On a certain Maximum Principle for Positively Invertible Matrices and Operators in an Ordered Normed Space*

A. Kalauch and M. R. Weber

Abstract

The equation $Ax = y$ for a positively invertible linear continuous operator in an ordered normed space or a similar matrix in $\mathbb{R}^n$ is considered. For a matrix A and a vector $y \geq 0$ the following thoroughly studied property is of interest: does the maximum of the solution $A^{-1}y = x = (x_1, \ldots, x_n)^T$ occur among those subscripts, where the corresponding components of y are strongly positive? If this is the case, then the matrix A is said to satisfy the inverse maximum principle. In the present paper this maximum principle is first generalized to a positively invertible operator in an ordered normed space $(X, K, \|\cdot\|)$ with an arbitrary closed cone K, which is assumed to have a nonempty interior, and second, it is characterized both analytically by some extreme point condition and geometrically by means of the behaviour under A^{-1} of the faces of the cone K. It is also shown that the conditions, which have been obtained for finite dimensional spaces in earlier related papers, are special cases of conditions presented in this paper. The relation of the introduced maximum pinciple and the one which is known for elliptic boundary value problems is demonstrated.

1 Introduction and Preliminaries

Let a mathematical model of some physical, technological, economical, biological or other problem lead to an equation

$$Ax = y \tag{1}$$

with a positive or an inverse monotone linear operator A in a normed space X (or a matrix A in a finite dimensional space) ordered by some cone K. In this paper for the equation (1) with some operator A we understand y to be a given arbitrary element of the cone K and look for the solution x and its properties. If $y \in K$, $y \neq 0$ (i. e. if the input y

*appeared in *Lecture Notes in Pure and Applied Mathematics: Function Spaces - the Fifth Conference*, **213**, 2000, 217-230. The paper is an extended version of a short lecture read at G. Y. Lozanovskij honour session at the conference "Function Spaces V" in September 1998 at the Adam Mickiewicz University of Poznan, Poland.

is positive) then besides the question whether (the output or response) x is positive, in the case of a finite dimensional space X and a matrix A it is also of interest, whether at least one of the maximal components of x is attained at the set of indices, where the corresponding components of y are strongly positive. More exactly, let A be an invertible (n,n)-matrix satisfying $A^{-1} \geq 0$ considered as a linear operator in the ordered normed space $(\mathbb{R}^n, \mathbb{R}^n_+, \|\cdot\|)$, where $\mathbb{R}^n_+$ denotes the cone of all vectors $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ with nonnegative components and $\|x\| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ the norm. Then the mentioned property can be expressed as

$$\max_{1 \leq i \leq n} x_i = \max_{i \in N^+(y)} x_i, \tag{2}$$

where x is assumed to be the solution of (1) and $N^+(y) = \{i \in \{1, \ldots, n\} \colon y_i > 0\}$.
In other words it is natural to ask whether the maximal response takes place in a part of the system where is a nonzero influence.
A similar problem arises for operators in spaces of continuous functions on some compact topological space.
As a further example we consider the following homogeneous Dirichlet (boundary value) problem for an elliptic operator

$$\left\{ \begin{array}{lcll} Lu & = & f & \text{in} \quad \Omega \\ u & = & 0 & \text{on} \quad \partial\Omega \end{array} \right. , \tag{3}$$

where Ω is an open bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with a sufficiently smooth (s. [5]) boundary $\partial\Omega$, and L denotes the operator

$$(Lu)(x) = -\sum_{i,j=1}^n a_{ij}(x) D_{ij} u + \sum_{i=1}^n b_i(x) D_i u + c(x) u$$

with the coefficient-functions $a_{ij} = a_{ij}(x)$ satisfying $a_{ij} = a_{ji}$. We assume that a_{ij}, b_i, c, f are Hölder-continuous functions with the exponent $\mu \in (0,1)$, i. e. belong to the space

$$C^\mu(\overline{\Omega}) = \{v \in C(\overline{\Omega}) : |v(x) - v(y)| \leq k\|x - y\|^\mu, \ x, y \in \overline{\Omega}\}$$

(k is some positive constant) and satisfy the conditions $c(x) \geq 0, f(x) \geq 0$ for $x \in \overline{\Omega}$, where $\overline{\Omega}$ denotes the closure of Ω with respect to the norm $\|\cdot\|$ in $\mathbb{R}^n$. Denote by $\mathrm{supp}(f)$ the closure of the set $\{x \in \Omega \colon f(x) > 0\}$ and, in order to avoid the trivial case, assume $\mathrm{supp}(f) \neq \overline{\Omega}$,.
The operator L is said to be *strongly (uniformly) elliptic*, if there is a constant $\lambda > 0$ such that

$$0 < \lambda\|\xi\|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j$$

for any $x \in \Omega$ and $\xi = (\xi_1, \ldots, \xi_n)^T \in \mathbb{R}^n, \ \xi \neq 0$.
In what follows for $k \in \mathbb{N}$ we denote by $C^k(\overline{\Omega})$ the vector space of all uniformly continuous functions $v \colon \Omega \longrightarrow \mathbb{R}$, for which all partial derivatives $D^\alpha v$ of order $|\alpha| \leq k$ exist and are

also uniformly continuous on Ω. Therefore they possess unique continuous extensions onto $\overline{\Omega}$. These extensions are again denoted by $D^\alpha v$.
Under the made assumptions for any function $f \in C^\mu(\overline{\Omega})$ the boundary value problem (3) has a unique solution u, which belongs to the HÖLDER-space (s. [2], [3])

$$C^{2,\mu}(\overline{\Omega}) = \{v \in C^2(\overline{\Omega}) : \; D^\alpha v \in C^\mu(\overline{\Omega}) \quad \text{for} \quad 0 \le |\alpha| \le 2, \; \mu \in (0,1)\}.$$

So the *solution operator*

$$\mathcal{S} \colon C^\mu(\overline{\Omega}) \longrightarrow C^{2,\mu}(\overline{\Omega})$$

is defined, where $\mathcal{S}f$ denotes the unique solution of the boundary value problem (3). As a consequence of the well known (usual) maximum principles for boundary value problems (s. [10], [5]) the operator $\mathcal{S}$ is strongly positive[1]. From this we get $(\mathcal{S}f)(x) = u(x) > 0$, $x \in \Omega$, and a standard argument shows that u satisfies the condition

$$\max_{x \in \overline{\Omega}} u(x) = \max_{x \in \operatorname{supp}(f)} u(x) \; . \tag{4}$$

Even the following is true.

Lemma 1 *If $c(x) > 0$ for all $x \in \Omega$ and if the solution u of the boundary value problem (3) has a local maximum at the point x^*, then $f(x^*) > 0$ and $u(x^*) \le \frac{f(x^*)}{c(x^*)}$.*

Proof. From the calculus it is known that at the point x^* one has $(D_i u)(x^*) = 0$ for $i = 1, \ldots, n$ and

$$\sum_{i,j=1}^{n} (D_{ij}u)(x^*)\xi_i\xi_j \le 0$$

for any vector $\xi \in \mathbb{R}^n$. Due to the (strong) ellipticity the inequality

$$\sum_{i,j=1}^{n} a_{ij}(x^*)\xi_i\xi_j \ge 0$$

holds, which yields (s. f. e. [6], § III.1.1)

$$\sum_{i,j=1}^{n} a_{ij}(x^*)(D_{ij}u)(x^*) \le 0 \, .$$

Therefore the left hand side of the equality

$$-\sum_{i,j=1}^{n} a_{ij}(x^*)(D_{ij}u)(x^*) = f(x^*) - c(x^*)u(x^*)$$

[1]This means any nonzero element of the cone of the nonnegative functions is mapped by $\mathcal{S}$ into the interior of this cone.

is nonnegative. Since $x^* \in \Omega$ the expression $-c(x^*)u(x^*)$ is strongly negative. So, a contradiction will be obtained if $f(x^*)$ is not large enough. In particular, this is the case if $f(x^*) = 0$. The estimate for $u(x^*)$ is immediate. ∎

If the derivatives in (3) are understood in the sense of distributions, then f can be an arbitrary element of $L^p(\Omega)$. By using the AGMON-DOUGLIS-NIRENBERG-estimation

$$\|\mathcal{S}f\|_{W_p^2(\Omega)} \leq k\|f\|_{L^p(\Omega)}$$

for $p > 1$ and $k > 0$ and the embedding $W_p^2(\Omega) \hookrightarrow C^{1,\mu}(\Omega)$, which holds if the boundary $\partial\Omega$ is sufficiently smooth and $p \geq \frac{n}{1-\mu}$, the solution operator $\mathcal{S}$ has a unique extension (s. [2], [8]) to a compact strongly positive linear operator (again denoted by $\mathcal{S}$)

$$\mathcal{S}\colon C(\overline{\Omega}) \longrightarrow C(\overline{\Omega})\,.$$

Similar properties hold for more general elliptic and parabolic boundary value problems.

Very often the discretization of a boundary value problem, which is the mathematical model of some physical problem, leads to an equation (1). It is natural then to expect that (1) preserves some intrinsic physical properties of the original process. This manifests itself in special kinds of matrices. So monotonicity properties and discrete maximum principles at their early stages are studied, f. e. in [4], [12], [17]. More systematically the property (2) was dealt with in [14] and [15] for positively invertible and other matrices.
Condition (2) as well as condition (4) (joint with the nonnegativity of x and u, respectively) and the conclusion of Lemma 1 use the natural order in the corresponding spaces. So, in the early ninetieth it was understood that positively invertible or positive operators in ordered normed spaces are the natural framework for posing and investigating the maximum principle (s. [16], [7]). This idea paved also the way for generalizations of (2) in some different directions:

- the notation of the weighted (inverse) maximum principle was introduced and studied in [16] and extended in [11],

- the cone $\mathbb{R}^n_+$ was replaced by a finitely generated and a circular cone in [11],

- matrices are replaced by positively invertible operators in normed spaces in the present paper and by positive operators in [7].

The purpose of the present paper is also to show the relations between these approaches.

2 The Maximum Principle and its Geometric Characterization in the Finite Dimensional Case

In the following $\mathrm{co}(M)$ denotes the convex hull[2] of the subset M of a real vector space X, i. e.

$$\mathrm{co}(M) = \{x \in X \colon x = \sum_{i=1}^{m} \lambda_i x^i,\ \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1,\ x^i \in M,\ i = 1, \ldots, m\},$$

where $m = m(x) \in \mathbb{N}$, and $\mathrm{c}(M)$ denotes the conical hull of M, i. e.

$$\mathrm{c}(M) = \{x \in X \colon x = \sum_{i=1}^{m} \lambda_i x^i,\ \lambda_i \geq 0,\ x^i \in M, i = 1, \ldots, m,\ m = m(x) \in \mathbb{N}\}.$$

We consider the equation (1) in the normed space $\mathbb{R}^n$ ordered by the cone $\mathbb{R}^n_+$ and assume A to be an invertible (n,n)-matrix such that $A^{-1} \geq 0$.

Definition 1 (s. [14]) A matrix A is said to satisfy the *inverse maximum principle*, if in the equation (1) there holds $x \geq 0$ and (2), whenever $y \geq 0$.

If $y \in \mathrm{int}(\mathbb{R}^n_+)$, i. e. all components of y are strongly positive, then $N^+(y) = \{1, \ldots, n\}$, and (2) is trivial. Therefore, (2) is a proper restriction only if y is a boundary point of the cone $\mathbb{R}^n_+$.
Denote the vector $(0, \ldots, 0, 1, 0, \ldots, 0)^T$ with 1 at the k-th position by $e^{(k)}$, by

$$\mathcal{D} = \mathrm{co}\{e^{(1)}, \ldots, e^{(n)}\}$$

the standard simplex in $\mathbb{R}^n$ and its subsets of all vectors one maximal component of which occurs at the k-th position by

$$\mathcal{D}_k = \{x = (x_1, \ldots, x_n)^T \in \mathcal{D} \colon x_k \geq x_i,\ i = 1, \ldots, n\}, \quad k = 1, \ldots, n.$$

By means of this partition the inverse maximum principle can be described as follows:

Theorem 1 *Let A be a positively invertible (n,n)-matrix. Then A satisfies the inverse maximum principle if and only if*

$$A^{-1}\left(\mathrm{co}\{e^{(i)} \colon i \in N'\}\right) \subset \bigcup_{i \in N'} \mathrm{c}(\mathcal{D}_i)$$

holds for any nonvoid subset $N' \subset \{1, \ldots, n\}$.

For $n = 3$ such a partition of $\mathcal{D}$ is illustrated in Figure 1.

[2]If M is written in the form $\{x \in X \colon \ldots\}$, then we use the simple notation $\mathrm{co}\{x \in X \colon \ldots\}$ and, somewhat later, also $\mathrm{c}\{x \in X \colon \ldots\}$.

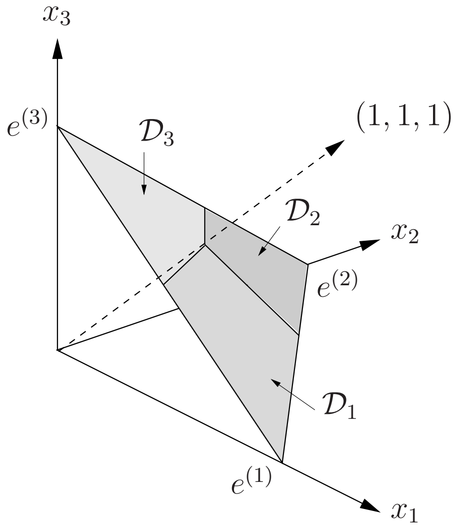

Figure 1: Decomposition of the base $\mathcal{D}$

Take (in Figure 1), for instance, $N' = \{2, 3\}$ and let y belong to the interval

$$[e^{(2)}, e^{(3)}] = \{x \in \mathcal{D} \colon x = \lambda e^{(2)} + (1 - \lambda)e^{(3)},\ 0 \leq \lambda \leq 1\}.$$

If A satisfies the maximum principle then $x = A^{-1}y$ must not belong to the subcone

$$\mathrm{c}(\mathcal{D}_1) = \{\lambda x \colon x \in \mathcal{D}_1,\ \lambda \geq 0\},$$

i. e. x has to belong to the union of the cones $\mathrm{c}(\mathcal{D}_2) \cup \mathrm{c}(\mathcal{D}_3)$.

3 The Infinite Dimensional Case and Arbitrary Cones

Taking on some more general point of view the maximum principle described in the previous section depends on a certain base $\mathcal{D}$ of the cone $\mathbb{R}^n_+$. A detailed analysis shows that a base of the dual cone to the given one is necessary in order to formulate this maximum principle. This is the starting point for generalizing the main idea into two directions: first, towards operators in an infinite dimensional (ordered normed) space, and second, a replacement of the standard cone in $\mathbb{R}^n$ by an arbitrary closed cone with nonempty interior. In the space $\mathbb{R}^n$ the dual cone to $\mathbb{R}^n_+$ coincides with itself and $\mathcal{D}$ serves as a base for the dual cone as well.

We need the notation of an extreme subset of some sets in a real vector space X. Let C be a nonempty convex subset of X. A nonempty convex subset $E \subset C$, $(E \neq C)$ is called an *extreme subset* or a *face* of C, if $y, z \in C$, $\lambda y + (1 - \lambda)z \in E$ for some $\lambda \in (0, 1)$ implies $y, z \in E$. An element $x \in C$ is called an *extreme point* of C, if $\{x\}$ is an extreme subset of C. The set of all extreme points of C is denoted by $\mathrm{ext}(C)$. Obviously, $x \in C$ belongs to $\mathrm{ext}(C)$ if and only if $x = \lambda y + (1-\lambda)z$ implies $x = y = z$, whenever $y, z \in C$ and $\lambda \in (0, 1)$.

Let K be a convex cone of a real vector space X. Any element $x \in K$, $x \neq 0$ defines a *ray* of K by

$$r(x) = \{\lambda x \colon \lambda \in \mathbb{R}_+\}.$$

A ray is called *extreme*, if $y \in r(x),\ z \in K,\ y - z \in K$ imply $z \in r(x)$.

Let $(X, K, \|\cdot\|)$ be an ordered normed space, where X is a real vector space, K a cone in X (i. e. $x, y \in K$, $\lambda, \mu \geq 0$ imply $\lambda x + \mu y \in K$ and $K \cap (-K) = \{0\}$), and $\|\cdot\|$ is a norm on X. We use the notations $x \in K$ and $x \geq 0$ synonymously and write $x > 0$ instead of $x \geq 0$, $x \neq 0$. The cone K is always assumed to be closed.

As usual X' denotes the vector space of all linear continuous functionals on X. On X' the weak* topology $\sigma(X', X)$ is considered.

If

$$A \colon X \longrightarrow X$$

is a linear positively invertible continuous operator (s. [9], [13], [20],) and the equation (1) is considered with $y \in K$, then in order to formulate the inverse maximum principle for A we look for some appropriate collection $\mathcal{F}$ of positive linear continuous functionals on X such that for any $x \in X$ the value $\alpha(x) = \sup\limits_{g \in \mathcal{F}} g(x)$ is finite and attained at some element of $\mathcal{F}$. Then for any $y \in K,\ y \neq 0$ and $x = A^{-1}y$ the sets

$$\mathcal{F}^{\max}(x) = \{f \in \mathcal{F} \colon\ f(x) = \alpha(x)\} \quad \text{and} \quad \mathcal{F}^{+}(y) = \{f \in \mathcal{F} \colon\ f(y) > 0\}$$

of all *maximizing* functionals for x and of all *strongly positive* functionals on y (provided such functionals exist) can be defined, respectively. Now replace the condition (2) by

$$\mathcal{F}^{\max}(x) \cap \mathcal{F}^{+}(y) \neq \varnothing. \tag{5}$$

A good candidate for $\mathcal{F}$ would be a base[3] of the dual cone K' of K which is compact in the locally convex topology $\sigma(X', X)$ in X'. The existence of a base for K' possessing the required properties is guaranteed if K additionally satisfies the condition $\mathrm{int}(K) \neq \varnothing$. Under this assumption K' is a cone[4] in X' (s. [18],§ II.2) and for each fixed $u \in \mathrm{int}(K)$ the set

$$\mathcal{F} = \{f \in K' \colon f(u) = 1\} \tag{6}$$

is a $\sigma(X', X)$-compact base for the cone K' (s.[19], Th.II.3.2). The Krein-Milman Theorem states that $\mathrm{ext}(\mathcal{F}) \neq \varnothing$ and that

$$\mathcal{F} = \overline{\mathrm{co}}^{\sigma(X', X)} \mathrm{ext}(\mathcal{F}). \tag{7}$$

Henceforth, it is assumed that an interior point u of K is fixed. This allows us to give the

[3] A base of the cone K in a vector space is a convex subset $\mathcal{F} \subset K$ such that each $g \in K$, $g \neq 0$ has a unique representation $g = \lambda f$, where $f \in \mathcal{F}$ and $\lambda > 0$.

[4] K' is always a wedge, but in general, does not satisfy the condition $K' \cap (-K') = \{0\}$. Moreover, $f \in K',\ f \neq 0$ and $x \in \mathrm{int}(K)$ implies $f(x) > 0$.

Definition 2 (s. [11]) Let be $A\colon\ X \longrightarrow X$. The pair $(A, \mathcal{F})$ is said to *satisfy the inverse maximum principle* (or synonymously, the operator A *satisfies the inverse maximum principle with respect to the base* $\mathcal{F}$), if for (1) the following is true: whenever $y \in K$, $y \neq 0$ then $x \geq 0$, and the relation (5) holds.

Since for each $y \in \operatorname{int}(K)$ one has $\mathcal{F}^+(y) = \mathcal{F}$ and for each $x \in K$ the set $\mathcal{F}^{\max}(x)$ is nonempty the relation $\mathcal{F}^{\max}(A^{-1}y) \cap \mathcal{F}^+(y) \neq \varnothing$ holds automatically for $y \in \operatorname{int}(K)$ and therefore (5) turns out to be a condition for boundary points y of K only (s. the remark after Definition 1).

For the finite dimensional space this notation was introduced[5] in [11] and applied to two special classes of cones in $\mathbb{R}^n$: finitely generated reproducing cones[6] and to circular cones. Notice that $X = \mathbb{R}^n,\ K = \mathbb{R}^n_+$ imply $\operatorname{int}(K) \neq \varnothing$ and $K' = \mathbb{R}^n_+$. Take now $u = (1, \ldots, 1) \in \operatorname{int}(K)$ and $\mathcal{F} = \operatorname{co}\{e^{(1)}, \ldots, e^{(n)}\}$ then (5) is exactly (2).

For any $x \in X$ the linear functional $\hat{x}\colon X' \longrightarrow \mathbb{R}$ defined by the element $x \in X$, i. e. $\hat{x}(f) = f(x)$ for each $f \in X'$, is $\sigma(X', X)$-continuous and achieves on $\mathcal{F}$ its maximum value $\alpha(x) = \max\limits_{f \in \mathcal{F}} f(x)$. So the set $\mathcal{F}^{\max}(x)$ is defined for any $x \in X$ and the set $\mathcal{F}^+(y)$ makes sense for any $y \in K, y \neq 0$.
The most important properties of the sets $\mathcal{F}^{\max}(x)$ and $\mathcal{F}^+(y)$ are gathered in Lemma 2.

Lemma 2 *The following statements hold:*

(a) $\mathcal{F}^+(x) \neq \mathcal{F}$ *if and only if* $x \in \partial K$.

(b) $\mathcal{F}^{\max}(y)$ *is for any* $y \in X$ *a nonempty* $\sigma(X', X)$*-compact extreme (convex) subset of* $\mathcal{F}$*, where* $\mathcal{F}^{\max}(y) \cap \operatorname{ext}(\mathcal{F}) \neq \varnothing$.

(c) $\mathcal{F}^{\max}(y) = \overline{\operatorname{co}}^{\sigma(X',X)} \{\mathcal{F}^{\max}(y) \cap \operatorname{ext}(\mathcal{F})\}$.

The proof of the lemma and also of some further properties of the mentioned sets and of the set $\mathcal{F} \setminus \mathcal{F}^+(x)$ is based on Bauer's Maximum Principle for real linear continuous functionals on a nonempty convex compact subset, and in its full length can be found in [7], (Lemma 3) (s. also [1], § 4.20).

Remark Looking in $\mathcal{F}$ for maximizing functionals for an element y the statement (b) of the Lemma tells us that we will find some of them already in $\operatorname{ext}(\mathcal{F})$. Therefore, we have

$$\alpha(x) = \max_{f \in \mathcal{F}} f(x) = \max_{f \in \operatorname{ext}(\mathcal{F})} f(x)\ .$$

The extreme points of $\mathcal{F}$ are used to represent the boundary ∂K of the cone K. For each

[5]This property was called there *maximum principle*. Since the operator A^{-1} is directly involved, we prefer the extended notation *inverse*, s. also [7].

[6]In finite dimensional spaces for a cone K the property to be reproducing is equivalent to $\operatorname{int}(K) \neq \varnothing$, s. [11], Th.1(b).

$f \in \text{ext}(\mathcal{F})$ denote $H_f = f^{-1}(0) \cap K$, and for any nonempty set $E \subset \text{ext}(\mathcal{F})$ denote

$$H_E = \bigcap_{f \in E} H_f.$$

It can be shown that $H_E \subset \partial K$ and that H_E is a closed extreme subset of the cone K for each such nonempty set E (s. [7]). In particular, the set H_f is extreme for any $f \in \text{ext}(\mathcal{F})$. Now we have (s. [7])

Lemma 3 *There holds the following representation:*

$$\partial K = \bigcup_{f \in \text{ext}(\mathcal{F})} H_f.$$

The sets H_f for $f \in \text{ext}(\mathcal{F})$ are also used to define subsets K_f of K, which allow us to formulate an equivalent condition for an operator to satisfy the inverse maximum principle with respect to the base $\mathcal{F}$. Set

$$K_f = \bigcup_{\lambda \geq 0} \{\lambda u - H_f\} \cap K$$

for each $f \in \text{ext}(\mathcal{F})$. We answer now the question for which $x \in K$ is the given functional $f \in \text{ext}(\mathcal{F})$ a maximizing one and represent the cone K as the union of the sets K_f for $f \in \text{ext}(\mathcal{F})$ (s. [7], Th.3).

Theorem 2 *The following statements hold:*

(a) Let $f \in \text{ext}(\mathcal{F})$ *and* $x \in K$. *Then*

$$f \in \mathcal{F}^{\max}(x) \quad \textit{if and only if} \quad x \in K_f.$$

(b)

$$K = \bigcup_{f \in \text{ext}(\mathcal{F})} K_f\ .$$

Let A be a linear positively invertible continuous operator in an ordered normed space $(X, K, \|\cdot\|)$ with a closed cone K that satisfies the condition $\text{int}(K) \neq \varnothing$. Let $u \in \text{int}(K)$ be chosen and $\mathcal{F}$ be constructed according to (6).
The inverse maximum principle is now characterized as follows.

Theorem 3 *The following statements are equivalent:*

(a) $\text{ext}(\mathcal{F}) \cap \mathcal{F}^{\max}(A^{-1}y) \cap \mathcal{F}^{+}(y) \neq \varnothing$ *for any* $y \in K$, $y \neq 0$.

(b) $(A, \mathcal{F})$ *satisfies the inverse maximum principle.*

(c) *For any $y \in K$, $y \neq 0$ there exists a functional $f \in \operatorname{ext}(\mathcal{F})$ such that $y \notin H_f$ and $A^{-1}y \in K_f$.*

(d) *The pair (A^{-1}, u) satisfies the following condition: for any nonempty set $E \subset \operatorname{ext}(\mathcal{F})$ there holds the inclusion*

$$A^{-1}(H_E) \subset \bigcup_{\substack{g \in \operatorname{ext}(\mathcal{F}) \\ g \notin E}} K_g \,.$$

The proof can be found in [7].

We come back to the solution $u = \mathcal{S}f$ of the boundary value problem (3), where $c(x) > 0$ is supposed for $x \in \Omega$. The property described in Lemma 1 refers in particular to the maximum of u on the compact set $\overline{\Omega}$. Let be $x^{(0)} \in \overline{\Omega}$ such that $u(x^{(0)}) = \max_{x \in \overline{\Omega}} u(x)$ and consider the positive linear continuous functional ε_0 on $C(\overline{\Omega})$ defined by

$$\varepsilon_0(v) = v(x^{(0)}), \quad v \in C(\overline{\Omega})\,.$$

If now the function $\mathbf{1}(x) = 1$, $\forall x \in \overline{\Omega}$ is chosen as the interior point of the cone of all nonnegative functions in $C(\overline{\Omega})$, and a base $\mathcal{F}$ in the dual cone is constructed according to (6), then the functional ε_0 belongs to $\operatorname{ext}(\mathcal{F})$ and satisfies the condition $\varepsilon_0(f) > 0$ (s. Lemma 1). Since the set $\operatorname{ext}(\mathcal{F})$ can be identified with the set $\overline{\Omega}$ one has

$$\varepsilon_0(\mathcal{S}f) = \varepsilon_0(u) = u(x^{(0)}) = \max_{\varepsilon \in \operatorname{ext}(\mathcal{F})} \varepsilon(\mathcal{S}f)\,.$$

From the Remark after Lemma 2 it follows that

$$\max_{\varepsilon \in \operatorname{ext}(\mathcal{F})} \varepsilon(\mathcal{S}f) = \max_{\varepsilon \in \mathcal{F}} \varepsilon(\mathcal{S}f) = \alpha(\mathcal{S}f).$$

This together with $\varepsilon_0(f) > 0$ shows that ε_0 belongs to $\mathcal{F}^+(f) \cap \mathcal{F}^{\max}(\mathcal{S}f)$. So the condition (a) of the last theorem holds and consequently, in particular, the pair $(\mathcal{S}, \mathbf{1})$ satisfies the condition (d) of Theorem 3. So, if $E \subset \overline{\Omega}$ then H_E is the set of all nonnegative functions $v \in C(\overline{\Omega})$ with $v(x) = 0$ for all $x \in E$. Since (s. [7])

$$K_x = \{v \in C(\overline{\Omega})\colon v \geq 0,\ \exists \lambda \geq 0,\ \text{ such that } v(y) \leq \lambda,\ \forall y \in \overline{\Omega},\ v(x) = \lambda\}$$

the condition (d) with A^{-1} replaced by $\mathcal{S}$ means that the solution u of (3) is a nonnegative function with maximum in $\overline{\Omega} \setminus E$ whenever $f \in H_E$. Moreover, from Lemma 1 there follows that its maximum is attained only on $\overline{\Omega} \setminus E$.

As an example we consider the boundary value problem

$$\left\{ \begin{array}{rcl} -(x_1 + x_2 + 5)\,\Delta u - \dfrac{\partial u}{\partial x_1} - \dfrac{\partial u}{\partial x_2} + u & = & f \ \text{ in } \ \Omega \\ u & = & 0 \ \text{ on } \ \partial\Omega\,, \end{array} \right.$$

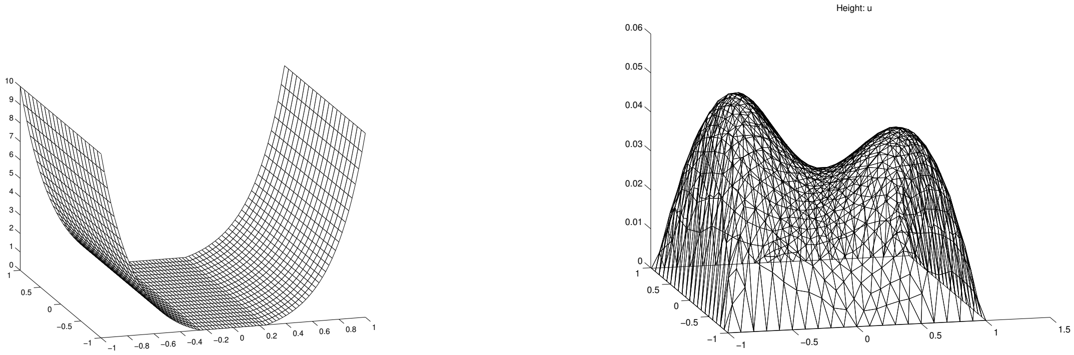

Figure 2: The function $f(x_1, x_2)$ and the solution $u(x_1, x_2)$

where $\Omega = \{(x_1, x_2)^T \in \mathbb{R}^2 \colon |x_1| \le 1,\ |x_2| \le 1\}$ and

$$f(x_1, x_2) = \begin{cases} 0\,, & \text{if} \quad |x_1| < 0.25 \\ 10\,\dfrac{\mathrm{e}^{4|x_1|} - \mathrm{e}}{\mathrm{e}^4 - \mathrm{e}}\,, & \text{if} \quad |x_1| \ge 0.25\,. \end{cases}$$

Its solution $u(x_1, x_2)$, which is illustrated in Figure 2, attains the maximum 0.053 at the point $(-0.66, -0.17) \in \mathrm{supp} f = \{(x_1, x_2)^T \in \Omega \colon |x_1| \ge 0.25\}$.

4 The Weighted Maximum Principle

The inverse maximum principle introduced by means of Definition 2 essentially depends on the base $\mathcal{F}$ and so according to (6) on the chosen point $u \in \mathrm{int}(K)$. We study now the alteration of the inverse maximum principle if the interior point $u = u^{(0)}$ is replaced by $u^{(1)} \in \mathrm{int}(K)$, where $u^{(1)}$ is not a positive multiple[7] of $u^{(0)}$.
Denote for $u^{(0)}$, $u^{(1)}$ according to (6)

$$\mathcal{F}_i = \{f \in K' \colon\ f(u^{(i)}) = 1\}, \quad i = 0, 1\,.$$

The extreme points of these bases of K' are exactly the intersections of $\mathcal{F}_i$ with the set of all extreme rays of K', $i = 0, 1$, respectively. Consequently the extreme points of $\mathcal{F}_0$ and $\mathcal{F}_1$ are in a one-to-one correspondence, and (with respect to footnote 2) a simple calculation shows

$$\mathrm{ext}(\mathcal{F}_0) = \left\{\frac{1}{h(u^{(0)})} h \colon\ h \in \mathrm{ext}(\mathcal{F}_1)\right\} \quad \text{and} \quad \mathrm{ext}(\mathcal{F}_1) = \left\{\frac{1}{f(u^{(1)})} f \colon\ f \in \mathrm{ext}(\mathcal{F}_0)\right\}\,.$$

[7]The case $u^{(1)} = \lambda u^{(0)}$ with $\lambda > 0$ is covered by the previous investigation.

It is clear that $f^{-1}(0) = h^{-1}(0)$ for $f \in \mathcal{F}_0$ and that its corresponding h belongs to $\mathcal{F}_1$. Therefore the sets H_f and H_h coincide and actually depend only on the extreme rays of K'.
Let the pair $(A, \mathcal{F}_0)$ satisfy the inverse maximum principle. Then for any $y \in K,\ y \neq 0$ there exists a functional $f \in \mathrm{ext}(\mathcal{F}_0)$ such that $f(A^{-1}y) = \max\limits_{g \in \mathrm{ext}(\mathcal{F}_0)} g(x)$ and $f(y) > 0$. Then for some $h \in \mathrm{ext}(\mathcal{F}_1)$

$$f(A^{-1}y) = \max_{g \in \mathrm{ext}(\mathcal{F}_0)} g(x) = \frac{1}{h(u^{(0)})} h(A^{-1}y) = \max_{k \in \mathrm{ext}(\mathcal{F}_1)} \frac{1}{k(u^{(0)})} k(A^{-1}y)$$

and $f(y) = h(u^{(0)})\, h(y) > 0$.
Summing up there holds the following lemma (s. [11]).

Lemma 4 *The operator A satisfies the inverse maximum principle with respect to the base $\mathcal{F}_0$ if and only if there exists a functional $h \in \mathrm{ext}(\mathcal{F}_1)$ satisfying the conditions*

$$\frac{1}{h(u^{(0)})} h(A^{-1}y) = \max_{k \in \mathrm{ext}(\mathcal{F}_1)} \frac{1}{k(u^{(0)})} k(A^{-1}y) \quad \textit{and} \quad h(y) > 0\ . \tag{8}$$

We come back to the finite dimensional case and establish a special version of Theorem 3 for that case. For this purpose consider the ordered normed space $(\mathbb{R}^n, \mathbb{R}^n_+)$ and fix some vector $\gamma = (\gamma_1, \ldots, \gamma_n)^T \in \mathbb{R}^n$ with $\gamma_i > 0$ for each $i = 1, \ldots, n$.

Definition 3 (s. [16]) A matrix A is said to satisfy the *weighted inverse maximum principle with respect to (the weight)* γ, if in the equation (1) there holds $x \geq 0$ and

$$\max_{1 \leq i \leq n} \gamma_i x_i = \max_{i \in N^+(y)} \gamma_i x_i\ , \tag{9}$$

whenever $y \geq 0$.

For $\gamma = \mathbf{1} = (1, \ldots, 1)^T$ we obtain the usual inverse maximum principle (2).
Since $K' = K = \mathbb{R}^n_+$ and $u^{(0)} = (\frac{1}{\gamma_1}, \ldots, \frac{1}{\gamma_n})^T \in \mathrm{int}(K)$, the set

$$\mathcal{F} = \mathcal{F}_0 = \{z \in \mathbb{R}^n_+ :\ \langle z, u^{(0)} \rangle = 1\} \tag{10}$$

is a base of the cone K' and $\mathrm{ext}(\mathcal{F}) = \{\gamma_i e^{(i)} :\ i = 1, \ldots, n\}$. Then $\mathrm{ext}(\mathcal{F}) \cap \mathcal{F}^{\max}(x) \neq \varnothing$ for each $x \in \mathbb{R}^n$ (s. Lemma 2(b)) implies

$$\alpha(x) = \max_{z \in \mathcal{F}} \{\langle x, z \rangle\} = \max_{z \in \mathrm{ext}(\mathcal{F})} \{\langle x, z \rangle\} = \max_{1 \leq i \leq n} \{\langle x, \gamma_i e^{(i)} \rangle\} = \max_{1 \leq i \leq n} \gamma_i x_i\ , \tag{11}$$

where $\langle x, z \rangle = \sum_{i=1}^n x_i z_i$ denotes the scalar product of the vectors $x, z \in \mathbb{R}^n$. Let A in (1) be a positively invertible matrix. If A satisfies the inverse maximum principle, then the relation of Theorem 3(a) holds. We show first

Lemma 5 *The matrix A satisfies the inverse maximum principle with respect to the base $\mathcal{F}$, defined in (10), if and only if it satisfies the weighted maximum principle with respect to the weight γ.*

Proof. $\Longrightarrow$: Let be $y \geq 0$ and $x = A^{-1}y$. There exists a number $k \in \{1, \ldots, n\}$ such that $\gamma_k e^{(k)} \in \mathcal{F}^{\max}(x) \cap \mathcal{F}^{+}(y)$. Then due to $\mathcal{F}^{\max}(x) = \{z \in \mathcal{F}\colon \langle x, z\rangle = \alpha(x)\}$ and with respect to (11) one has on one hand

$$\langle x, \gamma_k e^{(k)}\rangle = \gamma_k x_k = \max_{1\leq i\leq n} \gamma_i x_i \ .$$

On the other hand one has $\gamma_k e^{(k)} \in \mathcal{F}^{+}(y) = \{z \in \mathcal{F}\colon\ \langle y, z\rangle > 0\}$, which yields $\langle y, \gamma_k e^{(k)}\rangle = \gamma_k y_k > 0$ and, because of $\gamma_k > 0$ also $y_k > 0$. So $k \in N^{+}(y)$. Therefore (9) holds.
$\Longleftarrow$: From (9) there follows the existence of a number $k \in N^{+}(y)$ with

$$\gamma_k x_k = \max_{1\leq i\leq n} \gamma_i x_i = \alpha(x)$$

and $y_k > 0$. Then obviously

$$\gamma_k e^{(k)} \in \operatorname{ext}(\mathcal{F}) \cap \mathcal{F}^{\max}(x) \cap \mathcal{F}^{+}(y) \ .$$

■

The base $\mathcal{F}$, more exactly the set $\operatorname{ext}(\mathcal{F})$, of the cone $K' = \mathbb{R}^n_+$ defined by means of the element $u^{(0)}$ and used in Lemma 5 determines a decomposition of the cone $K = \mathbb{R}^n_+$ according to Theorem 2(b) of the general theory. This decomposition induces a decomposition of the standard base

$$\mathcal{F}_1 = \operatorname{co}\{e^{(1)}, \ldots, e^{(n)}\} = \{x \in \mathbb{R}^n_+\colon \langle x, u^{(1)}\rangle = 1\},$$

where $u^{(1)} = (1, \ldots, 1)^T$.
Let be $N = \{1, \ldots, n\}$. Using the description $\operatorname{ext}(\mathcal{F}) = \{\gamma_i e^{(i)}\colon\ i \in N\}$ for each nonempty $E \subset N$ we have

$$\begin{aligned} H_E &= \{x \in \mathbb{R}^n_+\colon\ \langle x, \gamma_k e^{(k)}\rangle = 0, \quad \forall\, k \in E\} \\ &= \{x\colon\ x_i \geq 0, \quad \forall\, i \in N \quad \text{and} \quad x_k = 0, \quad \forall\, k \in E\} \\ &= \operatorname{c}\{e^{(i)}\colon\ i \in N \setminus E\}. \end{aligned} \tag{12}$$

Denote the set $H_{\{l\}}$ by H_l, $l \in N$. Then an element $x = (x_1, \ldots, x_n)^T$ belongs to

$$K_l = \bigcup_{\lambda\geq 0} \{\lambda u^{(0)} - H_l\} \cap K$$

if there exist $\lambda \geq 0$, $z = (z_1, \ldots, z_n)^T \in K$ such that $z_i \geq 0$, $z_l = 0$ and

$$x_i = \lambda \frac{1}{\gamma_i} - z_i, \quad i \neq l. \tag{13}$$

Denote $\mathcal{F}_1$ by $\mathcal{D}$ and consider the partition of $\mathcal{D}$ into the sets

$$\mathcal{D}^{(\gamma)}_k = \{x = (x_1, \ldots, x_n)^T \in \mathcal{D}\colon \gamma_k x_k \geq \gamma_i x_i,\ i = 1, \ldots, n\} \quad k = 1, \ldots, n.$$

Notice that $\mathrm{ext}(\mathcal{D}) = \{e^{(1)}, \ldots, e^{(n)}\}$, $\mathrm{c}(\mathcal{D}_l^{(\gamma)}) = \bigcup_{\beta \geq 0} \beta \mathcal{D}_l^{(\gamma)}$ and that $x \in \mathrm{c}(\mathcal{D}_l^{(\gamma)})$ if and only if $\gamma_l x_l \geq \gamma_i x_i$, $i \in N$. Then

$$K_l = \mathrm{c}(\mathcal{D}_l^{(\gamma)}) \ . \tag{14}$$

Indeed, $x \in K_l$ has the representation (13), i. e. $\gamma_i x_i = \lambda - \gamma_i z_i$, from where $\gamma_l x_l = \lambda \geq \gamma_i z_i$ follows for all $i \in N$. So $x \in \mathrm{c}(\mathcal{D}_l^{(\gamma)})$. Conversely, for $x \in \mathrm{c}(\mathcal{D}_l^{(\gamma)})$ put

$$\lambda = \gamma_l x_l \quad \text{and} \quad z_i = \frac{\lambda}{\gamma_i} - x_i = \frac{\gamma_l}{\gamma_i} x_l - x_i, \ i \in N \ .$$

Then $\gamma_l x_l \geq \gamma_i x_i$ implies $z_i \geq 0$ and $z_l = 0$. This shows $z \in H_l$, and hence the equality

$$\lambda u_i - z_i = \frac{\gamma_l x_l}{\gamma_i} - \frac{\gamma_l x_l}{\gamma_i} + x_i = x_i$$

implies $x \in K_l$.

We are now able with the help of Lemma 5 to characterize the *weighted* inverse maximum principle similarly as the usual one[8] in Theorem 1.

Theorem 4 *Let A be a positively invertible (n,n)-matrix. Then A satisfies the weighted inverse maximum principle with respect to γ if and only if*

$$A^{-1}\left(\mathrm{co}\{e^{(i)} \colon i \in N'\}\right) \subset \bigcup_{i \in N'} \mathrm{c}(\mathcal{D}_i^{(\gamma)})$$

holds for any nonvoid subset $N' \subset \{1, \ldots, n\}$.

Proof. First of all remark that for any nonempty subset $N' \subset N$ one has

$$\mathrm{c}\left(A^{-1}(\mathrm{co}\{e^{(i)} \colon \ i \in N'\})\right) = A^{-1}\left(\mathrm{c}\{e^{(i)} \colon \ i \in N'\}\right) \ .$$

With respect to (12) and (14) the condition contained in Theorem 3(d) can be written as follows:

$$\text{For any} \quad E \subset N \quad \text{there holds} \quad A^{-1}\left(\mathrm{c}\{e^{(i)} \colon \ i \in N \setminus E\}\right) \subset \bigcup_{l \in N \setminus E} \mathrm{c}(\mathcal{D}_l^{(\gamma)}) \ .$$

After replacing its left side according to the remark, this is equivalent to

$$\text{For any} \quad E \subset N \quad \text{there holds} \quad A^{-1}\left(\mathrm{co}\{e^{(i)} \colon \ i \in N \setminus E\}\right) \subset \bigcup_{l \in N \setminus E} \mathrm{c}(\mathcal{D}_l^{(\gamma)}).$$

If $N \setminus E$ is denoted by N' the last inclusion is exactly the condition of Theorem 4. ∎

If $A = (a_{ij})_{i,j=1}^n$ is an M-matrix[9], i. e. there exists A^{-1}, $A^{-1} \geq 0$ and $a_{ij} \leq 0$ for $i \neq j$, then the following result was proved in [16].

[8] A raw version of Theorem 4 is already contained in [16] (Th.1).

[9] Although M-matrices are of interest by their own they are important in Numerical Analysis, s. [21].

Theorem 5 *Let A be an M-matrix of order n and $\gamma = (\gamma_1, \ldots, \gamma_n)^T \in \mathbb{R}^n$ with $\gamma_i > 0$, for $i = 1, \ldots, n$.*
Then A satisfies the weighted inverse maximum principle with respect to γ if and only if

$$A \left(\frac{1}{\gamma_1}, \ldots, \frac{1}{\gamma_n} \right)^T \geq 0 \,.$$

References

[1] Aliprantis, C.D., Border, K.C. *Infinite Dimensional Analysis.* Springer-Verlag, 1994.

[2] Amann, H. On the number of solutions of nonlinear equations in ordered Banach spaces. *J. Functional Analysis*, 11:346–384, 1972.

[3] Amann, H. Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. *SIAM Review*, 18, No.4:620–709, 1976.

[4] Ciarlet, P.G., Raviart, P.A. Maximum principle and uniform convergence for the finite element method. *Comput.Methods Appl. Mech. Engrg.*, 2:17–31, 1973.

[5] Gilbarg, D., Trudinger, N.S. *Elliptic Partial Differential Equations of Second Order.* (Russian), Nauka, Fismatlit, Moscow, 1989.

[6] Hellwig, G. *Partielle Differentialgleichungen.* B.G. Teubner Verlagsgesellschaft, Stuttgart, 1960.

[7] Kalauch, A., Weber, M.R. On some maximum principle for positive operators in an ordered normed space. *Preprint, TU Dresden. No. MATH-AN-13-1998*, 1998.

[8] Krasnosel'skij, M.A. *Positive Solutions of Operator Equations,.* P. Noordhoff, Groningen, The Netherlands, 1964.

[9] Peris, J.E. A new characterization of inverse-positive matrices. *Linear Algebra Appl.*, 154–156:45–58, 1991.

[10] Protter, M.H., Weinberger, H.F. *Maximum Principles in Differential Equations.* Englwood Cliffs, Prentice Hall, New York, 1967.

[11] Pühl, H., Schirotzek, W., Weber, M.R. On matrices satisfying a maximum principle with respect to a cone. *Linear Algebra Appl.*, 277/1-3:337–356, 1998.

[12] Samarskij, A.A. *Theorie der Differenzenverfahren.* Akademische Verlagsgesellschaft, Geest und Portig K.-G., Leipzig, 1984.

[13] Schröder, J. Lineare Operatoren mit positiver Inversen. *Arch. Rat. Mech. Analysis*, 8:408–434, 1961.

[14] Stoyan, G. On a maximum principle for matrices and on conservation of monotonicity with applications to discretization methods. *ZAMM*, 62:375–381, 1982.

[15] Stoyan, G. On the maximum principles for monotone matrices. *Linear Algebra Appl.*, 78:147–161, 1986.

[16] Türke, C., Weber, M.R. On a maximum principle for inverse monotone matrices. *Linear Algebra Appl.*, 218:47–57, 1995.

[17] Varga, R.S. On the discrete maximum principle. *SIAM J. Numer.Analysis*, 3:355–359, 1966.

[18] Vulikh, B.Z. *Introduction to the theory of cones in normed spaces.* (Russian). Izdat. Kalinin Univ., Kalinin, 1977.

[19] Vulikh, B.Z. *Special topics of geometry of cones in normed spaces.* (Russian). Izdat. Kalinin Univ., Kalinin, 1978.

[20] Weber, M.R. On the positiveness of the inverse operator. *Math. Nachr.*, 163:145–149, 1993. Erratum: *Math. Nachr.*, 171:325-326, 1995.

[21] Windisch, G. *M-matrices in Numerical Analysis.* Teubner-Texte Math. 115, Leipzig, 1989.

Keywords: Positive operator; ordered normed space; base of a cone;
maximum principle
AMS subject classification: primary 46A40, 46A55, secondary 47B60, 52A07
Address: Technical University of Dresden, Department of Mathematics,
Institute of Analysis. D - 01062 Dresden, Germany