Locking constraints for elastic rods and a curvature bound for spatial curves

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Abstract

The paper presents necessary conditions for curves in $\mathbb{R}^3$ subjected to the nonholonomic constraint of an upper bound for curvature and suitable boundary conditions. The proof essentially uses a reformulation of the problem by means of framed curves. The Euler-Lagrange equations for nonlinearly elastic Cosserat rods subjected to a general class of locking constraints is derived by similar methods.

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1 Introduction

A typical question in geometry is to look for a curve γ of minimal length subjected to suitable constraints. If both ends of the curve are fixed at given points, then we are led to the classical problem of geodesics that are characterized by a vanishing curvature along the curve. An analytically much more difficult problem is the study of length minimizing curves with prescribed positions and tangential directions at the ends and subjected to the nonholonomic constraint of an upper bound for curvature. A few years ago this problem was formulated by Y. Wang [17] for planar curves which, e.g., models the path of a car or robot where a lower bound for the turning radius is taken into account. While a solution of E. Verriest based on control theory was given in [14], it turned out that already L.E. Dubins [5] had solved the problem with much more elementary methods about 50 years ago (cf. also [7]). Though the existence of a solution is verified for curves in $\mathbb{R}^n$ ($n \in \mathbb{N}$) in [5], its characterization is studied for planar curves only. In the present paper we derive a characterization of such curves in $\mathbb{R}^3$.

A much more subtle constraint is obtained if the tubular neighborhood of a spatial curve γ with given radius $\vartheta > 0$ is prohibited to intersect itself. This can be formulated analytically by an upper bound on the global curvature $\mathcal{K}[\gamma]$ of γ (cf. Gonzalez et al. [6], Schuricht & v.d. Mosel [13]). A corresponding length minimizing solution within a given knot class is called an ideal knot in the case of a closed curve γ and it is called an ideal link in the case of several linked curves. For the characterization of ideal knots and links we are confronted with the difficulty that global curvature can express either a local curvature bound or a nonlocal "self-distance" property of the curve (or both). While the results in [13] are restricted to situations where only the nonlocal case occurs, the subsequent investigations supplement these results for the case where only the local case occurs.

The deformation of an elastic rod is usually described by some deformed reference curve supplemented with deformed configurations of the cross sections. We observe that elastic deformation becomes harder and harder the more the body is already deformed. This phenomenon can be modeled by a suitable growth of the elastic energy density and, if only strains in a given set can be realized, by so-called locking constraints (cf. Ciarlet & Nečas [2]). We shall use the Cosserat theory to describe the behavior of nonlinearly elastic rods that can extend, bend, and shear. Pointwise restrictions for the strains turn out to be quite similar to the nonholonomic curvature condition discussed for curves. For a general class of problems the Euler-Lagrange equation as necessary condition for constrained energy minimizing configurations is verified. For the special case of a homogeneous circular unshearable rod we observe that the mechanical requirement of non-interpenetration of matter locally corresponds to an upper bound on curvature for the deformed middle curve showing the affinity to the previous pure geometric problem. The more sophisticated question of global non-interpenetration of matter for rods is studied in Gonzalez et al. [6], Schuricht [9], Schuricht & v.d. Mosel [12], [11] where our investigations supplement these results in the local case.

The analytical difficulty due to the nonholonomic constraint is circumvented by a special formulation of the problems based on framed curves, i.e., the curve γ is equipped with an orthonormal frame at each point. This way the constraint forms a convex set and we can combine the direct methods of the calculus of variations with convex analysis in order to derive necessary minimality conditions. In the case of an elastic rod we get Lagrange multipliers as corresponding contributions in the balance of forces and moments. For curves subjected to a curvature bound we obtain an equation as necessary condition and can easily derive further properties characterizing such curves.

In Section 2 we first introduce the main ideas of the Cosserat theory for elastic rods. Then we briefly discuss the existence of energy minimizing configurations with respect to a general class of locking constraints and we formulate the corresponding Euler-Lagrange equation in Theorem 2.23. Curves of minimal length subjected to an upper curvature bound are considered in Section 3. Theorem 3.5 provides a general necessary condition and Corollary 3.9 collects a number of simple consequences. In contrast to the planar case where minimal curves are a composition of straight segments and circular arcs, for spatial curves also spirals can occur as it is shown by a simple example. The proofs are deferred to Section 4. Some results needed for our analysis about the dependence of ordinary differential equations on parameters, that are a slight modification of them in Schuricht & v.d. Mosel [10], can be found in the Appendix.

Finally I would like to remark that all results presented in v.d. Mosel [15] are joint results of H. v.d. Mosel and myself with comparable contributions (partially also with O. Gonzalez and J. Maddocks) that are published in [6], [11], [12] and where my contributions go far beyond "conversations" as it is stated in [15] (in particular, [15] contains proofs completely done by myself). The joint manuscripts of [11], [12] are not cited in [15], but they had already been available in almost final form (despite the gap mentioned below). Though the central transversality assertions (b), (c) are already stated in the main theorem [15, Th. 4.2.1], this longstanding open problem of our cooperation had not yet been solved at that time and it still took almost a year after submitting [15] that the essential gap in the proof could be closed by myself (cf. [12, Lemma 15]).

Notation.

We use $|a|$ for the Euclidean norm, $a \cdot b$ for the scalar product, and $a \times b$ for the cross product in $\mathbb{R}^n$. The Lebesgue measure of $M \subset \mathbb{R}^n$ is denoted by $|M|$. The open $\varepsilon-$ball of u is $B_\varepsilon(u)$ the boundary of a set M is ∂M. We write A^T for the transpose of matrix A. For a Banach space X and its dual space X^* the duality pairing is given by $\langle \cdot, \cdot \rangle$. $\mathcal{L}^p(\Omega)$, $1 \le p \le \infty$, stands for the Lebesgue space of p-integrable functions and we use $1/p + 1/p' = 1$ to identify the dual spaces. $\mathcal{W}^{k,p}(\Omega)$ is the space of all $\mathcal{L}^p$-functions having p-integrable weak derivatives up to order k.

2 Constrained elastic rods

We first give a brief introduction into the special Cosserat theory which describes the behavior of nonlinearly elastic rods that can undergo large deformations in space by suffering flexure, torsion, extension, and shear. For a more comprehensive presentation see Antman [1, Chap. VIII], Schuricht [8].

Let the deformed position field of a slender elastic body be given by

$$p(s, \zeta_1, \zeta_2) = \gamma(s) + \zeta_1 d_1(s) + \zeta_2 d_2(s) \quad \text{for } (s, \zeta_1, \zeta_2) \in \Omega\,, \tag{2.1}$$

where

$$\Omega = \{(s, \zeta_1, \zeta_2) |\ s \in I,\ (\zeta_1, \zeta_2) \in \mathcal{A}(s)\}\,, \quad I = [0, 1].$$

We interpret $\gamma : I \to \mathbb{R}^3$ as the deformed configuration of some material curve in the body, the so-called *base curve*, with length parameter s. $d_1(s)$, $d_2(s)$ are orthogonal unit vectors describing the orientation of the cross section at s and ζ_1, ζ_2 are corresponding thickness parameters. The parameter sets $\mathcal{A}(s) \subset \mathbb{R}^2$ characterize the shapes of the cross sections corresponding to s and do not have to be uniform. For the stress free reference configuration, which has not to be straight, we assume that s is the arc length of the base curve and that the cross sections are orthogonal to the base curve. Setting $d_3 := d_1 \times d_2$, the vectors $\{d_1, d_2, d_3\}$, which we call *directors*, form a right-handed orthonormal basis for each s and can be identified with an orthogonal matrix $D = (d_1|d_2|d_3) \in SO(3)$ (the right hand side denotes the matrix with columns d_1, d_2, d_3).

A deformed configuration of the rod is thus determined by mappings $\gamma : I \to \mathbb{R}^3$ and $D : I \to SO(3)$. It is reasonable to assume that $r \in \mathcal{W}^{1,q}(I, \mathbb{R}^3)$ and $D \in \mathcal{W}^{1,p}(I, \mathbb{R}^{3\times 3})$ for $p, q \geq 1$. In Gonzalez et al. [6] it is shown that each such configuration uniquely corresponds to an element

$$w = (u, v, r_0, D_0) \quad \text{with} \quad u = (u_1, u_2, u_3),\;\; v = (v_1, v_2, v_3)$$

belonging to

$$X = X^{p,q} = \mathcal{L}^p(I, \mathbb{R}^3) \times \mathcal{L}^q(I, \mathbb{R}^3) \times \mathbb{R}^3 \times SO(3) \tag{2.2}$$

by means of

$$d_k' \;=\; \Big(\sum_{j=1}^{3} u_j d_j\Big) \times d_k \quad \text{a.e. on } I\,, \quad k = 1, 2, 3 \tag{2.3}$$

$$\gamma' \;=\; \sum_{j=1}^{3} v_j d_j \quad \text{a.e. on } I\,, \tag{2.4}$$

$$\gamma(0) \;=\; \gamma_0\,, \quad D(0) \;=\; D_0\,.$$

u, v are called the *strains* of the problem. The notation $p[w]$, $\gamma[w]$, etc. indicates the dependence of p, γ, etc. on $w = (u, v, \gamma_0, D_0)$.

We obtain more special rod theories by fixing suitable components of the strains u, v. E.g., the requirement $v = (0, 0, 1)$ provides a theory for unshearable and inextensible rods.

We assume that the material of the rod is *hyperelastic*, i.e., there is a stored energy density $W : \mathbb{R}^3 \times \mathbb{R}^3 \times I \to \mathbb{R}$ depending on (u, v, s) where $W(\cdot, \cdot, s)$ is convex and such that the total elastic energy of the rod is given by

$$E_{\mathrm{s}}(w) = E_{\mathrm{s}}(u, v) = \int_0^1 W(u(s), v(s), s)\, ds\,.$$

The contact force $f(s) = \sum_{i=1}^3 f_i(s) d_i(s)$ and the contact couple $m(s) = \sum_{i=1}^3 m_i(s) d_i(s)$ exerted from the elastic material through the cross section at s are then given by

$$f_i(s) = W_{v_i}(u(s), v(s), s), \quad m_i(s) = W_{u_i}(u(s), v(s), s) \tag{2.5}$$

for almost all cross sections s (cf. Antman [1, Chap. VIII.7], Schuricht [8]). The respond of an external force described by a vector valued Radon measure $\mathfrak{f}_e$ on Ω enters the theory by means of an potential energy

$$E_{\mathrm{p}}(w) = -\int_\Omega p(s, \zeta_1, \zeta_2) \cdot d\mathfrak{f}_e(s, \zeta_1, \zeta_2) \tag{2.6}$$

(cf. Schuricht [8]). By

$$f_e(s) = \int_{\Omega_s} d\mathfrak{f}_e(s,\zeta_1,\zeta_2), \qquad \Omega_s := \{(\sigma,\zeta_1,\zeta_2) \in \Omega |\ \sigma \geq s\} \tag{2.7}$$

we denote the resultant external force acting on the material corresponding to Ω_s. The couple induced by the force $\mathfrak{f}_e$ is given by

$$m_{f_e}(s) = \int_{\Omega_s} \Big(\zeta_1 d_1(\sigma) + \zeta_2 d_2(\sigma)\Big) \times d\mathfrak{f}_e(\sigma,\zeta_1,\zeta_2)\,. \tag{2.8}$$

The mechanical requirement that configurations should preserve orientation and should be locally invertible is imposed by the analytical condition that

$$\det \frac{\partial p(s,\zeta_1,\zeta_2)}{\partial(s,\zeta_1,\zeta_2)} \geq 0 \quad \text{a.e. on } \Omega \tag{2.9}$$

By (2.1) this is equivalent to

$$v_3(s) \geq \zeta_1 u_2(s) - \zeta_2 u_1(s) \quad \text{for a.e. } (s,\zeta_1,\zeta_2) \in \Omega\,. \tag{2.10}$$

and leads to the one-dimensional inequality

$$v_3(s) \geq \tilde{V}(u_1(s),u_2(s),s) \quad \text{for a.e. } s \in I \tag{2.11}$$

whith

$$\tilde{V}(u_1,u_2,s) \equiv \max_{(\zeta_1,\zeta_2)\in\mathcal{A}(s)} \zeta_1 u_2 - \zeta_2 u_1\,. \tag{2.12}$$

As an upper envelope of a family of linear functions, $\tilde{V}(\cdot,\cdot,s)$ is convex and continuous (cf. also Antman [1, Chap. VIII.6]). We claim to study general constrints of the type (2.11).

Let us first discuss some special case of (2.11). If we take a uniform rod with circular cross sections of radius $\vartheta > 0$ and and if we choose the curve of centroids as base curve, then condition (2.11) has the special form

$$v_3(s) \geq \vartheta\sqrt{u_1(s)^2 + u_2(s)^2} \quad \text{a.e. on } I. \tag{2.13}$$

This condition has a concrete interpretation if we restrict our attention to unshearable extensible rods where $v = (0,0,v_3)$. If $v_3(s) > 0$ a.e. on I, then the arc length parametrization $\Gamma : [0,L] \to \mathbb{R}^3$ of the base curve γ (here $L = L(\gamma)$ is the length of the curve) has the weak second derivative $\Gamma'' = (u_2 d_1 - u_1 d_2)/v_3$. Thus the curvature κ of γ is given by

$$\kappa = |\Gamma''| = \frac{\sqrt{u_1^2 + u_2^2}}{v_3} \quad \text{a.e. on } I \tag{2.14}$$

(cf. Gonzalez et al. [6]). Hence (2.13) expresses an upper bound on the curvature of γ of the form $\kappa(s) \leq 1/\vartheta$ a.e. on I ($\vartheta > 0$) in that special case.

A restriction for the strains (u,v) of the form (2.11) is also called a *locking constraint* for the material (cf. Ciarlet & Nečas [2]). Other simple locking constraints might be, e.g., a bound for maximal extension

$$\sqrt{v_1^2(s) + v_2^2(s) + v_3^2(s)} \leq \alpha_1 \quad \text{a.e. on } I,$$

a bound for maximal torsion

$$|u_3(s)| \leq \alpha_2 \quad \text{a.e. on } I,$$

or a bound for a maximal shear angle

$$\frac{\sqrt{v_1^2(s)+v_2^2(s)}}{v_3(s)} \leq \alpha_3 \quad \text{a.e. on } I$$

for suitable constants $\alpha_i > 0$, $i = 1, 2, 3$.

All formulated constraints can be written in the form

$$V(u(s), v(s), s) \leq 0 \quad \text{a.e. on } I$$

for a suitable function V that is convex in (u, v). Below we study a variational problem containing that kind of side condition and we use the notation,

$$m^V(s) = \sum_{i=1}^{3} m_i^V(s) d_i(s), \quad f^V(s) = \sum_{i=1}^{3} f_i^V(s) d_i(s),$$

where

$$m_i^V(s) = V_{u_i}(u(s), v(s), s), \quad f_i^V(s) = V_{v_i}(u(s), v(s), s).$$

To avoid technicalities we assume that

$$(V_u(u,v,s), V_v(u,v,s)) \neq 0 \quad \text{for all } (u,v,s) \text{ with } V(u,v,s) = 0 \tag{2.15}$$

and define the normalized vectors

$$\bar{m}^V(s) = \frac{m^V(s)}{|(m^V(s), f^V(s))|}, \quad \bar{f}^V(s) = \frac{f^V(s)}{|(m^V(s), f^V(s))|}, \quad \nu(s) = (\bar{m}^V(s), \bar{f}^V(s)). \tag{2.16}$$

We now study the variational problem

$$E(w) := \int_I W(u(s), v(s), s)\, ds - \int_\Omega p[w] \cdot d\mathfrak{f}_e \;\rightarrow\; \text{Min!}, \quad w \in X^{p,q}, \tag{2.17}$$

$$\gamma[w](0) = \gamma_0, \;\; \gamma[w](1) = \gamma_1, \;\; D[w](0) = D_0, \;\; D[w](1) = D_1, \tag{2.18}$$

$$V(u(s), v(s), s) \leq 0 \quad \text{a.e. on } I \tag{2.19}$$

where $\gamma_0, \gamma_1 \in \mathbb{R}^3$, $D_0, D_1 \in SO(3)$, $1 \leq p, q \leq \infty$, and the Radon measure $\mathfrak{f}_e$ on Ω are given. For our subsequent analysis we always assume that:

$$\begin{array}{ll} V(\cdot,\cdot,s), \;\; W(\cdot,\cdot,s) & \text{are continuous for all } s \in I, \\ V(u,v,\cdot), \;\; W(u,v,\cdot) & \text{are Lebesgue measurable for all } (u,v) \in \mathbb{R}^3 \times \mathbb{R}^3. \end{array}$$

Moreover, notice that $X^{p,q}$ is not a linear space, but it is contained in the Banach space

$$Z = Z^{p,q} = \mathcal{L}^p(I, \mathbb{R}^3) \times \mathcal{L}^q(I, \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3}.$$

Let us first discuss the existence of a minimizer of (2.17) - (2.19). For that reason we assume that $1 < p, q < \infty$, that $V(\cdot,\cdot,s)$, $W(\cdot,\cdot,s)$ are convex for all $s \in I$ and we impose the standard growth condition

$$W(u,v,s) \geq c(|u|^p + |v|^q) + \psi(s) \quad \text{for all } u,v \in \mathbb{R}^3 \text{ and a.e. } s \in I \tag{2.20}$$

where $c > 0$ is a constant and $\psi \in \mathcal{L}^1(I)$. Then $E_{\mathrm{s}}(\cdot)$ is weakly lower semicontinuous on Z by standard arguments (cf. Dacorogna [4]). $E_{\mathrm{p}}(\cdot)$ is weakly continuous on Z, since $w_n \rightharpoonup w$ in Z implies that $\gamma[w_n] \to \gamma[w]$ and $D[w_n] \to D[w]$ in $\mathcal{C}(I)$ (cf. Gonzalez et al. [6]). Hence $E(\cdot)$ is weakly lower semicontinuous on Z. The subset of Z defined by (2.19) is convex and closed and, hence, weakly closed. Since the boundary conditions (2.18) are stable under weak convergence (cf. [6]) and since $SO(3)$ is a closed subset of $\mathbb{R}^{3\times 3}$, we readily obtain the following existence result by the direct methods of the calculus of variations.

Theorem 2.21 *Let (2.20) be satisfied with $1 < p, q < \infty$ and let the admissible set be nonempty. Then variational problem (2.17) - (2.19) possesses a minimizer.*

Now we are interested in the Euler-Lagrange equation as necessary condition for a minimizer. Here we again assume that $V(\cdot,\cdot,s)$ is convex and we suppose that $W(\cdot,\cdot,s)$, $V(\cdot,\cdot,s)$ are continuously differentiable for all $s \in I$. Furthermore we impose the usual growth restriction

$$|W_u(u,v,s)| + |W_v(u,v,s)| \leq \tilde{c}(|u|^p + |v|^q) + \tilde{\psi}(s) \quad \text{for all } u,v \in \mathbb{R}^3, \text{ a.e. } s \in I \tag{2.22}$$

where $\tilde{c} > 0$ is a constant and $\tilde{\psi} \in \mathcal{L}^1(I)$. For a minimizer w of the variational problem we define the set

$$I_a = \{s \in I \,|\, V(u(s),v(s),s) = 0\}$$

of active s in (2.19) (note that I_a is only determined up to a set of measure zero).

Theorem 2.23 *Let (2.15), (2.22) be satisfied and let $w \in X$ be a local minimizer of the variational problem (2.17) - (2.19) with $1 < p, q < 1$. Then there are multipliers $\lambda \geq 0$, $f^1, f^2, m^1, m^2 \in \mathbb{R}^3$, and a measurable function $\varrho : I \to [0,\infty)$, not all zero, such that $\varrho\bar{m}^V \in \mathcal{L}^{p'}(I)$, $\varrho\bar{f}^V \in \mathcal{L}^{q'}(I)$, $\varrho(s) = 0$ a.e. on $I \setminus I_a$, and*

$$\begin{aligned}
0 &= \lambda m[w](s) - \lambda \int_s^1 \gamma'[w](\sigma) \times (f_e(\sigma) - f^2)\, d\sigma - \lambda m_{f_e}(s) + m^2 - \varrho(s)\bar{m}^V(s) && (2.24)\\
0 &= \lambda f[w](s) - \lambda f_e(s) + f^2 - \varrho(s)\bar{f}^V(s) && (2.25)\\
0 &= -\lambda \int_0^1 \gamma[w]'(s) \times f_e(s)\, ds - \lambda m_{f_e}(0) + m^1 + m^2 && (2.26)\\
0 &= -\lambda f_e(0) + f^1 + f^2 && (2.27)
\end{aligned}$$

for a.e. $s \in I$. If $|I \setminus I_a| > 0$ or if both $\bar{m}^V$ and $\bar{f}^V$ are not constant on I, then we can choose $\lambda = 1$.

Notice that (2.24), (2.25) express the balance of moments and forces, respectively, and (2.26), (2.27) say that the sum of all external moments and forces, respectively, has to vanish.

The assertion can be extended to the case where $V(\cdot,\cdot,s)$ is not differentiable if we extend Lemma 4.11 below to that case and replace the gradients $m^V(s)$, $f^V(s)$ with suitable elements from the convex subdifferentials $\partial_u V$ and $\partial_v V$, respectively.

An application of the theorem to the special constraints discussed after (2.14) provides a contribution to the balance of forces (2.25) if we restrict extension or shear angle and we get a contribution to the balance of moments (2.24) if we restrict torsion.

3 Constrained curves

Let $\gamma \in \mathcal{W}^{2,p}([0,1],\mathbb{R}^3)$ be a curve and let $\Gamma \in \mathcal{W}^{2,p}([0,1],\mathbb{R}^3)$ be the corresponding reparametrization of constant speed, i.e., $|\Gamma'(s)| = L$ for all s where $L = L[\gamma]$ is the length of the curve (without danger of confusion we always assume that Γ, $\tilde{\Gamma}$, Γ_k etc. refers to γ, $\tilde{\gamma}$, γ_k). Denoting the (local) curvature of γ at s by $\kappa[\gamma](s)$ and using $I := [0,1]$ we consider the variational problem

$$L[\gamma] = \int_0^1 |\gamma'(s)|\, ds \quad \to \quad \text{Min!}, \quad \gamma \in \mathcal{W}^{2,p}(I), \tag{3.1}$$

$$\gamma(0) = \gamma_0, \;\; \gamma(1) = \gamma_1, \;\; \frac{\Gamma'(0)}{|\Gamma'(0)|} = t_0, \;\; \frac{\Gamma'(1)}{|\Gamma'(1)|} = t_1 \tag{3.2}$$

$$\kappa[\gamma](s) = \sqrt{\frac{|\gamma'|^2|\gamma''|^2 - (\gamma'\cdot\gamma'')^2}{|\gamma'|^6}} \le \kappa_0 \quad \text{a.e. on } I \tag{3.3}$$

where $\gamma_j, t_j \in \mathbb{R}^3$, $\kappa_0 > 0$ are given with $|t_j| = 1$, $j = 1,2$. The curvature κ can be expressed much simpler by means of the constant speed parametrization Γ as

$$\kappa[\gamma](s) = \frac{1}{L[\gamma]^2}|\Gamma''(s)|\,. \tag{3.4}$$

Note that, for each solution $\tilde{\gamma} \in \mathcal{W}^{2,p}(I)$ of the previous variational problem we always have the corresponding solution $\tilde{\Gamma} \in \mathcal{W}^{2,\infty}(I)$.

The existence of a solution in $\mathcal{W}^{2,\infty}(I)$ (with constant speed parametrization) was shown by L.E. Dubins [5] for curves in $\mathbb{R}^n$ (based on the theorem of Arzelà-Ascoli, but without explicit use of the Sobolev space $\mathcal{W}^{2,\infty}(I)$). One might ask how far the direct methods of the calclus of variations yield the existence of a solution in $\mathcal{W}^{2,p}(I)$, $1 < p < \infty$, for the nonholonomic problem. As usually we select a minimizing sequence γ_k where we can assume that $\gamma_k = \Gamma_k$. The sequence Γ_k has to be bounded in $\mathcal{W}^{2,p}(I)$ by (3.4) and, thus, has a subsequence (denoted the same way) with a weak limit $\gamma \in \mathcal{W}^{2,p}(I)$. Conditions (3.2) are certainly stable with respect to weak convergence and $|\gamma'(s)| = L$ a.e. on I for $L := \lim_{k\to\infty} L[\Gamma_k]$, i.e., γ has again constant speed parametrization and agrees with Γ. Using (3.4) and the fact that closed convex sets are weakly closed we find that γ solves the variational problem.

The next natural question is to ask for a necessary minimality condition, i.e., for the Euler-Lagrange equation of the problem. In the case of planar curves Dubins [5] has shown, by comprehensive investigations of circular arcs and the curvature of planar curves, that a length minimizing curve subjected to the considered constraints has to be a composition of straight segments S and circular arcs C having curvature κ_0. Furthermore, by geometric arguments combined with second order minimality conditions, it is verified that a solution curve has to be of the type CCC, CSC

or a subarc thereof. By an approach based on control theory, E. Verriest [14] has basically proved that the curvature vector for a solution curve γ, which is obviously orthogonal to the tangent, either has to have length κ_0 (circular arcs) or length 0 (straight segments). Moreover it follows that there is a fixed direction $\tilde{t} \in \mathbb{R}^2$ such that $\Gamma'(s) = \tilde{t}$ on all straight segments of γ. But the discussion in [14] about global minimizers γ only considers curves of the type CSC and misses the type CCC.

Using the same methods as before for elastic rods we are able to derive the Euler-Lagrange equation for our variational problem in $\mathbb{R}^3$. While the result is new for non-planar solution curves, we easily recover the previous results in the planar case. A standard difficulty in geometric problems is the freedom in reparametrization. We circumvent that problem for our curves γ by the use of framed curves where an orthonormal frame $D(s) \in SO(3)$ is attached to each point $\gamma(s)$ of the curve. This way we merely take into account curves with constant speed parametrization. Let $n(s)$ denote the (unit) binormal of Γ directed as $\Gamma'(s) \times \Gamma''(s)$ at points $\Gamma(s)$ where $\Gamma''(s) \neq 0$ and we set $n(s) = 0$ at points where $\Gamma''(s) = 0$. The next theorem provides the Euler-Lagrange equation as necessary condition for solutions of our variational problem.

Theorem 3.5 *Let γ be a solution of the variational problem (3.1) – (3.3) with constant speed parametrization, i.e., $|\gamma'(s)| = |\Gamma'(s)| = L$ for a.e. $s \in I$. Then there are Lagrange multipliers $\lambda \in \mathbb{R}^3$, $\varrho \in \mathcal{C}(I, \mathbb{R})$, $\varrho(s) \geq 0$, not both zero, such that*

$$\varrho(0)n(0) + (\gamma(s) - \gamma(0)) \times \lambda = \varrho(s)n(s) \qquad \text{for all } s \in I, \tag{3.6}$$

$$(\gamma(1) - \gamma(0)) \cdot \lambda \leq \kappa_0 L \int_0^1 \varrho(s)\, ds\,, \tag{3.7}$$

$$\varrho(s) = 0 \quad \text{for a.e. } s \text{ with } \quad \kappa(s) < \kappa_0\,. \tag{3.8}$$

Moreover, ϱ and n are continuously differentiable on the open set $I_+ := \{s \in I \,|\, \varrho(s) > 0\}$ and the second derivatives $\varrho''(s)$ and $n''(s)$ exist a.e. on I_+. (Note that $\varrho(s) = 0$ for all s with $n(s) = 0$.)

Quite easily we obtain further conclusions from the previous Euler-Lagrange equation.

Corollary 3.9 *Let γ be a solution of (3.1) – (3.3) as in the previous theorem satisfying (3.6) – (3.8). Then:*

(a) If $\lambda = 0$, then $\varrho(s) = \varrho(0) > 0$ for all $s \in I$ and $\gamma(I)$ is a planar circular arc.

(b) If $\varrho(s) = 0$ for some $s \in I$, then $\lambda \neq 0$ and there is a line ℓ in $\mathbb{R}^3$ parallel to λ such that $\gamma(s) \in \ell$ if and only if $\varrho(s) = 0$. In particular, if $\gamma(I)$ has a straight segment, then it belongs to ℓ. Moreover $\gamma(s) \in \ell$ for all s where n is discontinuous.

(c) If $\varrho(s) > 0$ on $(s_1, s_2) \subset I$, $s_1 < s_2$, and $\varrho(\tilde{s}) = 0$ for some $\tilde{s} \in I$, then $\gamma([s_1, s_2])$ is a planar circular arc.

(d) If the solution curve $\gamma(I)$ contains a straight segment, then $\gamma(I)$ consists of at most two planar circular arcs separated by the straight segment. In particular, $\gamma(I)$ cannot contain two separated straight segments.

Remark 3.10 The proof in [5] that a planar solution curve $\gamma(I)$ cannot contain four successive circular arcs (with switching n) uses second order arguments and cannot be seen directly from Theorem 3.5, but Corollary 3.9 c) provides at least a partial result contained in the corresponding proof in [5].

The next example demonstrates that minimizing curves in $\mathbb{R}^3$ do not have to be the composition of planar arcs.

Example: We fix

$$\gamma_0 = (\vartheta, 0, 0), \quad \gamma_1 = (\vartheta, 0, 2\pi\vartheta), \quad t_0 = t_1 = 2\pi\vartheta(0, 1, 1), \quad \kappa_0 = \frac{2\pi}{\sqrt{2}} \quad \text{where } \vartheta = \frac{1}{2\pi\sqrt{2}}.$$

For the curve

$$\gamma(s) = \vartheta(\cos 2\pi s, \sin 2\pi s, 2\pi s), \quad s \in I = [0, 1].$$

we have

$$\gamma'(s) = 2\pi\vartheta(-\sin 2\pi s, \cos 2\pi s, 1), \quad \gamma''(s) = 4\pi^2\vartheta(-\cos 2\pi s, -\sin 2\pi s, 0).$$

Thus $\gamma = \Gamma$ has arc length parametrization, $\kappa(s) = |\gamma''(s)| = \kappa_0$ for all $s \in I$, and

$$\gamma(0) = \gamma_0, \quad \gamma(1) = \gamma_1, \quad \gamma'(0) = t_0, \quad \gamma'(1) = t_1,$$

i.e., γ satisfies the side conditions (3.2), (3.3). Let us now check conditions (3.6), (3.7) for γ. For the binormal of γ we readily obtain

$$n(s) = \frac{1}{\sqrt{2}}(\sin 2\pi s, -\cos 2\pi s, 1).$$

Then (3.6) becomes

$$\frac{\varrho(0)}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \vartheta \begin{pmatrix} \cos 2\pi s - 1 \\ \sin 2\pi s \\ 2\pi s \end{pmatrix} \times \lambda \; = \; \frac{\varrho(s)}{\sqrt{2}} \begin{pmatrix} \sin 2\pi s \\ -\cos 2\pi s \\ 1 \end{pmatrix}.$$

For $s = 1$ we find that $\varrho(0) \neq \varrho(1)$ is impossible and, since we can normalize λ, we obtain that $\lambda = (0, 0, \pm 1)$. Thus

$$\frac{\varrho(0)}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \pm \vartheta \begin{pmatrix} \sin 2\pi s \\ 1 - \cos 2\pi s \\ 0 \end{pmatrix} \; = \; \frac{\varrho(s)}{\sqrt{2}} \begin{pmatrix} \sin 2\pi s \\ -\cos 2\pi s \\ 1 \end{pmatrix}.$$

From the last line we get $\varrho(s) = \varrho(0)$ for all $s \in I$. Hence the first line implies that $\varrho(s) = 1/2\pi$ on I and $\lambda = (0, 0, 1)$. This way the seond line is satisfied too. Moreover (3.7) becomes $2\pi\vartheta \leq \kappa_0/2\pi$ which is obviously true. Consequently γ satisfies (3.6) – (3.8).

4 Proofs

4.1 Proof of Theorem 2.23

Since we need a linear structure for our analysis, we represent small variations $D_0\check{D}_0 \in SO(3)$ of $D_0 \in SO(3)$ by variations $\check{a} \in \mathbb{R}^3$. For that reason we consider the mapping $\alpha : SO(3) \to \mathbb{R}^3$ assigning the rotation vector $\check{a} \in \mathbb{R}^3$ to $\check{D} \in SO(3)$, i.e., $\check{a}$ provides the rotation axis and $|\check{a}|$ the positively oriented rotation angle in $[0, \pi)$ for the element $\check{D}$. α is a bijective and continuously differentiable mapping from a neighborhood of the identity in $SO(3)$ on a neighborhood of the origin in $\mathbb{R}^3$ (cf. Schuricht & v.d. Mosel [12] for further details.)

Thus small perturbations $(u+\check{u}, v+\check{v}, \gamma_0+\check{\gamma}_0, D_0\check{D}_0) \in X$ of $(u, v, \gamma_0, D_0) \in X$ can be identified with the elements

$$(u+\check{u}, v+\check{v}, \gamma_0+\check{\gamma}_0, \alpha(\check{D})) \in \mathcal{L}^p(I, \mathbb{R}^3) \times \mathcal{L}^q(I, \mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3 =: Y$$

(note that $D_0 \in SO(3)$ corresponds to $\check{a} = 0$ in $\mathbb{R}^3$). Identifying the elements of X and Y in a neighborhood of the solution w we readily see that w is a local minimizer of the corresponding variational problem formulated in Y and we claim to derive the necessary minimality condition for w with respect to the latter problem.

Theorem 5.8 in the Appendix implies that $\check{w} \to d_k[\check{w}](\cdot)$ is continuously differentiable on a neighborhood $U(w) \subset Y$ of w and, in analogy to Schuricht & v.d. Mosel [12, Lemma 16], we obtain (D_w - derivative with repect to w)

$$D_w d_k[w](s)\, \check{w} = z[\check{u}, \check{a}](s) \times d_k[w](s), \quad k = 1, 2, 3, \tag{4.1}$$

for all $\check{w} = (\check{u}, \check{v}, \check{\gamma}_0, \check{a}) \in Y$, $s \in I$ where, for the fixed frame field $D[w](\cdot)$, the curves $z = z[\check{u}, \check{a}] \in \mathcal{W}^{1,p}(I)$ given by

$$z(s) = z(0) + \int_0^s \sum_{i=1}^3 \check{u}_i(\sigma) d_i[w](\sigma)\, d\sigma \qquad \text{with} \quad z(0) = \sum_{i=1}^3 \check{a}_i d_i[w](0)\,. \tag{4.2}$$

provide a unique correspondence between the perturbations $(\check{u}, \check{a}) \in \mathcal{L}^p(I) \times \mathbb{R}^3$ and the elements $z \in \mathcal{W}^{1,p}(I)$.

Note. (4.1), (4.2) first follow for small intervals $J \subset \mathbb{R}$ instead of I from Theorem 5.8. The uniform boundedness of the solutions $d_k[\check{w}](\cdot)$ and the compactness of I then allow the continuation on I. Furthermore we have to derive the explicit representation of $z(0)$ in (4.2) from the more implicit version [12, (A.166)]. This can be done by using that $D_0 \in SO(3)$ consists of the column vectors $d_j[w](0)$, $j = 1, 2, 3$, that $D_0^{-1} = D_0^T$, and that

$$(\alpha^{-1})'(0)\, a = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \quad \text{for} \quad a = (a_1, a_2, a_3) \in \mathbb{R}^3$$

(cf. [12, p. 79]).

We now imply that $\gamma[\cdot](s)$ is continuously differentiable on $U(w)$ with

$$D_w \gamma[w](s)\check{w} = \check{\gamma}_0 + \sum_{i=1}^3 \int_0^s \Big(\check{v}_i d_i[w](\sigma) + v_i z[\check{u}, \check{a}](\sigma) \times d_i[w](\sigma) \Big) d\sigma \tag{4.3}$$

for all $\check{w} = (\check{u}, \check{v}, \check{\gamma}_0, \check{a}) \in Y$, $s \in I$.

For the treatment of the boundary conditions (2.18) we introduce the functions

$$g_1(\tilde{w}) := \gamma[\tilde{w}](0) - \gamma_0, \quad g_2(\tilde{w}) := \gamma[\tilde{w}](1) - \gamma_1\,.$$

Note that the boundary conditions $D[w](0) = D_0$, $D[w](1) = D_1$ in (2.18) are overdetermined, since there are 9 scalar equations for 3 degrees of freedom in each case. Thus we replace them with the (locally) equivalent conditions

$$g_3(\tilde{w}) := \begin{pmatrix} d_1[\tilde{w}](0) \cdot d_{02} \\ d_3[\tilde{w}](0) \cdot d_{01} \\ d_3[\tilde{w}](0) \cdot d_{02} \end{pmatrix} = 0\,, \qquad g_4(\tilde{w}) := \begin{pmatrix} d_1[\tilde{w}](1) \cdot d_{12} \\ d_3[\tilde{w}](1) \cdot d_{11} \\ d_3[\tilde{w}](1) \cdot d_{12} \end{pmatrix} = 0$$

where $D_0 = (d_{01}|d_{02}|d_{03})$, $D_1 = (d_{11}|d_{12}|d_{13})$. From our previous considerations we get that all g_j are continuously differentiable on $U(w)$ and (4.1), (4.3) imply

$$\begin{aligned}
g_1'(w)\check{w} &= \check{\gamma}_0\,, \\
g_2'(w)\check{w} &= \check{\gamma}_0 + \sum_{i=1}^{3} \int_0^1 \Big(\check{v}_i d_i[w](s) + z[\check{u},\check{a}](s) \times v_i(s) d_i[w](s) \Big) ds\,, \\
&\overset{\text{(Fubini)}}{=} \check{\gamma}_0 + \sum_{i=1}^{3} \Big(\int_0^1 \check{v}_i d_i[w](s)\,ds + \int_0^1 z[\check{u},\check{a}]'(s) \times \int_s^1 v_i(\sigma) d_i[w](\sigma)\,d\sigma\,ds \Big) \\
g_3'(w)\check{w} &= (d_{03}|d_{02}| - d_{01})^T \cdot z[\check{u},\check{a}](0)\,, \\
g_4'(w)\check{w} &= (d_{13}|d_{12}| - d_{11})^T \cdot z[\check{u},\check{a}](1) \\
&= (d_{13}|d_{12}| - d_{11})^T \cdot \Big(z[\check{u},\check{a}](0) + \int_0^1 z[\check{u},\check{a}]'(s)\,ds \Big)
\end{aligned} \tag{4.4}$$

for all $\check{w} = (\check{u}, \check{v}, \check{\gamma}_0, \check{a}) \in Y$.

Standard arguments using (2.22) imply the continuous differentiability of $E_s(\cdot)$ on $U(w)$ with

$$\begin{aligned}
E_s'(w)\check{w} &= \int_0^1 \sum_{i=1}^{3} \Big(W_{u_i}(u(s), v(s), s)\, \check{u}_i(s) + W_{v_i}(u(s), v(s), s)\, \check{v}_i(s) \Big)\,ds \\
&= \int_0^1 \Big(z'[\check{u},\check{a}](s) \cdot \sum_{i=1}^{3} m_i(s) d_i[w](s) + \sum_{i=1}^{3} f_i(s) \check{v}_i(s) \Big)\,ds
\end{aligned} \tag{4.5}$$

for all $\check{w} = (\check{u}, \check{v}, \check{\gamma}_0, \check{a}) \in Y$ where we have used (2.5) and (4.2) for the last identity.

For $E_p(\cdot)$ we observe that the derivative of the integrand in (2.6) has an integrable majorant. Thus $E_p(\cdot)$ is continuously differentiable on $U(w)$ with

$$\begin{aligned}
E_p'(w)\check{w} &= -\int_\Omega \Big(\check{\gamma}_0 + \int_0^s \Big(\sum_{i=1}^{3} \check{v}_i(\sigma) d_i[w](\sigma) + v_i(\sigma) z[\check{u},\check{a}](\sigma) \times d_i[w](\sigma) \Big)\,d\sigma \Big) \cdot d\mathfrak{f}_e(s,\zeta_1,\zeta_2) \\
&\quad - \int_\Omega \Big(\zeta_1 z[\check{u},\check{a}](s) \times d_1[w](s) + \zeta_2 z[\check{u},\check{a}](s) \times d_2[w](s) \Big) \cdot d\mathfrak{f}_e(s,\zeta_1,\zeta_2)
\end{aligned} \tag{4.6}$$

for all $\check{w} = (\check{u}, \check{v}, \check{\gamma}_0, \check{a}) \in Y$. Using Fubini's Theorem and partial integration we obtain the following identities (arguments in brackets are omitted)

$$\begin{aligned}
&\int_\Omega \Big(\int_0^s z(\sigma) \times v_i(\sigma) d_i(\sigma)\,d\sigma \Big) \cdot d\mathfrak{f}_e(s,\zeta_1,\zeta_2) \\
&\qquad = \int_0^1 z'(s) \cdot \Big(\int_s^1 v_i(\sigma) d_i(\sigma) \times \int_{\Omega_\sigma} d\mathfrak{f}_e(\tau,\zeta_1,\zeta_2)\,d\sigma \Big)\,ds \\
&\qquad\quad + z(0) \cdot \Big(\int_0^1 v_i(\sigma) d_i(\sigma) \times \int_{\Omega_\sigma} d\mathfrak{f}_e(\tau,\zeta_1,\zeta_2) \Big)\,d\sigma\,,
\end{aligned} \tag{4.7}$$

$$\int_\Omega \Big(\zeta_i z(s)\times d_i(s)\Big)\cdot d\mathfrak{f}_e(s,\zeta_1,\zeta_2)$$
$$= \int_0^1 z'(s)\cdot\Big(\int_{\Omega_s}\zeta_i d_i(\sigma)\times d\mathfrak{f}_e(\sigma,\zeta_1,\zeta_2)\Big)\,ds$$
$$+\, z(0)\cdot\int_\Omega \zeta_i d_i(\sigma)\times d\mathfrak{f}_e(\sigma,\zeta_1,\zeta_2)\,, \tag{4.8}$$

$$\int_\Omega\int_0^s \check{v}_i(\sigma)d_i(\sigma)\,d\sigma\;\cdot d\mathfrak{f}_e(s,\zeta_1,\zeta_2)$$
$$= \int_0^1 \check{v}_i(s)d_i(s)\cdot\Big(\int_{\Omega_s} d\mathfrak{f}_e(\sigma,\zeta_1,\zeta_2)\Big)\,ds\,. \tag{4.9}$$

for $i=1,2,3$ (cf. also [12, (106), (107)]). Consequently, using the notation from (2.7), (2.8),

$$\begin{aligned} E_p'(w)\check{w} \;\;=\;\; & -\check{\gamma}_0 f_e(0) - \int_0^1\Big(\sum_{i=1}^3 \check{v}_i(s)d_i(s)\Big)\cdot f_e(s)\,ds \\ & -\int_0^1 z'(s)\cdot\Big(\int_s^1\sum_{i=1}^3 v_i(\sigma)d_i(\sigma)\times f_e(\sigma)\Big)\,ds \\ & -z(0)\cdot\Big(\int_0^1\sum_{i=1}^3 v_i(s)d_i(s)\times f_e(s)\Big)\,ds \\ & -\int_0^1 z'(s)\cdot m_{f_e}(s)\,ds - z(0)\cdot m_{f_e}(0)\,. \end{aligned} \tag{4.10}$$

Condition (2.19) describes a convex set $C\subset Y$ and we are interested in the structure of the normal cone $N_C(w)\subset Y^*$ of C at w where the dual space of Y is given by

$$Y^* = \mathcal{L}^{p'}(I)\times\mathcal{L}^{q'}(I)\times\mathbb{R}^3\times\mathbb{R}^3\,.$$

Lemma 4.11 *We have $w^*=(u^*,v^*,\gamma_0^*,a^*)\in N_C(w)$ with $u^*=(u_1^*,u_2^*,u_3^*)$, $v^*=(v_1^*,v_2^*,v_3^*)$, if and only if $\gamma_0^*=0$, $a^*=0$,*

$$u_i^*(s)=\begin{cases}\varrho(s)\bar{m}_i^V(s) & \text{a.e. on } I_a,\\ 0 & \text{otherwise,}\end{cases} \tag{4.12}$$

$$v_i^*(s)=\begin{cases}\varrho(s)\bar{f}_i^V(s) & \text{a.e. on } I_a,\\ 0 & \text{otherwise,}\end{cases} \tag{4.13}$$

for a nonnegative measurable function ϱ such that $\varrho\bar{m}^V\in\mathcal{L}^{p'}(I)$, $\varrho\bar{f}^V\in\mathcal{L}^{q'}(I)$ (cf. (2.16) for notation).

Proof . In this proof we use the abbreviation $\alpha=(u,v)$ for pairs (u,v) of points in $\mathbb{R}^6$ or functions in $\mathcal{L}^p(I)\times\mathcal{L}^q(I)$ and $\alpha_k=(u_k,v_k)$, $\tilde{\alpha}=(\tilde{u},\tilde{v})$ etc. Let us define the convex sets

$$C(s)=\{\tilde{\alpha}\in\mathbb{R}^6 \,|\, V(\tilde{\alpha},s)\le 0\}\,,\quad s\in I\,.$$

Standard arguments show that the normal cone $N_{C(s)}(\tilde{\alpha}) \subset \mathbb{R}^3 \times \mathbb{R}^3$ of $C(s)$ at $\tilde{\alpha}$ equals $\{0\}$ if $\tilde{\alpha} \in \operatorname{int} C(s)$ and $\{t\nu(s) |\, t \geq 0\}$ if $\tilde{\alpha} \in \partial C(s)$. Hence, if $\tilde{\alpha} \in \partial C(s)$ and $\tilde{\alpha} + \check{\alpha} \in C(s)$, then

$$\nu(s) \cdot \check{\alpha} \leq 0\,.$$

For $w^* \in Y^*$ having the form stated in the lemma we thus get for all $\check{w}$ with $w + \check{w} \in C$ that

$$\langle w^*, \check{w} \rangle = \int_0^1 \varrho(s)\, \nu(s) \cdot (\check{u}(s), \check{v}(s))\, ds \leq 0\,,$$

i.e., $w^* \in N_C(w)$.

Now let $w^* = (u^*, v^*, \gamma_0^*, a^*) \in Y^*$ be an arbitrary element in $N_C(w)$. Since $(u, v, \gamma_0 + \check{\gamma}_0, \check{a}) \in C$ for all $\check{\gamma}_0, \check{a} \in \mathbb{R}^3$, we readily see that $\gamma_0^* = 0$, $a^* = 0$. Next we claim that

$$\alpha^*(s) = (u^*(s), v^*(s)) = 0 \quad \text{a.e. on } I \setminus I_a. \tag{4.14}$$

Suppose that $\alpha^*(s) \neq 0$ on $I_1 \subset (I \setminus I_a)$ with $|I_1| > 0$. Since $\alpha(s) \in \operatorname{int} C(s)$ a.e. on I_1, we find $\delta > 0$ and $I_1' \subset I_1$ such that $|I_1'| > 0$ and $\alpha(s) + B_\delta(0) \subset C(s)$ for almost all $s \in I_1'$ ($B_\delta(0) \subset \mathbb{R}^6$). If we choose $\check{\alpha}(s) = \delta\alpha^*(s)/|\alpha^*(s)|$ on I_1', $\check{\alpha}(s) = 0$ on $I \setminus I_1'$, and $\check{w} = (\check{\alpha}, 0, 0)$, then we obtain the contradiction that $0 \geq \langle w^*, \check{w} \rangle > 0$ which verifies (4.14).

We now set

$$\psi(s) = \alpha^*(s) \cdot \nu(s)\,, \quad \check{\alpha}^*(s) = \alpha^*(s) - \psi(s)\nu(s)\,. \tag{4.15}$$

Obviously,

$$\check{\alpha}^*(s) \cdot \nu(s) = 0 \quad \text{for all } s \in I. \tag{4.16}$$

If $\psi(s) < 0$ a.e. on a set $I_2 \subset I$ with $|I_2| > 0$, then we find $\delta > 0$ and $I_2' \subset I_2$ with $|I_2'| > 0$ such that $\alpha(s) + \check{\alpha}(s) \in C(s)$ with $\check{\alpha}(s) = \delta\alpha^*(s)/|\alpha^*(s)|$ for all $s \in I_2'$ by the smoothness of $V(\cdot,\cdot,s)$. Choosing $\check{\alpha}(s) = 0$ on $I \setminus I_2'$ we get the contradicition that $0 \geq \langle w^*, \check{w} \rangle = \langle \alpha^*, \check{\alpha} \rangle > 0$. Hence

$$\psi(s) \geq 0 \quad \text{a.e. on } I.$$

Assume now that $\check{\alpha}^*(s) \neq 0$ on a set $I_3 \subset I_a$ with $|I_3| > 0$ and define

$$\check{\alpha}_1^*(s) = \frac{\check{\alpha}^*(s)}{|\check{\alpha}^*(s)|} \quad \text{a.e. on } I_3.$$

Since $\partial C(s)$, $s \in I$, are smooth surfaces in $\mathbb{R}^6$, for each $s \in I_3$ there is $r(s) > 0$ and a real function $\tau_s(\cdot)$ such that for all $|t| \leq r(s)$

$$\alpha(s) + t\check{\alpha}_1^*(s) + \tau_s(t)\nu(s) \in \partial C(s) \quad \text{with} \quad \tau_s(t) = o(t) \;\; (\text{as } t \to 0)$$

and $\tau_s(t) \leq 0$ by convexity of $C(s)$ (cf. Zeidler [18, Theorem 43.C]). We thus find $r_0 > 0$ and $I_3' \subset I_3$ with $|I_3'| > 0$ and $r(s) \geq r_0$ on I_3'. By Egoroff's theorem we can assume that $k\tau_s(1/k) \to 0$ as $k \to \infty$ uniformly for all $s \in I_3'$. We have that $w + \check{w}_t \in C$ for all $0 < t < r_0$ where $\check{w}_t = (\check{\alpha}_t, 0, 0)$ with

$$\check{\alpha}_t(s) = \begin{cases} t\check{\alpha}_1^*(s) + \tau_s(t)\nu(s) & \text{on } I_3', \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned}
0 \geq \langle w^*, \breve{w}_t\rangle &\overset{(4.15)}{=} \int_{I_3'} \langle \breve{\alpha}^*(s) + \psi(s)\nu(s), t\breve{\alpha}_1^*(s) + \tau_s(t)\nu(s)\rangle\, ds \\
&\overset{(4.16)}{=} \int_{I_3'} t|\breve{\alpha}^*(s)| + \psi(s)\tau_s(t)\, ds \\
&= t\int_{I_3'} \Big(|\breve{\alpha}^*(s)| + \psi(s)\frac{\tau_s(t)}{t}\Big)\, ds \quad \text{for all } 0 < t < r_0. \qquad (4.17)
\end{aligned}$$

But, for a sufficiently small $t > 0$ the right hand side becomes positive and yields a contradiction. Hence $\breve{\alpha}^*(s) = 0$ a.e. on I_a which implies that $\alpha^*(s)$ is directed as $\nu(s)$ a.e. on I_a, i.e., there is $\varrho(s) \geq 0$ such that (4.12) and (4.13) hold. The fact that $u^* \in \mathcal{L}^{p'}(I)$ and $v^* \in \mathcal{L}^{q'}(I)$ completes the assertion. $\diamondsuit$

We now consider the variational problem (2.17), (2.19) with the boundary conditions $g_j(\tilde{w}) = 0$, $j = 1, \ldots, 4$, in Y instead of X and, obviously, $w \in Y$ (which is identified with the minimizer $w \in X$) is a local minimizer. Note that $w \in Y$ minimizes a continuously differentiable functional with respect to equality constraints for a finite number of continuously differentiable real functions and the closed convex set $C \subset Y$. Thus we can apply a Lagrange multiplier rule (e.g., Clarke [3, Theorem 6.1.1]) and obtain that there are multipliers $\lambda \geq 0$, $\lambda_j \in \mathbb{R}^3$, $j = 1, \ldots, 4$, $w^* \in N_C(w)$, not all zero, such that for all $\breve{w} = (\breve{u}, \breve{v}, \breve{\gamma}_0, \breve{a}) \in Y$

$$\begin{aligned}
0 &= \langle \lambda E'(w) + \sum_{j=1}^{4} \lambda_j \cdot g_j'(w) + w^*, \breve{w}\rangle \\
&= \lambda \int_0^1 \Big(z'[\breve{u}, \breve{a}](\sigma) \cdot \sum_{i=1}^{3} m_i(u(\sigma), v(\sigma), \sigma) d_i[w](\sigma) + \sum_{i=1}^{3} f_i(\sigma)\breve{v}_i(\sigma)\Big)\, d\sigma \\
&\quad -\lambda\breve{\gamma}_0 \int_\Omega d\mathfrak{f}_e(s, \zeta_1, \zeta_2) - \lambda \int_0^1 \Big(\sum_{i=1}^{3} \breve{v}_i(s) d_i[w](s)\Big) \cdot \Big(\int_{\Omega_s} d\mathfrak{f}_e(\sigma, \zeta_1, \zeta_2)\Big)\, ds \\
&\quad -\lambda \int_0^1 z'[\breve{u}, \breve{a}](s) \cdot \Big(\int_s^1 \sum_{i=1}^{3} v_i(\sigma) d_i[w](\sigma) \times \int_{\Omega_\sigma} d\mathfrak{f}_e(\tau, \zeta_1, \zeta_2)\, d\sigma\Big)\, ds \\
&\quad -\lambda z[\breve{a}](0) \cdot \int_0^1 \Big(\sum_{i=1}^{3} v_i(\sigma) d_i[w](\sigma) \times \int_{\Omega_\sigma} d\mathfrak{f}_e(\tau, \zeta_1, \zeta_2)\Big)\, d\sigma \\
&\quad -\lambda \int_0^1 z'[\breve{u}, \breve{a}](s) \cdot \Big(\int_{\Omega_s} (\zeta_1 d_1[w](\sigma) + \zeta_2 d_2[w](\sigma)) \times d\mathfrak{f}_e(\sigma, \zeta_1, \zeta_2)\Big)\, ds \\
&\quad -\lambda z(0) \cdot \int_\Omega (\zeta_1 d_1[w](\sigma) + \zeta_2 d_2[w](\sigma)) \times d\mathfrak{f}_e(\sigma, \zeta_1, \zeta_2) + \lambda_1 \cdot \breve{\gamma}_0 \\
&\quad \lambda_2 \cdot \Big(\breve{\gamma}_0 + \sum_{i=1}^{3} \int_0^1 \Big(\breve{v}_i(s) d_i[w](s) + z'[\breve{u}, \breve{a}](s) \times \int_s^1 v_i(\sigma) d_i[w](\sigma)\, d\sigma\Big) ds\Big) \\
&\quad +\lambda_3 \cdot (d_{03}|d_{02}| - d_{01})^T \cdot z[\breve{a}](0) \\
&\quad +\lambda_4 \cdot (d_{13}|d_{12}| - d_{11})^T \cdot \Big(z[\breve{a}](0) + \int_0^1 z'[\breve{u}, \breve{a}](s)\, ds\Big)
\end{aligned}$$

$$-\int_0^1 \varrho(s)\sum_{i=1}^3 \bar{m}_i^V(s)\check{u}_i(s)\,ds - \int_0^1 \varrho(s)\sum_{i=1}^3 \bar{f}_i^V(s)\check{v}_i(s)\,ds \tag{4.18}$$

Taking special variations we readily imply that

$$\begin{aligned}
0 &= \lambda\sum_{i=1}^3 m_i(s)d_i[w](s) - \lambda\int_s^1 \Big(\sum_{i=1}^3 v_i(\sigma)d_i[w](\sigma)\times\int_{\Omega_\sigma} d\mathfrak{f}_e(\tau,\zeta_1,\zeta_2)\Big)\,d\sigma\\
&\quad -\lambda\int_{\Omega_s}(\zeta_1 d_1[w](\sigma)+\zeta_2 d_2[w](\sigma))\times d\mathfrak{f}_e(\sigma,\zeta_1,\zeta_2) + \int_s^1 v_i(\sigma)d_i[w](\sigma)\,d\sigma\times\lambda_2\\
&\quad +(d_{13}|d_{12}|-d_{11})\cdot\lambda_4 - \varrho(s)\bar{f}^V(s)\\
0 &= \lambda\sum_{i=1}^3 f_i(s)d_i[w](s) - \lambda\int_{\Omega_s} d\mathfrak{f}_e(\sigma,\zeta_1,\zeta_2) + \lambda_2 - \varrho(s)\bar{m}^V(s)\\
0 &= -\lambda f_e(0)+\lambda_1+\lambda_2\\
0 &= -\lambda\int_0^1\sum_{i=1}^3 v_i(s)d_i(s)\times f_e(s)\,ds - \lambda m_{f_e}(0)\\
&\quad +(d_{03}|d_{02}|-d_{01})\cdot\lambda_3 + (d_{13}|d_{12}|-d_{11})\cdot\lambda_4
\end{aligned}$$

for a.e. $s\in I$. We set $f^1 := \lambda_1$, $f^2 := \lambda_2$, $m^1 := (d_{03}|d_{02}|-d_{01})\cdot\lambda_3$, $m^2 := (d_{13}|d_{12}|-d_{11})\cdot\lambda_4$. Note that $(d_{i3}|d_{i2}|-d_{i1})\in SO(3)$ for $i=1,2$.

Let now $|I\setminus I_a|>0$ and assume that $\lambda=0$. Then $\lambda_2=\varrho(s)\bar{f}^V(s)$ a.e. on I. Since $\varrho(s)=0$ a.e. on $I\setminus I_a$, we get $\lambda_2=0$ and thus $\varrho(s)=0$ a.e. on I. Hence $m^2=0$, $\lambda_1=0$, $m^1=0$, and also $\lambda_3=\lambda_4=0$. But this contradicts the fact that not all multipliers are zero. Similarly we obtain a contradiction in the other case. Without any loss of generality we can thus take $\lambda=1$ in these cases which completes the proof.

4.2 Proof of Theorem 3.5 and Corollary 3.9

Proof of Theorem 3.5. First we recall a representation of curves $\gamma\in\mathcal{W}^{2,p}(I)$ $(1\le p\le\infty)$ having constant speed parametrization from [13]. A pair (γ,D) is said to be a *framed curve* if $\gamma\in\mathcal{W}^{2,p}(I,\mathbb{R}^3)$ is a curve with constant speed parametrization equipped with a frame field $D: I\to SO(3)$ of class $\mathcal{W}^{1,p}(I,\mathbb{R}^{3\times 3})$ such that $\gamma'(s)=vd_3(s)$ for some $v>0$ where $D(s)=(d_1(s)|d_2(s)|d_3(s))$ is the matrix with the orthonormal column vectors $d_j(s)$, $j=1,2,3$. Thus we can interpret a framed curve as a curve having an orthonormal frame $D(s)$ attached to each point $\gamma(s)$ and $|\gamma'(s)|=v$. Notice that such framed curves are in fact the same structure we had used for the description of elastic rods but with the special choice of $(v_1,v_2,v_3)=(0,0,v)$ (observe that, in contrast to Section 2 where v had been a vector-valued funcion, here $v\in\mathbb{R}$ is just a real number). In analogy to the statement surrounding (2.2) we find that a framed curve (γ,D) uniquely corresponds to an element

$$w=(u,v,\gamma_0,D_0)\in X := \mathcal{L}^p(I,\mathbb{R}^3)\times(0,\infty)\times\mathbb{R}^3\times SO(3)$$

with $u=(u_1,u_2,u_3)$ such that

$$d_k'(s) \;=\; \Big(\sum_{j=1}^3 u_j(s)d_j(s)\Big)\times d_k(s)\quad\text{for a.e. } s\in I,\quad k=1,2,3,$$

$$
\begin{aligned}
\gamma'(s) &= v d_3(s) \quad \text{for a.e. } s \in I, \\
\gamma(0) &= \gamma_0, \quad D(0) = D_0
\end{aligned}
\tag{4.19}
$$

(cf. Gonzalez et al. [6], Schuricht & v.d. Mosel [13]).

Clearly, each framed curve (γ, D) provides a curve $\gamma \in \mathcal{W}^{2,p}(I)$. The next lemma states that we also can assign a framed curve (γ, D) to each curve $\gamma \in \mathcal{W}^{2,p}(I)$ having constant speed parametrization.

Lemma 4.20 *Let $\gamma \in \mathcal{W}^{2,p}(I, \mathbb{R}^3)$, $1 \le p \le \infty$, with $|\gamma'(s)| = v > 0$ a.e. on I. Then there are $u_1, u_2 \in \mathcal{L}^p(I)$, $\gamma_0 \in \mathbb{R}^3$, $D_0 \in SO(3)$ such that $w = ((u_1, u_2, u_3), v, \gamma_0, D_0) \in X$ corresponds to a framed curve $(\gamma[w], D[w])$ with $\gamma = \gamma[w]$ for any $u_3 \in \mathcal{L}^p(I)$ via equations (4.19).*

The proof proceeds as that in Schuricht & v.d. Mosel [13, Prop. 6] for the case $p = \infty$.

Thus we can reformulate variational problem (3.1) – (3.3) in terms of elements $w \in X$ (recall (2.14)).

$$
\begin{aligned}
E(w) &:= \int_0^1 |\gamma'[w](s)|\, ds \;=\; v \;\to\; \text{Min!}, \quad w \in X, \\
\gamma[w](0) &= \gamma_0, \quad \gamma[w](1) \;=\; \gamma_1, \quad d_3[w](0) \;=\; t_0, \quad d_3[w](1) \;=\; t_1, && (4.21) \\
0 &\ge \sqrt{u_1(s)^2 + u_2(s)^2} - v\kappa_0 \quad \text{a.e. on } I. && (4.22)
\end{aligned}
$$

Note that the boundary conditions for the tangent vectors are overdetermined, since $d_3(s)$ is always a unit vector. To remove this we choose vectors $d_{j,1}, d_{j,2} \in \mathbb{R}^3$ such that $(d_{j1}, d_{j2}, t_j) \in SO(3)$, $j = 1, 2$, and replace (4.21) with the following equivalent equations.

$$
\begin{aligned}
g_1(w) &:= \gamma[w](0) - \gamma_0 = 0, & g_2(w) &:= \gamma[w](1) - \gamma_1 = 0, \\
g_3(w) &:= d_3[w](0) \cdot d_{0,1} = 0, & g_4(w) &:= d_3[w](0) \cdot d_{0,2} = 0, \\
g_5(w) &:= d_3[w](1) \cdot d_{1,1} = 0, & g_6(w) &:= d_3[w](1) \cdot d_{1,2} = 0.
\end{aligned}
\tag{4.23}
$$

Let now $w \in X$ correspond to a solution γ of the variational problem (3.1) – (3.3) with constant speed parametrization and note that $L[\gamma] = v$. In analogy to the beginning of the proof in Section 4.1 we identify small perturbations $(u + \check{u}, v + \check{v}, \gamma_0 + \check{\gamma}_0, D_0 \check{D}_0) \in X$ of w with elements

$$
(u + \check{u}, v + \check{v}, \gamma_0 + \check{\gamma}_0, \check{a}) \in \mathcal{L}^p(I, \mathbb{R}^3) \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 =: Y
$$

where Y is a Banach space.

We claim to apply a Lagrange multiplier rule and determine the needed derivatives. Obviously $E(\cdot)$ is continuously differentiable on Y with

$$
\langle E'(w), \check{w} \rangle = \check{v} \quad \text{for all } \check{w} \in Y.
$$

In analogy to (4.1) we obtain (for $1 < p < \infty$) that the mappings $\check{w} \to d_k[\check{w}](s)$, $k = 1, 2, 3$, $s \in I$ are continuously differentiable in Y near w and, with the notation $z(s) = z[\check{u}, \check{a}](s)$ from (4.2), we have that

$$
D_w d_k[w](s)\check{w} = z(s) \times d_k[w](s), \qquad k = 1, 2, 3, \;\; s \in I
$$

for all $\breve{w} = (\breve{u}, \breve{v}, \breve{\gamma}_0, \breve{a}) \in Y$. Therefore $\breve{w} \to \gamma[\breve{w}](s)$ is continuously differentiable with

$$D_w\gamma[w](s)\breve{w} = \breve{\gamma}_0 + v\int_0^s z(\sigma) \times d_3[w](\sigma)\,d\sigma + \breve{v}\int_0^s d_3[w](\sigma)\,d\sigma, \quad \text{for all } s \in I,\ \breve{w} \in Y.$$

Hence, all $g_j(\cdot)$ are continuously differentiable near w and we readily obtain for all $\breve{w} \in Y$

$$\begin{aligned}
g_1'(w)\breve{w} &= \breve{\gamma}_0\,, \\
g_2'(w)\breve{w} &= \breve{\gamma}_0 + v\int_0^1 z(\sigma) \times d_3[w](\sigma)\,d\sigma + \breve{v}\int_0^1 d_3[w](\sigma)\,d\sigma\,, \\
g_3'(w)\breve{w} &= (z(0) \times d_3[w](0)) \cdot d_{01} = z(0) \cdot d_{02}\,, \\
g_4'(w)\breve{w} &= (z(0) \times d_3[w](0)) \cdot d_{02} = -z(0) \cdot d_{01}\,, \\
g_5'(w)\breve{w} &= (z(1) \times d_3[w](1)) \cdot d_{11} = z(1) \cdot d_{12}\,, \\
g_6'(w)\breve{w} &= (z(1) \times d_3[w](1)) \cdot d_{12} = -z(1) \cdot d_{11}\,.
\end{aligned}$$

Condition (4.22) defines a convex set $C \subset Y$ and we want to determine the normal cone $N_C(w)$ of C at the point $w \in C$. Using the notation $V(u_1, u_2, u_3, v) := \sqrt{u_1^2 + u_2^2} - v\kappa_0$ we define the vector $n = \sum_{i=1}^3 n_i d_i$ with

$$n_1 := \frac{\partial V}{\partial u_1} = \frac{u_1}{\sqrt{u_1^2 + u_2^2}}, \quad n_2 := \frac{\partial V}{\partial u_2} = \frac{u_2}{\sqrt{u_1^2 + u_2^2}}, \quad n_3 := \frac{\partial V}{\partial u_3} = 0$$

for $\sqrt{u_1^2 + u_2^2} > 0$. Inserting the functions $u_j(s)$ on the right hand side we obtain $n(s)$. Note that $|n(s)| = 1$, $n(s) \cdot d_3(s) = n(s) \cdot \gamma'(s) = 0$, $n(s) \cdot \Gamma''(s) = 0$ where $\Gamma''(s)$ is the curvature vector of curve γ (recall the arguments after (2.13)). Hence we readily see that $n(s)$ is directed as $\gamma'(s) \times \Gamma''(s)$, i.e., $n(s)$ coincides with the (unit) binormal of γ at $\gamma(s)$.

The dual space of Y $(1 < p < \infty)$ is obviously given by

$$Y^* = \mathcal{L}^{p'}(I, \mathbb{R}^3) \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3\,, \qquad \tfrac{1}{p} + \tfrac{1}{p'} = 1\,.$$

Lemma 4.24 *Let $1 < p < \infty$. Then $w^* \in N_C(w) \subset Y^*$ if and only if there is $\varrho \in \mathcal{L}^{p'}(I, \mathbb{R})$ such that $\varrho(s) \geq 0$ a.e. on I, $\varrho(s) = 0$ for a.e. s satisfying $V(u_1(s), u_2(s), u_3(s), v) < 0$, and*

$$w^* = (u^*, v^*, \gamma_0^*, a^*) = (\varrho(n_1, n_2, n_3), -\varrho_0\kappa_0, 0, 0) \quad \textit{where} \quad \varrho_0 := \int_0^1 \varrho(\sigma)\,d\sigma\,. \tag{4.25}$$

Proof . We introduce the convex set

$$C_0 := \{(\breve{u}, \breve{v}) \in \mathbb{R}^3 \times \mathbb{R} \mid V(\breve{u}_1, \breve{u}_2, \breve{u}_3, \breve{v}) \leq 0\}$$

(which is even a cone). Standard arguments show that the normal cone $N_{C_0}(\breve{u}, \breve{v})$ of C_0 at $(\breve{u}, \breve{v}) \in \mathbb{R}^3 \times \mathbb{R}$ equals $\{0\}$ for $(\breve{u}, \breve{v}) \in \operatorname{int} C_0$ and $\operatorname{cone}\{(n, -\kappa_0)\}$ for $(\breve{u}, \breve{v}) \in \partial C_0$ with $\breve{v} > 0$ (cone - cone hull). Hence, if $(\breve{u}, \breve{v}) \in \partial C_0$ with $\breve{v} > 0$ and $(\breve{u} + \breve{u}, \breve{v} + \breve{v}) \in C_0$, then

$$(n, -\kappa_0) \cdot (\breve{u}, \breve{v}) \leq 0\,.$$

For w^* from (4.25) we thus have for any $\breve{w} \in Y$ with $w + \breve{w} \in C$ that

$$\langle w^*, \breve{w} \rangle = \int_0^1 \varrho(s)(\sum_{i=1}^3 n_i(s)\breve{u}_i(s) - \kappa_0\breve{v})\,ds \leq 0\,,$$

i.e., $w^* \in N_C(w)$.

Now let $w^* = (u^*, v^*, \gamma_0^*, a^*) \in Y^*$ be any element in $N_C(w)$. With

$$n^\perp := \frac{u_2 d_1 - u_1 d_2}{\sqrt{u_1^2 + u_2^2}}$$

we have an orthonormal frame $(n(s), n^\perp(s), d_3(s))$ for each $s \in I$ and we can decompose

$$\sum_{i=1}^{3} u_i^*(s) d_i(s) = \varrho_1(s) n(s) + \varrho_2(s) n^\perp(s) + u_3^*(s) d_3(s)\,. \tag{4.26}$$

If $\sqrt{u_1^2 + u_2^2} = 0$, then we take $n(s) = d_1(s)$ and $n^\perp(s) = d_2(s)$. First we observe that $((u_1, u_2, \breve{u}_3), v, \breve{\gamma}_0, \breve{a}) \in C$ for any $\breve{u}_3 \in \mathcal{L}^p(I)$, $\breve{\gamma}_0 \in \mathbb{R}^3$, $\breve{a} \in \mathbb{R}^3$ which implies that $u_3^* = 0$, $\gamma_0^* = 0$, $a^* = 0$.

Let $I^0 := \{s \in I |\ u_1(s) = u_2(s) = 0\}$, $\breve{u}_1, \breve{u}_2 \in \mathcal{L}^\infty$ with $\breve{u}_1(s) = \breve{u}_2(s) = 0$ a.e. on $I \setminus I^0$. Then $w + \breve{w} \in C$ for all $\breve{w} = ((\breve{u}_1, \breve{u}_2, 0), 0, 0, 0)$ with $\|\breve{u}_j\|_{\mathcal{L}^\infty}$ sufficiently small. Hence

$$0 \geq \langle w^*, \breve{w} \rangle = \int_0^1 \varrho_1 \breve{u}_1 + \varrho_2 \breve{u}_2 \, ds$$

(note that $u_j^*(s) = \varrho_j(s)$ on I^0, $j = 1, 2$). The arbitrariness of $\breve{u}_1$, $\breve{u}_2$ implies that $\varrho_1(s) = \varrho_2(s) = 0$ a.e. on I^0. Now let

$$I_\delta := \{s \in I |\ \sqrt{u_1(s)^2 + u_2(s)^2} - v\kappa_0 \leq -\delta\} \quad \text{for } \delta \geq 0.$$

A simple computation shows that, for $\tau < 1$,

$$(u(s), v) + (-\tau u(s), \breve{v}) \in C_0 \quad \text{as long as} \quad \breve{v} + \delta \geq -\tau(v - \delta), \quad s \in I_\delta. \tag{4.27}$$

For $\delta > 0$ we thus have $w + \breve{w} \in C$ for all $\breve{w} = (-\breve{\varrho} u, 0, 0, 0)$ with $\breve{\varrho} \in \mathcal{L}^\infty(I)$, $\breve{\varrho}(s) = 0$ a.e. on $I \setminus I_\delta$ and $\|\breve{\varrho}\|_{\mathcal{L}^\infty}$ sufficiently small. Hence

$$0 \geq \langle w^*, \breve{w} \rangle = -\int_0^1 \varrho_1 \breve{\varrho} \sqrt{u_1^2 + u_2^2} \, ds$$

The arbitrariness of $\breve{\varrho}$ implies $\varrho_1(s) = 0$ a.e. on I_δ for all $\delta > 0$. Since

$$\{s \in I |\ \sqrt{u_1^2(s) + u_2^2(s)} - v\kappa_0 < 0\} = \bigcup_{k \in \mathbb{N}} I_{1/k}\,,$$

we get $\varrho_1(s) = 0$ a.e. on this set. For $\delta = 0$ we obtain, by (4.27), that $w + \breve{w} \in C$ for all $\breve{w} = (-\breve{\varrho} u, 0, 0, 0)$ with $\breve{\varrho} \in \mathcal{L}^\infty(I)$, $\breve{\varrho}(s) \geq 0$ a.e. on I and $\|\breve{\varrho}\|_{\mathcal{L}^\infty}$ sufficiently small. Therefore

$$0 \geq \langle w^*, \breve{w} \rangle = -\int_0^1 \varrho_1 \breve{\varrho} \sqrt{u_1^2 + u_2^2} \, ds\,.$$

The arbitrariness of $\breve{\varrho}$ implies, combined with our previous results, $\varrho_1(s) \geq 0$ a.e. on I. Thus $\varrho_1(s) > 0$ implies essentially that s belongs to the set

$$I_a := \{s \in I |\ \sqrt{u_1(s)^2 + u_2(s)^2} - v\kappa_0 = 0\}$$

where the inequality constraint is active. Analogously as above we now get with variations $\check{w} = (-\tau u, -\tau v, 0, 0)$, $\tau \in \mathbb{R}$ small, $\delta = 0$

$$0 \geq \langle w^*, \check{w} \rangle = -\tau \int_0^1 \varrho_1 \sqrt{u_1^2 + u_2^2}\, ds - \tau v^* v = -\tau v \Big(\int_{I_a} \varrho_1 \kappa_0 \, ds + v^* \Big)$$

and, consequently,

$$v^* = -\kappa_0 \int_0^1 \varrho_1 \, ds \,. \tag{4.28}$$

In analogy to (4.27) we get by a further elementary computation that, for all $s \in I$,

$$(u(s), v) + \Big(\tau (u_2(s), -u_1(s), 0), \check{v} \Big) \in C_0 \quad \text{as long as} \quad \check{v} \geq (\sqrt{1+\tau^2} - 1) v \,.$$

We choose $\check{w} = (\check{u}, \check{v}, 0, 0)$ with $\check{u}_1 = \tau \check{\varrho} u_2$, $\check{u}_2 = -\tau \check{\varrho} u_1$, $\check{\varrho} = \operatorname{sign} \varrho_2$, $\check{v} = (\sqrt{1+\tau^2} - 1) v$, $\tau \in \mathbb{R}$, and derive as above

$$0 \geq \langle w^*, \check{w} \rangle = \tau \int_0^1 |\varrho_2| \sqrt{u_1^2 + u_2^2}\, ds + v^* v (\sqrt{1+\tau^2} - 1)$$

By (4.28) we get for $\tau > 0$

$$\int_0^1 |\varrho_2| \sqrt{u_1^2 + u_2^2}\, ds \leq \frac{\kappa_0 v (\sqrt{1+\tau^2} - 1)}{\tau} \int_0^1 \varrho_1 \, ds \;\overset{\tau \to 0}{\longrightarrow}\; 0$$

Hence $\varrho_2(s) = 0$ a.e. on I. Recalling (4.26) we verify the assertion of the lemma. $\diamondsuit$

As in the previous proof we are now able to apply a Lagrange multiplier rule (e.g., Clarke [3, Theorem 6.1.1]). Thus there exist multipliers $\lambda_0 > 0$, $\lambda_1, \lambda_2 \in \mathbb{R}^3$, $\lambda_3, \ldots, \lambda_6 \in \mathbb{R}$, $w^* \in N_C(w)$, not all zero, such that

$$0 = \lambda_0 \langle E'(w), \check{w} \rangle + \lambda_1 \cdot g_1'(w) \check{w} + \lambda_2 \cdot g_2'(w) \check{w} + \sum_{i=3}^{6} \lambda_i \langle g_i'(w), \check{w} \rangle + \langle w^*, \check{w} \rangle$$

for all $\check{w} \in Y$. Specifying all derivatives and using the notation $m_0 := \lambda_3 d_{02} - \lambda_4 d_{01}$, $m_1 := \lambda_5 d_{12} - \lambda_6 d_{11}$ we get

$$\begin{aligned} 0 \;&=\; \lambda_0 \check{v} + \lambda_1 \cdot \check{\gamma}_0 + \lambda_2 \cdot \check{\gamma}_0 + v \int_0^1 (z \times d_3) \cdot \lambda_2 \, ds + \check{v} \lambda_2 \cdot \int_0^1 d_3 \, ds \\ &\quad + z(0) \cdot m_0 + z(1) \cdot m_1 + \sum_{i=1}^{3} \int_0^1 \varrho n_i \cdot \check{u}_i \, ds - \varrho_0 \kappa_0 \check{v} \qquad \text{for all } \check{w} \in Y. \end{aligned} \tag{4.29}$$

Using Fubinis theorem we obtain

$$\begin{aligned} \int_0^1 (z \times d_3) \cdot \lambda_2 \, ds \;&=\; \int_0^1 \Big(z(0) + \int_0^s z'(\sigma)\, d\sigma \Big) \cdot (d_3(s) \times \lambda_2)\, ds \\ &=\; z(0) \cdot \int_0^1 d_3 \times \lambda_2 \, ds + \int_0^1 \int_s^1 z'(s) \cdot (d_3(\sigma) \times \lambda_2)\, d\sigma \, ds \,. \end{aligned}$$

Furthermore

$$\int_0^1 d_3 \, ds = \frac{1}{v} (\gamma(1) - \gamma(0)) \,, \qquad z(1) = z(0) + \int_0^1 z' \, ds \,.$$

Thus (4.29) implies for all $\check{w} \in Y$

$$\begin{aligned} 0 &= \check{v}\Big(\lambda_0 + \tfrac{1}{v}(\gamma(1)-\gamma(0))\cdot\lambda_2 - \varrho_0\kappa_0\Big) + \check{\gamma}_0\cdot(\lambda_1+\lambda_2) \\ & \quad + z(0)\cdot\Big(m_0 + m_1 + (\gamma(1)-\gamma(0))\times\lambda_2\Big) \\ & \quad + \int_0^1 z'(s)\cdot\Big(m_1 + \int_s^1 v d_3(\sigma)\times\lambda_2\, d\sigma + \varrho(s)n(s)\Big)\, ds\,. \end{aligned} \tag{4.30}$$

Recalling (4.2) and choosing special $\check{w} \in Y^*$ we get

$$0 = \lambda_0 + \tfrac{1}{v}(\gamma(1)-\gamma(0))\cdot\lambda_2 - \varrho_0\kappa_0\,, \tag{4.31}$$

$$0 = \lambda_1 + \lambda_2\,, \tag{4.32}$$

$$0 = m_0 + m_1 + (\gamma(1)-\gamma(0))\times\lambda_2\,, \tag{4.33}$$

$$0 = m_1 + \int_s^1 v d_3(\sigma)\times\lambda_2\, d\sigma + \varrho(s)n(s) \quad \text{a.e. on } I. \tag{4.34}$$

Since $\lambda_0 \geq 0$, $v = L[\gamma]$, we recover (3.7) from (4.31) with $\lambda := \lambda_2$. By continuity (4.34) has to be true for all $s \in I$ (with a suitable representative of ϱ). Taking $s = 0$ and $s = 1$ we obtain together with (4.33) that

$$m_0 = \varrho(0)n(0), \qquad m_1 = -\varrho(1)n(1)$$

and, therefore,

$$m_0 + v\int_0^s d_3\times\lambda\, d\sigma = \varrho(s)n(s) \quad \text{for all } s \in I$$

which obviously equals (3.6). Assume that $\lambda = 0$ and $\varrho = 0$. Thus $\varrho_0 = 0$ and, hence, $\lambda_0 = \lambda_1 = 0$ and $m_0 = m_1 = 0$ where the latter implies $\lambda_3 = \ldots = \lambda_6 = 0$. But this contradicts the fact that not all multipliers are simultaneously zero, i.e., λ, ϱ cannot both be zero.

$\varrho(\cdot)n(\cdot)$ has to be continuous on I by (3.6) and $|n(s)| = 1$ for all s with $\varrho(s) > 0$. Hence $\varrho(\cdot)$ is even continuous on I and $n(\cdot)$ is continuous on the open set I_+. Obviously,

$$\varrho(s) = |\varrho(0)n(0) + (\gamma(s)-\gamma(0))\times\lambda| \quad \text{on } I_+$$

by (3.6) and $|n(s)| = 1$ there. Thus

$$\varrho'(s) = \frac{\varrho(0)n(0) + (\gamma(s)-\gamma(0))\times\lambda}{\varrho(s)}\cdot(\gamma'(s)\times\lambda) \quad \text{on } I_+, \tag{4.35}$$

i.e., ϱ is continuously differentiable on I_+. Since the left hand side in (3.6) is continuously differentiable, we conclude by standard arguments that also n has to be continuously differentiable on I_+. Moreover, the right hand side in (4.35) and the left hand side in (3.6) have a derivative a.e. on I_+. Consequently, $\varrho''(s)$ and $n''(s)$ exist a.e. on I_+. This way we have verified the theorem.

Proof of Corollary 3.9.

(a) If $\lambda = 0$, then $\varrho(s)n(s) = \varrho(0)n(0)$ for all $s \in I$. Since $\varrho \neq 0$ in that case and since $|n(s)| = 1$ where $\varrho(s) > 0$, we conclude that $n(s) = n(0) \neq 0$ and $\varrho(s) = \varrho(0) \neq 0$ for all $s \in I$. By $d_3(s)\cdot n(s) = 0$ we get that the curve is planar. Furthermore we always have that $\kappa(s) = \kappa_0$. Since $n(s)$ is constant, also the curvature vector $\frac{1}{v^2}\Gamma''(s)$ has to be constant which means that $\gamma(I)$ is a planar circular arc.

(b) If $I \setminus I_+ \neq \emptyset$, then $\lambda \neq 0$ by (a) and, by (3.6),

$$(\gamma(s) - \gamma(0)) \times \lambda = -\varrho(0)n(0) \quad \text{for all } \; s \in I \setminus I_+.$$

Since the right hand side is constant, all $\gamma(s)$ satisfying the previous equation belong to a uniquely determined line ℓ in $\mathbb{R}^3$ parallel to λ and vice versus. Note that $\varrho(s) = 0$ on straight segments of $\gamma(I)$ and at s where n is discontinuous.

(c) Note that $\lambda \neq 0$ in that case by (b). Subtracting (3.6) with $s = \tilde{s}$ from (3.6) we get

$$(\gamma(s) - \gamma(\tilde{s})) \times \lambda = \varrho(s)n(s) \quad \text{on } \; I. \tag{4.36}$$

Hence, $n(s) \cdot \lambda = 0$ and $n'(s) \cdot \lambda = 0$ on (s_1, s_2). By the Frenet formulas,

$$n'(s) = -\tau(s)\frac{\Gamma''(s)}{|\Gamma''(s)|} \quad \text{on } \; (s_1, s_2)$$

(τ - torsion). Thus $\tau(s)\Gamma''(s) \cdot \lambda = 0$. If $\tau(s) \neq 0$, then $n(s)$ and $\Gamma''(s)$ have to be orthogonal to λ and, whence, $\gamma'(s)$ is parallel to λ. Therefore, by (4.36), $0 = \gamma' \times \lambda = \varrho' n + \varrho n'$ at that point s. The dot product with $n(s)$ yields $\varrho'(s) = 0$ and, thus, $n'(s) = 0$ contradicting $\tau(s) \neq 0$. Consequently, $\tau(s) = 0$ on (s_1, s_2). Hence $\gamma([s_1, s_2])$ is planar and, since $n(s)$ and $\kappa(s)$ are constant, it has to be a circular arc.

(d) If the curve $\gamma(I)$ contains a straight segment $\gamma([s_1, s_2])$, $s_1 < s_2$, then $\lambda \neq 0$ and $\gamma([s_1, s_2]) \subset \ell$ by (b). Let now $\gamma(s_0) \notin \ell$ for some $s_0 < s_1$. Then there is a smallest $\tilde{s} \in (s_0, s_1]$ with $\gamma(\tilde{s}) \in \ell$. We have $\varrho(\tilde{s}) = 0$ by (b) and $\gamma([s_0, \tilde{s}])$ is a planar circular arc by (c). We now observe that a composition of consecutive planar circular arcs having all the same curvature κ_0 intersect ℓ always at the same angle. Since such arcs have always the same length, there are only finitely many of them. In particular, a circular arc next to the straight segment has to intersect ℓ tangentially. But such an arc has only one intersection point with ℓ and, by the minimality of length, γ can "pass" a full circle at most once, i.e., $\gamma([0, s_1])$ can have at most one circular arc. Applying the same argument also on $[s_2, 1]$, we obtain the first assertion. This result certainly excludes minimizing curves having two disjoint separated straight segments.

5 Appendix

The results in Schuricht & v.d. Mosel [10] can be used to study the dependence of the solutions of the differential equation (2.3) on the parameter u for $u \in \mathcal{L}^\infty(I)$, but we need it for $u \in \mathcal{L}^p(I)$ with $1 < p < \infty$. Here we present a version adopted to our special situation.

Let $f : M \subset \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, M open, be continuous and let $I \subset \mathbb{R}$ be an open bounded interval. For given $\tau \in I$, $\xi \in M$, $u \in \mathcal{L}^p(I, \mathbb{R}^m)$, $1 \leq p \leq \infty$ we consider the initial value problem

$$x'(t) = f(x(t), u(t)), \quad x(\tau) = \xi\,. \tag{5.1}$$

Notice that this differential equation has to be interpreted in the sense of Caratheodory, since the right-hand side is merely measurable in t (cf. Walter [16]). We are interested in the differentiable dependence of $x(t) = x(t; \xi, u)$ on the parameters (ξ, u).

Let $\tau \in I$, $\tilde{\xi} \in M$, and $\tilde{u} \in \mathcal{L}^p(I)$ be fixed. For suitable open balls $B_\delta(\tilde{\xi}) \subset M$ and $B_\delta(\tilde{u}) \subset \mathcal{L}^p(I)$ we assume that there is some $c > 0$ such that

$$|f(x,u)| \le |u| \quad \text{for all } x \in B_\delta(\tilde{\xi}),\ u \in \mathbb{R}^m, \tag{5.2}$$

$$|f(x,u) - f(y,u)| \le c|u|\,|x-y| \quad \text{for all } x,y \in B_\delta(\tilde{\xi}),\ u \in \mathbb{R}^m, \tag{5.3}$$

$$|f(x,u) - f(x,v)| \le c|u-v| \quad \text{for all } x \in B_\delta(\tilde{\xi}),\ u,v \in \mathbb{R}^m. \tag{5.4}$$

Theorem 5.5 *Let the continuous function f satisfy (5.2) - (5.4) with $1 \le p \le \infty$. Then there is some open interval $J \subset I$, $\tau \in J$, such that for each $(\xi,u) \in B_{\delta/2}(\tilde{\xi}) \times B_\delta(\tilde{u})$ there is a unique solution $x = x(t;\xi,u)$ on J of (5.1). Furthermore, the mapping $(t,\xi,u) \to x(t;\xi,u)$ is continuous on $J \times B_{\delta/2}(\tilde{\xi}) \times B_\delta(\tilde{u})$.*

The proof is a straightforward adoption of the proof of [10, Theorem 1.1].

For differentiable dependence we assume in addition that f is continuously differentiable and that there is some constant $c > 0$ such that

$$|f_x(x,u)\check{x}| + |f_u(x,u)\check{u}| \le c(|u|\,|\check{x}| + |\check{u}|)\,, \tag{5.6}$$

$$|(f_x(x,u) - f_x(y,u))\check{x}| + |(f_u(x,u) - f_u(y,u))\check{u}| \le c|x-y|\,(|u||\check{x}| + |\check{u}|) \tag{5.7}$$

for all $x \in B_\delta(\tilde{\xi})$, $\check{x} \in \mathbb{R}^n$, $u, \check{u} \in \mathbb{R}^m$. We use the notation $w = (\xi,u) \in B_\delta(\tilde{\xi}) \times B_\delta(\tilde{u})$.

Theorem 5.8 *Let the continuously differentiable function f satisfy (5.2) - (5.4), (5.6), (5.7) with $1 \le p \le \infty$. Then the solution $x = x(t;\xi,u)$ of (5.1) according to Theorem 5.5 is continuouusly differentiable with respect to (t,ξ,u) on $J \times B_{\delta/2}(\tilde{\xi}) \times B_\delta(\tilde{u})$. The derivative with respect to $w = (\xi,u)$ is given by*

$$D_w x(t;w)\check{w} = \check{\xi} + \int_\tau^t \Big(f_x(x(s;w),u(s)) D_w x(s;w)\check{w} + f_u(x(s;w),u(s))\check{u} \Big)\, ds$$

for all $\check{w} = (\check{\xi},\check{u}) \in \mathbb{R}^n \times \mathcal{L}^p(J)$. Moreover

$$\frac{d}{dt} D_w x(t;w)\check{w} = D_w\Big(\frac{d}{dt}x(t;w)\Big)\check{w}$$

for a.e. $t \in J$ and all $w = (\xi,u) \in B_{\delta/2}(\tilde{\xi}) \times B_\delta(\tilde{u})$, $\check{w} = (\check{\xi},\check{u}) \in \mathbb{R}^n \times \mathcal{L}^p(J)$.

The proof is a straightforward modification of the proof of [10, Theorem 2.1, Corollary 2.2].

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