Straight configurations of shearable nonlinearly elastic rods

Friedemann Schuricht *
Max-Planck-Institute for Mathematics in the Sciences
Inselstr. 22-26, 04103 Leipzig, Germany

1 Introduction

Investigating obstacle problems for elastic rods we are sometimes confronted with the question to look for a solution which has a prescribed shape along some part of it. In the simplest case the rod is enforced to be straight along some contact area (cf., e.g., Gastaldi & Kinderlehrer [3]). Motivated by such applications we study straight configurations of elastic rods in this paper. More precisely, as in the case of frictionless contact, we consider equilibrium configurations of planar rods which are enforced to be straight by special external forces that are orthogonal to the straight axis.

The reader might have in mind the mostly used Euler elastica (or simplifications of it), which neglects shear, extension, and thickness, and our subject appears to be very boring, since in that case only the trivial solution is straight, and even a school boy would not spend some attention to such a triviality. However, based on the Cosserat theory which describes planar deformations of nonlinearly elastic rods that can bend, strech, and shear and which takes into acccount an exact two- or three-dimensional geometry, the problem becomes much more subtle. In contrast to the more primitive models, we shall obtain an interesting richness of structure of this appearently simple problem.

An important observation for our general rod theory, which does not neglect thickness, is that we must say more precisely what we mean by a straight configuration. A Cosserat rod describes a "slender" two- or three-dimensional elastic body and, by the nontrivial interaction of flexure, shear, and extension, originally parallel material curves of the rod do not remain parallel under deformations in general. Thus some special material curve which we want to be straight in the deformed configuration has to be selected. Usually this will be some curve of centroids or a certain boundary curve. The last case is from particular interest for contact problems with a straight obstacle.

After introducing the underlying Cosserat theory for planar rods in Section 2, we show in Section 3 how the problem of straight configurations can be reduced to a second order system of ordinary differential equations. Based on some basic transformation rules for the reference

*Partially supported by Deutscher Akademischer Austauschdienst

curve we also discuss some alternative approaches. In Section 4 we restrict our attention to the case where the system of differential equations is autonomous, and we investigate its qualitative properties by phase plane analysis. Depending on the special choice of the constitutive law and of the external reactions as, e.g., terminal loads and weight, we verify a very rich behavior of straight equilibrium configuration. In Section 5 we finally apply our abstract results to some special examples. This way we illuminate a number of interesting effects which cannot be verified within an unshearable rod model.

2 Rod theory

In this section we formulate the Cosserat or director theory describing planar deformations of nonlinearly elastic rods which can bend, stretch and shear. For a more comprehensive presentation the reader is referred to Antman [2] and Schuricht [4].

Geometry of deformation. Let $\{\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ be a fixed right-handed orthonormal basis in $\mathbb{R}^3$. We consider a slender three-dimensional body $\mathcal{B}$ that is symmetric with respect to the $\{\boldsymbol{i},\boldsymbol{j}\}$-plane and we study deformations that preserve this symmetry. The deformed rod is identified with the region occupied by its intersection with the $\{\boldsymbol{i},\boldsymbol{j}\}$-plane. A *configuration* of the rod is described by a pair of absolutely continuous vector-valued functions $\{\boldsymbol{r}(\cdot),\boldsymbol{b}(\cdot)\}$, lying in the $\{\boldsymbol{i},\boldsymbol{j}\}$-plane, and we assume that the *position field* $\boldsymbol{p}$ of the deformed material points has the form

$$\boldsymbol{p}(s,\zeta) \;=\; \boldsymbol{r}(s)+\zeta\boldsymbol{b}(s) \quad \text{for} \;\; (s,\zeta)\in\Omega, \tag{2.1}$$

where

$$\Omega \;\equiv\; \{(s,\zeta)|\, s\in[0,L],\; h_-(s)\le\zeta\le h_+(s)\}.$$

We interprete $\boldsymbol{r}$ as the deformed configuration of some material curve in the body $\mathcal{B}$, the so-called *base curve* (e.g., the curve of centroids or a suitable boundary curve). The *director* $\boldsymbol{b}(s)$ is a unit vector which describs the orientation of the cross-section at s. We interpret s as length parameter and ζ as thickness parameter. $h_\pm$ are given bounded functions such that $h_-(s)\le 0\le h_+(s)$ and $h_-(s)<h_+(s)$.

We introduce $\boldsymbol{a}\equiv -\boldsymbol{k}\times\boldsymbol{b}$ and the angle θ from $\boldsymbol{i}$ to $\boldsymbol{a}$ such that

$$\boldsymbol{a}=\cos\theta\,\boldsymbol{i}+\sin\theta\,\boldsymbol{j}, \quad \boldsymbol{b}=-\sin\theta\,\boldsymbol{i}+\cos\theta\,\boldsymbol{j}. \tag{2.2}$$

Hence a configuration can be alternatively described by a pair $\{\boldsymbol{r}(\cdot),\theta(\cdot)\}$.

We define the strains $\boldsymbol{\xi}\equiv(\nu,\eta,\mu)$ for a configuration by

$$\boldsymbol{r}' \;=\; \nu\boldsymbol{a}+\eta\boldsymbol{b}, \quad \mu \;\equiv\; \theta',$$

By the absolute continuity of $\boldsymbol{r}(\cdot)$ and $\theta(\cdot)$, the strains must be integrable functions on $[0,L]$. With $\boldsymbol{r}_0\equiv\boldsymbol{r}(0)$, $\theta_0\equiv\theta(0)$ we have the representation

$$\begin{aligned}\boldsymbol{r}(s) &= \boldsymbol{r}_0+\int_0^s\Big[(\nu\cos\theta-\eta\sin\theta)\boldsymbol{i}+(\nu\sin\theta+\eta\cos\theta)\boldsymbol{j}\Big]\,d\tau,\\ \theta(s) &= \theta_0+\int_0^s\mu\,d\tau.\end{aligned}$$

The natural undeformed state of the rod is called its *reference configuration.* Corresponding variables are identified by a superposed circle and, usually, it is assumed that

$$\mathring{\nu} = 1, \ \mathring{\eta} = 0, \quad \text{i.e.,} \quad \mathring{\boldsymbol{r}}' = \mathring{\boldsymbol{a}} \,. \tag{2.3}$$

This expresses that the cross-sections are orthogonal to the base curve and that s is the arc-length of the base curve. For an originally straight rod we obviously have that $\mathring{\mu} = 0$.

The requirement that deformations be locally *orientation-preserving* can be expressed by the condition that

$$\nu(s) > V(\mu(s), s) \ \text{ for } \ s \in [0, L], \tag{2.4}$$

where

$$V(\mu, s) \ \equiv \ \begin{cases} h_+(s)\mu & \text{for } \mu \geq 0, \\ h_-(s)\mu & \text{for } \mu \leq 0. \end{cases}$$

Forces and equilibrium conditions. We identify subbodies $\breve{\mathcal{B}} \subset \mathcal{B}$ with the corresponding subset $\breve{\Omega} \subset \Omega$. In particular we define

$$\Omega_{\mathcal{I}} \ \equiv \ \{(\tau, \zeta) \in \Omega |\ \tau \in \mathcal{I}\} \quad \text{for } \mathcal{I} \subset [0, L] \,, \quad \Omega_s \ \equiv \ \Omega_{[s,L]} \,.$$

The material of $\Omega_{[s,L]}$ exerts the *resultant force* and the *resultant couple*

$$\boldsymbol{n}(s) \ \equiv \ N(s)\boldsymbol{a}(s) + H(s)\boldsymbol{b}(s) \quad \text{and} \quad \boldsymbol{m}(s) \ \equiv \ M(s)\boldsymbol{k} \,,$$

respectively, on the material of $\Omega_{[0,s[}$ for $s \in]0, L]$. For technical convenience we set

$$\boldsymbol{n}(0) \ \equiv \ \boldsymbol{0} \,, \quad \boldsymbol{m}(0) \ \equiv \ \boldsymbol{0} \,. \tag{2.5}$$

All other *external forces* acting at the body may be described by a finite vector-valued Borel measure $\mathfrak{f}$ on Ω, i.e., $\mathfrak{f}(\breve{\Omega})$ lies in the $\{\boldsymbol{i}, \boldsymbol{j}\}$-plane and assigns the resultant force to subbodies which correspond to Borel sets $\breve{\Omega} \subset \Omega$. It produces the *induced couple*

$$\mathfrak{l}_{\mathfrak{f}}(\breve{\Omega}) \ \equiv \ \int_{\breve{\Omega}} \Big(\boldsymbol{p}(s, \zeta) - \boldsymbol{r}(s) \Big) \times d\mathfrak{f}(s, \zeta) \ = \ \int_{\breve{\Omega}} \zeta \boldsymbol{b}(s) \times d\mathfrak{f}(s, \zeta) \,.$$

All *external couples* different from $\boldsymbol{m}$ and $\mathfrak{l}_{\mathfrak{f}}$ may correspond to a finite vector-valued Borel measure $\mathfrak{l}$ on Ω where $\mathfrak{l}(\breve{\Omega})$ is orthogonal to the $\{\boldsymbol{i}, \boldsymbol{j}\}$-plane. Using the distribution functions

$$\boldsymbol{f}(s) \ \equiv \ \int_{\Omega_s} d\mathfrak{f}(\tau, \zeta) \,, \quad \boldsymbol{l}(s) \ \equiv \ \int_{\Omega_s} d\mathfrak{l}(\tau, \zeta) \,, \quad \boldsymbol{l}_f(s) \ \equiv \ \int_{\Omega_s} d\mathfrak{l}_{\mathfrak{f}}(\tau, \zeta) \,,$$

we can formulate the *equilibrium equations* for the rod

$$\begin{aligned} \boldsymbol{n}(s) \ - \ \boldsymbol{f}(s) \quad &= \quad \boldsymbol{0} \quad \text{for } \ s \in [0, L] \,, \\ \boldsymbol{m}(s) \ - \ \int_s^L \boldsymbol{r}'(\tau) \times \boldsymbol{n}(\tau)\, d\tau \ - \ \boldsymbol{l}_f(s) \ - \ \boldsymbol{l}(s) \quad &= \quad \boldsymbol{0} \quad \text{for } \ s \in [0, L] \,. \end{aligned}$$

Observe that (2.5) just implies that the external reaction exerted on the whole rod has to vanish.

An important external force is weight. Let $\varrho(s, \zeta) > 0$ be an integrable mass density of the rod and let $\boldsymbol{g} = g_1 \boldsymbol{i} + g_2 \boldsymbol{j}$ be the acceleration of gravity. Then the corresponding entries $\boldsymbol{f}$ and $\boldsymbol{l}_f$ for the rod theory are

$$\boldsymbol{f}(s) \ = \ \boldsymbol{g} \int_s^L \varrho_0(\tau)\, d\tau \,, \quad \boldsymbol{l}_f(s) \ = \ -\boldsymbol{k} \int_s^L \Big(g_1 \cos \vartheta(\tau) + g_2 \sin \vartheta(\tau) \Big) \varrho_1(\tau)\, d\tau \,, \tag{2.6}$$

where

$$\varrho_i(\tau) \;=\; \int_{h_-(\tau)}^{h_+(\tau)} \zeta^i \varrho(\tau,\zeta)\, d\zeta\,, \quad i = 1,2.$$

Constitutive functions. The material of the rod is taken to be *elastic*, i.e., there exist *constitutive functions* $\hat{\boldsymbol{\Xi}} = (\hat{N}, \hat{H}, \hat{M})$ depending on (ν, η, μ, s) such that the stress resultants are determined by the strains through

$$N = \hat{N}(\nu,\eta,\mu,s)\,, \quad H = \hat{H}(\nu,\eta,\mu,s)\,, \quad M = \hat{M}(\nu,\eta,\mu,s)\,. \tag{2.7}$$

The domain of definition is restricted by (2.4). To avoid technicalities we adopt the usual restriction that $\boldsymbol{\xi} \to \hat{\boldsymbol{\Xi}}(\boldsymbol{\xi}, s)$ is continuously differentiable for all s. Since the shape of a configuration is not influenced by a change of the strains $\boldsymbol{\xi}(\cdot)$ on a set of measure zero, the constitutive relations in (2.7) can provide the correct resultant reactions at best at almost every cross-section s.

As a consequence of the Strong Ellipticity Condition we require that the Jacobian $\hat{\boldsymbol{\Xi}}_{\boldsymbol{\xi}}(\boldsymbol{\xi}, s)$ is strictly positive definite for all $\boldsymbol{\xi}$ and all s. Consequently $\boldsymbol{\xi} \to \hat{\boldsymbol{\Xi}}(\boldsymbol{\xi}, s)$ is strictly monotone for each s. The natural condition that extreme strains are accompanied by extreme reactions is ensured by the infinity conditions

$$\begin{aligned}
&\hat{N}(\nu,\eta,\mu,s) \to \begin{Bmatrix} +\infty \\ -\infty \end{Bmatrix} && \text{as} \quad \nu \to \begin{Bmatrix} +\infty \\ V(\mu,s) \end{Bmatrix}, \\
&\hat{H}(\nu,\eta,\mu,s) \to \pm\infty && \text{as} \quad \eta \to \pm\infty\,, \\
&\hat{M}(\nu,\eta,\mu,s) \to \pm\infty && \text{as} \quad \mu \ \text{approaches its positive and negative} \\
& && \qquad \text{extremes of the region (2.4).}
\end{aligned}$$

In general we have the symmetry properties that

$$\hat{N}(\nu,\cdot,\mu,s), \ \ \hat{M}(\nu,\cdot,\mu,s) \quad \text{are even,} \quad \hat{H}(\nu,\cdot,\mu,s) \quad \text{is odd.} \tag{2.8}$$

The additional symmetry conditions that

$$\hat{N}(\nu,\eta,\cdot,s), \ \ \hat{H}(\nu,\eta,\cdot,s) \quad \text{are even,} \quad \hat{M}(\nu,\eta,\cdot,s) \quad \text{is odd} \tag{2.9}$$

are satisfied for an originally straight rod with respect to the base curve of centroids.

The strict monotonicity of $\hat{\boldsymbol{\Xi}}(\cdot, s)$ and the coercivity condition that

$$\|\hat{\boldsymbol{\Xi}}(\boldsymbol{\xi}, s)\| \to \infty \quad \text{as} \quad \|\boldsymbol{\xi}\| \to \infty \ \text{ or } \ \nu - V(\mu,s) \to 0\,, \tag{2.10}$$

which is slightly stronger than the above infinity conditions, support a global implicit function theorem which ensures a unique solution of (2.7), i.e., there exist functions $\hat{\boldsymbol{\xi}} = (\hat{\nu}, \hat{\eta}, \hat{\mu})$ such that

$$\nu = \hat{\nu}(N,H,M,s)\,, \quad \eta = \hat{\eta}(N,H,M,s)\,, \quad \mu = \hat{\mu}(N,H,M,s)\,. \tag{2.11}$$

The mapping $\boldsymbol{\Xi} \to \hat{\boldsymbol{\xi}}(\boldsymbol{\Xi}, s)$ is also strictly monotone and inherits analog infinity, smoothness, and symmetry properties of $\hat{\boldsymbol{\Xi}}$. Henceforth we simply assume that (2.7) and (2.11) are equivalent. We readily also obtain equivalent sets of constitutive functions of the form

$$N = \breve{N}(\nu,\eta,M,s)\,, \quad H = \breve{H}(\nu,\eta,M,s)\,, \quad \mu = \breve{\mu}(\nu,\eta,M,s)\,, \tag{2.12}$$

or

$$\nu = \breve{\nu}(N, H, \mu, s)\,, \quad \eta = \breve{\eta}(N, H, \mu, s)\,, \quad M = \breve{M}(N, H, \mu, s)\,, \tag{2.13}$$

which again have properties analogous to the original constitutive functions.

A material is called hyperelastic if there exists a *stored energy function* $\hat{W}$ depending on (ν, η, μ, s) such that

$$\hat{N} \ = \ \hat{W}_\nu\,, \quad \hat{H} \ = \ \hat{W}_\eta\,, \quad \hat{M} \ = \ \hat{W}_\mu\,.$$

Usually the natural growth condition that

$$\hat{W}(\nu, \eta, \mu, s) \ \to \ \infty \quad \text{as} \quad \|\boldsymbol{\xi}\| \to \infty \ \text{ or } \ \nu - V(\mu, s) \to 0$$

is adopted for hyperelastic materials. With

$$\breve{W}(\nu, \eta, M, s) \ \equiv \ \hat{W}(\nu, \eta, -\breve{\mu}(\nu, \eta, M, s), s) + M\breve{\mu}(\nu, \eta, M, s)$$

we get an alternative stored energy function such that

$$\breve{N} \ = \ \breve{W}_\nu\,, \quad \breve{H} \ = \ \breve{W}_\eta\,, \quad \breve{\mu} \ = \ \breve{W}_M\,,$$

and we naturally invoke that

$$\breve{W}(\nu, \eta, M, s) \ \to \ \infty \quad \text{as} \quad \|(\nu, \eta, M)\| \to \infty \ \text{ or } \ \nu \to 0\,. \tag{2.14}$$

We say that we have a *material without shear instabilities* if

$$\Big(\nu \hat{H}(\nu, \eta, \mu, s) - \eta \hat{N}(\nu, \eta, \mu, s)\Big)\eta \ > \ 0 \quad \text{for all } \ (\nu, \eta, \mu, s) \ \text{ with } \ \eta \neq 0\,.$$

This condition just prevents the appearance of shear instabilities as described by Antman (cf. Antman [2]).

Transformation of the base curve. The presented formulation of the rod theory obviously depends on the choice of a special reference curve. In a forthcoming paper it will be shown that we obtain an equivalent formulation of the rod theory by a transformation of the base curve of the form

$$\boldsymbol{r}_*(s_*(s)) \ = \ \boldsymbol{r}(s) + h\boldsymbol{b}(s)\,.$$

Here $h \in \mathbb{R}$ is some constant with $\sup_{s\in[0,L]} h_-(s) \le h \le \inf_{s\in[0,L]} h_+(s)$, and $s_*(\cdot)$ is an absolutely continuous mapping such that $s_*'(s) = 1 - h \overset{\circ}{\mu}(s)$ a.e. on $[0, L]$. Identifying all values with respect to the new base curve by a subscript '$_*$', we have the following transformation rules

$$\boldsymbol{a}_* \ = \ \boldsymbol{a}, \quad \boldsymbol{b}_* \ = \ \boldsymbol{b}, \quad \vartheta_* \ = \ \vartheta,$$

$$\nu_* \ = \ \frac{\nu - h\mu}{1 - h\overset{\circ}{\mu}}\,, \quad \eta_* \ = \ \frac{\eta}{1 - h\overset{\circ}{\mu}}\,, \quad \mu_* \ = \ \frac{\mu}{1 - h\overset{\circ}{\mu}}\,,$$

$$N_* \ = \ N, \quad H_* \ = \ H, \quad M_* \ = \ M + hN.$$

The constitutive functions transform correspondingly.

3 Straight configurations

3.1 Formulation of the problem

Let us now study deformed equilibrium configurations of the rod where the base curve $s \to \boldsymbol{r}(s)$ is enforced to be straight by suitable external forces. Note that, in contrast to unshearable models where the cross-sections are always supposed to remain orthogonal to the base curve, this does not imply that all other curves $s \to \boldsymbol{r}(s) + \zeta \boldsymbol{b}(s)$ with $\zeta \neq 0$ are straight. This fact makes our appearently simple problem nontrivial and interesting. To be precise we assume that we have some mechanism which enforces the base curve to be straight by exerting a distributed force along this straight base curve and that force be directed orthogonal to the straight axis. We could, e.g., imagine that we remove all material points in a cylindrical neighborhood of the base curve of the rod, put a rigid wire into this hole, and neglect friction between the rod and that wire. This way the rigid wire exerts suitable contact forces to the rod such that it remains straight. If the base curve is some boundary curve, then we model the situation where the rod is in frictionless contact with a straight obstacle.

It is convenient to decompose the external force according to

$$\mathfrak{f} \;=\; \mathfrak{f}_{\mathrm{p}} + \mathfrak{f}_{\mathrm{u}}$$

where $\mathfrak{f}_{\mathrm{p}}$ is some prescribed known force as, e.g., weight and $\mathfrak{f}_{\mathrm{u}}$ is the unknown force which is necessary to enforce the base curve to be straight. Accordingly we define the distribution functions $\boldsymbol{f}_{\mathrm{p}}$, $\boldsymbol{f}_{\mathrm{u}}$, $\boldsymbol{l}_{f_{\mathrm{p}}}$, and $\boldsymbol{l}_{f_{\mathrm{u}}}$. Since $\mathfrak{f}_{\mathrm{u}}$ acts along the base curve, the couple $\mathfrak{l}_{\mathfrak{f}_{\mathrm{u}}}$ vanishes and, thus, $\boldsymbol{l}_{f_{\mathrm{u}}}$ is identical zero. We restrict our considerations to cases where $\boldsymbol{f}_{\mathrm{p}} = \boldsymbol{f}_{\mathrm{p}}(s)$ is a prescribed function depending at most on s and where there exist functions $\bar{\boldsymbol{l}}_{f_{\mathrm{p}}}$, $\bar{\boldsymbol{l}}$ depending on (ϑ, s) such that

$$\boldsymbol{l}'_{f_{\mathrm{p}}}(s) \;=\; \bar{\boldsymbol{l}}_{f_{\mathrm{p}}}(\vartheta(s), s)\,, \quad \boldsymbol{l}'(s) \;=\; \bar{\boldsymbol{l}}(\vartheta(s), s)$$

for all $s \in]0, L[$ along solutions. This way we cover, e.g., the case of a general inhomogeneous weight distribution and of general terminal loads (cf. Section 2).

Without loss of generality we assume that the base curve be parallel to the $\boldsymbol{i}$-axis. Then we can describe our restriction for the rod by the conditions

$$\boldsymbol{r}'(s) \cdot \boldsymbol{j} \;=\; 0\,, \quad \boldsymbol{f}_{\mathrm{u}}(s) \cdot \boldsymbol{i} \;=\; 0 \quad \text{for all } \; s \in [0, L]\,.$$

By (2.2) and the balance of forces this is equivalent to

$$N \cos\vartheta - H \sin\vartheta \;=\; \boldsymbol{f}_{\mathrm{p}} \cdot \boldsymbol{i}\,, \quad \nu \sin\vartheta + \eta\cos\vartheta \;=\; 0\,. \tag{3.1}$$

Using the constitutive relations (2.12), we obtain that

$$\breve{N}(\nu, \eta, M, s)\cos\vartheta - \breve{H}(\nu, \eta, M, s)\sin\vartheta \;=\; \boldsymbol{f}_{\mathrm{p}}(s) \cdot \boldsymbol{i} \tag{3.2}$$

$$\nu \sin\vartheta + \eta\cos\vartheta \;=\; 0\,. \tag{3.3}$$

We claim to solve this system for each fixed (ϑ, M, s). By $\nu > 0$ the second equation obviously excludes $\cos\vartheta = 0$ (recall (2.4)). Thus, by the periodicity in ϑ, we can restrict our attention to

$$\vartheta \;\in\,]-\tfrac{\pi}{2}, \tfrac{\pi}{2}[\,.$$

Hence the system is equivalent to the single equation

$$\breve{\varphi}(\nu,\vartheta,M,s) \;\equiv\; \breve{N}(\nu,-\nu\tan\vartheta,M,s)\cos\vartheta - \breve{H}(\nu,-\nu\tan\vartheta,M,s)\sin\vartheta \;=\; \boldsymbol{f}_{\mathrm{p}}(s)\cdot\boldsymbol{i}\,. \tag{3.4}$$

By the strict positive definiteness of $\frac{\partial(\breve{N},\breve{H},\breve{\mu})}{\partial(\nu,\eta,M)}$, we obtain that

$$\begin{aligned}\breve{\varphi}_\nu \;&=\; \breve{N}_\nu\cos\vartheta - \breve{N}_\eta\tan\vartheta\cos\vartheta - \breve{H}_\nu\sin\vartheta + \breve{H}\tan\vartheta\sin\vartheta\\ &=\; \frac{1}{\cos\vartheta}\begin{pmatrix}\breve{N}_\nu & \breve{N}_\eta\\ \breve{H}_\nu & \breve{H}_\eta\end{pmatrix}\begin{pmatrix}\cos\vartheta\\ \sin\vartheta\end{pmatrix}\cdot\begin{pmatrix}\cos\vartheta\\ \sin\vartheta\end{pmatrix} \;>\; 0\,.\end{aligned}$$

Hence $\breve{\varphi}(\cdot,\vartheta,M,s)$ is increasing. Below we show that

$$\breve{\varphi}(\nu,\vartheta,M,s) \;\to\; \begin{Bmatrix}+\infty\\ -\infty\end{Bmatrix} \quad \text{as} \quad \nu \;\to\; \begin{Bmatrix}+\infty\\ 0\end{Bmatrix} \tag{3.5}$$

under some mild additional constitutive assumptions which exclude certain singular cases. This implies the existence of a unique solution

$$\nu \;=\; \tilde{\nu}(\vartheta,M,s)\,, \quad \eta \;=\; \tilde{\eta}(\vartheta,M,s) \;\equiv\; -\tilde{\nu}(\vartheta,M,s)\tan\vartheta$$

of the system (3.2), (3.3). By the Implicit Function Theorem and our smoothness assumption for the constitutive rules, the functions $\tilde{\nu}(\cdot,\cdot,s)$, $\tilde{\eta}(\cdot,\cdot,s)$ are continuously differentiable for all $s\in[0,L]$. Let us set

$$\begin{aligned}\tilde{N}(\vartheta,M,s) \;&\equiv\; \breve{N}(\tilde{\nu}(\vartheta,M,s),\tilde{\eta}(\vartheta,M,s),M,s)\,,\\ \tilde{H}(\vartheta,M,s) \;&\equiv\; \breve{H}(\tilde{\nu}(\vartheta,M,s),\tilde{\eta}(\vartheta,M,s),M,s)\,,\\ \tilde{\mu}(\vartheta,M,s) \;&\equiv\; \breve{\mu}(\tilde{\nu}(\vartheta,M,s),\tilde{\eta}(\vartheta,M,s),M,s)\,.\end{aligned}$$

The functions $\tilde{N}$, $\tilde{H}$, and $\tilde{\mu}$ are also continuously differentiable with respect to (ϑ,M). By the symmetry condition (2.8) we easily verify that

$$\tilde{\nu}(\cdot,M,s),\;\; \tilde{\mu}(\cdot,M,s),\;\; \tilde{N}(\cdot,M,s) \;\text{ are even, }\; \tilde{\eta}(\cdot,M,s),\;\; \tilde{H}(\cdot,M,s) \;\text{ are odd.} \tag{3.6}$$

Moreover, $\tilde{\eta}(\vartheta,M,s)$ and $\tilde{H}(\vartheta,M,s)$ have the opposite sign of ϑ.

Using the balance of moments we obtain the equilibrium equations for straight configurations

$$M' \;=\; -\tilde{\nu}(\vartheta,M,s)\tilde{H}(\vartheta,M,s) + \tilde{\eta}(\vartheta,M,s)\tilde{N}(\vartheta,M,s) + \tilde{l}(\vartheta,s) \tag{3.7}$$

$$\vartheta' \;=\; \tilde{\mu}(\vartheta,M,s) \tag{3.8}$$

where

$$\tilde{l}(\vartheta,s) \;\equiv\; \Big(\bar{\boldsymbol{l}}_{f_{\mathrm{p}}}(\vartheta,s) + \bar{\boldsymbol{l}}(\vartheta,s)\Big)\cdot\boldsymbol{k}\,.$$

We readily see that, up to a translation along the $\boldsymbol{i}$-axis, a solution $\{\vartheta(\cdot),M(\cdot)\}$ of the system (3.7), (3.8) uniquely provides a straight equilibrium configuration of the rod where the unknown external force $\mathfrak{f}_{\mathrm{u}}$, which acts along the reference curve, is given by

$$\begin{aligned}\boldsymbol{f}_{\mathrm{u}}(s) \;&=\; \boldsymbol{n}(s) - \boldsymbol{f}_{\mathrm{p}}(s)\\ &=\; \tilde{N}(\vartheta(s),M(s),s)\,\boldsymbol{a}(\vartheta(s)) + \tilde{H}(\vartheta(s),M(s),s)\,\boldsymbol{b}(\vartheta(s)) - \boldsymbol{f}_{\mathrm{p}}(s) \quad \text{on } [0,L]\,.\end{aligned} \tag{3.9}$$

In the next section we study qualitative properties of the system (3.7), (3.8) for the autonomous case.

Justification of (3.5). For this reason we adopt the growth condition that

$$\left| \begin{pmatrix} \breve{N}(\nu,\eta,M,s) \\ \breve{H}(\nu,\eta,M,s) \end{pmatrix} \cdot \begin{pmatrix} \nu \\ \eta \end{pmatrix} \right| \;\ge\; c(M,s)(\nu^2+\eta^2) \tag{3.10}$$

where $c(M,s) > 0$ is a constant which can depend on M and s. This is a slightly stronger condition than (2.10). Furthermore we exclude certain singular cases by demanding that

$$\liminf_{\tau\to\infty} \hat{\eta}(-\tau,\tau\cot\vartheta,M,s) \;\equiv\; \varepsilon(\vartheta,M,s) \;>\; 0 \tag{3.11}$$

for $\vartheta \in]0,\frac{\pi}{2}[\,,\ M \in \mathbb{R}\,,\ s \in [0,L]$. Below we show that this condition is always satisfied for hyperelastic materials, which is the mechanically most important case.

(3.10) readily implies that

$$\begin{aligned} |\breve{\varphi}(\nu,\vartheta,M,s)| \;&=\; \frac{\cos\vartheta}{\nu}\left|\breve{N}(\nu,-\nu\tan\vartheta,M,s)\nu - \breve{H}(\nu,-\nu\tan\vartheta,M,s)\nu\tan\vartheta\right| \\ &\ge\; c(M,s)\,\nu\,(1+\tan^2\vartheta)\cos\vartheta\,. \end{aligned}$$

Since $\breve{\varphi}(\cdot,\vartheta,M,s)$ is increasing,

$$\breve{\varphi}(\nu,\vartheta,M,s) \;\to\; +\infty \quad \text{as} \quad \nu \;\to\; +\infty\,.$$

Let us now show that $\breve{\varphi}(\nu,\vartheta,M,s) \to -\infty$ as $\nu \to 0$. For $\vartheta = 0$ this easily follows from our basic infinity conditions for the constitutive functions stated in Section 2. By the symmetry (2.8) we thus can restrict our attention to $\vartheta \in]0,\frac{\pi}{2}[$. Let us fix (ϑ,M,s). We consider the sets

$$\mathcal{A}_r \;\equiv\; \{(N,H)|\, N < -r,\ |H| < |N|\cot\vartheta\}\,, \quad r > 0\,,$$

which are open in $\mathbb{R}^2$. Since $\hat{\boldsymbol{\xi}}(\cdot,s)$ is an homeomorphism, the images

$$\mathcal{A}_r^\dagger \;\equiv\; \{(\nu,\eta)|\, \nu = \hat{\nu}(N,H,M,s),\ \eta = \hat{\eta}(N,H,M,s),\ (N,H) \in \mathcal{A}_r\}\,, \quad r > 0\,,$$

are open in the $(\nu,\eta)-$plane. By (3.11) there exist sequences $r_n \to \infty$, $\nu_n \to 0$ such that

$$(\nu_n,-\nu_n\tan\vartheta) \;\in\; \mathcal{A}_{r_n}^\dagger\,.$$

To see this we assume, for contradiction, that there is some large $r > 0$ such that $|\eta| \ne \nu\tan\vartheta$ for all $(\nu,\eta) \in \mathcal{A}_r^\dagger$ with $\nu < \frac{1}{r}$. By the growth properties of $\breve{N}(\cdot,\eta,M,s)$ and with $\eta_r \equiv \frac{1}{2r}\tan\vartheta$ we then could choose $0 < \nu_r < \frac{1}{2r}$ such that $N_r \equiv \breve{N}(\nu_r,\eta_r,M,s) < -2r$. Since we had choosen $r > 0$ large, we can suppose that $\hat{\eta}(N_r,-N_r\cot\vartheta,M,s) > \frac{1}{2}\varepsilon(\vartheta,M,s)$ and $\eta_r < \frac{1}{2}\varepsilon(\vartheta,M,s)$. By the monotonicity of $\breve{H}(\nu,\cdot,M,s)$, we get that $H_r \equiv \breve{H}(\nu_r,\eta_r,M,s) < |N_r|\cot\vartheta$. Hence $(N_r,H_r) \in \mathcal{A}_r$ and $(\nu_r,\eta_r) \in \mathcal{A}_r^\dagger$. For $H \in [0,H_r]$ the points $(\hat{\nu}(N_r,H,M,s),\hat{\eta}(N_r,H,M,s))$ form a contiuous curve in $\mathcal{A}_r^\dagger$ containing the point $(\hat{\nu}(N_r,0,M,s),0)$. Since $|\eta_r| > \nu_r\tan\vartheta$ and $0 < \hat{\nu}(N_r,0,M,s)\tan\vartheta$, there exists some $H_0 \in]0,H_r[$ such that $\eta_0 = \nu_0\tan\vartheta$ where $\nu_0 \equiv \hat{\nu}(N_r,H_0,M,s)$ and $\eta_0 \equiv \hat{\eta}(N_r,H_0,M,s)$. By the monotonicity of $\hat{\eta}(N_r,\cdot,M,s)$ we readily see that $\nu_0 < \frac{1}{r}$. But this contradicts our above assumption. Using the symmetry of $\hat{\eta}$ in H we finally obtain the existence of our desired sequences.

Consequently, with

$$\begin{aligned} N_n &\equiv \hat{N}(\nu_n, -\nu_n \tan\vartheta, \breve{\mu}(\nu_n, -\nu_n \tan\vartheta, M, s), s) \;=\; \breve{N}(\nu_n, -\nu_n \tan\vartheta, M, s) \\ H_n &\equiv \hat{H}(\nu_n, -\nu_n \tan\vartheta, \breve{\mu}(\nu_n, -\nu_n \tan\vartheta, M, s), s) \;=\; \breve{H}(\nu_n, -\nu_n \tan\vartheta, M, s) \end{aligned}$$

we obtain that $(N_n, H_n) \in \mathcal{A}_{r_n}$ and $N_n \to -\infty$ by the definition of $\mathcal{A}_{r_n}$. We must even have that $|H_n| < |N_n| \cot\bar{\vartheta}$ for some $\bar{\vartheta} \in]\vartheta, \frac{\pi}{2}[$ and all large $n \in \mathbb{N}$. Otherwise, by the monotonicity of $\hat{\eta}(N, \cdot, M, s)$ and with the notation $\eta_n \equiv -\nu_n \tan\vartheta$,

$$|\eta_n| \;=\; \hat{\eta}(N_n, |H_n|, M, s) \;\geq\; \hat{\eta}(N_n, |N_n| \cot\bar{\vartheta}, M, s) \quad \text{for all } \bar{\vartheta} \in]\vartheta, \tfrac{\pi}{2}[\,, \;\; n \text{ large.}$$

This gives a contradiction by (3.11) (with $\bar{\vartheta}$ instead of ϑ) and by $\eta_n \to 0$. With $\delta \equiv \cot\vartheta - \cot\bar{\vartheta} > 0$ we thus obtain that

$$N_n \cos\vartheta - H_n \sin\vartheta \;<\; N_n \delta \sin\vartheta\,.$$

By $N_n \to -\infty$ and the monotonicity of $\breve{\varphi}(\cdot, \vartheta, M, s)$ we finally get that

$$\breve{\varphi}(\nu, \vartheta, M, s) \;\to\; -\infty \quad \text{as} \quad \nu \;\to\; -\infty\,.$$

But this justifies (3.5).

Sketch that (3.11) *holds true in the hyperelastic case.* For fixed (ϑ, M, s) with $\vartheta \in]0, \frac{\pi}{2}[$ we suppose that, in contrary to (3.11), there exists a sequence $\tau_n \to \infty$ such that

$$\eta_n \;\equiv\; \hat{\eta}(-\tau_n, \tau_n \cot\vartheta, M, s) \;\to\; 0\,. \tag{3.12}$$

Obviously $\eta_n > 0$ for all $n \in \mathbb{N}$. Assume that

$$\nu_n \;\equiv\; \hat{\nu}(-\tau_n, \tau_n \cot\vartheta, M, s) \;\to\; \nu_0 \;>\; 0$$

(at least for a subsequence). Then, by the continuity of the constitutive functions, $-\tau_n = \breve{N}(\nu_n, \eta_n, M, s)$ must converge to a finite value. But this contradicts $\tau_n \to \infty$. If $\nu_n \to \infty$, then we easily obtain a contradition to the monotonicity of $\hat{\boldsymbol{\xi}}(\cdot, s)$. Thus

$$\nu_n \;\to\; 0\,.$$

Let us introduce the level sets

$$\omega_n \;\equiv\; \{(\nu, \eta) |\, \breve{W}(\nu, \eta, M, s) \leq \breve{W}(\nu_n, \eta_n, M, s)\}\,.$$

By (2.14), $\breve{W}(\nu_n, \eta_n, M, s) \to \infty$ as $n \to \infty$. Thus, at least for a subsequence,

$$\omega_n \;\subset\; \omega_{n+1}\,, \qquad \bigcup_{n \in \mathbb{N}} \omega_n \;=\; \{(\nu, \eta) |\, \nu > 0\}\,.$$

Hence there exists some large $n_0 \in \mathbb{N}$ such that the line $\ell : \sigma \to (\sigma, 1 + \sigma \tan\vartheta)$, $\sigma \in \mathbb{R}$, intersects the interior of ω_{n_0}. Obviously $-\tau_n = \breve{W}_\nu(\nu_n, \eta_n, M, s)$ and $\tau_n \cot\vartheta = \breve{W}_\eta(\nu_n, \eta_n, M, s)$. Consequently, $(-\tau_n, \tau_n \cot\vartheta)$ is an outer normal of ω_n at (ν_n, η_n) and it is also normal to ℓ. By the convexity of $\breve{W}(\cdot, \cdot, M, s)$ the level sets ω_n are convex and we readily see that $\eta_n > 1$ for all $n > n_0$. But this contradicts (3.12) and, hence, (3.11) must be satisfied.

3.2 Discussion of alternative formulations

Let us discuss some alternative ways to derive equilibrium conditions for straight configurations of rods in this section.

Instead of (2.12) we could employ (2.11) for the solution of (3.1). We then get

$$\hat{\nu}(H\tan\vartheta + \beta(\vartheta,s), H, M, s)\sin\vartheta + \hat{\eta}(H\tan\vartheta + \beta(\vartheta,s), H, M, s)\cos\vartheta \;=\; 0$$

where $\beta(\vartheta,s) \equiv \boldsymbol{f}_{\mathrm{p}}(s)\cdot\boldsymbol{i}/\cos\vartheta$. By similar arguments as above we obtain a unique solution $\bar{H}(\vartheta,M,s)$. This then again leads to a system of ordinary differential equations for (ϑ,M) which should be identical with (3.7), (3.8).

In our previous considerations we always studied the case where the reference curve is enforced to be straight. However, sometimes it seems to be favorable to formulate the problem with respect to the curve of centroids, where the constitutive functions enjoy additional symmetry properties, though a different material curve is enforced to be straight. This situation naturally occurs in contact problems where a boundary curve of the rod is in contact with a straight obstacle. For this more general approach we choose some fixed $h\in\mathbb{R}$ such that $h\in[h_-(s),h_+(s)]$ for all $s\in[0,L]$ and we demand that

$$\boldsymbol{p}(s,h)\cdot\boldsymbol{j} \;=\; \Big(\boldsymbol{r}(s)+h\boldsymbol{b}(s)\Big)\cdot\boldsymbol{j} \;=\; 0\,, \quad \boldsymbol{f}_{\mathrm{u}}(s)\cdot\boldsymbol{i} \;=\; 0 \quad \text{on } [0,L].$$

Differentiation of the first equation with respect to s gives that $(\boldsymbol{r}'-h\mu\boldsymbol{a})\cdot\boldsymbol{j}=0$. By the balance of forces the previous conditions thus imply

$$N\cos\vartheta - H\sin\vartheta \;=\; \boldsymbol{f}_{\mathrm{p}}(s)\cdot\boldsymbol{i}\,, \quad \nu\sin\vartheta+\eta\cos\vartheta - h\mu\sin\vartheta \;=\; 0\,. \tag{3.13}$$

Using the constitutive functions (2.11) we obtain

$$\varphi_1 \;\equiv\; N\cos\vartheta - H\sin\vartheta - \boldsymbol{f}_{\mathrm{p}}(s)\cdot\boldsymbol{i} \;=\; 0\,, \tag{3.14}$$

$$\varphi_2 \;\equiv\; \hat{\nu}(N,H,M,s)\sin\vartheta + \hat{\eta}(N,H,M,s)\cos\vartheta - h\hat{\mu}(N,H,M,s)\sin\vartheta \;=\; 0\,. \tag{3.15}$$

We are looking for a solution $N^\star(\vartheta,M,s)$, $H^\star(\vartheta,M,s)$ of this system. By (2.4) the case $\cos\vartheta=0$ can be excluded, i.e., we can again restrict our attention to $\vartheta\in\,]-\frac{\pi}{2},\frac{\pi}{2}[\,$. That we can apply the Implicit Function Theorem we have to study

$$\det\left(\frac{\partial(\varphi_1,\varphi_2)}{\partial(N,H)}\right) \;=\; \begin{pmatrix} \hat{\nu}_N - h\hat{\mu}_N & \hat{\nu}_H - h\hat{\mu}_H \\ \hat{\eta}_N & \hat{\eta}_H \end{pmatrix}\begin{pmatrix}\sin\vartheta\\ \cos\vartheta\end{pmatrix}\cdot\begin{pmatrix}\sin\vartheta\\ \cos\vartheta\end{pmatrix}.$$

Unfortunately, for $h\neq 0$ we cannot expect that this expression is different from zero for all (ϑ,M,s) in general. (Note that the expression is always positive for $h=0$ by the monotonicity of the constitutive functions.) This means that, in contrast to (3.2), (3.3), given values (ϑ,M,s) do not uniquely determine the force vector (N,H) through (3.14), (3.15). Hence we cannot reduce the problem to equilibrium conditions in terms of (ϑ,M) in analogy to (3.7), (3.8). Similar difficulties arise if we would use the constitutive functions (2.12) in (3.13). Then we would need that

$$\begin{pmatrix}\check{N}_\nu & \check{H}_\nu\\ \check{N}_\eta & \check{H}_\eta\end{pmatrix}\begin{pmatrix}\cos\vartheta\\ -\sin\vartheta\end{pmatrix}\cdot\begin{pmatrix}\cos\vartheta\\ -\sin\vartheta\end{pmatrix} - h\sin\vartheta\begin{pmatrix}\check{N}_\nu & \check{H}_\nu\\ \check{N}_\eta & \check{H}_\eta\end{pmatrix}\begin{pmatrix}\cos\vartheta\\ -\sin\vartheta\end{pmatrix}\cdot\begin{pmatrix}\check{\mu}_\eta\\ -\check{\mu}_\nu\end{pmatrix}$$

does not equal zero, which also cannot be expected in general.

Recalling the formulas for the transformation of the base curve we could suspect that it is awkward to describe the problem in terms of (ϑ, M), since M changes under this transformation. Though (ϑ, M) are natural from the physical point of view for typical initial or boundary value problems, let us try to formulate the problem in terms of (ϑ, μ) which is independent of the special base curve. For this reason we adopt the constitutive functions (2.13) in (3.13) which gives

$$\varphi_3 \equiv N\cos\vartheta - H\sin\vartheta - \boldsymbol{f}_{\mathrm{p}}(s)\cdot\boldsymbol{i} = 0 \tag{3.16}$$

$$\varphi_4 \equiv \breve{\nu}(N,H,\mu,s)\sin\vartheta + \breve{\eta}(N,H,\mu,s)\cos\vartheta - h\mu\sin\vartheta = 0\,. \tag{3.17}$$

We readily get

$$\det\left(\frac{\partial(\varphi_3,\varphi_4)}{\partial(N,H)}\right) = \begin{pmatrix} \breve{\nu}_N & \breve{\nu}_H \\ \breve{\eta}_N & \breve{\eta}_H \end{pmatrix}\begin{pmatrix} \sin\vartheta \\ \cos\vartheta \end{pmatrix}\cdot\begin{pmatrix} \sin\vartheta \\ \cos\vartheta \end{pmatrix} > 0\,.$$

Thus, under some reasonable growth conditions, we would obtain a unique solution $N^\flat(\vartheta,\mu,s)$, $H^\flat(\vartheta,\mu,s)$ and, similar as before, we could define

$$M^\flat(\vartheta,\mu,s) \equiv \breve{M}(N^\flat(\vartheta,\mu,s),H^\flat(\vartheta,\mu,s),\mu,s)\,,\;\; \nu^\flat(\vartheta,\mu,s) \equiv \breve{\nu}(\ldots)\,,\;\; \eta^\flat(\vartheta,\mu,s) \equiv \breve{\eta}(\ldots)\,.$$

That we can write down the equilibrium equations we have to observe that, for $h \neq 0$, the unknown force $\mathfrak{f}_{\mathrm{u}}$ gives a nonvanishing contribution $\boldsymbol{l}_{f_{\mathrm{u}}}$ to the balance of moments (remember that $\mathfrak{f}_{\mathrm{u}}$ is supposed to act along the straight axis). Let us assume that $\boldsymbol{f}_{\mathrm{u}}$ and $\boldsymbol{l}_{f_{\mathrm{u}}}$ are smooth on $]0,L[$, which is reasonable at least in the case where the constitutive functions are smooth with respect to s (cf. Schuricht [5]). We then have that

$$\boldsymbol{l}_{f_{\mathrm{u}}}(s) = \int_{\Omega_s}\zeta\boldsymbol{b}(\tau)\times d\mathfrak{f}_{\mathrm{u}}(\tau,\zeta) = -h\int_s^L \boldsymbol{b}(\tau)\times\boldsymbol{f}'_{\mathrm{u}}(\tau)\,d\tau\,.$$

(3.9) implies that

$$\begin{aligned}\boldsymbol{l}'_{f_{\mathrm{u}}} &= h\boldsymbol{b}\times\left(\boldsymbol{n}' - \boldsymbol{f}'_{\mathrm{p}}\right) = h\boldsymbol{b}\times\left((N'-\mu H)\boldsymbol{a} - \boldsymbol{f}'_{\mathrm{p}}\right)\\ &= -h\Big(N^\flat_\vartheta\vartheta' + N^\flat_\mu\mu' + N^\flat_s - \mu H^\flat - \boldsymbol{f}'_{\mathrm{p}}\cdot\boldsymbol{a}\Big)\boldsymbol{k} \equiv \bar{\boldsymbol{l}}_{f_{\mathrm{u}}}(\vartheta,\vartheta',\mu,\mu',s)\,.\end{aligned}$$

Using $\vartheta' = \mu$, the balance of moments reads as

$$\begin{aligned}\tfrac{d}{ds}M^\flat(\vartheta,\vartheta',s) &= -\nu^\flat(\vartheta,\vartheta',s)H^\flat(\vartheta,\vartheta',s) + \eta^\flat(\vartheta,\vartheta',s)N^\flat(\vartheta,\vartheta',s)\\ &\quad + \tilde{l}(\vartheta,s) + \bar{\boldsymbol{l}}_{f_{\mathrm{u}}}(\vartheta,\vartheta',\vartheta',\vartheta'',s)\cdot\boldsymbol{k}\,.\end{aligned}$$

This is a quasi-linear second order ordinary differential equation for ϑ. The coefficient of ϑ'' is

$$\breve{M}_N N^\flat_\mu + \breve{M}_H H^\flat_\mu + \breve{M}_\mu + hN^\flat_\mu\,.$$

Again we cannot expect that this term is different from zero in general and, thus, we encounter similar difficulties as in the previous approach. Furthermore we do not have a function providing μ in terms of (ϑ, M) which would raise serious problems in formulating initial or boundary value problems where certain values of M are prescribed.

Now one could ask whether the intrinsic coordinates (ν,η) and (N,H) are convenient for a problem where we invoke a constraint with respect to the fixed $\{\boldsymbol{i},\boldsymbol{j}\}$-system. For this reason let us introduce the decompositions

$$\boldsymbol{r} \;=\; x\boldsymbol{i} + y\boldsymbol{j}\,, \quad \boldsymbol{n} \;=\; A\boldsymbol{i} + B\boldsymbol{j}\,.$$

We redily see that

$$\nu \;=\; \nu^\sharp(x',y',\vartheta) \;\equiv\; x'\cos\vartheta + y'\sin\vartheta\,, \quad \eta \;=\; \eta^\sharp(x',y',\vartheta) \;\equiv\; -x'\sin\vartheta + y'\cos\vartheta\,.$$

By $A = N\sin\vartheta + H\cos\vartheta$, $B = N\cos\vartheta - H\sin\vartheta$, and (2.7), we obtain constitutive relations of the form

$$A \;=\; A^\sharp(x',y',\vartheta',\vartheta,s)\,, \quad B \;=\; B^\sharp(x',y',\vartheta',\vartheta,s)\,, \quad M \;=\; M^\sharp(x',y',\vartheta',\vartheta,s)\,.$$

The constraints (3.13), which have a very simple form in this formulation, then lead to the equation

$$B^\sharp(x',h\vartheta'\sin\vartheta,\vartheta',\vartheta,s) - \boldsymbol{f}_{\mathrm{p}}(s)\cdot\boldsymbol{i} \;=\; 0\,.$$

A straightforward computation shows that

$$B^\sharp_{x'} \;=\; \begin{pmatrix} \hat{N}_\nu & \hat{N}_\eta \\ \hat{H}_\nu & \hat{H}_\eta \end{pmatrix} \begin{pmatrix} \cos\vartheta \\ -\sin\vartheta \end{pmatrix} \cdot \begin{pmatrix} \cos\vartheta \\ -\sin\vartheta \end{pmatrix} \;>\; 0\,.$$

Thus under some reasonable growth conditions we obtain a solution

$$x' \;=\; a^\sharp(\vartheta,\vartheta',s)\,.$$

To formulate the equilibrium equations we are, however, confronted with the same difficulties as in the last case.

Our discussion shows that there are serious difficulties in formulating the problem with respect to a base curve which is different from the curve that is enforced to be straight. In the rest of this paper we restrict our attention to qualitative investigations of the system (3.7), (3.8) which gives configurations where the base curve is straight.

4 Qualitative analysis

The qualitative properties of the system (3.7), (3.8) can be very different. They clearly depend on the prescribed force $\boldsymbol{f}_{\mathrm{p}}$ and on the choice of the constitutive functions describing some special material. In this section we want to demonstrate by phase plane analysis how rich the structure of this system can be. Here we restrict our attention to a situation where the system is autonomous.

Let us consider homogeneous rods which are straight in the undeformed stress free configuration, i.e., $\mathring{\mu} = 0$. More precisely, we assume that the constitutive functions and $h_\pm$ do not explicitely depend on s. That the system (3.7), (3.8) becomes homogeneous, $\boldsymbol{f}_{\mathrm{p}}\cdot\boldsymbol{i}$ and $\tilde{l}$ may not depend explicitly on s at least on $]0,L[$ (observe that terminal loads or boundary conditions for ϑ, which we want to cover, can cause certain discontinuities at the ends). We choose $\bar{l}(s) = 0$ on

$]0,L[$. This in fact implies that $\boldsymbol{f}_{\mathrm{p}}'(s) = g\boldsymbol{j}$ on $]0,L[$ for some constant $g \in \mathbb{R}$. Note that we cover (homogeneous) weight and terminal loads as external prescribed forces with this situation. As we shall see these cases already create a very rich analysis.

Without danger of confusion we use the same notation as in the previous section and simply omit the variable s in functions which do not explicitly depend on it in this special case. Thus we are lead to the autonomous system

$$M' = -\tilde{\nu}(\vartheta, M)\tilde{H}(\vartheta, M) + \tilde{\eta}(\vartheta, M)\tilde{N}(\vartheta, M) + \tilde{l}(\vartheta) \tag{4.1}$$

$$\vartheta' = \tilde{\mu}(\vartheta, M) \tag{4.2}$$

instaed of (3.7), (3.8). The function $\tilde{l}$ has the form

$$\tilde{l}(\vartheta) = g\sin\vartheta\,, \quad \text{for } \vartheta \in]-\tfrac{\pi}{2}, \tfrac{\pi}{2}[\tag{4.3}$$

(cf. (2.6)). Obviously $g < 0$ is the case of weight which we have in mind if we speak about small and large weight below. Let us set

$$\tilde{\kappa}(\vartheta, M) \equiv -\tilde{\nu}(\vartheta, M)\tilde{H}(\vartheta, M) + \tilde{\eta}(\vartheta, M)\tilde{N}(\vartheta, M)\,.$$

The symmetry of $\tilde{\nu}$, $\tilde{\eta}$, $\tilde{N}$, and $\tilde{H}$ readily implies that $\tilde{\kappa}(\cdot, M)$ is odd. Though $\tilde{\kappa}(\cdot, M)$ is not increasing in general, it is reasonable to suppose that

$$\tilde{\kappa}(\vartheta, M) \to \pm\infty \quad \text{as} \quad \vartheta \to \pm\tfrac{\pi}{2} \tag{4.4}$$

for fixed $M \in \mathbb{R}$.

Case 1. Let us start with the case where the base curve is a curve of centroids. For phase plane analysis we first have to determine the stationary points (ϑ^s, M^s) of our system. By the symmetry (2.9) the condition $\tilde{\mu}(\vartheta, M) = 0$ is equivalent with $M = 0$. Thus the stationary points must satisfy

$$\tilde{\kappa}(\vartheta^s, 0) = -\tilde{l}(\vartheta^s)\,, \quad M^s = 0\,.$$

Since $\tilde{\kappa}(\cdot, M)$ and $\tilde{l}(\cdot)$ are odd, $(\vartheta^s, M^s) = (0,0)$ is always a stationary solution. The appearance of further stationary points, that obviously appear in pairs $(\pm\vartheta^s, 0)$, essentially depends on the constitutive functions and on the value of $g \in \mathbb{R}$. Fig. 1 and Fig. 2 show some typical cases. Here the fat (solid and dashed) graphes represent $\tilde{\kappa}(\vartheta, 0)$ and the thin graphes provide $-\tilde{l}(\vartheta) = -g\sin\vartheta$ (the solid curves correspond to small weight and the dashed curve to large weight). We readily see that, in the case of small weight, there is only the trivial stationary point for a material without shear instabilities and there is at least one pair of nontrivial stationary points for materials with shear instabilities. In the case of large weight we have always at least one pair of nontrivial stationary points.

To determine the typ of the stationary points, we have to analyze the eigenvalues of the matrix

$$\mathcal{M}(\vartheta, M) \equiv \begin{pmatrix} \tilde{\kappa}_M & \tilde{\kappa}_\vartheta + \tilde{l}_\vartheta \\ \tilde{\mu}_M & \tilde{\mu}_\vartheta \end{pmatrix}$$

at the stationary points (ϑ^s, M^s) (cf. Amann [1]). Since $\tilde{\mu}(\vartheta, 0) = 0$ for all ϑ, we get

$$\tilde{\mu}_\vartheta(\vartheta, 0) = 0 \quad \text{for all } \vartheta\,.$$

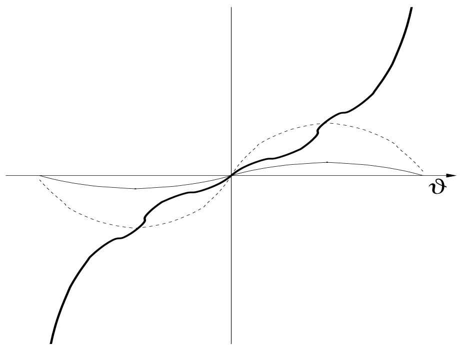

Fig. 1. Typical cases for materials without shear instabilities.

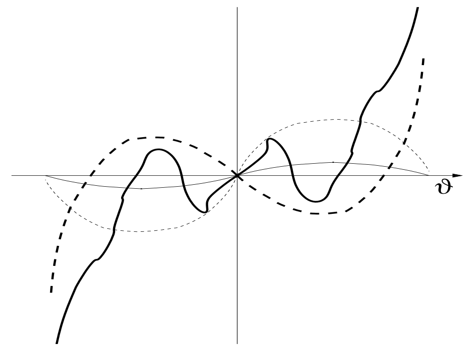

Fig. 2. Typical cases for materials with shear instabilities.

From (3.4) we readily derive that $\tilde{\nu}(\vartheta,\cdot)$ is even and, thus, also $\tilde{\eta}(\vartheta,\cdot)$, $\tilde{N}(\vartheta,\cdot)$, $\tilde{H}(\vartheta,\cdot)$ are even. Hence $\tilde{\nu}_M(\vartheta,0)=\tilde{\eta}_M(\vartheta,0)=0$ for all ϑ. Consequently, by the strict monotonicity of $\breve{\mu}(\nu,\eta,\cdot)$,

$$\tilde{\mu}_M(\vartheta,0) \;=\; \breve{\mu}_\nu(\ldots)\tilde{\nu}_M(\vartheta,0)+\breve{\mu}_\eta(\ldots)\tilde{\eta}_M(\vartheta,0)+\breve{\mu}_M(\ldots) \;>\; 0\,. \tag{4.5}$$

Furthermore $\tilde{\kappa}(\vartheta,\cdot)$ is even and, therefore,

$$\tilde{\kappa}_M(\vartheta,0) \;=\; 0\,.$$

Hence the eigenvalues λ of $\mathcal{M}(\vartheta,M)$ are given by

$$\lambda_\pm \;=\; \pm\sqrt{\alpha\beta} \quad \text{where} \quad \alpha\equiv\tilde{\mu}_M>0\,,\;\; \beta\equiv\tilde{\kappa}_\vartheta+\tilde{l}_\vartheta\,.$$

Consequently we have a saddle for $\beta>0$ and a center for $\beta<0$. By the oddness of $\tilde{\kappa}(\cdot,0)+\tilde{l}(\cdot)$, both stationary points of a pair $(\pm\vartheta^s,0)$ have the same type. Fig. 3 shows some typical phase portraits for the situations presented in Fig. 1 and Fig 2.

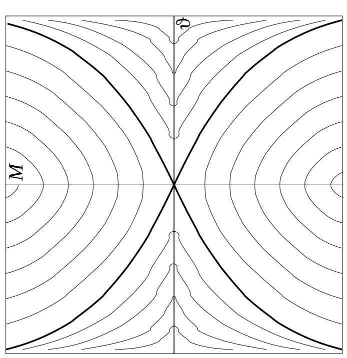

Fig. 3a. Material without shear instability and small weight.

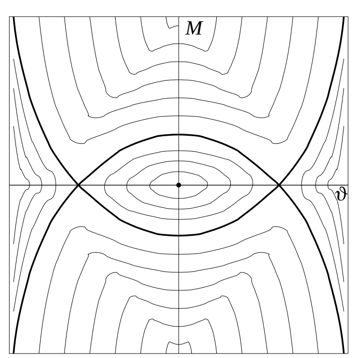

Fig. 3b. Material without shear instability and large weight or material with shear instability (dashed curve).

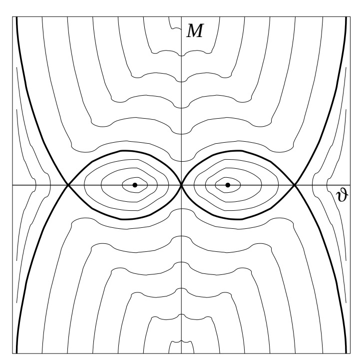

Fig. 3c. Material with shear instability (solid curve) and small weight.

Case 2. We now analyze the case where the base curve is not necessarily a curve of centroids. Let us again first look for stationary solutions (ϑ^s,M^s). In contrast to the previous case it is now not completely obvious whether $\tilde{\mu}(\vartheta,M)=0$ has a solution M for each $\vartheta\in\,]-\frac{\pi}{2},\frac{\pi}{2}[$. Though the whole problem cannot be formulated with respect to the curve of centroids (cf. the discussion in the previous section), we can use the formulas describung a change of the base curve for that special question. Let us identify all values with respect to the curve of centroids by a subscript '$_*$'. Since we consider homogeneous and originally straight rods, there always exists some $h_0 \in \mathbb{R}$ such that

$$\boldsymbol{r}_*(s) \;=\; \boldsymbol{r}(s) + h_0 \boldsymbol{b}(s) \quad \text{on } [0, L]\,.$$

We can obviously solve (3.2), (3.3) for the case that all entries are taken with respect to the curve of centroids. This way we get $\tilde{\nu}_*(\vartheta_*, M_*)$, $\tilde{\eta}_*(\vartheta_*, M_*)$, and also $\tilde{N}_*(\vartheta_*, M_*)$, $\tilde{H}_*(\vartheta_*, M_*)$. Let us set

$$\tilde{M}(\vartheta) \;\equiv\; -h_0 \tilde{N}_*(\vartheta, 0)\,. \tag{4.6}$$

Note that $\tilde{M}(\cdot)$ is even by the symmetry of $\tilde{N}_*(\cdot, M)$ (cf. (3.6)). Since $\mu = 0$ for $M_* = 0$, we can use the argumentation surrounding (3.14), (3.15) and the transformation formulas for the change of the base curve to see that

$$\tilde{\nu}(\vartheta, \tilde{M}(\vartheta)) \;=\; \tilde{\nu}_*(\vartheta, 0)\,, \quad \tilde{\eta}(\vartheta, \tilde{M}(\vartheta)) \;=\; \tilde{\eta}_*(\vartheta, 0)\,,$$

$$\tilde{N}(\vartheta, \tilde{M}(\vartheta)) \;=\; \tilde{N}_*(\vartheta, 0)\,, \quad \tilde{H}(\vartheta, \tilde{M}(\vartheta)) \;=\; \tilde{H}_*(\vartheta, 0)\,,$$

and that these values correspond to each other according to the transformation formulas. Hence

$$\begin{aligned} 0 \;&=\; \breve{\mu}_*(\tilde{\nu}_*(\vartheta, 0), \tilde{\eta}_*(\vartheta, 0), 0) \;=\; \breve{\mu}(\tilde{\nu}(\vartheta, \tilde{M}(\vartheta)), \tilde{\eta}(\vartheta, \tilde{M}(\vartheta)), \tilde{M}(\vartheta)) \\ &=\; \tilde{\mu}(\vartheta, \tilde{M}(\vartheta))\,, \end{aligned} \tag{4.7}$$

i.e., $\tilde{M}(\vartheta)$ solves the equation $\tilde{\mu}(\vartheta, M) = 0$. Let us check whether $\tilde{M}(\vartheta)$ is a unique solution for each ϑ. Unfortunately the symmetry arguments used in (4.5) do not apply in this more general case. But we can argue in a different way. For fixed $\vartheta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ and $M_1, M_2 \in \mathbb{R}$ we set

$$\nu_i \;\equiv\; \tilde{\nu}(\vartheta, M_i)\,, \;\; \eta_i \;\equiv\; \tilde{\eta}(\vartheta, M_i)\,, \;\; \mu_i \;\equiv\; \tilde{\mu}(\vartheta, M_i)\,, \;\; N_i \;\equiv\; \tilde{N}(\vartheta, M_i)\,, \;\; H_i \;\equiv\; \tilde{H}(\vartheta, M_i)\,,$$

$i = 1, 2$. Since $(\nu_i, \eta_i, N_i, H_i)$ satisfy (3.1), the vectors $(N_2 - N_1)\boldsymbol{a}(\vartheta) + (H_2 - H_1)\boldsymbol{b}(\vartheta)$ and $(\nu_2 - \nu_1)\boldsymbol{a}(\vartheta) + (\eta_2 - \eta_1)\boldsymbol{b}(\vartheta)$ are orthogonal. The monotonicity of the constitutive functions $(\breve{N}, \breve{H}, \breve{\mu})$ thus implies that

$$\begin{pmatrix} N_2 - N_1 \\ H_2 - H_1 \\ \mu_2 - \mu_1 \end{pmatrix} \cdot \begin{pmatrix} \nu_2 - \nu_1 \\ \eta_2 - \eta_1 \\ M_2 - M_1 \end{pmatrix} \;=\; (\mu_2 - \mu_1)(M_2 - M_1) \;>\; 0\,.$$

Therefore, by the arbitrariness of M_1, M_2, the function $\tilde{\mu}(\vartheta, \cdot)$ must be strictly increasing for each ϑ. Hence $\tilde{M}(\vartheta)$ is the unique solution of the equation $\tilde{\mu}(\vartheta, M) = 0$ for each $\vartheta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$. Consequently the stationary solutions must satisfy

$$\tilde{\kappa}(\vartheta^s, \tilde{M}(\vartheta^s)) + \tilde{l}(\vartheta^s) \;=\; 0\,, \quad M^s \;=\; \tilde{M}(\vartheta^s)\,.$$

By symmetry the stationary points again occur in pairs $(\pm\vartheta^s, \tilde{M}(\pm\vartheta^s))$. In particular we always have the stationary solution $(\vartheta^s, M^s) = (0, \tilde{M}(0))$. Observe that $\tilde{M}(0) \neq 0$ in general. In Fig. 1

and Fig. 2 we can consider the fat (solid or dashed) curve as graph of $\vartheta \to \tilde\kappa(\vartheta, \tilde M(\vartheta))$ in order to obtain some typical cases for this more general situation.

To determine the type of the stationary points we again have to study the eigenvalues of the matrix $\mathcal{M}(\vartheta^s, M^s)$. Using the notation

$$\alpha \equiv \tilde\mu_M\,, \quad \beta \equiv \tilde\kappa_\vartheta + \tilde l_\vartheta\,, \quad \gamma \equiv \tilde\kappa_M\,, \quad \delta \equiv \tilde\mu_\vartheta\,,$$

the eigenvalues are given by

$$\lambda_\pm \;=\; \frac{\gamma+\delta}{2} \pm \sqrt{\Big(\frac{\gamma+\delta}{2}\Big)^2 + \alpha\beta}\,.$$

By the monotonicity of $\tilde\mu(\vartheta,\cdot)$ we can suppose that always $\alpha > 0$. If $\beta > 0$, then $\lambda_\pm \in \mathbb{R}$ and we readily see that $\lambda_- < 0 < \lambda_+$, i.e., we have a saddle.

The case $\beta < 0$ needs some more effort. Recall that $\tilde M(\cdot)$ is even. Differentiating the identity $\tilde\mu(\vartheta, \tilde M(\vartheta)) = 0$ we readily obtain that

$$\tilde\mu_\vartheta(\vartheta, \tilde M(\vartheta)) + \tilde\mu_M(\vartheta, \tilde M(\vartheta))\tilde M_\vartheta(\vartheta) \;=\; 0\,. \tag{4.8}$$

Let us first consider the stationary point $(0, \tilde M(0))$. Obviously $\delta = \tilde\mu_\vartheta(0, M^s) = 0$ by (3.6). Since $\tilde\kappa(0, M) = 0$ for all M, we easily get that $\gamma = \tilde\kappa_M(0, M^s) = 0$. Hence, the stationary solution $(0, \tilde M(0))$ is a center in the case of $\beta < 0$.

In contrast to the case of a base curve of centroids, $\tilde M(\cdot)$ will not be constant in our more general situation. Thus, by the positivity of $\tilde\mu_M$, equation (4.8) implies that $\delta = \tilde\mu_\vartheta \neq 0$ for ϑ with $\tilde M_\vartheta(\vartheta) \neq 0$. Since for each $\vartheta \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ there exists a suitable $g \in \mathbb{R}$ in (4.3) such that $(\vartheta, \tilde M(\vartheta))$ is a stationary point, we do not have in general that $\delta = 0$ or even $\gamma + \delta = 0$ at stationary points with $\vartheta^s \neq 0$. Thus we obtain the following cases for stationary points with $\beta < 0$:

$$\begin{array}{lccl}
\text{If} & \gamma+\delta = 0, & \text{then it is a center,} \\
\text{if } (\frac{\gamma+\delta}{2})^2 + \alpha\beta \;\ge\; 0 \text{ and} & \left\{\begin{array}{l}\gamma+\delta > 0, \\ \gamma+\delta < 0,\end{array}\right\} & \text{then} & \left\{\begin{array}{l}\lambda_\pm > 0\,,\ \text{i.e., it is an unstable node,} \\ \lambda_\pm < 0\,,\ \text{i.e., it is a stable node,}\end{array}\right. \\
\text{if } (\frac{\gamma+\delta}{2})^2 + \alpha\beta \;<\; 0 \text{ and} & \left\{\begin{array}{l}\gamma+\delta > 0, \\ \gamma+\delta < 0,\end{array}\right\} & \text{then} & \left\{\begin{array}{l}\text{it is an unstable spiral,} \\ \text{it is a stable spiral.}\end{array}\right.
\end{array}$$

From the symmetry properties of $\tilde\kappa$, $\tilde\mu$, $\tilde l$, we get that $\alpha = \tilde\mu_M$, $\beta = \tilde\kappa_\vartheta + \tilde l_\vartheta$ are even in ϑ and $\gamma = \tilde\kappa_M$, $\delta = \tilde\mu_\vartheta$ are odd in ϑ. Consequently, for $\beta \neq 0$, the stationary points of the pair $(\pm\vartheta^s, \tilde M(\pm\vartheta^s))$ are either both saddles or both centers or we have a stable and an unstable node or a stable and an unstable spiral. Let us discuss some typical cases.

(i) Material without shear instability and small weight (cf. Fig. 1, solid graph for $-\tilde l$). We have exactly one stationary point where $\beta > 0$, i.e., it is a saddle (cf. Fig. 4a).

(ii) Material without shear instability and large weight (cf. Fig. 1, dotted graph for $-\tilde l$). The stationary point $(0, \tilde M(0))$ is obviously a center. Moreover we have a pair $(\pm\vartheta^s, \tilde M(\pm\vartheta^s))$ of stationary solutions which are both saddles (cf. Fig. 4b).

(iii) Material with shear instability (cf. Fig. 2). Obviously the case of the dashed curve for $\tilde\kappa(\vartheta, \tilde M(\vartheta))$ in Fig. 2 leads to the same situation as in Fig. 4b, both for small and large weight. In the case of the solid curve for $\tilde\kappa(\vartheta, \tilde M(\vartheta))$ in Fig. 2 and large weight (dashed graph for

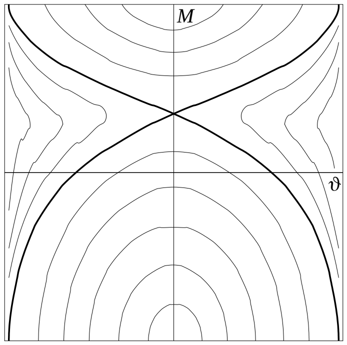

Fig. 4a. Material without shear instability and small weight.

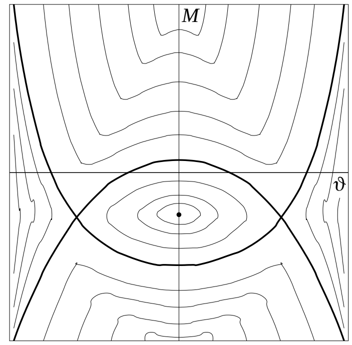

Fig. 4b. Material without shear instability and large weight.

$-\tilde{l}$), we again obtain the situation of Fig. 4b. For small weight, however, we have 5 stationary solutions $(0, \tilde{M}(0))$, $(\pm\vartheta_1^s, \tilde{M}(\pm\vartheta_1^s))$, $(\pm\vartheta_2^s, \tilde{M}(\pm\vartheta_2^s))$ (choose $0 < \vartheta_1^s < \vartheta_2^s$). Clearly $(0, \tilde{M}(0))$ and $(\pm\vartheta_2^s, \tilde{M}(\pm\vartheta_2^s))$ are saddles. At $(\pm\vartheta_1^s, \tilde{M}(\pm\vartheta_1^s))$ we have that $\beta < 0$ and, in general, we have that $\gamma+\delta \neq 0$. Fig. 5a shows the case where $(-\vartheta_1^s, \tilde{M}(-\vartheta_1^s))$ is an unstable node and $(\vartheta_1^s, \tilde{M}(\vartheta_1^s))$ is a stable node. In Fig. 5b we have an unstable spiral at $(-\vartheta_2^s, \tilde{M}(-\vartheta_2^s))$ and a stable spiral at $(\vartheta_2^s, \tilde{M}(\vartheta_2^s))$.

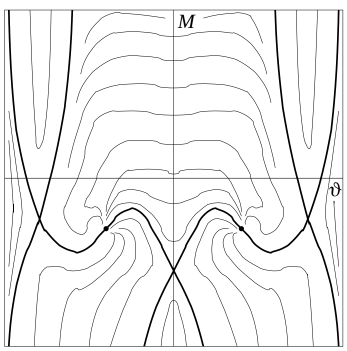

Fig. 5a. Material with shear instabilities and small weight. Case with unstable and stable node.

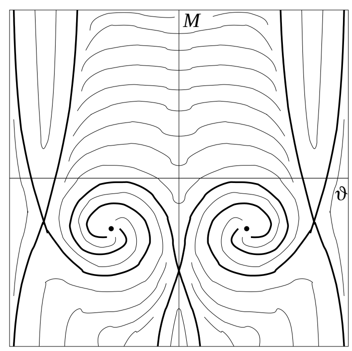

Fig. 5b. Material with shear instabilities and small weight. Case with unstable and stable spiral.

Observe that in Fig. 1 and Fig. 2 only the simplest possible cases are presented. We clearly can obtain much more complicated situations in general. In particular we can get the behavior as in Fig. 5a and Fig. 5b even for materials without shear instabilities.

The above phase plane analysis demonstrates how rich the structure of straight configurations can be. In the next section we apply our qualitative analysis to special problems.

5 Applications

Let us illuminate the previous qualitative analysis by some typical examples in this section. Here we restrict our attention to the most important case where the rod is in contact with some straight obstacle which enforces some lateral boundary curve of the rod to be straight. Thus, without loss of generality, we always choose the bottom curve as reference curve (i.e., $h_0 > 0$ in (4.6)). Furthermore we fix the point $\boldsymbol{r}(0)$ at the origin and apply a terminal load $\Lambda \boldsymbol{i}$, $\Lambda \in \mathbb{R}$, at the point $\boldsymbol{r}(L)$. To avoid technicalities we restrict our attention to materials without shear instabilities which already exhibit interesting effects that are not observable within a nonshearable theory.

Example 1. We start with the case where we neglect weight. Thus the only stationary solution is $(\vartheta^s, M^s) = (0, \tilde{M}(0)) = (0, -h_0\Lambda)$ and the phase portrait has the structure as in Fig. 4a. Note that $M^s < 0$ for a streching force, i.e., for $\Lambda > 0$. The phase portrait shows us that, for $\Lambda \neq 0$ and for suitable values L, there exist nontrivial solutions of the boundary value problem $M(0) = M(L) = 0$ (cf. Fig. 6).

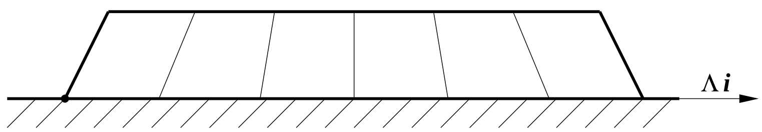

Fig. 6. Streched rod without weight.

Example 2. We again neglect weight and, in addition, we choose $\Lambda = 0$. Thus the phase portrait has the structure as in Fig. 4a with $(\vartheta^s, M^s) = (0, 0)$. The "initial value problem" $\vartheta(L) = 0$, $M(L) = 0$ has obviously the trivial stationary solution $\vartheta = 0$, $M = 0$. Hence, for a nontrivial solution with $M(L) = 0$ we conclude that $\vartheta(L) \neq 0$. On the other hand $M(L) = 0$ and $\vartheta(L) \neq 0$ imply that $M(\cdot)$ and $\vartheta(\cdot)$ cannot change sign along such a solution (observe that $M' = \tilde{\kappa}(\vartheta, M)$ has always the same sign of ϑ). Furthermore $\vartheta(L) \neq 0$ readily implies that $\eta(L) \neq 0$ and, hence, $H(L) \neq 0$. Consequently the straight obstacle must exert a concentrated force $\boldsymbol{n}(L) = n_L \boldsymbol{j} \neq \boldsymbol{0}$ to the last right cross-section of the rod if $\vartheta(L) \neq 0$ (cf. Fig. 7a).

Let us now consider a nontrivial solution with $M(L) = 0$ and $\vartheta(L) < 0$ as shown in Fig. 7a. Then, from phase plane analysis, we readily see that $M(s) > 0$ and $\vartheta(s) < 0$ for $s \in [0, L[$, and we expect that μ is non-negative along the rod. Obviously $\boldsymbol{n}(s) \cdot \boldsymbol{i} = 0$ on $[0, L]$ and $\boldsymbol{n}(L) \neq \boldsymbol{0}$. Thus $N(L) < 0$ and, therefore, $M_*(L) = h_0 N(L) < 0$. Since μ has the same sign as M_*, we conclude that $\mu(L) < 0$. By standard regularity arguments $\mu(\cdot)$ is continuous along a solution and we even get $\mu(s) < 0$ in a small neighborhood of $s = L$. This unexpected behavior near the right end is shown in Fig. 7b. In the case of $\Lambda > 0$ this effect obviously disappears as long as

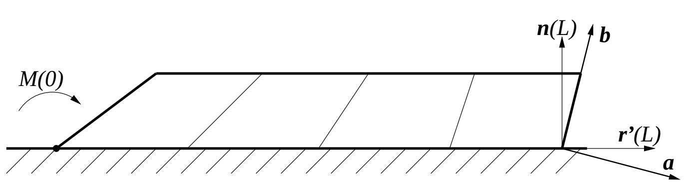

Fig. 7a. Nontrivial solution with $M(L) = 0$.

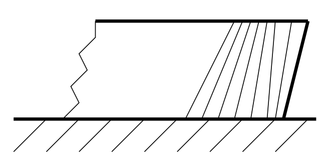

Fig. 7b. Unexpected effect.

$\vartheta(L)$ is close enough to zero.

Example 3. We now choose $\Lambda = 0$, but asssume that the rod is subjected to large weight. Thus we have essentially the situation of Fig. 4b. However, let us analyze the situation in more detail. Obviously $\vartheta = 0$ implies that $N = 0$ by $\Lambda = 0$. Hence $\tilde{M}(0) = 0$, i.e., we have the trivial stationary solution $(\vartheta^s, M^s) = (0, 0)$. Since $\boldsymbol{n}(s) \cdot \boldsymbol{i} = 0$ along solutions, we readily get that $M^s > 0$ at the other stationary points. Thus the phase portrait has the structure as in Fig. 8a. We readily see that the boundary value problem $M(0) = M(L) = 0$ can have different types

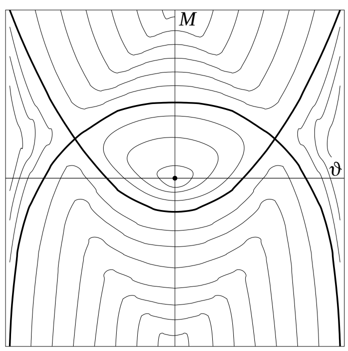

Fig. 8a. Large weight and $\Lambda = 0$.

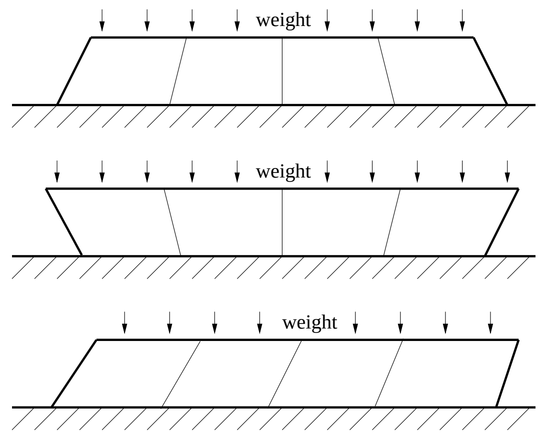

Fig. 8b. $M(0) = M(L) = 0$.

of solutions (at least for suitable values of L). Some typical cases are presented in Fig. 8b (note that the last variant is not stationary). This behavior presents some new kind of shear instability, which is different from the kind observed by Antman [2, Chap. IV] under tensil forces. We readily recognize from our phase plane analysis that some bifurcation takes place by passing from small to large weight. This obviously also implies a bifurcation for the solutions of the boundary value problem $M(0) = M(L) = 0$ with respect to the weight parameter g.

The observed behavior of the rod, which occurs for large weight, can be explained by some simple well-known physical effect. If we consider a vertical column subjected to its own weight, then it is known that it buckles as soon as the weight is large enough. Now if we consider our horizontal rod as an object consisting of a number of vertical columns connected with suitable springs, then the just described behavior seems to be reasonable from the physical point of view. Furthermore it appears to be natural that shearing becomes larger under increasing weight.

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