$

Let $y \in D$, $||y - u^0|| = \varrho$. We consider two cases.

1. $\mathbf{A}^{-1}y = \lambda u^0$ for some $\lambda > 0$. With $F^{\max}(\mathbf{A}^{-1}y) = F^{\max}(u^0)$ and $F^+(y) \neq \emptyset$ (which immediately follows from Lemma 9(a)), we get

$$F^{\max}(\mathbf{A}^{-1}y) \cap F^+(y) = F \cap F^+(y) \neq \emptyset.$$

2. $\mathbf{A}^{-1}y = \lambda[u^0 + \alpha(y' - u^0)]$ for some $\lambda > 0$, $\alpha \in (0, 1]$ and $y' \in D$, $||y' - u^0|| = \varrho$. Then, again by Lemma 9(a), we conclude

$$F^{\max}(\mathbf{A}^{-1}y) = F^{\max}(u^0 + \alpha(y' - u^0)) = \left\{ u^0 + \frac{1}{\varrho^2}(y' - u^0) \right\}.$$

According to Lemma 9(b), $F^+(y) = F \setminus \{u^0 - \frac{1}{\varrho^2}(y - u^0)\}$ holds, and therefore the condition $F^{\max}(\mathbf{A}^{-1}y) \cap F^+(y) \neq \emptyset$ is satisfied if and only if $y' - u^0 \neq -(y - u^0)$. ■

Compare the geometric meaning of condition (b) in Theorem 10 with condition (G^*), which is related to polyhedral cones. Both of them describe, indirectly, "forbidden regions" for the preimages of each relative extremal point y of D. In (G^*), these are certain parts of the decomposition of D. In the present case, this region degenerates to the segment on D that is "opposite" (viewed from u^0) to y.

5 The weighted maximum principle

In this section we discuss how the property of a matrix to satisfy a maximum principle reacts to a change of the base F for K°. A generalization of the maximum principle as considered so far enables us to do this.

Let K be a closed reproducing cone in $\mathbb{R}^n$. Then K° is a cone with a compact base F (see Theorem 1(b),(c)). Further let $w : K^\circ \setminus \{0\} \to \mathbb{R}_+$ be a continuous function. For each $x \in K$, $x \neq 0$, define

$$F_w^{\max}(x) = \{z \in F : \ w(z)\langle x, z\rangle = \sup_{z' \in F} w(z')\langle x, z'\rangle\}, \tag{13}$$

the *set of maximizing elements with respect to* F *and (the weight function)* w. Since F is compact and the objective function $z' \mapsto w(z')\langle x, z'\rangle$ is continuous, the supremum is attained and $F_w^{\max}(x)$ is nonempty. The *set of positive elements with respect to* F *and* w is defined as

$$F_w^+(x) = \{z \in F : \ w(z)\langle x, z\rangle > 0\}. \tag{14}$$

We say that the (n,n)-matrix $\mathbf{A}$ satisfies the *weighted maximum principle with respect to F and w* if

1*. $\mathbf{A}$ is inverse monotone in $(\mathbb{R}^n, K)$ and
2*. $F_w^{\max}(\mathbf{A}^{-1}y) \cap F_w^{+}(y) \neq \emptyset$ for each $y \in K \setminus \{0\}$.

Theorem 11 *Let $u^0 \in \mathrm{int}(K)$. Then for every (n,n)-matrix $\mathbf{A}$ the following statements are equivalent:*
(a) $\mathbf{A}$ *satisfies the maximum principle with respect to the base*

$$G = \{z \in K^\circ : \ \langle z, u^0 \rangle = 1\}.$$

(b) *For an arbitrary base F of K°, $\mathbf{A}$ satisfies the weighted maximum principle with respect to F and w, where $w(z) = \frac{1}{\langle z, u^0 \rangle}$ for $z \in K^\circ \setminus \{0\}$.*

Proof. Since $u^0 \in \mathrm{int}(K)$, Theorem 1(d) yields $\langle z, u^0 \rangle > 0$ on $K^\circ \setminus \{0\}$, thus w is well-defined and continuous there.
The inverse monotony of $\mathbf{A}$ holds independently of any bases or weight functions considered. Therefore it has to be proved only that condition 2^* is equivalent to condition 2 (see the Introduction) of the respective maximum principles.
Each element $z \in F$ has a corresponding positive multiple $\tilde{z} \in G$; $\tilde{z}$ is uniquely determined by

$$\tilde{z} = \frac{z}{\langle z, u^0 \rangle}.$$

Obviously, for an arbitrary element $y \in K$, $y \neq 0$, there holds

$$F_w^{\max}(\mathbf{A}^{-1}y) \cap F_w^{+}(y) \neq \emptyset \ \text{ if and only if } \ G_1 \cap G_2 \neq \emptyset, \tag{15}$$

wherein $G_1 = \{\tilde{z} \in G : \ z \in F_w^{\max}(\mathbf{A}^{-1}y)\}$ and $G_2 = \{\tilde{z} \in G : \ z \in F_w^{+}(y)\}$.

For each $x \in K$, $x \neq 0$, it follows that

$$\begin{aligned} F_w^{\max}(x) &= \{z \in F : \ \frac{\langle x, z \rangle}{\langle z, u^0 \rangle} = \max_{z' \in F} \frac{\langle x, z' \rangle}{\langle z', u^0 \rangle}\} \\ &= \{z \in F : \ \frac{\langle x, z \rangle}{\langle z, u^0 \rangle} = \max_{z' \in F} \langle x, \widetilde{z'} \rangle\} \\ &= \{z \in F : \ \langle x, \tilde{z} \rangle = \max_{z' \in G} \langle x, z' \rangle\} \end{aligned}$$

and thus with $x = \mathbf{A}^{-1}y$

$$G_1 = \{z \in G : \ \langle x, z \rangle = \max_{z' \in G} \langle x, z' \rangle\} = G^{\max}(\mathbf{A}^{-1}y).$$

Since $w(z) > 0$ for all $z \in K^\circ$, $z \neq 0$, we get $F_w^+(y) = F^+(y)$ and consequently $G_2 = G^+(y)$.

Using these characterizations of G_1 and G_2 in the equivalence (15) and taking into account that $y \in K$, $y \neq 0$ was arbitrary completes the proof. ■

Remarks
1) If $\mathbf{A}$ satisfies the weighted maximum principle with respect to F, then — in general — $\mathbf{A}$ need not satisfy the (non-weighted) maximum principle with respect to this base (for an example see [6]).

2) Since the investigation of the weighted maximum principle can be reduced to the non-weighted case with a suitably chosen base, the geometric condition (G^*) also applies to the weighted maximum principle — only it has to be investigated in a transformed base.

Example
The following example points out how the choice of the base $F \subset K^\circ$ influences on the respective maximum principle.

Let $K = \mathbb{R}^n_+$, and thus $K^\circ = \mathbb{R}^n_+$. Choose $F = \mathrm{co}\{\gamma_1 e^1, \ldots, \gamma_n e^n\}$ with $\gamma_i > 0$ for each $i = 1, \ldots, n$. Define

$$\begin{aligned} I^{\max}(x) &= \{i : \ \gamma_i x_i = \max_{1 \le j \le n} \gamma_j x_j\} \\ I^+(x) &= \{i : \ \gamma_i x_i > 0\} = \{i : \ x_i > 0\}. \end{aligned}$$

As a consequence of Lemma 2, $F^{\max}(x) \cap F^+(y) \neq \emptyset$ holds if and only if there exists $z^i = \gamma_i e^i$ such that $z^i \in F^{\max}(x) \cap F^+(y)$. This implies $\langle x, z^i \rangle = \gamma_i x_i = \max_{1 \le j \le n} \gamma_j x_j$ and $\langle x, z^i \rangle > 0$, that is $i \in I^+(y)$. This means (with i being the index where the correspondence is realized)

$$\max_{1 \le j \le n} \gamma_j x_j = \max_{j \in I^+(y)} \gamma_j x_j.$$

Consequently, for the natural ordering cone, the matrix $\mathbf{A}$ satisfies the maximum principle with respect to the base $F = \mathrm{co}\{\gamma_1 e^1, \ldots, \gamma_n e^n\}$, where $\gamma_i > 0$ for $i = 1, \ldots, n$, if and only if it has the (γMP)-property with $\gamma = (\gamma_1, \ldots, \gamma_n)^T$ in the sense of Türke and Weber [6].

6 An infinite dimensional example

It is obvious how to extend the maximum principle to linear operators on normed vector spaces of arbitrary dimension.

Let E be a normed real vector space and K a wedge in E. By definition, the polar of K is

$$K^\circ = \{u \in E' :\ \langle x, u\rangle \geq 0 \text{ for each } x \in K\},$$

where E' denotes the topological dual of E and $\langle\cdot,\cdot\rangle$ here denotes the dual pairing of E and E'. Suppose K° possesses a base F. Then the sets $F^{\max}(x)$ and $F^+(y)$ and the property of a linear operator $\mathbf{A} : E \to E$ being inverse monotone or satisfying the maximum principle with respect to F are formally defined as in the Introduction. Here we only indicate how to proceed in a special example.

Given a locally compact Hausdorff space T, we consider the vector space $C_0(T)$ of all continuous real-valued functions on T that vanish at infinity. Recall that $C_0(T)$ is a Banach lattice with respect to the norm $\|x\|_\infty = \max_{t\in T}|x(t)|$ and the order relation generated by the cone K of all nonnegative functions in $C_0(T)$. Notice that K contains interior points if and only if T is compact.

Let $M(T)$ denote the vector space of all finite signed regular Borel measures μ on T normed by $|\mu|(T)$. According to the Riesz representation theorem, there is a one-to-one correspondence between $u \in (C_0(T))'$ and $\mu \in M(T)$ such that $\|u\| = |\mu|(T)$ and

$$\langle x, u\rangle = \int_T x \mathrm{d}\mu \quad \text{for each } x \in C_0(T).$$

We therefore identify $(C_0(T))'$ with $M(T)$. The polar cone K° then consists of the nonnegative measures in $M(T)$, and a base F for K° is given by

$$F = \{\mu \in M(T) :\ \mu \geq 0,\ \mu(T) = 1\}. \tag{16}$$

With respect to this base, we have for each $x, y \in K$,

$$\begin{aligned} F^{\max}(x) &= \{\mu \in F :\ \int_T x\mathrm{d}\mu = \sup_{\nu\in F} \int_T x\mathrm{d}\nu\}, \\ F^+(y) &= \{\mu \in F :\ \int_T y\mathrm{d}\mu > 0\}. \end{aligned}$$

Recall that a signed measure $\mu \in M(T)$ is said to be *concentrated* on the Borel set B if $|\mu|(T \setminus B) = 0$. For each $x, y \in K$, define the sets

$$\begin{aligned} T^{\max}(x) &= \{t \in T :\ x(t) = \|x\|_\infty\}, \\ \mathcal{B}^+(y) &= \{B \subseteq T :\ B \text{ is a Borel set, } y(t) > 0 \text{ for each } t \in B\}. \end{aligned}$$

The following lemma is easily verified.

Lemma 12 *For each $x, y \in K$, the following statements hold.*
(a) $\sup_{\nu \in F} \int_T x d\nu = \|x\|_\infty$, *and the supremum is attained, e. g., by the* Dirac *measure ε_t, where $t \in T^{\max}(x)$,*
(b) $F^{\max}(x) = \{\mu \in F : \mu \text{ is concentrated on } T^{\max}(x)\}$,
(c) $F^+(y) = \{\mu \in F : \mu(B) > 0 \text{ for some } B \in \mathcal{B}^+(y)\}$.

As an immediate consequence we now obtain

Theorem 13 *For each inverse monotone operator $\mathbf{A} : C_0(T) \to C_0(T)$, the following statements are equivalent.*
(a) $\mathbf{A}$ *satisfies the maximum principle with respect to the base F given by* (16).
(b) *For each $y \in K \setminus \{0\}$, there exists $t \in T^{\max}(\mathbf{A}^{-1}y)$ such that $y(t) > 0$.*

The theorem shows that condition (b), which might be considered as the "natural" maximum property of linear operators in $C_0(T)$, can be regained from our general concept by choosing an appropriate base F for K°.

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