A link between local projection stabilizations and the continuous interior penalty method for convection-diffusion problems

June 13, 2007

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Abstract

We study stabilization methods for the discretization of convection-dominated elliptic convection-diffusion problems by linear finite elements. It turns out that there exist close relations between a new version of stabilization via local projection and the continuous interior penalty method.

AMS Subject Classifications: 65 N15, 65N30, 65N12
Keywords: Convection-diffusion, singular perturbation, finite element method, stabilization

1 Introduction

Stabilized finite element methods are formed by adding to the standard Galerkin method terms that are mesh-dependent, in many cases (but not necessary) consistent and numerically stabilizing. Starting with the streamline upwind Petrov-Galerkin (SUPG) or streamline diffusion finite element method (SDFEM) [1] today there exist many different stabilization techniques.
In the recent survey [2] the authors discuss SUPG and its variants GLS and USFEM, the variational multiscale method and bubble enriched methods,

but they do not mention subgrid modelling [3], local projection stabilization [4], the continuous interior penalty method [6] and discontinuous Galerkin [9].
It is well known that there exist close relations between SDFEM and variational multiscale methods, moreover, in [7] and [5] the authors verify relations between local projection stabilizations and subgrid modelling introduced by Guermond and the variational multiscale method. In our note we want to show that as well a close relation between global and local projection stabilization (LPS) as the continuous interior penalty method (CIP) exists.

2 A new variant of the local projection stabilization

Let us consider the convection-diffusion problem

$$\begin{aligned} L_\epsilon u : \quad = -\epsilon\Delta u + b\cdot\nabla u + cu &= f \text{ in } \Omega \subset R^2, \\ u &= 0 \text{ on } \partial\,\Omega. \end{aligned}$$

We assume Ω to be polygonal, $0 < \epsilon << 1$, $c - \frac{1}{2}\nabla\cdot b \geq \gamma > 0$ and b, c, f to be sufficiently smooth.
For simplicity we discretize the problem using the space V_h of linear finite elements with $V_h \subset H_0^1(\Omega)$. Introducing the Galerkin bilinear form

$$a_G(w,v) := \epsilon(\nabla w, \nabla v) + (b\cdot\nabla w + cw, v)$$

the local projection stabilization is characterized by

$$a_G(u_h, v_h) + S(u_h, v_h) = (f, v_h) \text{ for all } v_h \in V_h \tag{1}$$

and a special form of the stabilization term $S(\cdot,\cdot)$ while we shall explain in a minute. Remark that other stabilization techniques can also be written in this form, for instance

- SDFEM : $S(u_h, v_h) := \sum\limits_k \delta_k (Lu_h - f, b\cdot\nabla v_h)_k$
- continuous IP: $S(u_h, v_h) := h^2 \sum\limits_e \delta_e \int\limits_e [b\cdot\nabla u_h]_e\,[b\cdot\nabla v_h]_e\,ds$

Here $[\cdot]_e$ denotes the jump over the edge e.
The local projection stabilization in the general form introduced in [7] uses a second finite element space M_h (with possibly discontinuous elements) on a macro mesh with elements $M \in T_M$.
Based on a projection $\pi_h : L_2 \to M_h$ the stabilization term is defined by

$$S(u_h, v_h) := \sum_M \delta_M \left(b \cdot \nabla u_h - \pi_h(b \cdot \nabla u_h), b \cdot \nabla v_h - \pi_h(b \cdot \nabla v_h)\right)_M . \qquad (2)$$

In contrast to SDFEM or continuous IP, LPS is not consistent. But nevertheless its error analysis uses standard arguments, we shall sketch the basic ideas. Let us introduce the norm

$$\|w\|_E^2 := \epsilon |w|_1^2 + \|w\|_0^2 + S(w, w)$$

and some "interpolant" $u^I \in V_h$ from u. Then with $\xi = u^I - u_h, \eta = u - u^I$ we obtain

$$\begin{aligned} \|\xi\|_E^2 &\leq a_G(\xi, \xi) \\ &= a_G(u - u_h, \xi) + a_G(u^I - u, \xi) + S(\xi, \xi) \\ &= S(u_h, \xi) + a_G(u^I - u, \xi) + S(\xi, \xi) \\ &= a_G(\eta, \xi) + S(u^I, \xi). \end{aligned} \qquad (3)$$

Based on inequality (3) and additional properties of π_h and the interpolant u^I used the choice $\delta_M = O(h)$ then leads to the typical error estimate for every stabilization method based on linear elements on a quasi- uniform mesh (see [7])

$$\left\| u^I - u_h \right\|_E \leq c \left(\epsilon^{1/2} h + h^{3/2} + h^2\right) |u|_2 . \qquad (4)$$

To introduce our new variant of a local projection stabilization, we use the discrete scalar product

$$(w, v)_h := \sum_K \frac{1}{3} meas K \sum_{j=1}^{3} (wv)(P_{K_j})) \qquad (5)$$

Here the P_{K_j} are the three vertices of the element K.
Based on the scalar product (5) we define for a piecewise continuous function w its projection $\pi_h w \in V_h$ by

$$(\pi_h w, v_h) = (w, v_h)_h \text{ for all } v_h \in V_h. \qquad (6)$$

Let for a given knot x_i denote by Λ_i the index set characterizing all triangles adjacent to x_i and $w_{i,j}$ for $j \in \Lambda_i$ the value of $w|_{K_j}$ in the point x_i. Then, the orthogonality of the nodal basis functions φ_l of V_h with respect to the scalar product $(\cdot,\cdot)_h$ implies

$$(\pi_h w)(x_i) = \sum_{j\in\Lambda_i} \alpha_j w_{i,j} \quad \text{with } \alpha_j = \frac{meas K_j}{\sum\limits_{j\in\Lambda_i} meas K_j}\,. \tag{7}$$

Our new local projection stabilization method reads

$$\begin{array}{c} a_G(u_h, v_h) + S(u_h, v_h) = (f, v_h) \quad \text{with} \\ S(u_h, v_h) := \delta\,(b\cdot\nabla u_h - \pi_h(b\cdot\nabla u_h), b\cdot\nabla v_h - \pi_h(b\cdot v_h))_h\,. \end{array} \tag{8}$$

Remarks: (i) The new method improves the so called orthogonal subscale stabilization proposed by Codina [8] who uses the global L_2 projection onto V_h instead of our discrete version.

(ii) The method (8) is consistent if $b\cdot\nabla u \in C(\overline{\Omega})$ because due to (7) in a continuity point x_k of w it holds (x_k is a knot of the triangulation as well)

$$(\pi_h w)(x_k) = w(x_k).$$

Theorem 1 *Assume $u \in W_2^\infty(\Omega)$ and $b\cdot\nabla u \in C(\bar{\Omega})$. Then, for $\delta = \delta_0 h$ the error of the method (8) on a quasi-uniform mesh can be estimated by*

$$\|u - u_h\|_E \le C\left\{\epsilon^{1/2}h + h^{3/2} + h^2\right\}|u|_{2,\infty}\,. \tag{9}$$

Proof: We use the splitting

$$u - u_h = u^I - u_h + u - u^I$$

and choose for the interpolant u^I the L_2 projection of u onto the finite element space. The consistency of the method allows us to start instead of (3) from

$$\|\xi\|_E^2 \le a_G\left(u^I - u, \xi\right) + S\left(u^I - u, \xi\right). \tag{10}$$

First, using $u \in W_2^\infty(\Omega)$, we get

$$\begin{aligned} \left|S\left(u^I - u, \xi\right)\right| &\le \left(S\left(u^I - u, u^I - u\right)\right)^{1/2}(S(\xi,\xi))^{1/2} \\ &\le ch^{3/2}\,|u|_{2,\infty}\,\|\xi\|_E\,. \end{aligned}$$

The estimate of $a_G\left(u^I-u,\xi\right)$ is quite standard with exception of the convective term. Integrating by parts, one has to estimate $\left(u-u^I, b\cdot\nabla\xi\right)$. Let $\tilde{b}$ denote a piecewise linear approximation of b. Then

$$|(u-u^I,b\cdot\nabla\xi)|\le|(u-u^I,(b-\tilde{b})\cdot\nabla\xi|+|(u-u^I,\tilde{b}\cdot\nabla\xi-\pi_h(\tilde{b}\cdot\nabla\xi))|$$

(because u^I is the L_2 projection, $(u-u^I,v_h)=0$ for all $v_h\in V_h$). It follows say for $b\in W_1^\infty(\Omega)$ the estimate

$$|(u-u^I,b\cdot\nabla\xi)|\le c_1h^2|u|_2\|\xi\|_0+\|u-u^I\|_0\|\tilde{b}\cdot\nabla\xi-\pi_h(\tilde{b}\cdot\nabla\xi)\|_0.$$

Because $\tilde{b}\cdot\nabla\xi$ is piecewise linear the norms $\|\cdot\|_0$ and $\|\cdot\|_h$ are equivalent (the norm $\|\cdot\|_h$ is generated by the discrete scalar product (5)), which leads to

$$|(u-u^I,b\cdot\nabla\xi)|\le c_1h^2|u|_2\|\xi\|_0+c_2h^{3/2}|u|_2[S(\tilde{b}\cdot\nabla\xi,\tilde{b}\cdot\nabla\xi)]^{1/2}\ .$$

Choosing $\tilde{b}$ to interpolate b in the mesh points we can replace $\tilde{b}$ by b in the last estimate and obtain

$$|(u-u^I,b\cdot\nabla\xi)|\le ch^{3/2}|u|_2\|\xi\|_E.\quad\blacksquare$$

Let us now consider the simplest case: a one-dimensional problem with piecewise constant b an an equidistant mesh. If say $\pi_h(bu_h')=p_i$ on (x_{i-1},x_i), then

$$(\pi_h(bu_h'))(x_i)=\frac{p_i+p_{i+1}}{2}.$$

Consequently,(with $b\,v_h'=q$)

$$\begin{aligned}(p-\pi_hp,q-\pi_hq)\quad&=\quad\sum_i\frac{h}{2}[(p-\pi_hp)(x_{i-1})(q-\pi_hq)(x_{i-1})+(p-\pi_hp)(q-\pi_hq)(x_i)]\\&=\quad\sum_i\frac{h}{2}[\frac{1}{4}[p]_{i-1}[q]_{i-1}+\frac{1}{4}[p]_i[q]_i]\,.\end{aligned}$$

That means we recover the continuous interior penalty method (because the parameter δ is of order $\delta_0\,h$, the jump terms are scaled with h^2, as usual). Let us in the two-dimensional case as well assume that b is piecewise constant, i.e., $b\cdot\nabla u_h=p_l$ on the triangle K_l. Then, the representation

$$(\pi_h(b\cdot\nabla u_h))(x_i)=\sum_{j\in\Lambda_i}\alpha_jp_j$$

implies

$$\sum_K \frac{1}{3} \text{ meas } K \sum_{l=1}^{3} (b \cdot \nabla u_h - \pi_h(b \cdot \nabla u_h))(x_i^l) = \sum_K \frac{1}{3} \text{ meas } K \sum_{l=1}^{3} (p_i - \sum_{j \in \Lambda_{i,l}} \alpha_j p_j)$$

$$= \sum_K \frac{1}{3} \text{ meas } K \sum_{l=1}^{3} \sum_{j \in \Lambda_{i,l}} \alpha_j (p_i - p_j).$$

Introducing

$$p_i - p_j = \sum_\mu [p]_{e,\mu}$$

with $[p]$ denoting the jump across element boundaries and the sum is taken over the shortest "path" from element K_i to element K_j, we recognize that our stabilization term (8) admits the form

$$S(u_h, v_h) = \delta_0 \, h^3 \sum_i \left(\sum_{\mu(i)} \beta_\mu [p]_{e,\mu} ds \right) \left(\sum_{\mu(i)} \beta_\mu [q]_{e,\mu} ds \right).$$

If one cancels the products of jumps of p and q on different edges, we recover the continuous interior penalty method with

$$S(u_h, v_h) = \delta_0 h^2 \sum_e \int_e [p_e][q_e] ds.$$

Remark: If b is piecewise constant, $\pi_h(b \cdot \nabla u_h)$ is equal to a Clément interpolant $\Pi(b \cdot \nabla u_h)$ of $b \cdot \nabla u_h$. Consequently, a modified version of (8) reads

$$S(u_h, v_h) := \delta(b \cdot \nabla u_h - \Pi(b \cdot u_h), b \cdot \nabla v_h - \Pi(b \cdot \nabla v_h))_0. \tag{11}$$

The stabilization method generated by (11) is not consistent. However, the error analysis based on (3) allows to prove the estimate (4) for $u \in H^2(\Omega)$ due to the known properties of the Clément interpolant. We prefer the method (8) because $\pi_h(b \cdot \nabla u_h)$ is easier to compute than $\Pi(b \cdot \nabla u_h)$. ■

Let us finally remark that the local projection stabilization with a discontinuous finite element space M_h, in our case with piecewise constants on a coarser mesh, yields for piecewise constant b also a scheme related to the continuous interior penalty method. But for triangles the necessary macro mesh (see [7]) requires that every triangle of the given triangulation arises from the decomposition of a macro triangle into subtriangles with the barycenter as a knot. In our approach we avoid this restrictive assumption.

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