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## Some special aspects related to the 1-Laplace operator

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# Some special aspects related to the 1-Laplace operator

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**Abstract**

The eigenvalue problem for the 1-Laplace operator, which is considered to be the Euler-Lagrange equation for an associated variational problem in $BV(\Omega)$, is formally given by

$$-\mathrm{Div}\,\frac{Du}{|Du|} = \lambda \frac{u}{|u|} \quad \text{on } \Omega\,.$$

However the undetermined expressions $Du/|Du|$ and $u/|u|$ have to be replaced with an appropriate vector field $z$ related to $u$ and a measurable selection $s$ of the set-valued sign function $\mathrm{Sgn}\,(u(\cdot))$, respectively, such that $-\mathrm{Div}\, z = \lambda s$. For the special case of a square $\Omega \subset \mathbb{R}^2$ and the known minimizer $u = \chi_C$ of the related variational problem, the paper presents a somehow explicit construction of corresponding vector fields $z$. In particular it is shown that, for a fixed selection $s$, the field $z$ is not determined by means of the differential equation and its coupling conditions with $u$, but there are infinitely many continuous vector fields $z$ that even differ on the boundary of $\Omega$.

# 1 Introduction

Let us consider the constrained minimization problem

$$\int_\Omega d|Du| + \int_{\partial\Omega} |u|\, dx \to \text{Min!}, \quad u \in BV(\Omega), \tag{1.1}$$

$$\int_\Omega |u|\, dx = 1 \tag{1.2}$$

for an open, bounded $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary. This problem can be considered as limit of the variational problem associated with the eigenvalue problem for the $p$-Laplace operator by taking $p \to 1$. In particular, the surface integral replaces the boundary condition and implies homogeneous Dirichlet data in a generalized way. Problem (1.1), (1.2) has always a solution which, however, does not have to be unique (cf. Kawohl & Schuricht [16]).

The formal Euler-Lagrange equation of (1.1), (1.2) as necessary condition for a minimizer is given by

$$-\text{Div}\left(\frac{Du}{|Du|}\right) = \lambda \frac{u}{|u|} \quad \text{on } \Omega\,, \tag{1.3}$$

which is also called the eigenvalue problem for the 1-Laplace operator. However, this equation is not well defined if we have in mind that typical minimizers of (1.1), (1.2) are characteristic functions, i.e. they are piecewise constant and even vanish on a set of positive measure. In order to give meaning to (1.3), variational problem (1.1), (1.2) has to be studied more carefully. The lack of differentiability of the functionals involved requires more general tools than usually used in the calculus of variations. A direct treatment of this singular problem has been proposed in [16]. Roughly speaking, for a minimizer $u \in BV(\Omega)$ there is a vector field $z : \Omega \to B_1(0) \subset \mathbb{R}^n$ replacing $Du/|Du|$ and satisfying certain coupling conditions to $u$ such that

$$-\text{Div}\, z(x) = \lambda s(x) \quad \text{on } \Omega \tag{1.4}$$

where $s : \Omega \to [-1, 1]$ is a measurable selection of the set-valued sign function $\text{Sgn}\,(u(x))$ replacing $u/|u|$. More precisely, for any measurable selection $s$ of $\text{Sgn}\,(u(\cdot))$ there exists a corresponding vector field $z$ coupled to $u$ such that (1.4) is satisfied. This is remarkable, since it provides infinitely many Euler-Lagrange equations (1.4) as replacement of (1.3).

The purpose of the present paper is the investigation of the vector fields $z$ that are associated with a fixed minimizer $u$ by means of the necessary condition just formulated. In particular, if we also fix a measurable selection $s$, then we have to look for a vector field $z$ with prescribed divergence according to (1.4). This is a classical question and, of course, $z$ is not determined by the differential equation on its own. But we still have to take into account the coupling of $z$ with the minimizer $u$ which roughly demands that $\|z\|_\infty = 1$ and that $z$ has to respect certain boundary conditions. Besides the interest to know how such vector fields look like, the question of to what extent the coupling conditions combined with a fixed $s$ determine $z$ is fundamental. Since it seems that $|z(x)| < 1$ at least on some ball $B \subset \Omega$ for a typical solution $z$, we clearly can add any divergence free vector field with sufficiently small amplitude and support on $B$ in order to get a further solution $z$. This way we will always obtain infinitely many vector fields satisfying the necessary condition for a fixed selection $s$. However, a much more subtle question is how much freedom we have for $z$ on the boundary of $\partial\Omega$ where, in general, the coupling conditions do not completely prescribe $z$.

We will analyze the questions discussed in the previous paragraph for the special case of a square $\Omega \subset \mathbb{R}^2$. It is known that a suitable multiple of the characteristic function $\chi_C$ of the Cheeger set $C$ of $\Omega$, which minimizes the ratio $|\partial D|/|D|$ among all subdomains $D \subset \Omega$ (cf. [12]), is a minimizer of (1.1), (1.2). In this special case of a square $\Omega$ the Cheeger set $C$ is a "rounded square" as shown in Figure 1 below. We present a construction of vector fields $z$ that, for each fixed selection $s$, provides infinitely many solutions $z$ differing in particular on $\partial\Omega \setminus \partial C$. Clearly, $z$ has to satisfy $-\mathrm{Div}\, z = \lambda$ on $C$ and, as in [16], we construct $z$ on $C$ by means of a suitable solution of the mean curvature equation. This vector field $z$ is then always used on $C$. On the "corners" $\hat{\Omega} := \Omega \setminus \bar{C}$ we have to fix a selection $s$. For the construction of the corresponding $z$ we first consider a unit normal field $\nu$ on $\hat{\Omega}$ that is associated with a suitable foliation of $\hat{\Omega}$ by circular arcs and, then, we construct a scalar function $a : \hat{\Omega} \to [0,1]$ such that $z := -a\nu$ meets the desired conditions on $\hat{\Omega}$. The composition of the fields $z$ on $C$ and $\hat{\Omega}$ then gives a solution $z$ on $\Omega$ for a prescribed selection $s$. The variety of foliations that can be used independent of the special $s$ provides the diversity of solutions $z$ for a fixed $s$.

In Section 2 we summarize the existence of a minimizer and the corresponding necessary condition for variational problem (2.1), (2.2). Consequences of the coupling condition between a minimizer $u$ and the vector field $z$ are investigated in Section 3. In Section 4 the existence of infinitely many continuous vector fields $z$ for fixed $s$, that differ in particular on the boundary $\partial\Omega \setminus \partial C$, is formulated as main result of the paper. But notice that their (to some extend) explicit construction is very instructive by its own. While the basic ideas of this construction are briefly discussed in Section 4, the details are carried out in the subsequent Section 5 that essentially consists of two parts. At first foliations of the corners $\hat{\Omega}$ by circular arcs are constructed. Then, using the ansatz $z = -a\nu$, we show the existence of a scalar function $a$ by solving a corresponding linear inhomogeneous partial differential equation of first order for $a$. Since the inhomogeneity is a multiple of the merely measurable function $s$, the classical results based on characteristics are not applicable. However, it turns out that we can carry out the method of characteristics explicitly if consider the ordinary differential equations along the characteristics in the sense of Carathéodory. This way we obtain continuous vector fields $z$ that satisfy (1.4) in the sense of distributions. A special solution $z$ is discussed in some more detail in Section 6.

Let us still mention that some of the results have already been announced in Milbers [17].

**Notation.** We denote the boundary of a set $A$ by $\partial A$ and its closure by $\overline{A}$ or $\mathrm{cl}\, A$. We define its characteristic function $\chi_A$ by

$$\chi_A(x) := \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{otherwise}. \end{cases}$$

$\mathrm{Div}\, u$ is the divergence of $u$ in the distributional sense. The set-valued sign function on $\mathbb{R}$ is given by

$$\mathrm{Sgn}\, \alpha := \begin{cases} 1 & \text{if } \alpha > 0, \\ [-1,1] & \text{if } \alpha = 0, \\ -1 & \text{if } \alpha < 0. \end{cases}$$

The space of $q$-integrable functions on $\Omega$ is denoted by $L^q(\Omega)$ and its dual by $L^{q'}(\Omega)$ where $\frac{1}{q} + \frac{1}{q'} = 1$. The Sobolev space $W^{1,q}(\Omega)$ consists of $q$-integrable functions having $q$-integrable weak derivatives. $BV(\Omega)$ stands for the functions of bounded variation. $C_c^k(\Omega)$ are $k$-times continuously differentiable functions with compact support in $\Omega$. The $k$-dimensional Hausdorff measure is denoted by $\mathcal{H}^k$. In particular, $|\partial\Omega|$ and $|\Omega|$ denote the $(n-1)$- and the $n$-dimensional Hausdorff measure of $\partial\Omega$ and $\Omega$, respectively. $\mu\lfloor\Omega$ is the restriction of a measure $\mu$ to the set $\Omega$ and $f\mu$ is the measure having density $f$ with respect to $\mu$. For a Banach space $X$ its dual is $X^*$ and $\langle\cdot,\cdot\rangle$ is the duality form on $X \times X^*$. By $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ we denote the weak and the weak$^*$ convergence, respectively.

## 2 Eigenvalue problem for the 1-Laplace operator

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary. We consider the constrained variational problem

$$E(u) := \int_\Omega d|Du| + \int_{\partial\Omega} |u| \, d\mathcal{H}^{n-1} \to \text{Min!}, \quad u \in BV(\Omega)\,, \tag{2.1}$$

$$\int_\Omega |u| \, dx = 1\,. \tag{2.2}$$

Since usual Dirichlet boundary conditions on $\partial\Omega$ are unsuitable in $BV(\Omega)$, there is a surface integral included in (2.1) which implies homogeneous boundary conditions in a weak sense, cf. Kawohl & Schuricht [16]. The existence of a minimizer is shown, e.g., in Fridman & Kawohl [11].

**Theorem 2.3** *Problem* (2.1), (2.2) *has a solution* $u \in BV(\Omega)$.

But it turns out that the solution does not have to be unique in general (cf. Kawohl & Schuricht [16]). The Euler-Lagrange equation as necessary condition for a minimizer of (2.1), (2.2) is formally given by

$$-\text{Div}\left(\frac{Du}{|Du|}\right) = \lambda \, \frac{u}{|u|} \quad \text{on } \Omega \tag{2.4}$$

where $\lambda > 0$ is a Lagrange multiplier. Since minimizers $u$ are typically constant or even vanish on a set of positive measure (see Fridman & Kawohl [11] or the arguments below), the expressions in (2.4) are not well defined in general. The following theorem of Kawohl & Schuricht [16] provides a suitable substitute for (2.4) (cf. also Demengel [7] for a partial result).

**Theorem 2.5** *Let $u \in BV(\Omega)$ be a minimizer of* (2.1), (2.2)*. Then for each measurable selection $s(x) \in \text{Sgn}\,(u(x))$ a.e. on $\Omega$ there is some vector field $z \in L^\infty(\Omega, \mathbb{R}^n)$ satisfying*

$$\|z\|_{L^\infty} = 1, \quad \text{Div}\, z \in L^n(\Omega)\,, \tag{2.6}$$

$$\int_\Omega d|Du| + \int_{\partial\Omega} |u| \, d\mathcal{H}^{n-1} = -\int_\Omega u \, \text{Div}\, z \, dx \tag{2.7}$$

*such that*

$$-\text{Div}\, z = \lambda s \quad \textit{a.e. on } \Omega, \quad \lambda = E(u)\,. \tag{2.8}$$

Notice that, in general, there are infinitely many Euler-Lagrange equations (2.8) as necessary condition for a minimizer $u$ of (1.1), (1.2), since (2.8) has to be satisfied not only for one but for any measurable selection $s$. We call (2.8) combined with the coupling conditions (2.6), (2.7) relating $z$ to $u$ the *eigenvalue problem* for the 1-Laplace operator. A solution $u$ is said to be an *eigenfunction* according to the *eigenvalue* $\lambda$.

The previous theorem states the existence of vector fields $z$ satisfying (2.6)-(2.8). For a deeper understanding of the problem or for a direct solution of the eigenvalue problem, it would certainly be useful to have some information about vector fields $z$ with prescribed divergence and about the relevance of the coupling conditions (2.6), (2.7) for that vector field. Unfortunately it turns out that not that much is known about such vector fields. For a fixed eigenfunction $u$ and a given selection $s$ it is even unclear how far the vector field $z$ is determined by (2.6)-(2.8). The intention of the present paper is to answer some of these questions for the simple special case of a square $\Omega \subset \mathbb{R}^2$.

Let us still illuminate some relation of the eigenvalue problem to a geometric question. It can be shown that the smallest eigenvalue $\lambda$ of the 1-Laplace operator equals the *Cheeger constant* of a nonempty open bounded set $\Omega \subset \mathbb{R}^n$ given by

$$h(\Omega) := \inf_{D \subset \Omega} \frac{|\partial D|}{|D|} \tag{2.9}$$

with $D$ varying over all nonempty sets $D \subset \Omega$ of finite perimeter, cf. Alter & Caselles [1]. The set $C \subset \Omega$ is called a *Cheeger set* of $\Omega$ if $|\partial C|/|C| = h(\Omega)$. Originally the Cheeger constant has been defined by Cheeger [6] in a slightly different manner. The multiple $u = \frac{1}{|C|}\chi_C$ of the characteristic function of $C$ is an eigenfunction corresponding to the smallest eigenvalue $\lambda = h(\Omega)$ by Kawohl & Fridman [11]. For a Cheeger set $C$ of $\Omega$ we know that the surface $\partial C \cap \Omega$ has constant mean curvature $h(\Omega)$, cf. Gonzalez et al. [14, Theorem 2]. In the case $\Omega \subset \mathbb{R}^2$ this readily implies that $\partial C \cap \Omega$ consists of circular arcs. If $\Omega$ is convex, then the Cheeger set has to be convex too and it is known that the Cheeger set is unique in that case, cf. Alter & Caselles [1]. These properties allow the description of the Cheeger set $C$ of a convex $\Omega \subset \mathbb{R}^2$ as the union of all balls contained in $\Omega$ that have radius $R = 1/h(\Omega) = 1/\lambda$.

# 3 Consequences of the coupling conditions

Let us start with some preliminary investigations of the coupling conditions (2.6), (2.7). As in the previous section we assume that $\Omega \subset \mathbb{R}^n$ is an open bounded set with Lipschitz boundary where $\nu$ denotes its outer unit normal. For a function $u \in BV(\Omega)$ and a vector field

$$z \in L^\infty_n := \{z \in L^\infty(\Omega, \mathbb{R}^n) |\, \operatorname{Div} z \in L^n(\Omega)\}$$

there is a function $[z, \nu] \in L^\infty(\partial\Omega, \mathbb{R})$, called normal trace of $z$ on $\partial\Omega$, and a Radon measure on $\Omega$ denoted by $(z, Du)$ such that

$$\|[z,\nu]\|_{L^\infty(\partial\Omega)} \le \|z\|_{L^\infty(\Omega)}\,, \qquad \int_{\hat\Omega} d(z, Du) \le \|z\|_{L^\infty(\Omega)} \int_{\hat\Omega} d|Du| \tag{3.1}$$

for all Borel sets $\hat\Omega \subset \Omega$ and

$$\int_\Omega u \operatorname{Div} z \, dx + \int_\Omega d(z, Du) = \int_{\partial\Omega} [z,\nu]\, u \, d\mathcal{H}^{n-1}\,. \tag{3.2}$$

If $z$ is continuous on $\Omega$, then

$$[z,\nu] = z \cdot \nu \quad \text{a.e. on} \quad \partial\Omega \tag{3.3}$$

and

$$\int_{\hat\Omega} d(z, Du) = \int_{\hat\Omega} z \cdot d(Du)$$

for all Borel sets $\hat\Omega \subset \Omega$ (cf. Anzellotti [4] or Kawohl & Schuricht [16]).

**Proposition 3.4** *Let $u \in BV(\Omega)$ and let $z \in L^\infty_n(\Omega, \mathbb{R}^n)$ satisfy $\|z\|_{L^\infty} \le 1$. Then (2.7) is equivalent to the two conditions*

$$-[z, \nu](x) \in \operatorname{Sgn}(u(x)) \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega \tag{3.5}$$

*and*

$$(z, Du) = |Du| \tag{3.6}$$

*in the sense of measures on $\Omega$.*

Proof. By (3.2) we have

$$\begin{aligned}\int_\Omega d|Du| + \int_{\partial\Omega} |u|\, d\mathcal{H}^{n-1} &= -\int_\Omega u \mathrm{Div}\, z\, dx \\ &= \int_\Omega d(z, Du) - \int_{\partial\Omega} [z,\nu] u\, d\mathcal{H}^{n-1}.\end{aligned}$$

Since

$$\int_\Omega d|Du| \ge \int_\Omega d(z, Du) \quad \text{and} \quad \int_{\partial\Omega} |u|\, d\mathcal{H}^{n-1} \ge -\int_{\partial\Omega} [z,\nu] u\, d\mathcal{H}^{n-1}$$

by (3.1), condition (2.7) is equivalent to

$$\int_\Omega d|Du| = \int_\Omega d(z, Du) \quad \text{and} \quad \int_{\partial\Omega} |u|\, d\mathcal{H}^{n-1} = -\int_{\partial\Omega} [z,\nu] u\, d\mathcal{H}^{n-1}.$$

Since $(z, Du)(\hat\Omega) \le |Du|(\hat\Omega)$ for all Borel sets $\hat\Omega \subset \Omega$ by (3.1), the first identity is equivalent to (3.6). Since $\|[z,\nu]\|_{L^\infty(\partial\Omega)} \le 1$ by (3.1), the second identity is equivalent to (3.5). ♢

Let us still consider the case of a characteristic function $u$.

**Proposition 3.7** *Let $u = \chi_C \in BV(\Omega)$ for some open $C \subset \Omega$ having Lipschitz boundary and satisfying*

$$\chi_C = 1 \quad \mathcal{H}^{n-1}\text{-a.e. on} \quad \partial\Omega \cap \partial C \tag{3.8}$$

*in the sense of trace. Moreover let $z \in L^\infty_n(\Omega, \mathbb{R}^n)$ satisfy $\|z\|_{L^\infty} \le 1$. Then $z$ has a normal trace $[z,\nu]$ on $\partial C$ and (2.7) is equivalent to*

$$[z, \nu] = -1 \quad \mathcal{H}^{n-1}\text{-a.e. on} \quad \partial C\,. \tag{3.9}$$

*If $z$ is continuous, then (3.9) can be written as*

$$z = -\nu \quad \mathcal{H}^{n-1}\text{-a.e. on} \quad \partial C\,. \tag{3.10}$$

Proof. Clearly,

$$\int_\Omega d|D\chi_C| = \int_{\partial C\cap\Omega} d\mathcal{H}^{n-1}\,.$$

Since we have (3.2) also with $C$ instead of $\Omega$, we get by $\int_C d|Du| = 0$ and by (3.1) that

$$\int_C \operatorname{Div} z\,dx = \int_{\partial C} [z,\nu]\,d\mathcal{H}^{n-1}\,.$$

Thus,

$$\begin{aligned}\int_\Omega d(z, D\chi_C) &= -\int_\Omega \chi_C \operatorname{Div} z\,dx + \int_{\partial\Omega} [z,\nu]\,\chi_C\,d\mathcal{H}^{n-1}\\ &= -\int_C \operatorname{Div} z\,dx + \int_{\partial\Omega} [z,\nu]\,\chi_C\,d\mathcal{H}^{n-1}\\ &= -\int_{\partial C} [z,\nu]\,d\mathcal{H}^{n-1} + \int_{\partial\Omega\cap\partial C} [z,\nu]\,d\mathcal{H}^{n-1} \;=\; -\int_{\partial C\cap\Omega} [z,\nu]\,d\mathcal{H}^{n-1}\,.\end{aligned}$$

Consequently, (3.6) is equivalent to

$$\int_{\partial C\cap\Omega} d\mathcal{H}^{n-1} = -\int_{\partial C\cap\Omega} [z,\nu]\,d\mathcal{H}^{n-1}\,,$$

i.e., $[z,\nu] = -1$ $\mathcal{H}^{n-1}$-a.e. on $\partial C\cap\Omega$. Now we notice that (3.5) is automatically satisfied on $\partial\Omega\setminus\partial C$. Hence the equivalence of (2.7) and (3.9) follows from Proposition 3.4. The equivalence on $\partial\Omega\cap\partial C$ follows by (3.8). The specialization for continuous $z$ readily follows with (3.3). $\diamondsuit$

# 4 Special case of a square in $\mathbb{R}^2$

Let us now consider the particular case that $\Omega\subset\mathbb{R}^2$ is a square of the form $(0,b)\times(0,b)$. Since $\Omega$ is convex, we have that $u = \frac{1}{|C|}\chi_C$ is a minimizer of (2.1), (2.2) where $C$ is the Cheeger set of $\Omega$. According to our previous arguments, the Cheeger set $C$ of $\Omega$ has the shape of a square with "round corners" as shown in Figure 1.

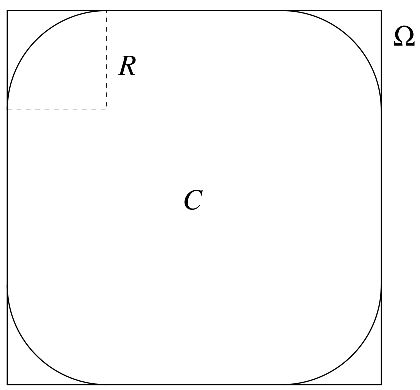


Figure 1: The Cheeger set $C$ of the square $\Omega$.

The necessary condition from Theorem 2.5 implies for this special case that for any measurable function

$$s(x)\in[-1,1]\quad\text{a.e. on }\;\Omega\setminus C \tag{4.1}$$

there is a vector field $z \in L^\infty(\Omega, \mathbb{R}^2)$ satisfying the coupling condition

$$\|z\|_{L^\infty} = 1\,, \quad \operatorname{Div} z \in L^2(\Omega)\,, \quad E(u) = -\int_\Omega u \operatorname{Div} z \, dx\,, \tag{4.2}$$

such that

$$-\operatorname{Div} z = \lambda \quad \text{a.e. on } C \tag{4.3}$$

and

$$-\operatorname{Div} z = \lambda s \quad \text{a.e. on } \Omega \setminus C \tag{4.4}$$

whith $\lambda = E(u)$. Notice that the two equations (4.3), (4.4) are equivalent to the single equation (2.8), since we do not have to evaluate $\operatorname{Div} z$ on the zero set $\partial C \cap \Omega$ by the demand $\operatorname{Div} z \in L^2(\Omega)$. The next result shows that the vector field $z$ is not specified by the previous conditions for a fixed selection $s$.

**Theorem 4.5** *Let $\Omega \subset \mathbb{R}^2$ be a square, let $u \in BV(\Omega)$ be the minimizer of* (2.1), (2.2) *on $\Omega$, and let $s(x) \in \operatorname{Sgn}(u(x))$ a.e. on $\Omega$ be a measurable selection. Then there exist infinitely many vector fields $z \in C(\bar{\Omega})$ satisfying* (4.2) - (4.4) *that pairwise differ on $\Omega \setminus C$ and, in particular, also on $\partial\Omega \setminus \partial C$.*

**Remark 4.6** We will see in the proof that, for fixed $s$, we can parametrize different solutions $z$ by means of real numbers belonging to an interval in $\mathbb{R}$, i.e., we in fact obtain even a continuum of vector fields $z \in C(\bar{\Omega})$ for each fixed $s$.

Let us start with some preliminary considerations before we carry out the proof in the next section. Since we are looking for continuous vector fields $z$ we can replace (4.2) with the equivalent condition

$$\|z\|_{L^\infty} = 1\,, \quad \operatorname{Div} z \in L^2(\Omega)\,, \quad z = -\nu \quad \mathcal{H}^{n-1}\text{-a.e. on} \quad \partial C \tag{4.7}$$

according to Proposition 3.7. Notice that the most right equation provides boundary conditions for the differential equations (4.3), (4.4). We now intend to solve (4.3) and (4.4) separately. More precisely, below in this section we provide a solution for (4.3) and we discuss a strategy to determine solutions of (4.4). Then, in the next section, we carry out the construction of infinitely many solutions $z$ on $\Omega \setminus C$ in full detail. Using on $C$ always the “fixed” solution from below, we finally obtain the assertion of the theorem.

**Solution $z$ on $C$.** As demonstrated by Kawohl & Schuricht in [16], we use the mean curvature equation

$$\operatorname{div}\left(\frac{Dw}{\sqrt{1+|Dw|^2}}\right) = \lambda \quad \text{on} \quad C \tag{4.8}$$

for the construction of a solution $z$ of (4.3) with the boundary condition

$$z = -\nu \quad \mathcal{H}^{n-1}\text{-a.e. on} \quad \partial C \tag{4.9}$$

and the additional constraint $\|z\|_{L^\infty} = 1$. Since the curved part of the boundary $\partial C$ has curvature $\lambda$, the curvature of $\partial C$ is less than or equal to $\lambda = h(\Omega) = |\partial C|/|C|$ everywhere. Hence there exists a solution $w$ of (4.8) on $C$ such that

$$\lim_{y \to x} \frac{Dw(y)}{\sqrt{1+|Dw(y)|^2}} = \nu(x) \quad \text{for all} \quad x \in \partial C$$

where $\nu(x)$ is the exterior normal of $C$ and where the limit is uniform on $\partial C$ (cf. Giusti [13] where one can also find that the solution $w$ is unique up to an additive constant). Obviously

$$z := -\frac{Dw}{\sqrt{1+|Dw|^2}} \quad \text{on } C\,, \tag{4.10}$$

provides a solution of (4.3) with boundary condition (4.9). Moreover, $\|z\|_{L^\infty} = 1$ and $\operatorname{div} z \in L^2(C)$. As already mentioned, for the subsequent vector fields $z$ on $\Omega$ we always use $z$ satisfying (4.10) on $C$ and we merely vary $z$ on $\Omega \setminus C$.

**Special solution $z$ on $\Omega \setminus C$.** Now we want to demonstrate how solutions $z$ of (4.4) can be constructed. By symmetry we can restrict our attention to the set

$$\tilde{\Omega} := \left\{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \in (0, R), \left|x - \overline{R}\right| > R\right\}$$

with $\overline{R} := (R, R) \in \mathbb{R}^2$ and $R = 1/\lambda = 1/h(\Omega) = 1/E(u)$, cf. Figure 2. Notice that $\tilde{\Omega}$ is just one "corner" of the set $\Omega \setminus C$.

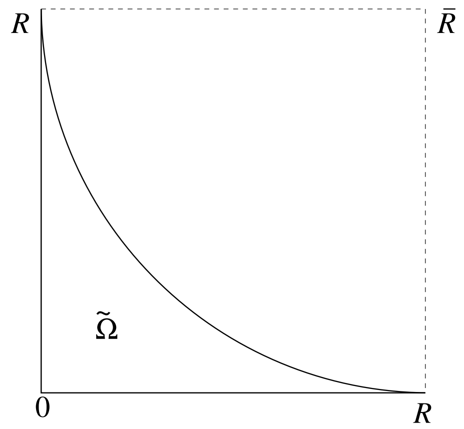


Figure 2: The set $\tilde{\Omega}$.

For the construction of $z$ we exploit a result relating the normal field of a family of plane curves to their curvature. More precisely, let $D \subset \mathbb{R}^2$ be an open set that is covered by a foliation of $C^1$-curves that do not intersect or touch each other and such that the associated field $\nu(x)$ of unit normals is of class $C^1$. Then

$$\operatorname{div} \nu(x) = \pm\kappa(x) \quad \text{on} \quad D \tag{4.11}$$

where $\kappa(x)$ denotes the (nonnegative) curvature of the curve through point $x$ at $x$ and we have to take the positive sign if the normal $\nu(x)$ points away from the center of the corresponding osculating circle and the negative one otherwise.

Thus, in order to find solutions $z$ of (4.4), we can look for suitable foliations of $\tilde{\Omega}$ and take the normal field $\nu$ as $z$. For a simple example of a foliation of $\tilde{\Omega}$ we shift the circular arc forming the curved part of $\partial\tilde{\Omega}$ diagonally outward, cf. Figure 3. This way each point $x \in \tilde{\Omega}$ lies on a circular arc of radius $R = 1/\lambda$. Hence the unit normal field $\nu$ to this foliation pointing away from $C$ satisfies $\operatorname{div} \nu = \lambda$. Therefore

$$z(x) := -\nu(x) \quad \text{on} \quad \Omega \setminus C$$

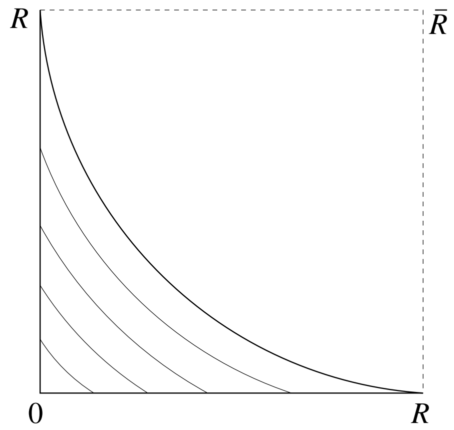


Figure 3: Foliation of $\tilde{\Omega}$ by arcs with radius $R$.

satisfies (4.4) for $s \equiv 1$ and, clearly, also boundary condition (4.9) and $\|z\|_{L^\infty} = 1$, cf. [16]. Summarizing,

$$z(x) := \begin{cases} -\frac{Dw(x)}{\sqrt{1+|Dw(x)|^2}} & \text{if } x \in C\,, \\ -\nu(x) & \text{if } x \in \Omega \setminus C \end{cases}$$

satisfies the necessary conditions (4.2), (4.3), and (4.4) for $s \equiv 1$.

Notice that such a construction always gives a vector field $z$ of unit vectors. In our subsequent constructions we also modify the length of the normals $\nu$ in order to construct more general solutions $z$. More precisely, for a given foliation with normal field $\nu$ we will use the ansatz

$$z(x) = -\nu(x)a(x) \tag{4.12}$$

in order to construct a solution $z$ of (4.4) for some given $s$, i.e., we construct a suitable function $a : \tilde{\Omega} \to [0, 1]$ such that $z$ is a solution.

# 5 Proof of Theorem 4.5

In a first step we construct infinitely many foliations of $\tilde{\Omega}$. Then we fix some $s$ with (4.1) and construct a function $a$ such that $z$ according to (4.12) gives a solution.

## 5.1 Construction of infinitely many foliations

We now provide a general construction for a foliation of $\tilde{\Omega}$ with circular arcs having their center on the bisector connecting the origin with the point $\bar{R} = (R, R) \in \mathbb{R}^2$. In fact we consider foliations that even cover the closure of $\tilde{\Omega}$ and that always contain the curved part of $\partial\tilde{\Omega}$

$$\Gamma := \left\{ x = (x_1, x_2) \in \tilde{\Omega} \mid x_1, x_2 \in [0, R], \left|x - \overline{R}\right| = R \right\}$$

as an element of the foliation. A simple computation shows that the intersection $\bar{\gamma} = (\gamma, \gamma)$ of $\Gamma$ with the straight segment connecting the origin with $\bar{R}$ is obtained for

$$\gamma = \left(1 - \frac{1}{\sqrt{2}}\right) R\,,$$

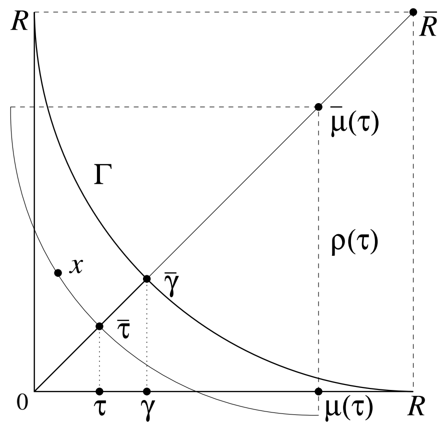


Figure 4: Construction of a foliation in $\tilde{\Omega}$.

cf. Figure 4. We consider foliations of the described type where each point $\bar{\tau} = (\tau, \tau)$ on the straight segment connecting the origin with $\bar{\gamma}$ uniquely identifies an arc of the foliation and where each circular arc touches the straight boundary of $\tilde{\Omega}$ with its ends. Hence it is reasonable to parametrize the curves of the foliation by means of $\tau \in [0, \gamma]$ such that the arc associated with $\tau$ contains the point $\bar{\tau} = (\tau, \tau)$. Consequently, a foliation is uniquely described by a function $\rho : [0, \gamma] \to \mathbb{R}^+$ giving the radius of the corresponding circular arc. An important question now is to determine functions $\rho : [0, \gamma] \to \mathbb{R}^+$ that really give a foliation of $\tilde{\Omega}$. Moreover we have to compute from $\rho$ the relevant quantities needed in our subsequent analysis.

Let us discuss how we can recognize whether a family of curves given by means of some function $\rho$ is a foliation of $\tilde{\Omega}$. We readily verify that the center of the circular arc containing $(\tau, \tau)$ and having radius $\rho(\tau)$ is given by $\bar{\mu}(\tau) = (\mu(\tau), \mu(\tau))$ with

$$\mu(\tau) = \tau + \frac{\rho(\tau)}{\sqrt{2}} .$$

Thus we can identify the points on the circular arc corresponding to $\tau$ by assigning some $\sigma$ such that

$$x_1(\tau, \sigma) = \mu(\tau) + \rho(\tau) \cos \sigma \; = \; \tau + \frac{\rho(\tau)}{\sqrt{2}} + \rho(\tau) \cos \sigma \tag{5.13}$$

$$x_2(\tau, \sigma) = \mu(\tau) + \rho(\tau) \sin \sigma \; = \; \tau + \frac{\rho(\tau)}{\sqrt{2}} + \rho(\tau) \sin \sigma \tag{5.14}$$

where $\sigma$ belongs to a suitable subinterval $I_\tau \subset [\pi, 3\pi/2]$ dependent on $\tau$. If we can find $\varrho$ such that (5.13), (5.14) defines a smooth change of coordinates between $(\tau, \sigma)$ and $(x_1, x_2)$ on $\tilde{\Omega}$, then $\varrho$ obviously provides a foliation of $\tilde{\Omega}$. In that case we denote the inverse transformation by $\tau = \tau(x)$, $\sigma = \sigma(x)$ and we introduce the functions

$$r(x) := \rho(\tau(x)) , \quad m(x) := \mu(\tau(x)) \tag{5.15}$$

for our further analysis. The unit normal field corresponding to the foliation and pointing away from $\Gamma$ is obviously given by

$$\nu(x) = \begin{pmatrix} \cos \sigma(x) \\ \sin \sigma(x) \end{pmatrix} . \tag{5.16}$$

In order to construct different foliations we take $R_0 \in [0, R]$ as parameter and look for

$$\rho : [0, \gamma] \to [R_0, R] \quad \text{with} \quad \rho(0) = R_0\,, \;\; \rho(\gamma) = R\,.$$

The next lemma provides sufficient conditions ensuring that $\varrho$ defines a foliation of $\tilde{\Omega}$.

**Lemma 5.17** *Let $\rho : [0, \gamma] \to [R_0, R]$ with $\rho \in C^1([0, \gamma])$ satisfy the following conditions:*

(i) $\rho(0) = R_0\,,\ \rho(\gamma) = R\,,$

(ii) $\rho(\tau) \geq \big(\sqrt{2} + 2\big)\,\tau$ *for all* $\tau \in [0, \gamma]\,,$

(iii) $0 \leq \rho'(\tau) \leq \sqrt{2} + 2$ *for all* $\tau \in [0, \gamma]$ .

*Then* (5.13),(5.14) *defines a change of variables of class $C^1$ on $\tilde{\Omega}$ and, thus, $\rho$ determines a foliation of $\tilde{\Omega}$ such that, with the corresponding unit normal field $\nu$ according to* (5.16)*, relation* (4.11) *is applicable.*

Examples given in Section 6 show that for any $R_0 \in [0, R]$ there is at least one foliation satisfying conditions (i)-(iii) of Lemma 5.17.

Proof. According to our previous discussion, $\rho$ defines a family of circular arcs parametrized by $\tau$ on $\tilde{\Omega}$. We readily obtain that condition (ii) is equivalent with

$$\rho(\tau) \geq \mu(\tau) \quad \text{on } [0, \gamma]\,,$$

i.e., each arc touches the straight part of $\partial\tilde{\Omega}$ with its ends. Using representation (5.13),(5.14) we can explicitly determine the intervals $I_\tau = [\sigma_1(\tau), \sigma_2(\tau)] \subset [\pi, \frac{3}{2}\pi]$ by solving the equations

$$\tau + \frac{\rho(\tau)}{\sqrt{2}} + \rho(\tau)\cos\sigma_1(\tau) = 0\,,$$

$$\tau + \frac{\rho(\tau)}{\sqrt{2}} + \rho(\tau)\sin\sigma_2(\tau) = 0\,,$$

and we obtain

$$\sigma_1(\tau) = 2\pi - \arccos\left(-\frac{\tau}{\rho(\tau)} - \frac{1}{\sqrt{2}}\right)\,,$$
$$\sigma_2(\tau) = \pi - \arcsin\left(-\frac{\tau}{\rho(\tau)} - \frac{1}{\sqrt{2}}\right)\,.$$

Thus (5.13), (5.14) is defined on

$$M := \{(\sigma, \tau)|\ \tau \in (0, \gamma),\ \ \sigma \in (\sigma_1(\tau), \sigma_2(\tau))\}\,.$$

Let us now analyze transformation (5.13), (5.14) by means of the inverse function theorem. We calculate

$$\det\begin{pmatrix} \frac{\partial x_1}{\partial \tau}(\tau,\sigma) & \frac{\partial x_1}{\partial \sigma}(\tau,\sigma) \\ \frac{\partial x_2}{\partial \tau}(\tau,\sigma) & \frac{\partial x_2}{\partial \sigma}(\tau,\sigma)\end{pmatrix} = \left(1 + \frac{1}{\sqrt{2}}\rho'(\tau) + \rho'(\tau)\cos\sigma\right)\rho(\tau)\cos\sigma$$
$$+ \left(1 + \frac{1}{\sqrt{2}}\rho'(\tau) + \rho'(\tau)\sin\sigma\right)\rho(\tau)\sin\sigma$$

$$= \rho(\tau)\left(\left(1+\frac{1}{\sqrt{2}}\rho'(\tau)\right)(\cos\sigma+\sin\sigma)+\rho'(\tau)\right). \tag{5.18}$$

Obviously condition (iii) implies that

$$\frac{\rho'(\tau)}{\sqrt{2}+\rho'(\tau)} \le \frac{1}{\sqrt{2}} \quad \text{on } [0,\gamma]\,.$$

Since

$$-\frac{\cos+\sin}{\sqrt{2}}\left(\left(\pi,\frac{3}{2}\pi\right)\right)=\left(\frac{1}{\sqrt{2}},1\right),$$

we obtain

$$\frac{\rho'(\tau)}{\sqrt{2}+\rho'(\tau)} < -\frac{\cos\sigma+\sin\sigma}{\sqrt{2}}$$

and, thus,

$$\left(1+\frac{1}{\sqrt{2}}\rho'(\tau)\right)(\cos\sigma+\sin\sigma)+\rho'(\tau)<0 \quad \text{on } M\,.$$

We have $\varrho(\tau)>0$ on $(0,\gamma)$ by (ii) and, therefore, the expression in (5.18) is negative on $M$. Hence transformation (5.13), (5.14) is locally a $C^1$-diffeomorphism.

Let us denote by

$$\varphi_1(\tau) := x_1(\tau,\sigma_2(\tau)) = \tau + \frac{\rho(\tau)}{\sqrt{2}} + \rho(\tau)\cos\sigma_2(\tau)$$

the $x_1$-coordinates of the end points of the arcs on the $x_1$-axis. Hence

$$\begin{aligned}
\varphi_1' &= 1+\rho'\Big(\frac{1}{\sqrt{2}}+\cos\sigma_2\Big)-\rho\sigma_2'\sin\sigma_2\\
&= 1+\rho'\Big(\frac{1}{\sqrt{2}}+\cos\sigma_2\Big)+\frac{\frac{\tau}{\rho}+\frac{1}{\sqrt{2}}}{\sqrt{1-(\frac{\tau}{\rho}+\frac{1}{\sqrt{2}})^2}}\\
&\ge 1+\rho'\Big(\frac{1}{\sqrt{2}}-1\Big) \overset{\text{(iii)}}{\ge} 0\,.
\end{aligned} \tag{5.19}$$

Using the symmetry of $\tilde{\Omega}$ and of the circular arcs with respect to the bisector, this implies that the circular arcs do not intersect or touch each other. Therefore the transformation (5.13), (5.14) is bijective.

We still have to show that the image of $M$ under transformation (5.13), (5.14) coincides with $\tilde{\Omega}$. Obviously, the image of $M$ is a subset of $\tilde{\Omega}$ and let us assume that there is some point of $\tilde{\Omega}$ that does not belong to the image. Then the image of $M$ must also have a boundary point $(\hat{x}_1,\hat{x}_2)\in\tilde{\Omega}$. Clearly, $(\hat{x}_1,\hat{x}_2)$ has to be contained in the image of the closure of $M$ and, thus, it has to be in the image of the boundary $\partial M$. But this is a contradiction, since $\partial M$ is mapped on $\partial\tilde{\Omega}$. $\diamondsuit$

## 5.2 Vector fields $z$ on $\Omega\setminus C$

We now intend to construct continuous vector fields $z$ on $\tilde{\Omega}$ satisfying (4.4) and (4.7) by using the ansatz (4.12). For that we fix a foliation of $\tilde{\Omega}$ as constructed in the previous section that corresponds to some $\rho$ satisfying the assumptions of Lemma 5.17. By $\nu(\cdot)$ we denote the unit normal field associated to the foliation and pointing away from $\Gamma$. Thus we are looking for vector fields $z$ of the form

$$z(x) = -\nu(x)a(x) \tag{5.20}$$

with a scalar function $a : \tilde{\Omega} \to \mathbb{R}$. Since

$$\operatorname{div} \nu(x) = \frac{1}{r(x)} \in (1/R, \infty)$$

by (4.11), (5.15), we get for a differentiable $a$

$$\begin{aligned} \operatorname{div}\, (\nu(x)a(x)) &= \nu(x) \cdot Da(x) + a(x) \operatorname{div} \nu(x) \\ &= \nu(x) \cdot Da(x) + \frac{a(x)}{r(x)}\,. \end{aligned}$$

Moreover we have $\lambda = 1/R$. Thus (4.4) is equivalent to

$$\nu(x) \cdot Da(x) + \frac{a(x)}{r(x)} = \frac{s(x)}{R} \quad \text{on } \tilde{\Omega} \tag{5.21}$$

and (4.7) provides the boundary condition

$$a(x) = 1 \quad \text{on } \Gamma\,. \tag{5.22}$$

This is a linear inhomogeneous partial differential equation of first order for $a$ which can be solved by the method of characteristics as long as the right hand side of (5.21) is continuous (cf. [9, p. 97]). In our case the right hand side is merely measurable. We will show explicitly that the method of characteristics can be applied in this case as well in order to verify the next lemma.

**Lemma 5.23** *Let $s : \tilde{\Omega} \to [-1, 1]$ be a measurable function satisfying (4.1) and let $\nu$ be the unit normal field according to (5.16) that is associated with a foliation of $\tilde{\Omega}$ satisfying the assumptions of Lemma 5.17 and pointing away from $\Gamma$. Then there is a continuous function $a : \operatorname{cl} \tilde{\Omega} \to [0, 1]$ such that*

$$a(x) = 1 \quad \textit{on } \Gamma$$

*and*

$$\operatorname{Div} a\nu = \lambda s \quad \textit{a.e. on } \tilde{\Omega}$$

*in the sense of distributions.*

At the end of this section we will use the vector field $z := -a\nu$ on $\tilde{\Omega}$ in order to verify Theorem 4.5.

Proof. We construct the function $a$ by explicitly carrying out the method of characteristics for the linear partial differential equation (5.21) with boundary condition (5.22).

*Characteristics.* Let us start with the construction of the characteristics. We parametrize the boundary curve $\Gamma$ by

$$\begin{aligned} x_1^0(\xi) &= R + R\cos\xi\,, \\ x_2^0(\xi) &= R + R\sin\xi \end{aligned}$$

with $\xi \in \left(\pi, \frac{3}{2}\pi\right)$. Thus the characteristics, parametrized by $t$, must satisfy

$$\dot{x}_1(t,\xi) = \nu_1(x(t,\xi))\,, \quad x_1(0,\xi) = x_1^0(\xi) = R + R\cos\xi \tag{5.24}$$

$$\dot{x}_2(t,\xi) = \nu_2(x(t,\xi))\,, \quad x_2(0,\xi) = x_2^0(\xi) = R + R\sin\xi\,, \tag{5.25}$$

where $\dot{x}$ denotes the derivative of $x$ with respect to $t$. Using (5.15), we have

$$\nu(x) = \frac{x - \overline{m}(x)}{|x - \overline{m}(x)|} = \frac{x - \overline{m}(x)}{r(x)} \quad \text{on } \tilde{\Omega}\,.$$

Therefore (5.24), (5.25) can be written as

$$\begin{aligned}
\dot{x}_1(t,\xi) &= \frac{x_1(t,\xi) - m(x(t,\xi))}{r(x(t,\xi))}\,, \quad x_1(0,\xi) = R + R\cos\xi\,,\\
\dot{x}_2(t,\xi) &= \frac{x_2(t,\xi) - m(x(t,\xi))}{r(x(t,\xi))}\,, \quad x_2(0,\xi) = R + R\sin\xi\,.
\end{aligned}$$

The vector valued function

$$f(x) := \frac{x - \overline{m}(x)}{r(x)}$$

is locally Lipschitz continuous on $\mathbb{R} \times \tilde{\Omega}$, since $r$, $m$ are of class $C^1$ and $r(x) > 0$ on $\tilde{\Omega}$, cf. Lemma 5.17 (ii). Consequently, for any $\xi \in (\pi, \frac{3}{2}\pi)$ the initial value problem has a unique local solution which can be extended up to the boundary of $\mathbb{R} \times \tilde{\Omega}$.

Let $x(\cdot,\xi)$ exist on the interval $[0, t(\xi))$ for some $t(\xi) \in (0,\infty]$. We want to show that $t(\xi) < \infty$ and $x(t(\xi),\xi) \in \partial\tilde{\Omega} \setminus \Gamma$. For that we consider the orthogonal linear projection $P : \mathbb{R}^2 \to B$ on the linear subspace

$$B := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 = x_2\}$$

generated by the vector $(1,1)$. Obviously,

$$Px = \frac{(x_1 + x_2)}{2}\begin{pmatrix}1\\1\end{pmatrix} \quad \text{for all } x = (x_1, x_2)\,.$$

We apply $P$ on both sides of the differential equations in (5.24), (5.25) and get

$$P\dot{x}(t,\xi) = P\nu(x(t,\xi))\,.$$

Since $P$ is linear, this can be written as

$$\frac{d}{dt}\left(Px(t,\xi)\right) = P\nu(x(t,\xi))\,.$$

By $|\nu(x(t,\xi))| = 1$ and $\nu_1(x), \nu_2(x) \le 0$ on $\tilde{\Omega}$,

$$\begin{aligned}
\left|\frac{d}{dt}(Px(t,\xi))\right| &= |P\nu(x(t,\xi))|\\
&= \frac{1}{\sqrt{2}}\,|\nu_1(x(t,\xi)) + \nu_2(x(t,\xi))|\\
&= \frac{1}{\sqrt{2}}\,(|\nu_1(x(t,\xi))| + |\nu_2(x(t,\xi))|)
\end{aligned}$$

$$\geq \frac{1}{\sqrt{2}}\sqrt{\nu_1^2(x(t,\xi)) + \nu_2^2(x(t,\xi))}$$
$$= \frac{1}{\sqrt{2}}\,.$$

Using also $\dot{x}(t,\xi) = \nu(x(t,\xi))$, we can conclude that, for each $\xi$, the projection of the characteristic "moves" at least with speed $1/\sqrt{2} > 0$ towards the origin. Therefore all characteristics have to leave $\tilde{\Omega}$ through $\partial\tilde{\Omega} \setminus \Gamma$, cf. Figure 5. This way we in particular obtain that $t(\xi) < \infty$.

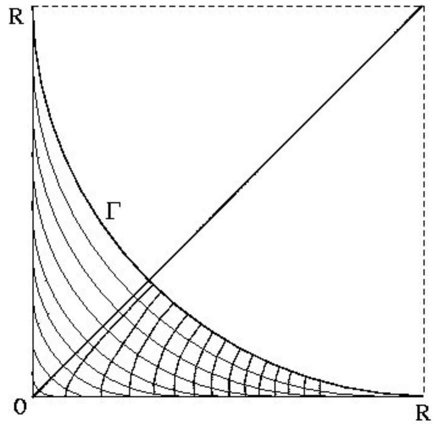


Figure 5: Characteristics in $\tilde{\Omega}$.

Since $\nu \in C^1(\tilde{\Omega})$ by (5.16) and Lemma 5.17, solution $x$ is of class $C^2$ as function of $t$ and the initial value $x^0 = x^0(\xi)$, cf. [19]. By $x^0 \in C^\infty(\mathbb{R})$ we conclude that $(t,\xi) \to x(t,\xi)$ is of class $C^2$ too. Below we show that

$$\frac{d}{dt}\,|\det Dx(t,\xi)| = \frac{|\det Dx(t,\xi)|}{r(x(t,\xi))}\,, \quad \det Dx(0,\xi) > 0 \tag{5.26}$$

which implies $\det Dx(t,\xi) > 0$. Since the characteristics cannot cross each other, they have to cover the whole set $\tilde{\Omega}$. Consequently, $(t,\xi) \to x(t,\xi)$ has to be invertible on

$$\{(t,\xi)|\,\xi \in (\pi, \tfrac{3}{2}\pi),\ t \in [0, t(\xi))\}$$

and there are functions $t(\cdot)$, $\xi(\cdot)$ on $\tilde{\Omega}$ of class $C^1$ such that $x = x(t(x), \xi(x))$.

For the verification of (5.26) we introduce the notation

$$x_t^i := \dot{x}_i\,, \quad x_\xi^i := \frac{d}{d\xi}x_i\,, \quad \nu_{x_j}^i := \frac{\partial}{\partial x_j}\nu_i\,, \quad i,j = 1,2\,.$$

Hence

$$\det Dx(t,\xi) = \det\begin{pmatrix} x_t^1(t,\xi) & x_\xi^1(t,\xi) \\ x_t^2(t,\xi) & x_\xi^2(t,\xi)\end{pmatrix} = x_t^1(t,\xi)x_\xi^2(t,\xi) - x_t^2(t,\xi)x_\xi^1(t,\xi)\,.$$

By $\nu \in C^1$ and $x \in C^2$ we derive from (5.24), (5.25)

$$x_{tt}(t,\xi) = \frac{d}{dt}\nu(x(t,\xi)) = D\nu(x(t,\xi)) \cdot x_t(t,\xi)\,,$$

$$x_{t\xi}(t,\xi) = \frac{d}{d\xi}\nu(x(t,\xi)) = D\nu(x(t,\xi)) \cdot x_\xi(t,\xi)\,,$$

and calculate

$$\begin{aligned}
\frac{d}{dt}\det Dx(t,\xi) &= \frac{d}{dt}\left(x_t^1(t,\xi)x_\xi^2(t,\xi) - x_t^2(t,\xi)x_\xi^1(t,\xi)\right)\\
&= x_{tt}^1(t,\xi)x_\xi^2(t,\xi) + x_t^1(t,\xi)x_{\xi t}^2(t,\xi) - x_{tt}^2(t,\xi)x_\xi^1(t,\xi) - x_t^2(t,\xi)x_{\xi t}^1(t,\xi)\\
&= x_\xi^2(t,\xi)\left(\nu_{x_1}^1(x(t,\xi))x_t^1(t,\xi) + \nu_{x_2}^1(x(t,\xi))x_t^2(t,\xi)\right)\\
&\quad + x_t^1(t,\xi)\left(\nu_{x_1}^2(x(t,\xi))x_\xi^1(t,\xi) + \nu_{x_2}^2(x(t,\xi))x_\xi^2(t,\xi)\right)\\
&\quad - x_\xi^1(t,\xi)\left(\nu_{x_1}^2(x(t,\xi))x_t^1(t,\xi) + \nu_{x_2}^2(x(t,\xi))x_t^2(t,\xi)\right)\\
&\quad - x_t^2(t,\xi)\left(\nu_{x_1}^1(x(t,\xi))x_\xi^1(t,\xi) + \nu_{x_2}^1(x(t,\xi))x_\xi^2(t,\xi)\right)\\
&= \nu_{x_1}^1(x(t,\xi))\left(x_t^1(t,\xi)x_\xi^2(t,\xi) - x_\xi^1(t,\xi)x_t^2(t,\xi)\right)\\
&\quad + \nu_{x_2}^2(x(t,\xi))\left(x_t^1(t,\xi)x_\xi^2(t,\xi) - x_\xi^1(t,\xi)x_t^2(t,\xi)\right)\\
&= \left(\nu_{x_1}^1(x(t,\xi)) + \nu_{x_2}^2(x(t,\xi))\right)\left(x_t^1(t,\xi)x_\xi^2(t,\xi) - x_\xi^1(t,\xi)x_t^2(t,\xi)\right)\\
&= \operatorname{div}\nu(x(t,\xi))\det Dx(t,\xi)\\
&= \frac{\det Dx(t,\xi)}{r(x(t,\xi))}\,. \qquad (5.27)
\end{aligned}$$

Notice that this gives the equation in (5.26) without modulus.

For $t = 0$ we have $x_\xi(0,\xi) = x_\xi^0(\xi)$, i.e. $x_\xi(0,\xi)$ is tangent to the boundary curve $\Gamma$. Thus $\nu(x(0,\xi)) = x_t(0,\xi)$ and $x_\xi^\perp(0,\xi) := (x_\xi^2(0,\xi), -x_\xi^1(0,\xi))$ point into the same direction and

$$\begin{aligned}
\det Dx(0,\xi) &= x_t^1(0,\xi)x_\xi^2(0,\xi) - x_t^2(0,\xi)x_\xi^1(0,\xi)\\
&= x_t(0,\xi)\cdot x_\xi^\perp(0,\xi)\\
&= |x_\xi^\perp(0,\xi)| > 0\,.
\end{aligned}$$

Using (5.27) we see that $t \to \det Dx(t,\xi)$ has to be increasing for all $t \geq 0$. But this verifies (5.26). Since $x(t,\xi)$ is invertible and $t(x), \xi(x) \in C^1$ are bijective, we have a change of coordinates of class $C^1$.

*Differential equation.* In order to determine the unknown function $a$ on the characteristics we have to compute $\alpha(t,\xi) := a(x(t,\xi))$ by solving the initial value problem

$$\dot\alpha(t,\xi) = \frac{s(x(t,\xi))}{R} - \frac{\alpha(t,\xi)}{r(x(t,\xi))}\,, \quad \alpha(0,\xi) = 1 \qquad (5.28)$$

for all $\xi$. Since $s$ is merely measurable, we cannot solve (5.28) in the classical sense. Instead, we solve it in the sense of Carathéodory and apply some corresponding extension of the existence and uniqueness Theorem of Picard-Lindelöf, cf. [19]

We consider the right hand side

$$f(t,\alpha) := \frac{s(x(t,\xi))}{R} - \frac{\alpha}{r(x(t,\xi))}$$

of (5.28) on a set $J \times \mathbb{R}$ where $J \subset [0, t(\xi))$ is a nonempty closed interval. Clearly, $f$ is continuous in $\alpha$ for any fixed $t \in J$ and, since $1/r(x(\cdot,\xi))$ is bounded on the closed interval $J$, the function $f$ is measurable and integrable in $t$ on $J$ for any fixed $\alpha \in \mathbb{R}$. Since

$$|f(t,\alpha) - f(t,\tilde\alpha)| = \left|\frac{\alpha}{r(x(t,\xi))} - \frac{\tilde\alpha}{r(x(t,\xi))}\right|$$

$$= \frac{1}{r(x(t,\xi))}|\alpha - \tilde{\alpha}| \qquad \text{for all } \; t \in J, \;\; \alpha, \tilde{\alpha} \in \mathbb{R}\,,$$

$f$ satisfies a Lipschitz condition in $\alpha$ with a Lipschitz constant

$$l(t) := \frac{1}{r(x(t,\xi))}$$

that is measurable and integrable on $J$. Therefore equation (5.28) has a unique local solution in the sense of Carathéodory for any $\xi \in (\pi, \frac{3}{2}\pi)$.

Let us prove that $|\alpha(\cdot,\xi)| \le 1$ on any closed $J \subset [0, t(\xi))$ in order to justify that $\alpha(\cdot,\xi)$ can be extended on the whole interval $[0, t(\xi))$. For any fixed $J$ we assume that there is some $t_+ \in J$ with $\alpha(t_+, \xi) > 1$. Since $\alpha$ is continuous in $t$, there exists a largest $t_1 \in [0, t_+)$, such that $\alpha(t_1) = 1$. By

$$\alpha(t_+,\xi) = \alpha(t_1,\xi) + \int_{t_1}^{t_+} \frac{s(x(\tau,\xi))}{R} - \frac{\alpha(\tau,\xi)}{r(x(\tau,\xi))}\, d\tau$$

and

$$\frac{s(x(\tau,\xi))}{R} - \frac{\alpha(\tau,\xi)}{r(x(\tau,\xi))} < \frac{1}{R} - \frac{1}{r(x(\tau,\xi))} \le 0 \quad \text{for all } \; \tau \in (t_1, t_+)$$

we obtain the contradiction $\alpha(t_+,\xi) \le 1$. Hence $\alpha(\cdot,\xi) \le 1$ and, analogously, $\alpha(\cdot,\xi) \ge -1$ on any $J$. Therefore we have solutions $\alpha(\cdot,\xi)$ of (5.28) on $[0, t(\xi))$.

Defining $a(x) := \alpha(t(x), \xi(x))$, we clearly have $a(x) = 1$ on $\Gamma$ and it remains to show that

$$\operatorname{Div}\,(a(x)\nu(x)) = \frac{s(x)}{R} \quad \text{on } \; \tilde{\Omega}$$

in the sense of distributions, i.e.,

$$-\int_{\tilde{\Omega}} a(x)\nu(x) \cdot D\varphi(x)\, dx = \int_{\tilde{\Omega}} \frac{s(x)}{R}\varphi(x)\, dx$$

for all $\varphi \in C_c^1(\tilde{\Omega})$. Notice that $\varphi(x(\cdot,\cdot))$ has compact support on

$$M := \left\{(t,\xi) \in \mathbb{R}^2 \;\middle|\; \xi \in \left(\pi, \tfrac{3}{2}\pi\right),\, t \in [0, t(\xi))\right\}$$

for $\varphi \in C_c^1(\tilde{\Omega})$. Thus we obtain for any $\varphi \in C_c^1(\tilde{\Omega})$

$$\begin{aligned}
-&\int_{\tilde{\Omega}} a(x)\nu(x) \cdot D\varphi(x)\, dx \\
&= -\int_M a(x(t,\xi))\nu(x(t,\xi)) \cdot D\varphi(x(t,\xi))\, |\det Dx(t,\xi)| \; d(t,\xi) \\
&= -\int_M \alpha(t,\xi)\, \dot{x}(t,\xi) \cdot D\varphi(x(t,\xi))\, |\det Dx(t,\xi)| \; d(t,\xi) \\
&= -\int_\pi^{\frac{3}{2}\pi} \int_0^{t(\xi)} \alpha(t,\xi)\, |\det Dx(t,\xi)| \; \frac{d}{dt}\varphi(x(t,\xi))\, dt\, d\xi \\
&= \int_\pi^{\frac{3}{2}\pi} \int_0^{t(\xi)} \frac{d}{dt}\Big(\alpha(t,\xi)\, |\det Dx(t,\xi)|\Big)\, \varphi(x(t,\xi))\, dt\, d\xi \\
&= \int_M \bigg(\dot{\alpha}(t,\xi)\, |\det Dx(t,\xi)| + \alpha(t,\xi)\frac{d}{dt}\, |\det Dx(t,\xi)|\bigg)\, \varphi(x(t,\xi))\, d(t,\xi)
\end{aligned}$$

$$
\begin{aligned}
&= \int_M \left(\dot\alpha(t,\xi) + \frac{\alpha(t,\xi)}{|\det Dx(t,\xi)|}\,\frac{d}{dt}\,|\det Dx(t,\xi)|\right)\varphi(x(t,\xi))\,|\det Dx(t,\xi)|\;d(t,\xi)\\
&\overset{(5.26)}{=} \int_M \left(\dot\alpha(t,\xi) + \frac{\alpha(t,\xi)}{r(x(t,\xi))}\right)\varphi(x(t,\xi))\,|\det Dx(t,\xi)|\;d(t,\xi)\\
&\overset{(5.28)}{=} \int_M \frac{s(x(t,\xi))}{R}\,\varphi(x(t,\xi))\,|\det Dx(t,\xi)|\;d(t,\xi)\\
&= \int_{\tilde\Omega} \frac{s(x)}{R}\,\varphi(x)\,dx
\end{aligned}
$$

which verifies Lemma 5.23. ♢

Proof of Theorem 4.5. Let $s$ be fixed as in the theorem. Then we choose a foliation of $\tilde\Omega$ satisfying the assumptions of Lemma 5.17 with the associated unit normal field $\nu$ pointing away from $\Gamma$. Let $a$ be the corresponding function according to Lemma 5.23. Then we extend $\nu$, $a$ on all of $\Omega\setminus C$ by symmetry. Using (4.10), we see that

$$
z(x) := \begin{cases} -\frac{Dw(x)}{\sqrt{1+|Dw(x)|^2}} & \text{if } x\in C\,,\\ -a(x)\nu(x) & \text{if } x\in\Omega\setminus C \end{cases} \tag{5.29}
$$

satisfies equations (4.3), (4.4). Clearly $\|z\|_\infty \le 1$ and the boundary condition in (4.7) is satisfied by (4.9) and $a = 1$ on $\Gamma$. Hence it remains to show $\operatorname{Div} z \in L^2(\Omega)$ in order to verify (4.7) which is equivalent to (4.2). We know that $\operatorname{Div} z \in L^2(C)$ and $\operatorname{Div} z \in L^2(\tilde\Omega)$ for the corresponding restrictions of $z$. For any $v \in C_0^\infty(\Omega) \subset BV(\Omega)$ and the outer unit normal $\nu_C$ on $\partial C$ we can apply (3.2) on $C$ and $\Omega\setminus C$ to obtain

$$
\begin{aligned}
0 &= \int_{\partial C\cap\Omega} vz\cdot\nu_C\,d\mathcal{H}^1 + \int_{\partial C\cap\Omega} vz\cdot(-\nu_C)\,d\mathcal{H}^1\\
&= \int_{\partial C} vz\cdot\nu_C\,d\mathcal{H}^1 + \int_{\partial(\Omega\setminus C)} vz\cdot\nu_{\Omega\setminus C}\,d\mathcal{H}^1\\
&= \int_C z\cdot Dv\,dx + \int_C v\operatorname{Div} z\,dx + \int_{\Omega\setminus C} z\cdot Dv\,dx + \int_{\Omega\setminus C} v\operatorname{Div} z\,dx\\
&= \int_\Omega z\cdot Dv\,dx + \int_\Omega v\operatorname{Div} z\,dx\,.
\end{aligned}
$$

This implies that $\operatorname{Div} z$ on $\Omega$ in the sense of distributions is just the composition of $\operatorname{Div} z$ on $C$ and on $\Omega\setminus C$. Hence $\operatorname{Div} z\in L^2(\Omega)$ and $z$ satisfies (4.2) - (4.4).

The previous construction works for any foliation of the considered type. As seen in the previous section we can find infinitely many foliations such that the corresponding fields $\nu$ of unit normals differ pairwise. By (5.29) we then obtain vector fields $z$ that differ pairwise on $\Omega\setminus C$ and, in particular, on $\partial\Omega\setminus\partial C$.

♢

# 6 Special example

Here we show that a function with linear growth satisfies conditions (i)-(iii) of Lemma 5.17. For fixed $R_0\in[0,R]$ we consider

$$
\rho(\tau) := \left(\sqrt{2}+2\right)\left(1-\frac{R_0}{R}\right)\tau + R_0\,. \tag{6.1}
$$

Since

$$\rho(0) = R_0 \,, \qquad \rho(\gamma) = R$$

for $\gamma = 1 - 1/\sqrt{2}$, function $\rho$ satisfies (i). Condition

$$\rho(\tau) \geq \left(\sqrt{2} + 2\right) \tau$$

is obviously equivalent to $\tau \leq \gamma$ and, thus, $\rho$ satisfies (ii). From

$$\rho'(\tau) = \left(\sqrt{2} + 2\right) (1 - R_0/R)$$

we conclude that $0 \leq \rho'(\tau) \leq \sqrt{2} + 2$ and, thus, (iii) is satisfied. Consequently, for any $R_0 \in [0, R]$ we obtain at least one function $\rho$ satisfying the assumptions of Lemma 5.17 by (6.1).

Let us analyze the particular foliation of $\tilde{\Omega}$ corresponding to $\rho$ in (6.1) with $R_0 = 0$ in some more detail. It turns out that all circular arcs are tangential to both the $x_1$-axis and the $x_2$-axis at the touching points, cf. Figure 6. The center of each circular arc is the point $\overline{r} := (r, r)$ where

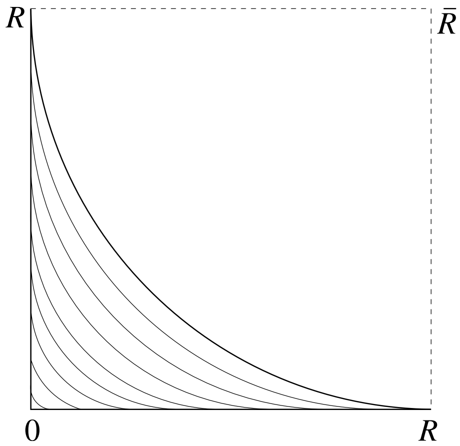


Figure 6: Foliation of $\tilde{\Omega}$ by arcs with decreasing radii.

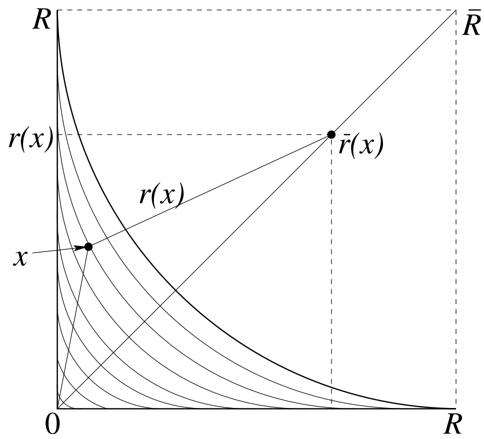


Figure 7: Construction of $r(x)$ in $\tilde{\Omega}$.

$r$ is the radius of the arc, cf. Figure 7. Clearly, $r(x)$ has to satisfy

$$|x - \overline{r}(x)| = r(x) \quad \text{and} \quad |x| \leq r(x)$$

and a simple computation implies that

$$r(x) = x_1 + x_2 + \sqrt{2x_1x_2} \quad \text{on } \tilde{\Omega}\,.$$

To get some information about the vector field $z$, let us calculate $\alpha$ explicitly along the characteristic corresponding to $\sigma = \frac{5}{4}\pi$ for $s \equiv 1$. From (5.24), (5.25), and (5.16) we readily obtain for that characteristic

$$x_1(0) = R + R\cos\left(\frac{5}{4}\pi\right) = \gamma\,, \quad x_2(0) = R + R\sin\left(\frac{5}{4}\pi\right) = \gamma\,,$$

and

$$x_1(t) = x_2(t)\,, \qquad \dot{x}_1(t) = \dot{x}_2(t) = -\frac{1}{\sqrt{2}}\,.$$

Thus

$$x_1(t) = x_2(t) = -\frac{1}{\sqrt{2}}t + \gamma\,.$$

Since

$$r(x(t)) = -\left(\sqrt{2}+1\right)t + R\,,$$

we have to solve the initial value problem

$$\dot{\alpha}(t) = \frac{1}{R} - \frac{\alpha(t)}{-\left(\sqrt{2}+1\right)t + R}\,, \qquad \alpha(0) = 1$$

to get $\alpha$ along the special characteristic. As explicit solution of the linear problem we obtain

$$\alpha(t) = \left(1 + \frac{1}{\sqrt{2}}\right)\left(-\frac{1+\sqrt{2}}{R}t + 1\right)^{\sqrt{2}-1} - \frac{1}{\sqrt{2}}\left(-\frac{1+\sqrt{2}}{R}t + 1\right),$$

see Figure 8. Moreover we obtain that $x_1(t) = x_2(t) = 0$ for $t = \left(\sqrt{2}-1\right)R = \sqrt{2}\gamma$, i.e. $t\left(\frac{5}{4}\pi\right) =$

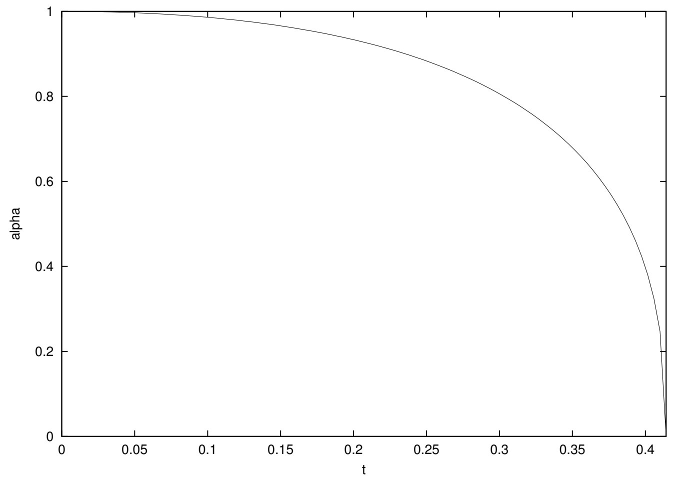


Figure 8: Graph of $\alpha$ for $R = 1$.

$\sqrt{2}\gamma$. Hence

$$\alpha(\sqrt{2}\gamma) = \alpha\left(\left(\sqrt{2}-1\right)R\right) = 0\,,$$

i.e., the length of the vectors $z(x)$ approaches zero if $x$ approaches the origin.

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