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Abstract

The paper verifies the existence of a sequence of eigenfunctions for the 1-Laplace operator by showing that the corresponding variational problem has a sequence of critical points. Since the functionals entering the variational problem are not differentiable, critical points are defined by means of the weak slope.

1 Introduction

For the eigenvalue problem of the Laplace operator

$$-\Delta u = \lambda u \quad \text{on } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

($\Omega \subset \mathbb{R}^n$) there is a sequence $(u_k)_{k \in \mathbb{N}}$ of eigenfunctions that are, for $p = 2$, critical points of

$$E(u) := \int_{\Omega} |Du|^p dx \quad \text{in } W_0^{1,p}(\Omega)$$

subject to the constraint

$$G(u) := \int_{\Omega} |u|^p dx = 1.$$

As a natural generalization it turns out that, for each $p \in (1, \infty)$, there is a sequence $(u_k)_{k \in \mathbb{N}}$ of critical points that solve the eigenvalue problem for the p -Laplace operator

$$-\operatorname{div} |Du|^{p-2} Du = \lambda |u|^{p-2} u \quad \text{on } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

In the limit case $p = 1$ we formally obtain the equation

$$-\operatorname{div} \frac{Du}{|Du|} = \lambda \frac{u}{|u|} \quad \text{on } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \quad (1.1)$$

and, typically, a minimizer of

$$E(u) = \int_{\Omega} |Du| dx \quad \text{subject to } G(u) = \int_{\Omega} |u| dx = 1 \quad (1.2)$$

is a characteristic function $u = \chi_C$ vanishing on a set of positive measure (cf. Fridman & Kawohl [12]). Since such u do not belong to $W_0^{1,1}(\Omega)$, the problem has to be studied in the larger space $BV(\Omega)$ of functions of bounded variation. Moreover (1.1) is not well-defined for such solutions and needs some appropriate interpretation. It turns out that $Du/|Du|$ has to be replaced with a vector field $z : \Omega \rightarrow \mathbb{R}^n$ and $u/|u|$ with a sign function $s : \Omega \rightarrow [-1, 1]$ satisfying suitable coupling conditions relating z and s to u such that

$$-\operatorname{Div} z = \lambda s \quad \text{a.e. on } \Omega \quad (1.3)$$

(cf. Theorem 2.8 below). Equation (1.3) (combined with the coupling conditions) is usually considered as eigenvalue problem for the 1-Laplace operator and a minimizer u of (1.2) corresponds to the smallest eigenvalue λ . Now the question for higher eigensolutions, which is the subject of this paper, is natural but confronts us with fundamental difficulties.

Let us start with a closer look at (1.3) in order to clarify what an eigensolution (u, λ) of the 1-Laplace operator should be. The sign function s is related to u by

$$s(x) \in \operatorname{Sgn}(u(x)) \quad \text{a.e. on } \Omega \quad (1.4)$$

where Sgn denotes the set valued sign function (cf. (1.5) below). As a first necessary condition for a minimizer u of (1.2) we have that there is *one* sign function s satisfying (1.4) and a vector field $z \in L^\infty(\Omega, \mathbb{R}^n)$ (related to u) such that (1.3) is satisfied with $\lambda = E(u)$. A more refined analysis shows that for *any* sign function s satisfying (1.4) there is a vector field z (which may depend on s) such that (1.3) is satisfied. Now it turns out that the first condition with one s is too weak for a definition of eigensolutions, since it is

satisfied by “very many” u . On the other hand, there is some analytical evidence that the second condition, taking into account any s , might be true merely for a minimizer u of (1.2) and, hence, it would be too strong for a definition of eigensolutions. Thus we cannot decide for which functions s equation (1.3) should be satisfied for a higher eigenfunction u . Consequently, we do not have a precise replacement for the formal eigenvalue problem (1.1) as basis for a definition of higher eigensolutions.

Recall that, in the case $p > 1$, eigenfunctions u are critical points of E subject to the constraint $G(u) = 1$, i.e., there is some $\lambda \in \mathbb{R}$ with $E'(u) - \lambda G'(u) = 0$. However, we cannot define critical points in that way in the limit case $p = 1$, since E and G are not differentiable. But it turns out that the notion of weak slope (cf. Section 3), which allows a definition of critical points for certain lower semicontinuous functions and which is consistent with the usual definition in the smooth case, is applicable to our setting in the singular case $p = 1$. Thus we define eigenfunctions of the 1-Laplace operator as critical points by means of the weak slope. As in the classical smooth case we verify the existence of a sequence $(u_k)_{k \in \mathbb{N}}$ of critical points of (1.2) and we show that each u_k has to satisfy the eigenvalue equation (1.3) at least for one sign function s_k . In this sense we verify the existence of a sequence of eigensolutions of the 1-Laplace operator.

In Section 2 we recall the results concerning the first eigenvalue of the 1-Laplace operator. In Section 3 the notion of weak slope is introduced and we define critical points for lower semicontinuous functions. The main results concerning critical points, that are considered as eigenfunctions of the 1-Laplace operator, are stated in Section 4, while the proofs are presented in Section 5.

Notation. For a set A we denote the boundary by ∂A and the closure by \bar{A} . Its indicator function I_A is given by

$$I_A(x) := \begin{cases} 0 & \text{for } x \in A, \\ \infty & \text{otherwise.} \end{cases}$$

$B_r(u)$ stands for the open ball with center u and radius r . We write $\text{Div } u$ for the divergence of a function u in the distributional sense. The set-valued sign function on \mathbb{R} is given by

$$\text{Sgn } \alpha := \begin{cases} 1 & \text{for } \alpha > 0, \\ [-1, 1] & \text{for } \alpha = 0, \\ -1 & \text{for } \alpha < 0. \end{cases} \quad (1.5)$$

The space of q -integrable functions on Ω is denoted by $L^q(\Omega)$ and its dual by $L^{q^*}(\Omega)$ where $\frac{1}{q} + \frac{1}{q^*} = 1$. The space $BV(\Omega)$ is the space of functions of bounded variation and $|Du|$ is the total variation measure for these functions. The k -dimensional Hausdorff measure is denoted by \mathcal{H}^k . For a Banach space X its dual is X^* and $\langle \cdot, \cdot \rangle$ is the duality form on $X \times X^*$. We write $\partial E(u)$ both for the subdifferential of a convex function E and for Clarke’s generalized gradient of a locally Lipschitz continuous function E . By $E^0(u; v)$ we denote Clarke’s generalized directional derivative of u in direction v , which coincides with the usual directional derivative if E is convex.

2 First Eigensolution

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. For $u \in BV(\Omega)$ we consider the functional

$$E(u) := \int_{\Omega} d|Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{n-1} \quad (2.6)$$

under the constraint

$$G(u) := \int_{\Omega} |u| dx - 1 = 0. \quad (2.7)$$

The surface integral in (2.6), which we did not take into account in (1.2), is a replacement for homogeneous boundary conditions in $BV(\Omega)$. It is known that a minimizer of (2.6), (2.7) exists, but it is not necessarily unique (cf. Kawohl & Schuricht [13]). Moreover, any minimizer satisfies the following Euler-Lagrange equation as necessary condition, as has been shown in [13, Corollary 4.18].

Theorem 2.8 *Let $u \in BV(\Omega)$ be a minimizer of (2.6), (2.7). Then for each measurable selection $s(x) \in \text{Sgn}(u(x))$ a.e. on $x \in \Omega$, there is some vector field $z \in L^\infty(\Omega, \mathbb{R}^n)$ with*

$$\|z\|_{L^\infty} = 1, \text{ Div } z \in L^n(\Omega),$$

$$E(u) = - \int_{\Omega} u \text{ Div } z dx$$

such that

$$-\text{Div } z = \lambda s \quad \text{a.e. on } \Omega, \quad \lambda = E(u).$$

We call (u, λ) the (first) eigensolution of the 1-Laplace operator. It is remarkable that the Euler-Lagrange equation has to be satisfied not only for one but for any measurable selection s , i.e. in general infinitely many Euler-Lagrange equations have to be satisfied.

Our goal is to show that higher critical points of (2.6), (2.7) exist and that they can be considered as eigensolutions of the 1-Laplace operator. However, first we have to clarify what critical points of a nondifferentiable functional such as E with respect to constraint (2.7) should be. We use a definition of critical points by means of weak slope, which works for continuous and even some classes of lower semicontinuous functions.

3 Tools of Nonsmooth Analysis

The notion of weak slope has been introduced in Degiovanni & Marzocchi [8]. Let X be a metric space endowed with metric m and let $E : X \rightarrow \mathbb{R}$ be a continuous function. For every $u \in X$ we denote by $|dE|(u)$ the supremum of all $\sigma \in [0, \infty)$ for which there exist $\delta > 0$ and a continuous map $\mathcal{H} : B_\delta(u) \times [0, \delta] \rightarrow X$ such that for all $v \in B_\delta(u)$ and all $t \in [0, \delta]$

$$m(\mathcal{H}(v, t), v) \leq t, \quad (3.1)$$

$$E(\mathcal{H}(v, t)) \leq E(v) - \sigma t. \quad (3.2)$$

The extended real number $|dE|(u)$ is called the *weak slope* of E at u . Note that for differentiable functions the weak slope corresponds to the norm of the gradient.

Now we consider a lower semicontinuous function $E : X \rightarrow \mathbb{R} \cup \{\infty\}$. We define the *domain* of E by

$$\mathcal{D}(E) := \{u \in X \mid E(u) < \infty\} \quad (3.3)$$

and the *epigraph* of E by

$$\text{epi}(E) := \{(u, \xi) \in X \times \mathbb{R} \mid E(u) \leq \xi\}.$$

The set $X \times \mathbb{R}$ will be endowed with the metric

$$m((u, \xi), (v, \mu)) = (m(u, v)^2 + (\xi - \mu)^2)^{1/2} \quad (3.4)$$

and $\text{epi}(E)$ with the induced metric. Using the continuous function

$$\mathcal{G}_E : \text{epi}(E) \rightarrow \mathbb{R}, \quad \mathcal{G}_E(u, \xi) = \xi, \quad (3.5)$$

we define the *weak slope* of E at $u \in \mathcal{D}(E)$ as

$$|dE|(u) := \begin{cases} \frac{|d\mathcal{G}_E|(u, E(u))}{\sqrt{1 - |d\mathcal{G}_E|(u, E(u))^2}} & \text{for } |d\mathcal{G}_E|(u, E(u)) < 1, \\ \infty & \text{for } |d\mathcal{G}_E|(u, E(u)) = 1. \end{cases}$$

When E is finite and continuous on X this definition is consistent with the definition of the weak slope for continuous functions. Occasionally we denote the weak slope of E at u by $|dE|^X(u)$ in order to indicate that it is taken in the metric space X . The idea of this definition is to reduce the study of the lower semicontinuous function E to that of the Lipschitz continuous function \mathcal{G}_E .

We say that $u \in \mathcal{D}(E)$ is a *critical point* of E if $|dE|(u) = 0$. The value $c \in \mathbb{R}$ is called a *critical value* of E if there exists a critical point $u \in \mathcal{D}(E)$ of E with $E(u) = c$. Note that if $(u, E(u)) \in \text{epi}(E)$ is a critical point of \mathcal{G}_E then u is also a critical point of E . The bijective correspondence between the critical points of E and those of \mathcal{G}_E is given if

$$\inf\{|d\mathcal{G}_E|(u, \xi) \mid E(u) < \xi\} > 0, \quad (3.6)$$

cf. Canino & Degiovanni [3, Theorem 1.5.5]. If E is finite and continuous, we have $|d\mathcal{G}_E|(u, \xi) = 1$ whenever $E(u) < \xi$. The same property holds for some important classes of lower semicontinuous functions (cf. Canino & Perri [4], Corvellec et al. [7], Degiovanni & Marzocchi [8]).

We are interested in critical points of E under a constraint $G(u) = 0$ where $G : X \rightarrow \mathbb{R}$ is a locally Lipschitz continuous function. We set

$$K := \{u \in X \mid G(u) = 0\} \quad (3.7)$$

and call $u \in \mathcal{D}(E)$ a *critical point of E with respect to K* if u is a critical point of E on the metric space K with induced metric m of X .

The following result gives us a useful characterization of critical points of E with respect to K .

Lemma 3.8 *Let $E : X \rightarrow \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function and let $u \in \mathcal{D}(E) \cap K$. Then $|dE|^K(u) = 0$ if and only if $|d(E + I_K)|^X(u) = 0$.*

PROOF. For the calculation of the weak slope of $E + I_K$ on X and of E with respect to K , we must turn to functions

$$\mathcal{G}_{E+I_K} : \text{epi}(E + I_K) \rightarrow \mathbb{R} \quad \text{and} \quad \mathcal{G}_E : \text{epi}(E) \cap (K \times \mathbb{R}) \rightarrow \mathbb{R},$$

respectively, as defined in (3.5). First we observe that

$$\begin{aligned} \text{epi}(E + I_K) &= \{(v, \mu) \in X \times \mathbb{R} \mid (E + I_K)(v) \leq \mu\} \\ &= \{(v, \mu) \in K \times \mathbb{R} \mid E(v) \leq \mu\} \\ &= \text{epi}(E) \cap (K \times \mathbb{R}). \end{aligned}$$

Thus \mathcal{G}_{E+I_K} and \mathcal{G}_E are defined on the same metric space, since the metric m as defined in (3.4) is the same in both cases. By definition (3.5) we have

$$\mathcal{G}_{E+I_K} \equiv \mathcal{G}_E.$$

Hence,

$$|d\mathcal{G}_{E+I_K}|(u, (E + I_K)(u)) = |d\mathcal{G}_E|(u, E(u))$$

for each $u \in \mathcal{D}(E) \cap K$. Therefore

$$|d(E + I_K)|^X(u) = |dE|^K(u) \quad \text{for all } u \in \mathcal{D}(E) \cap K$$

which implies the assertion. \diamond

4 Higher Eigensolutions

We intend to verify the existence of a sequence of pairs $(u_k, -u_k)_{k \in \mathbb{N}}$ of critical points of (2.6), (2.7) and derive an Euler-Lagrange equation as necessary condition for critical points.

The functionals E and G according to (2.6), (2.7) are convex. If we want to characterize their subdifferentials, which are subsets of the dual space $BV(\Omega)^*$, we are confronted with the difficulty that not much is known about the structure of the space $BV(\Omega)^*$. In order to be able to calculate the subdifferential of (2.6) and (2.7) we trivially extend the problem to the space $L^q(\Omega)$ for any fixed $1 < q < \frac{n}{n-1}$. Note that this is basically done in Kawohl & Schuricht [13] where, by formal reasons, only the case $\frac{n}{n-1} \leq q < \infty$ is considered. However, all corresponding results in [13] can be extended to the general case $1 < q < \infty$ (cf. also Andreu-Vaillio et al. [1]). The choice of $q < \frac{n}{n-1}$ makes sure that $BV(\Omega)$ is compactly embedded into $L^q(\Omega)$, which will be crucial for the proof of the Palais-Smale condition. More precisely, we consider

$$E(u) := \begin{cases} \int_{\Omega} d|Du| + \int_{\partial\Omega} |u| d\mathcal{H}^{n-1} & \text{for } u \in BV(\Omega), \\ \infty & \text{for } u \in L^q(\Omega) \setminus BV(\Omega), \end{cases} \quad (4.1)$$

and

$$G(u) := \int_{\Omega} |u| dx - 1 = 0 \quad \text{for } u \in L^q(\Omega), \quad (4.2)$$

while we identify E with its extension on $L^q(\Omega)$. Based on this setting we say that (2.6) has a critical point under the constraint (2.7) if and only if (4.1) has a critical point under the constraint (4.2).

Now we are able to state the main result.

Theorem 4.3 *There exists a sequence $(u_k, -u_k)_{k \in \mathbb{N}}$, $u_k \in BV(\Omega)$, of pairs of critical points of (2.6) under the constraint (2.7) with $E(u_k) < \infty$ for all $k \in \mathbb{N}$. Moreover, for each critical point $u_k \in BV(\Omega)$ and any $1 < q < \frac{n}{n-1}$ there exists a measurable selection $s_k(x) \in \text{Sgn}(u_k(x))$ for a.e. $x \in \Omega$ and a vector field $z_k \in L^\infty(\Omega, \mathbb{R}^n)$ with*

$$\|z_k\|_{L^\infty} = 1, \quad \text{Div } z_k \in L^{q^*}(\Omega),$$

$$E(u_k) = - \int_{\Omega} u_k \text{Div } z_k \, dx$$

such that

$$-\text{Div } z_k = \lambda_k s_k \quad \text{a.e. on } \Omega, \quad \lambda_k = E(u_k). \quad (4.4)$$

We also have $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$.

Notice that the Euler-Lagrange equation might not be satisfied for all measurable selections $s_k(x) \in \text{Sgn}(u_k(x))$ as in the case of Theorem 2.8.

5 Proofs

In this section we carry out the proof of Theorem 4.3. After some preliminary considerations we apply Proposition 5.2 from below, which is due to Degiovanni & Marzocchi [8], to our setting. We claim to verify the existence of a sequence of critical points of (4.1) under the constraint (4.2) which, by definition, are considered as critical points of (2.6) under the constraint (2.7). According to Lemma 3.8 it is sufficient to verify critical points of the functional $E + I_K$ on $L^q(\Omega)$. Since $E + I_K$ is merely lower semicontinuous, cf. Kawohl & Schuricht [13, Proposition 4.23], we have to determine critical points of the corresponding function \mathcal{G}_{E+I_K} according to (3.5). More precisely, we show the existence of critical points of

$$\mathcal{G}_{E+I_K} : \text{epi}(E + I_K) \rightarrow \mathbb{R}, \quad \mathcal{G}_{E+I_K}(u, \xi) = \xi$$

with

$$\text{epi}(E + I_K) = \{(u, \xi) \in L^q(\Omega) \times \mathbb{R} \mid u \in BV(\Omega) \cap K, E(u) \leq \xi\}, \quad (5.1)$$

where $\text{epi}(E + I_K)$ is endowed with the metric according to (3.4) that is induced by the L^q -norm. In Lemma 5.8 below we see that (3.6) is satisfied in general and, thus, critical points of \mathcal{G}_{E+I_K} are also critical points of $E + I_K$. But, before we formulate the details of the proof, let us introduce some notions.

Let X be a metric space endowed with metric m and let $A \subset X$ be a closed nonempty set. The *category of A in X* , denoted by $\text{cat}(A, X)$, is defined as the least integer k such that A can be covered by k open subsets of X , each of which is contractible in X . If no such integer k exists, we set $\text{cat}(A, X) = \infty$. We also set $\text{cat}(\emptyset, X) = 0$.

A metric space X is said to be *weakly locally contractible* if for every $u \in X$ there exists a neighborhood U of u which is contractible in X .

Let X be a topological space and $A \subset X$. Then a continuous function $r : X \rightarrow A$ is a *retraction* if $r(a) = a$ for all $a \in A$. A subspace A is called a *retract* of X if such a retraction exists.

Let $F : X \rightarrow \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous function defined on a metric space X and let $c \in \mathbb{R}$. A sequence $(u_k)_{k \in \mathbb{N}}$ in $\mathcal{D}(F)$, cf. (3.3), is said to be a *Palais-Smale sequence* at level c for F , if

$$F(u_k) \rightarrow c \quad \text{and} \quad |dF|(u_k) \rightarrow 0.$$

We say that F satisfies the *Palais-Smale condition* at level c , if every Palais-Smale sequence $(u_k)_{k \in \mathbb{N}}$ at level c for F has a convergent subsequence in X .

Let A be a convex subset of a Banach space X and $u \in A \cap K$ with K as in (3.7). Then A and K are said to be *transversal* at u , if there exist $u_-, u_+ \in A$ such that

$$G^0(u; u_- - u) < 0 \quad \text{and} \quad G^0(u; u - u_+) < 0.$$

We claim to apply the following result, which can be found in Degiovanni & Marzocchi [8, Theorem 3.10] and which is an adaptation of a classical result to a nonsmooth setting.

Proposition 5.2 *Let X be a weakly locally contractible complete metric space and let $F : X \rightarrow \mathbb{R}$ be a continuous function which is bounded from below and which satisfies the Palais-Smale condition at level c for all $c \in \mathbb{R}$. Moreover, let*

$$\sup\{\text{cat}(A, X) \mid A \subset X \text{ compact}\} = \infty.$$

Then F has infinitely many critical points $(u_k)_{k \in \mathbb{N}}$ with critical values

$$c_k = \inf_{A \in \mathcal{A}_k} \sup_{u \in A} F(u), \quad c_k \rightarrow \infty$$

where

$$\mathcal{A}_k = \{A \subset X \mid A \text{ compact, } \text{cat}(A, X) \geq k\},$$

and $\sup_X F = \infty$.

In order to get a nontrivial setting for the application of Proposition 5.2 we have to reformulate our problem again. We transfer our considerations from the Banach space Y to the *projective space* \mathcal{P}_Y obtained from Y by identifying every $u \in Y \setminus \{0\}$ with its antipodal point $-u$. The relation $u \sim v$, if $u = v$ or $u = -v$, is an equivalence relation and $\mathcal{P}_Y = (Y \setminus \{0\})/\sim$ is the corresponding quotient space. We denote the elements of \mathcal{P}_Y by $[u]$ and endowe \mathcal{P}_Y with the induced metric

$$m_{\mathcal{P}_Y}([u], [v]) := \min\{m(\bar{u}, \bar{v}) \mid \bar{u} \in [u], \bar{v} \in [v]\} = \min\{m(u, v), m(u, -v)\}.$$

We call a set $A \subset Y$ *symmetric*, if for each $u \in A$ also $-u \in A$. For a symmetric set $A \subset Y \setminus \{0\}$ we denote by $\mathcal{P}_Y(A)$ the corresponding set in the projective space \mathcal{P}_Y .

In our case $Y := L^q(\Omega)$ and we denote the corresponding projective space by \mathcal{P} for simplicity. Since E and G according to (4.1), (4.2) are even functionals, we can think of them as mappings from \mathcal{P} to $\mathbb{R} \cup \{\infty\}$, i.e. without danger of confusion we set $E : \mathcal{P} \rightarrow \mathbb{R} \cup \{\infty\}$ and $G : \mathcal{P} \rightarrow \mathbb{R} \cup \{\infty\}$ as

$$E([u]) := E(\bar{u}) \quad \text{and} \quad G([u]) := G(\bar{u})$$

for any representative $\bar{u} \in [u]$.

We intend to apply Proposition 5.2 to the map

$$\mathcal{G}_{E+I_K}([u], \xi) = \xi$$

defined on

$$X := \text{epi}_{\mathcal{P}}(E + I_K) := \{([u], \xi) \in \mathcal{P}(BV(\Omega) \cap K) \times \mathbb{R} \mid E([u]) \leq \xi\}$$

as a subset of the space $\mathcal{P} \times \mathbb{R}$ which is endowed with the metric

$$m_{\mathcal{P}}([u], \xi), ([v], \mu) = (m_{\mathcal{P}}([u], [v])^2 + (\xi - \mu)^2)^{1/2}$$

and $\text{epi}_{\mathcal{P}}(E + I_K)$ with the induced metric. Note that $0 \notin BV(\Omega) \cap K$ and $BV(\Omega) \cap K$ is a symmetric set. According to the following Lemma the critical points $([u], E([u]))$ of \mathcal{G}_{E+I_K} on $\text{epi}_{\mathcal{P}}(E + I_K)$ correspond to pairs $((u, E(u)), (-u, E(-u)))$ of critical points of \mathcal{G}_{E+I_K} on $\text{epi}(E + I_K)$.

Lemma 5.3 *Let Y be a Banach space with metric m , let \mathcal{P}_Y be the corresponding projective space, and let $F : Y \rightarrow \mathbb{R} \cup \{\infty\}$ be a lower semicontinuous even functional. If $[u] \in \mathcal{P}_Y$ with $u \in \mathcal{D}(F)$ is a critical point of F with respect to \mathcal{P}_Y , then u and $-u$ are critical points of F with respect to Y .*

PROOF. It is enough to show that u is a critical point of F for any fixed representative $\bar{u} \in [u]$, since F is even. We assume first that F is continuous on Y , which implies that F is also continuous on \mathcal{P}_Y . Let us assume that

$$|dF|^{\mathcal{P}_Y}([u]) = 0 \quad \text{and} \quad |dF|^Y(\bar{u}) > 0 \quad \text{for any } \bar{u} \in [u]. \quad (5.4)$$

Then, by the definition of the weak slope, there exist constants $\sigma > 0$, $\delta > 0$ and a continuous map $\mathcal{H} : B_\delta(\bar{u}) \times [0, \delta] \rightarrow Y$ such that for all $v \in B_\delta(\bar{u})$ and all $t \in [0, \delta]$

$$m(\mathcal{H}(v, t), v) \leq t \quad (5.5)$$

and

$$F(\mathcal{H}(v, t)) \leq F(v) - \sigma t. \quad (5.6)$$

Since $\bar{u} \neq 0$, we achieve that $\text{dist}(B_\delta(\bar{u}), B_\delta(-\bar{u})) > 0$ and $\mathcal{H}(v, t) \neq 0$ for all $v \in B_\delta(\bar{u})$ and all $t \in [0, \delta]$ by choosing a smaller $\delta > 0$ if necessary. We see that

$$B_\delta([u]) = \mathcal{P}_Y(B_\delta(\bar{u}) \cup -B_\delta(\bar{u}))$$

is a neighborhood of $[u]$ in \mathcal{P}_Y .

Now we define a deformation $\tilde{\mathcal{H}} : B_\delta([u]) \times [0, \delta] \rightarrow \mathcal{P}_Y$ by

$$\tilde{\mathcal{H}}([v], t) := [\mathcal{H}(\bar{v}, t)]$$

where the representative $\bar{v} \in [v]$ is chosen such that $\bar{v} \in B_\delta(\bar{u})$. We must show that $\tilde{\mathcal{H}}$ is continuous. Let $([v], t) \in B_\delta([u]) \times [0, \delta]$ and $([v_k], t_k)_{k \in \mathbb{N}} \in B_\delta([u]) \times [0, \delta]$ with $m_{\mathcal{P}_Y}([v], t), ([v_k], t_k) \rightarrow 0$. Note that, if we pick the representatives $\bar{v} \in [v]$, $\bar{v}_k \in [v_k]$ such that $\bar{v}, \bar{v}_k \in B_\delta(\bar{u})$ we have that $m((\bar{v}, t), (\bar{v}_k, t_k)) \rightarrow 0$, since $m(\bar{v}, -\bar{v}_k) \geq \text{dist}(B_\delta(\bar{u}), B_\delta(-\bar{u})) > 0$. For $\bar{v}, \bar{v}_k \in B_\delta(\bar{u})$ and $t, t_k \in [0, \delta]$ we get

$$\begin{aligned} m_{\mathcal{P}_Y}(\tilde{\mathcal{H}}([v], t), \tilde{\mathcal{H}}([v_k], t_k)) &= m_{\mathcal{P}_Y}([\mathcal{H}(\bar{v}, t)], [\mathcal{H}(\bar{v}_k, t_k)]) \\ &= \min\{m(\mathcal{H}(\bar{v}, t), \mathcal{H}(\bar{v}_k, t_k)), m(\mathcal{H}(\bar{v}, t), -\mathcal{H}(\bar{v}_k, t_k))\} \\ &\leq m(\mathcal{H}(\bar{v}, t), \mathcal{H}(\bar{v}_k, t_k)) \rightarrow 0 \end{aligned}$$

for $m_{\mathcal{P}_Y}([v], t), ([v_k], t_k) \rightarrow 0$, since \mathcal{H} is continuous on $B_\delta(\bar{u}) \times [0, \delta]$.

Moreover,

$$m_{\mathcal{P}_Y}(\tilde{\mathcal{H}}([v], t), [v]) = m_{\mathcal{P}_Y}([\mathcal{H}(\bar{v}, t)], [v]) \leq m(\mathcal{H}(\bar{v}, t), \bar{v}) \leq t$$

by (5.5) and

$$F(\tilde{\mathcal{H}}([v], t)) = F([\mathcal{H}(\bar{v}, t)]) = F(\mathcal{H}(\bar{v}, t)) \leq F(\bar{v}) - \sigma t$$

with a $\sigma > 0$ by (5.6). This contradicts (5.4). Therefore u and $-u$ are critical points of F on Y .

Now let us assume, that $F : \mathcal{P}_Y \rightarrow \mathbb{R} \cup \{\infty\}$ is merely lower semicontinuous and has a critical point $[u] \in \mathcal{P}_Y$. Then, by definition, the function $\mathcal{G}_F : \text{epi}_{\mathcal{P}_Y}(F) \rightarrow \mathbb{R}$ has a critical point $([u], F([u])) = ([u], F(u))$. Since \mathcal{G}_F is continuous, we can follow the previous argumentation step by step in the first component of the variable of \mathcal{G}_F . Thus, \mathcal{G}_F must also have critical points $(u, F(u))$ and $(-u, F(u))$ on $\text{epi}(F)$. But then F also has critical points u and $-u$ on Y . \diamond

For the proof of Lemma 5.9 below we have to show transversality of the convex set $\mathcal{D}(E) = BV(\Omega)$ and K in $L^q(\Omega)$ at each $u \in BV(\Omega) \cap K$. But let us first recall a characterization of the subdifferential of G from Kawohl & Schuricht [13, Proposition 4.23].

Proposition 5.7 *The functional G according to (4.2) is convex and Lipschitz continuous on $L^q(\Omega)$. Moreover we have $u_G^* \in \partial G(u)$ for $u \in L^q(\Omega)$ if and only if*

$$u_G^*(x) \in \text{Sgn}(u(x)) \quad \text{a.e. on } \Omega.$$

Note that this result has been shown in [13] for $u \mapsto \int_{\Omega} |u| dx$ but, obviously, it is also valid for our G .

Lemma 5.8 *The sets $BV(\Omega)$ and K are transversal in $L^q(\Omega)$ at each $u \in BV(\Omega) \cap K$. Moreover, (3.6) is satisfied.*

PROOF. Let $u \in BV(\Omega) \cap K$. We define

$$u_- := 0 \quad \text{and} \quad u_+ := 2u.$$

Then $u_-, u_+ \in BV(\Omega)$ and, using Clarke [5, Proposition 2.1.2] and Proposition 5.7, we get

$$G^0(u; u_- - u) = G^0(u; u - u_+) = G^0(u; -u) = \max_{u_G^* \in \partial G(u)} -\langle u_G^*, u \rangle = -G(u) - 1 = -1,$$

cf. [13, Proof of Theorem 4.6], i.e. transversality is satisfied.

By Degiovanni & Schuricht [9, Theorem 3.4] it is enough to show transversality of the convex set $\mathcal{D}(E) = BV(\Omega)$ and K in $L^q(\Omega)$ at each $u \in \mathcal{D}(E) \cap K$ for (3.6), as we just did. \diamond

Now we are able to prove the following result.

Lemma 5.9 *The set $\text{epi}_{\mathcal{P}}(E + I_K)$ is a complete weakly locally contractible metric space.*

PROOF. We show the assertion for the set $\text{epi}(E + I_K) \subset L^q(\Omega) \times \mathbb{R}$. Since $0 \notin \text{epi}(E + I_K)$ and since with $(u, \xi) \in \text{epi}(E + I_K)$ also $(-u, \xi) \in \text{epi}(E + I_K)$, we can choose the neighborhoods of (u, ξ) and $(-u, \xi)$ to be disjoint. Thus the result follows for $\text{epi}_{\mathcal{P}}(E + I_K)$.

First we show that $\text{epi}(E + I_K) \subset L^q(\Omega) \times \mathbb{R}$ is a complete metric space. Let $(u_j, \xi_j)_{j \in \mathbb{N}} \in \text{epi}(E + I_K)$ with $(u_j, \xi_j) \rightarrow (u, \xi)$ in $L^q(\Omega) \times \mathbb{R}$. By (5.1) we have

$u_j \in BV(\Omega) \cap K$ and $E(u_j) \leq \xi_j$ for all $j \in \mathbb{N}$. Since $BV(\Omega) \cap K$ is closed and E is lower semicontinuous on $L^q(\Omega)$, cf. [13, Proposition 4.23], we get

$$E(u) \leq \liminf_{j \rightarrow \infty} E(u_j) \leq \lim_{j \rightarrow \infty} \xi_j = \xi.$$

Then $(u, \xi) \in \text{epi}(E + I_K)$ and therefore $\text{epi}(E + I_K)$ is complete.

We will show that $\text{epi}(E + I_K)$ is an ANR (absolute neighborhood retract), see Borsuk [2, p. 85] for definition. Moreover, we use [2, p. 99, Corollary 10.4], which states that for separable metric spaces the property of being an ANR is a local property. Since $\text{epi}(E + I_K)$ is separable, in our case it is enough to show, that every $(u, \xi) \in \text{epi}(E + I_K)$ has a neighborhood in $L^q(\Omega) \times \mathbb{R}$, which is an ANR. This will be shown below. Then, by [2, p. 28; p. 87, Corollary 3.3], the space $\text{epi}(E + I_K)$ is weakly locally contractible.

(a) First, for each $u \in BV(\Omega) \cap K$, we construct a neighborhood of u in $L^q(\Omega) \cap K$ which is a retract of a neighborhood of u in $L^q(\Omega)$.

We use functions u_- and u_+ as defined in the proof of Lemma 5.8. The directional derivative $G^0(u; z)$ is upper semicontinuous as a function of (u, z) , cf. Clarke [5, Proposition 2.1.1], therefore, using Lemma 5.8, we can find a $\delta > 0$ such that for all $v, w \in B_\delta(u)$ (a neighborhood of u in $L^q(\Omega)$)

$$G^0(w; u_- - v) < 0 \quad \text{and} \quad G^0(w; v - u_+) < 0.$$

By replacing u_- and u_+ by $u + t_0(u_- - u) = (1 - t_0)u$ and $u + t_0(u_+ - u) = (1 + t_0)u$ for a suitable $t_0 \in (0, 1]$ if necessary, we can assume that $u_-, u_+ \in B_\delta(u)$ (we keep the notation for simplicity). Since $B_\delta(u)$ is convex, we get

$$G^0(v + t(u_- - v); u_- - v) < 0 \quad \text{and} \quad G^0(v + t(u_+ - v); v - u_+) < 0$$

for all $v \in B_\delta(u)$ and all $t \in [0, 1]$. By a property of the generalized directional derivative, cf. [5, Proposition 2.1.1], we have

$$G^0(v + t(u_+ - v); v - u_+) = (-G)^0(v + t(u_+ - v); u_+ - v),$$

thus

$$G^0(v + t(u_- - v); u_- - v) < 0 \quad \text{and} \quad (-G)^0(v + t(u_+ - v); u_+ - v) < 0.$$

Note that by [5, Proposition 2.1.2] we have

$$G^0(v + t(u_- - v); u_- - v) = \max\{\langle \zeta^*, u_- - v \rangle \mid \zeta^* \in \partial G(v + t(u_- - v))\} < 0$$

and

$$(-G)^0(v + t(u_+ - v); u_+ - v) = \max\{\langle \zeta^*, u_+ - v \rangle \mid \zeta^* \in \partial(-G)(v + t(u_+ - v))\} < 0.$$

Now we apply [5, p. 41, Lemma] to the Lipschitz continuous functions

$$g_-(t) := G(v + t(u_- - v)) \quad \text{and} \quad -g_+(t) := -G(v + t(u_+ - v))$$

and get that

$$\partial g_-(t) \subset \{\langle \zeta^*, u_- - v \rangle \mid \zeta^* \in \partial G(v + t(u_- - v))\}$$

and

$$\partial(-g_+)(t) \subset \{\langle \zeta^*, u_+ - v \rangle \mid \zeta^* \in \partial(-G)(v + t(u_+ - v))\}.$$

Thus g_- and $-g_+$ are strictly decreasing on $[0, 1]$. In particular

$$g_+(t) = G(v + t(u_+ - v))$$

is strictly increasing on $[0, 1]$. Note that $g_-(1) = -t_0$ and $g_+(1) = t_0$. Thus, for all $v \in B_\delta(u)$ with $G(v) \geq 0$ there exists a unique $\tau_-(v) \in [0, 1]$ such that

$$g_-(\tau_-(v)) = G(v + \tau_-(v)(u_- - v)) = 0$$

and, analogously, for all $v \in B_\delta(u)$ with $G(v) \leq 0$ there exists a unique $\tau_+(v) \in [0, 1]$ such that

$$g_+(\tau_+(v)) = G(v + \tau_+(v)(u_+ - v)) = 0.$$

The function

$$\tau(v) := \begin{cases} \tau_-(v) & \text{for } G(v) \geq 0 \\ \tau_+(v) & \text{for } G(v) \leq 0 \end{cases}$$

is continuous on $B_\delta(u)$ and if $v \in B_\delta(u) \cap K$, then $\tau(v) = 0$. We define a function $r : B_\delta(u) \rightarrow B_\delta(u) \cap K$ in the following way

$$r(v) := \begin{cases} v + \tau(v)(u_- - v) & \text{for } G(v) \geq 0, \\ v + \tau(v)(u_+ - v) & \text{for } G(v) \leq 0. \end{cases}$$

We observe that r is continuous and if $v \in K$ then $r(v) = v$. Moreover, $r(B_\delta(u)) = B_\delta(u) \cap K$ and $B_\delta(u) \cap K$ is a neighborhood of u in $L^q(\Omega) \cap K$. Thus $r : B_\delta(u) \rightarrow B_\delta(u) \cap K$ is a retraction. Note that if $v \in BV(\Omega)$, then also $r(v) \in BV(\Omega)$.

(b) Let now $(u, \xi) \in \text{epi}(E + I_K)$, i.e. $u \in BV(\Omega) \cap K$ and $E(u) \leq \xi$. We construct a neighborhood of (u, ξ) in $\text{epi}(E + I_K)$ which is a retract of a neighborhood of (u, ξ) in $\text{epi}(E)$.

We consider the neighborhood $(B_\delta(u) \times \mathbb{R}) \cap \text{epi}(E)$ of (u, ξ) in $\text{epi}(E)$ and construct a retraction of it onto $(B_\delta(u) \times \mathbb{R}) \cap \text{epi}(E + I_K)$. For all $(v, \mu) \in (B_\delta(u) \times \mathbb{R}) \cap \text{epi}(E)$ we define the continuous function

$$\tilde{r}(v, \mu) := (r(v), (1 - \tau(v))\mu + \tau(v)E(u_+)).$$

If $v \in K$, then $\tau(v) = 0$ and therefore

$$\tilde{r}(v, \mu) = (v, \mu) \quad \text{on} \quad (B_\delta(u) \times \mathbb{R}) \cap \text{epi}(E + I_K).$$

We show that $\tilde{r}(v, \mu) \in \text{epi}(E + I_K)$. By definition of r we have $r(v) \in B_\delta(u) \cap BV(\Omega) \cap K$. Moreover, since E is 1-homogeneous, we get

$$E(u_-) = E((1 - t_0)u) = (1 - t_0)E(u) < (1 + t_0)E(u) = E((1 + t_0)u) = E(u_+).$$

If $G(v) \geq 0$ we deduce, by using convexity and 1-homogeneity of E ,

$$\begin{aligned} E(r(v)) &= E(v + \tau(v)(u_- - v)) \\ &= E((1 - \tau(v))v + \tau(v)u_-) \\ &\leq (1 - \tau(v))E(v) + \tau(v)E(u_-) \\ &\leq (1 - \tau(v))E(v) + \tau(v)E(u_+) \end{aligned}$$

and, analogously, for $G(v) \leq 0$ we get

$$E(r(v)) \leq (1 - \tau(v))E(v) + \tau(v)E(u_+).$$

Therefore

$$E(r(v)) \leq (1 - \tau(v))\mu + \tau(v)E(u_+)$$

i.e. $\tilde{r}(v, \mu) \in \text{epi}(E + I_K)$ and

$$\tilde{r} : (B_\delta(u) \times \mathbb{R}) \cap \text{epi}(E) \rightarrow (B_\delta(u) \times \mathbb{R}) \cap \text{epi}(E + I_K)$$

is a retraction. Thus, the neighborhood $(B_\delta(u) \times \mathbb{R}) \cap \text{epi}(E + I_K)$ of (u, ξ) in $\text{epi}(E + I_K)$ is a retract of the neighborhood $(B_\delta(u) \times \mathbb{R}) \cap \text{epi}(E)$ of (u, ξ) in $\text{epi}(E)$.

(c) Since E is convex, $\text{epi}(E)$ is also convex. Then also $(B_\delta(u) \times \mathbb{R}) \cap \text{epi}(E)$ is convex as intersection of convex sets and by Dugundji [10, Corollary 4.2] an ANR. Thus, by (b), the neighborhood $(B_\delta(u) \times \mathbb{R}) \cap \text{epi}(E + I_K)$ is a retract of an ANR and by [2, p. 87, Corollary 3.2] also an ANR. \diamond

Lemma 5.10 *The functional \mathcal{G}_{E+I_K} satisfies the Palais-Smale condition at level c for any $c \in \mathbb{R}$ as function on $\text{epi } \mathcal{P}(E + I_K)$.*

PROOF. Let us first show that $E + I_K$ satisfies the Palais-Smale condition at level c as a function on $L^q(\Omega)$. Let $(u_j)_{j \in \mathbb{N}} \subset BV(\Omega) \cap K$ be any Palais-Smale sequence at level c for $E + I_K$, i.e.

$$(E + I_K)(u_j) \rightarrow c \quad \text{and} \quad |d(E + I_K)|(u_j) \rightarrow 0. \quad (5.11)$$

Since $(u_j)_{j \in \mathbb{N}}$ is bounded in $BV(\Omega)$ by $(E + I_K)(u_j) < \infty$, $(u_j)_{j \in \mathbb{N}}$ is also bounded in $L^q(\Omega)$. Since $BV(\Omega)$ is compactly embedded in $L^q(\Omega)$, there exists a convergent subsequence of $(u_j)_{j \in \mathbb{N}}$ in $L^q(\Omega)$.

Now we show that \mathcal{G}_{E+I_K} satisfies the Palais-Smale condition on $\text{epi}(E + I_K)$. Let $(u_j, \xi_j) \subset \text{epi}(E + I_K)$ be a Palais-Smale sequence for \mathcal{G}_{E+I_K} at level c , i.e.

$$\xi_j = \mathcal{G}_{E+I_K}(u_j, \xi_j) \rightarrow c \quad \text{and} \quad |d\mathcal{G}_{E+I_K}|(u_j, \xi_j) \rightarrow 0. \quad (5.12)$$

Since (3.6) is satisfied, cf. Lemma 5.8, there exists a $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ we have $\xi_j = (E + I_K)(u_j)$. Moreover, by the definition of the weak slope

$$|d(E + I_K)|(u_j) = \frac{|d\mathcal{G}_{E+I_K}|(u_j, (E + I_K)(u_j))}{\sqrt{1 - |d\mathcal{G}_{E+I_K}|(u_j, (E + I_K)(u_j))}} \rightarrow 0$$

and therefore $(u_j)_{j \in \mathbb{N}}$ satisfies (5.11), i.e. $(u_j)_{j \in \mathbb{N}}$ is a Palais-Smale sequence for $E + I_K$ at level c . Since $E + I_K$ satisfies the Palais-Smale condition at level c , there exists a convergent subsequence of $(u_j)_{j \in \mathbb{N}}$ in $L^q(\Omega)$. Since $(\xi_j)_{j \in \mathbb{N}}$ is convergent by (5.12) in \mathbb{R} , we deduce that $(u_j, \xi_j)_{j \in \mathbb{N}}$ has a convergent subsequence in the complete metric space $\text{epi}(E + I_K)$.

Now we can easily transfer the result to \mathcal{G}_{E+I_K} defined on $\text{epi } \mathcal{P}(E + I_K)$. \diamond

Lemma 5.13 *We have*

$$\sup \{ \text{cat}(A, \text{epi } \mathcal{P}(E + I_K)) \mid A \subset \text{epi } \mathcal{P}(E + I_K) \text{ compact} \} = \infty.$$

PROOF. The idea of the proof is to construct compact subsets of $\text{epi}_{\mathcal{P}}(E + I_K)$ with arbitrarily large category.

In the proof of Proposition 5.9 we have shown that $\text{epi}_{\mathcal{P}}(E + I_K)$ is an ANR. In this case we can take closed covers instead of open covers in the definition of category (cf. Cornea et al. [6, Proposition 1.10]) and, in this proof, we use the definition of category by means of closed sets.

We fix any $k \in \mathbb{N}$ and linearly independent functions $v_1, \dots, v_k \in BV(\Omega)$. We set

$$V_k := \text{span} \{v_i \mid i = 1, \dots, k\} \cap K.$$

By $S^{k-1} \subset \mathbb{R}^k$ we denote the $(k-1)$ -dimensional unit sphere and \mathbb{R}^k shall be endowed with the canonical basis $\{e_i \in \mathbb{R}^k \mid i = 1, \dots, k\}$. We construct a homeomorphism $\psi : S^{k-1} \rightarrow V_k$ by setting

$$x = \sum_{i=1}^k x_i e_i \mapsto \frac{\sum_{i=1}^k x_i v_i}{\left\| \sum_{i=1}^k x_i v_i \right\|_{L^1(\Omega)}}.$$

We have

$$\left\| \sum_{i=1}^k x_i v_i \right\|_{L^1(\Omega)} \geq C > 0, \quad (5.14)$$

since not all $x_i = 0$, since the v_i are linearly independent, and since $x \mapsto \left\| \sum_{i=1}^k x_i v_i \right\|_{L^1(\Omega)}$ is a continuous function defined on the compact set S^{k-1} . We see that ψ is a homeomorphism, since its inverse $\psi^{-1} : \psi(S^{k-1}) \rightarrow S^{k-1}$ is given by

$$u = \sum_{i=1}^k \xi_i v_i \mapsto \frac{\sum_{i=1}^k \xi_i e_i}{\left(\sum_{i=1}^k |\xi_i|^2 \right)^{1/2}}$$

and ψ^{-1} is continuous, since it is the inverse of a continuous function on a compact set. Clearly ψ is odd, i.e. it satisfies $\psi(-x) = -\psi(x)$, and the set $\psi(S^{k-1})$ is compact and symmetric.

For $u \in \psi(S^{k-1})$ we obtain

$$\begin{aligned} E(u) &= E\left(\frac{\sum_{i=1}^k \xi_i v_i}{\left\| \sum_{i=1}^k \xi_i v_i \right\|_{L^1(\Omega)}}\right) \\ &= \frac{1}{\left\| \sum_{i=1}^k \xi_i v_i \right\|_{L^1(\Omega)}} E\left(\sum_{i=1}^k \xi_i v_i\right) \\ &= \frac{1}{\left\| \sum_{i=1}^k \xi_i v_i \right\|_{L^1(\Omega)}} \left(\int_{\Omega} d \left| D \sum_{i=1}^k \xi_i v_i \right| + \int_{\partial\Omega} \left| \sum_{i=1}^k \xi_i v_i \right| d\mathcal{H}^{n-1} \right) \\ &\leq \frac{1}{\left\| \sum_{i=1}^k \xi_i v_i \right\|_{L^1(\Omega)}} \sum_{i=1}^k |\xi_i| \left(\int_{\Omega} d|Dv_i| + \int_{\partial\Omega} |v_i| d\mathcal{H}^{n-1} \right) \\ &\stackrel{(5.14)}{\leq} \frac{1}{C} \sum_{i=1}^k |\xi_i| E(v_i) \\ &\leq \frac{k}{C} \max_{i=1, \dots, k} E(v_i) =: C_k. \end{aligned}$$

Thus

$$\sup_{u \in \psi(S^{k-1})} E(u) \leq C_k \quad (5.15)$$

for a constant $C_k \in \mathbb{R}$ depending only on k .

Let us now consider the real projective space \mathcal{P}^{k-1} which is defined as the projective space according to $\mathbb{R}^k \setminus \{0\}$. By Zeidler [14, p. 347] we know that

$$k = \text{cat} \left(\mathcal{P}(S^{k-1}), \mathcal{P}(S^{k-1}) \right). \quad (5.16)$$

By a straightforward argument following from the definition of category one readily obtains that

$$\text{cat} \left(\mathcal{P}(S^{k-1}), \mathcal{P}(S^{k-1}) \right) = \text{cat} \left(\mathcal{P}^{k-1}, \mathcal{P}^{k-1} \right). \quad (5.17)$$

Now we use Fadell [11, Proposition 2.10] with S^{k-1} and $\psi(S^{k-1})$ as subsets of $\mathbb{R}^k \setminus \{0\}$ and $L^q(\Omega) \setminus \{0\}$, respectively, and get

$$\text{cat} \left(\mathcal{P}^{k-1}, \mathcal{P}^{k-1} \right) = \text{cat} \left(\mathcal{P} \left(\psi \left(S^{k-1} \right) \right), \mathcal{P} \right). \quad (5.18)$$

Note that we get equality, since ψ is a homeomorphism.

Moreover, by the definition of category, we get

$$\text{cat} \left(\mathcal{P} \left(\psi \left(S^{k-1} \right) \right), \mathcal{P} \right) \leq \text{cat} \left(\mathcal{P} \left(\psi \left(S^{k-1} \right) \right), \mathcal{P}(BV(\Omega) \cap K) \right), \quad (5.19)$$

since each covering of $\mathcal{P}(\psi(S^{k-1}))$ by closed sets in $\mathcal{P}(BV(\Omega) \cap K)$ is always also a covering of $\mathcal{P}(\psi(S^{k-1}))$ by closed sets in \mathcal{P} .

With the constant C_k as defined in (5.15) we now assume that

$$\text{cat} \left(\mathcal{P} \left(\psi \left(S^{k-1} \right) \right) \times \{C_k\}, \text{epi}_{\mathcal{P}}(E + I_K) \right) = m$$

for some $m \in \mathbb{N}$, i.e. there exists a covering of the form

$$\mathcal{P} \left(\psi \left(S^{k-1} \right) \right) \times \{C_k\} \subset \bigcup_{i=1}^m \mathcal{P}(B_{k,i}) \times \{C_k\}$$

with closed, symmetric, and contractible sets $B_{k,i} \subset BV(\Omega) \cap K$ such that

$$\mathcal{P}(B_{k,i}) \times \{C_k\} \subset \text{epi}_{\mathcal{P}}(E + I_K).$$

Note that $\{C_k\}$ is a covering of itself and contractible in itself. But then we also have

$$\mathcal{P} \left(\psi \left(S^{k-1} \right) \right) \subset \bigcup_{i=1}^m \mathcal{P}(B_{k,i})$$

and therefore

$$\begin{aligned} & \text{cat} \left(\mathcal{P} \left(\psi \left(S^{k-1} \right) \right), \mathcal{P}(BV(\Omega) \cap K) \right) \\ & \leq \text{cat} \left(\mathcal{P} \left(\psi \left(S^{k-1} \right) \right) \times \{C_k\}, \text{epi}_{\mathcal{P}}(E + I_K) \right). \end{aligned} \quad (5.20)$$

By analogous arguments we even get equality.

Combining equations (5.16), (5.17), (5.18), (5.19), and (5.20), we get

$$\begin{aligned}
k &= \text{cat} \left(\mathcal{P}(S^{k-1}), \mathcal{P}(S^{k-1}) \right) \\
&= \text{cat} \left(\mathcal{P}^{k-1}, \mathcal{P}^{k-1} \right) \\
&= \text{cat} \left(\mathcal{P} \left(\psi \left(S^{k-1} \right) \right), \mathcal{P} \right) \\
&\leq \text{cat} \left(\mathcal{P} \left(\psi \left(S^{k-1} \right) \right), \mathcal{P}(BV(\Omega) \cap K) \right) \\
&= \text{cat} \left(\mathcal{P} \left(\psi \left(S^{k-1} \right) \right) \times \{C_k\}, \text{epi}_{\mathcal{P}}(E + I_K) \right).
\end{aligned}$$

Since $k \in \mathbb{N}$ can be chosen arbitrarily large and since $\mathcal{P} \left(\psi \left(S^{k-1} \right) \right) \times \{C_k\} \subset \text{epi}(E + I_K)$ by (5.15), we have thus shown that $\text{epi}_{\mathcal{P}}(E + I_K)$ contains compact sets of arbitrarily large category. \diamond

For the derivation of equation (4.4) in Theorem 4.3 we use a suitable Lagrange multiplier rule derived in Degiovanni & Schuricht [9, Corollary 3.6] and the characterization of the subdifferential of E given in Kawohl & Schuricht [13, Proposition 4.23].

Proposition 5.21 *Let $E : X \rightarrow \mathbb{R}$ be a lower semicontinuous convex function and let $G : X \rightarrow \mathbb{R}$ be locally Lipschitz continuous. Let $u \in \mathcal{D}(E) \cap K$ be a critical point of E under the constraint K . If $\mathcal{D}(E)$ and K are transversal at u in $L^q(\Omega)$, then $\partial E(u) \neq \emptyset$ and there exist*

$$u_E^* \in \partial E(u), \quad u_G^* \in \partial G(u), \quad \lambda \in \mathbb{R}$$

such that

$$u_E^* + \lambda u_G^* = 0.$$

Proposition 5.22 *The functional E is convex and lower semicontinuous on $L^q(\Omega)$. Moreover, $u_E^* \in \partial E(u)$ for $u \in L^q(\Omega)$ if and only if there exists a vector field $z \in L^\infty(\Omega, \mathbb{R}^n)$ with*

$$\begin{aligned}
\|z\|_{L^\infty} &\leq 1, \quad u_E^* = -\text{Div} z \in L^{q^*}(\Omega), \\
E(u) &= \langle u_E^*, u \rangle = - \int_{\Omega} u \text{Div} z \, dx.
\end{aligned}$$

If $E(u) > 0$, then $\|z\|_{L^\infty} = 1$.

Now we combine the provided tools in the final proof.

PROOF of Theorem 4.3. We use Lemma 5.9, 5.10 and 5.13 and Proposition 5.2 in order to get the existence of infinitely many critical points $([u]_k, \xi_k)_{k \in \mathbb{N}}$ of \mathcal{G}_{E+I_K} on the space $\text{epi}_{\mathcal{P}}(E + I_K)$. Note that \mathcal{G}_{E+I_K} is bounded from below by $E(u) \geq 0$.

By going back to $\text{epi}(E + I_K)$ we thus have the existence of infinitely many pairs $((u_k, \xi_k), (-u_k, \xi_k))_{k \in \mathbb{N}}$ of critical points for \mathcal{G}_{E+I_K} on $\text{epi}(E + I_K)$ by Lemma 5.3. By Lemma 5.8 we know that \mathcal{G}_{E+I_K} satisfies (3.6) and, therefore, the critical points of \mathcal{G}_{E+I_K} are in fact $((u_k, E(u_k)), (-u_k, E(u_k)))_{k \in \mathbb{N}}$ and thus $(u_k, -u_k)_{k \in \mathbb{N}}$ are critical points of $E + I_K$. Since $u_k \in BV(\Omega) \cap K$ for each $k \in \mathbb{N}$, we also get $E(u_k) < \infty$.

Now, since $BV(\Omega)$ and K are transversal by Lemma 5.8, we may apply Proposition 5.21 by using Propositions 5.22 and 5.7. Then, for each critical point $u_k \in BV(\Omega)$ there exists a

measurable selection $s_k(x) \in \text{Sgn}(u_k(x))$ for a. e. $x \in \Omega$ and a vector field $z_k \in L^\infty(\Omega, \mathbb{R}^n)$ with

$$\|z_k\|_{L^\infty} = 1, \quad \text{Div } z_k \in L^q(\Omega),$$

$$E(u_k) = - \int_{\Omega} u_k \text{Div } z_k \, dx$$

such that

$$-\text{Div } z_k = \lambda_k s_k \text{ a.e. on } \Omega.$$

Moreover, by multiplying this equation with u_k and integrating over Ω we get

$$E(u_k) = - \int_{\Omega} u_k \text{Div } z_k \, dx = \lambda_k \int_{\Omega} u_k s_k \, dx = \lambda_k.$$

Going back to Proposition 5.2 and using that $\lambda_k = E(u_k) = \mathcal{G}_{E+I_k}(u_k, E(u_k))$ we deduce that $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$.

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