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Sebastian Franz

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**Abstract**

Recently, a new method for computing a numerical solution of singularly perturbed problems, the combination method was introduced. In [1] it was shown, that the numerical approximations using $\mathcal{O}\left(N^{3/2}\right)$ degrees of freedom are comparable to those of standard Galerkin using $\mathcal{O}\left(N^2\right)$ degrees.

Moreover, the numerical results given in that paper indicate a supercloseness property of the method. In this new paper we develop a postprocessing operator that achieves superconvergence, if the supercloseness property holds.

*AMS subject classification (2000):* 65N12, 65N15, 65N30
*Key words:* combination method, postprocessing, recovery

## 1 Introduction

In [1] a method is described that combines standard Galerkin solutions on different meshes. Let $u_{N_x,N_y}$ denote the standard Galerkin solution on a mesh with $N_x$ cells in $x$-direction and $N_y$ cells in $y$-direction. Moreover, let $\hat{N} = \sqrt{N}$ be integer. Then the final numerical approximation resulting from the combination method reads

$$u^N_{\hat{N},\hat{N}} = u_{N,\hat{N}} + u_{\hat{N},N} - u_{\hat{N},\hat{N}}.$$

This method will be applied to singularly perturbed problems, thus we consider the model problem

$$Lu := -\varepsilon\Delta u - \mathbf{b}\cdot\nabla u + cu = f \quad \text{in } \Omega = (0,1)^2, \tag{1a}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1b}$$

where $0 < \varepsilon \ll 1$ is a small positive parameter and

$$c(x,y) + \frac{1}{2}\,\text{div }\mathbf{b}(x,y) \geq c_0 > 0 \text{ on } \bar{\Omega}, \tag{2}$$

*Institut für Numerische Mathematik, Technische Universität Dresden, D-01062, Germany; e-mail: sebastian.franz@tu-dresden.de

where $c_0$ is some constants. Let us assume that the functions $\mathbf{b}$, $c$ and $f$ are sufficiently smooth. These hypotheses ensure that (1) has a unique solution in $H_0^1(\Omega) \cap H^2(\Omega)$ for all $f \in L^2(\Omega)$.
Due to the perturbation parameter, the solution exhibits layers. In order to resolve them, we use an adapted piecewise equidistant mesh, a so called Shishkin mesh, see [1].

# 2 Interpolation and Postprocessing

Let $V^{N_x,N_y}$ be the usual space of piecewise bilinear elements on the mesh $T^{N_x,N_y}$ and $I_{N_x,N_y}$ the standard nodal interpolation operator into $V^{N_x,N_y}$. The solution $u^N_{\hat{N},\hat{N}}$ lies in a subspace of $V^{N,N}$ namely

$$u^N_{\hat{N},\hat{N}} \in V^N_{\hat{N},\hat{N}} := V^{N,\hat{N}} + V^{\hat{N},N} \subset V^{N,N}.$$

The two-scale interpolation operator $I^N_{\hat{N},\hat{N}} : C(\bar{\Omega}) \to V^N_{\hat{N},\hat{N}}$ is defined by

$$I^N_{\hat{N},\hat{N}} u = I_{N,\hat{N}} u + I_{\hat{N},N} u - I_{\hat{N},\hat{N}} u.$$

For the analysis we need the following assumption.

**Assumption 2.1.** *Assume that*

$$u = S + E_{21} + E_{12} + E_{22}, \tag{3}$$

*where there exists a constant $C$ such that for all $(x,y) \in \Omega$ and $i+j = 0,1$ we have*

$$\left|\frac{\partial^{i+j} S}{\partial x^i \partial y^j}(x,y)\right| \le C, \quad \left|\frac{\partial^{i+j} E_{22}}{\partial x^i \partial y^j}(x,y)\right| \le C\varepsilon^{-(i+j)} e^{-(\beta_1(1-x)+\beta_2(1-y))/\varepsilon},$$
$$\left|\frac{\partial^{i+j} E_{21}}{\partial x^i \partial y^j}(x,y)\right| \le C\varepsilon^{-i} e^{-\beta_1(1-x)/\varepsilon}, \quad \left|\frac{\partial^{i+j} E_{12}}{\partial x^i \partial y^j}(x,y)\right| \le C\varepsilon^{-j} e^{-\beta_2(1-y)/\varepsilon}$$

*and for $i+j = 3,4,5$ the $L_2$ bounds*

$$\left\|\frac{\partial^{i+j} S}{\partial x^i \partial y^j}(x,y)\right\|_0 \le C, \qquad \left\|\frac{\partial^{i+j} E_{21}}{\partial x^i \partial y^j}(x,y)\right\|_0 \le C\varepsilon^{-i+1/2}, \tag{4a}$$
$$\left\|\frac{\partial^{i+j} E_{12}}{\partial x^i \partial y^j}(x,y)\right\|_0 \le C\varepsilon^{-j+1/2}, \qquad \left\|\frac{\partial^{i+j} E_{22}}{\partial x^i \partial y^j}(x,y)\right\|_0 \le C\varepsilon^{1-i-j}. \tag{4b}$$

In [1] it was shown, that the difference between the nodal interpolation on the fine mesh and the two-scale interpolation is small, i.e.

**Theorem 2.2** (Theorem 2.5 in [1])**.** *There exists a constant $C$ such that*

$$\left|\left|\left| I^N_{\hat{N},\hat{N}} u - I_{N,N} u \right|\right|\right|_\varepsilon \le C(\varepsilon^{1/2} N^{-\sigma} + N^{-\sigma} \ln^{1/2} N + N^{-1}).$$

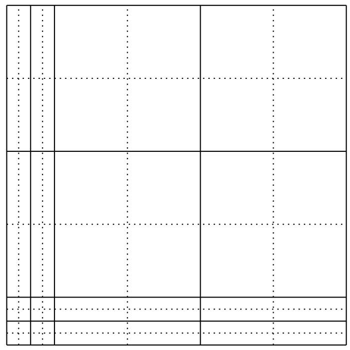

Figure 1: Macroelements of $M^{\hat{N},\hat{N}}$ constructed from $T^{\hat{N},\hat{N}}$ for $\hat{N} = 8$

As a corollary we have an interpolation error estimate

**Corollary 2.3** (Interpolation error)**.** *Let* $\sigma \geq 2$*. Then*

$$\left|\left|\left|u - I^N_{\hat{N},\hat{N}} u\right|\right|\right|_\varepsilon \leq C N^{-1} \ln N.$$

For the postprocessing operator let $\hat{N}$ be divisible by two and $M^{N_x,N_y}$ be a macro mesh, consisting of $2 \times 2$ neighbouring cells from $T^{N_x,N_y}$, such that the transition line of the Shishkin mesh is not crossed, see Figure 1. In order to derive a postprocessing operator, consider for a macro $M \in M^{\hat{N},\hat{N}}$ the set

$$F_M := \left\{(x, y) \in M | I^N_{\hat{N},\hat{N}} u(x, y) = u(x, y) \, \forall u \in C(\bar{M})\right\}.$$

A closer look to the identity reveals for $M = M_{i,j} := ([x_{i-1}, x_i] \cup [x_i, x_{i+1}]) \times ([y_{j-1}, y_j] \cup [y_j, y_{j+1}])$ with

$$\begin{aligned} F_{\hat{N},\hat{N}} :=& \{(x, y) : x \in \{x_{i-1}, x_i, x_{i+1}\}, y \in \{y_{j-1}, y_j, y_{j+1}\}\} \\ F_{N,\hat{N}} :=& \{(x, y) : x \in \{x_{i-1} : h_M/(2\hat{N}) : x_{i+1}\}, y \in \{y_{j-1}, y_j, y_{j+1}\}\} \\ F_{\hat{N},N} :=& \{(x, y) : x \in \{x_{i-1}, x_i, x_{i+1}\}, y \in \{y_{j-1} : k_M/(2\hat{N}) : y_{j+1}\}\} \end{aligned}$$

that this set can be rewritten into

$$F_{M_{i,j}} = F_{N,\hat{N}} \cup F_{\hat{N},N} \cup F_{\hat{N},\hat{N}}$$

see Figure 2. In other words, a function $v^N \in V^N_{\hat{N},\hat{N}}$ is uniquely defined over a macro element $M$ by its function values in the points of $F_M$.
A postprocessing operator that is consistent with the two-scale interpolation needs to use the same degrees of freedom on the macro element. Moreover, it should be piecewise biquadratic on $M^{N,N}$. The biquadratic postprocessing operator $P_{N,N}$ presented in [2] creates piecewise biquadratic functions on $M^{N,N}$, but is not consistent with $I^N_{\hat{N},\hat{N}}$. Therefore, we propose an operator as combination of standard piecewise biquadratic recovery operators. Let

$$P^N_{\hat{N},\hat{N}} u := P_{N,\hat{N}} u + P_{\hat{N},N} u - P_{\hat{N},\hat{N}} u.$$

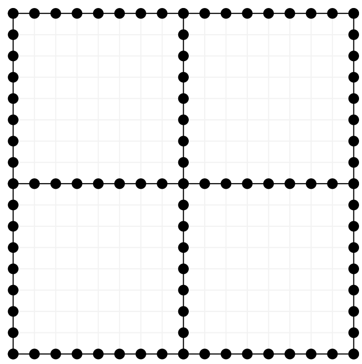

Figure 2: The set $F_M$ for $\hat{N} = 8$ on a macro $M$

**Theorem 2.4.** *The postprocessing operator* $P^N_{\hat{N},\hat{N}}$ *is consistent with* $I^N_{\hat{N},\hat{N}}$*, i.e.*

$$P^N_{\hat{N},\hat{N}} u = P^N_{\hat{N},\hat{N}} I^N_{\hat{N},\hat{N}} u. \tag{5}$$

*Furthermore, we have stability in the* $L_\infty$*-norm of the operator*

$$\|P^N_{\hat{N},\hat{N}} u\|_{L_\infty(M)} \le C\|u\|_{L_\infty(M)}, \,\forall u \in C(M) \tag{6}$$

*and its derivative, if* $\nabla u \in C(M)$

$$\|\nabla(P^N_{\hat{N},\hat{N}} u)\|_{L_\infty(M)} \le C\|\nabla u\|_{L_\infty(M)}, \,\forall \nabla u \in C(M). \tag{7}$$

*Moreover, stability the energy norm holds*

$$\left|\!\left|\!\left| P^N_{\hat{N},\hat{N}} u^N \right|\!\right|\!\right|_\varepsilon \le C\, |||u^N|||_\varepsilon, \,\forall u^N \in V^N_{\hat{N},\hat{N}} \tag{8}$$

*and the interpolation error for* $\sigma \ge 5/2$ *can be estimated by*

$$\left|\!\left|\!\left| P^N_{\hat{N},\hat{N}} u - u \right|\!\right|\!\right|_\varepsilon \le C(\varepsilon^{1/2} N^{1-\sigma} + N^{-2}\ln^2 N + \varepsilon^{1/2} N^{-2} \ln^4 N). \tag{9}$$

*Proof.* In order to prove (5) we only need to look at one macro element $M_{i,j} \in M^{\hat{N},\hat{N}}$. On this macro element $u$ and $I^N_{\hat{N},\hat{N}} u$ coincide in $F_M$, see Figure 2. On the other hand, $P_{\hat{N},\hat{N}}$ uses the values in the nine nodes of $F_{\hat{N},\hat{N}}$, $P_{N,\hat{N}}$ those in $F_{N,\hat{N}}$ and $P_{\hat{N},N}$ those in $F_{\hat{N},N}$. Consistency (5) follows.

The stability estimates (6) and (7) follow directly from the stability of $P_{N,N}$. Stability (8) can be proven similarly as for $P_{N,N}$ in [2]. We start on a macro element by

$$\|P^N_{\hat{N},\hat{N}} u^N\|_0 = 0 \;\Leftrightarrow\; u^N(x,y) = 0, \,\forall (x,y) \in F_M \;\Leftrightarrow\; u^N \equiv 0.$$

Thus the mapping $v \to \|P^N_{\hat{N},\hat{N}} v\|_0$ is a norm on $V^N_{\hat{N},\hat{N}}$. Similarly we have

$$\left| P^N_{\hat{N},\hat{N}} u^N \right|_1 = 0 \;\Leftrightarrow\; P^N_{\hat{N},\hat{N}} u^N \equiv c = \text{const.} \;\Leftrightarrow\; u^N(x,y) = c, \,\forall (x,y) \in F_M \;\Leftrightarrow\; u^N \equiv c.$$

Thus the mapping $v \to \left|P^N_{\hat{N},\hat{N}} v\right|_1$ is a norm on the quotient space $V^N_{\hat{N},\hat{N}} \setminus \mathbb{R}$. Then (8) follows from the equivalence of norms in finite dimensional spaces.
Finally, we adapt the proof of [1, Lemma 2.3 and 2.4] to get the error estimates. We start by

$$\left|\left|\left|u - P^N_{\hat{N},\hat{N}} u\right|\right|\right|_\varepsilon \le |||u - P_{N,N} u|||_\varepsilon + \left|\left|\left|P_{N,N} u - P^N_{\hat{N},\hat{N}} u\right|\right|\right|_\varepsilon \tag{10}$$

The first term in (10) is bounded by the interpolation error result of [2] for $\sigma \ge 5/2$

$$|||u - P_{N,N} u|||_\varepsilon \le C(\varepsilon N^{1-\sigma} + N^{-2} \ln^2 N).$$

Imitating the proof of [1, Lemma 2.3] gives

$$\|P^N_{\hat{N},\hat{N}} u - P_{N,N} u\|_0 \le C(N^{-\sigma} + \hat{N}^{-4}(1 + \varepsilon^{1/2} \ln^2 N + \varepsilon \ln^4 N)).$$

Using the ideas of [1, Lemma 2.4] we assemble

$$\varepsilon^{1/2} \|(P^N_{\hat{N},\hat{N}} u - P_{N,N} u)_x\|_0 \le C(\varepsilon^{1/2} N^{1-\sigma} + N^{-\sigma} \ln^{1/2} N + \hat{N}^{-4} \ln N + \varepsilon^{1/2} \hat{N}^{-4} \ln^4 N)$$

and similarly for the $y$-derivative. Combining these estimates gives (9). □

**Remark 2.5.** *Let us assume $\sigma \ge 5/2$ and $\varepsilon \le CN^{-1} \ln^4 N$ or $\sigma \ge 3$ and $\varepsilon^{1/2} \ln^2 N \le C$. Then the interpolation error can be estimated by*

$$\left|\left|\left|P^N_{\hat{N},\hat{N}} u - u\right|\right|\right|_\varepsilon \le CN^{-2} \ln^2 N.$$

*Application of this operator to the combination-method solution $u^N$ of (1) can now be estimated by*

$$\begin{aligned} \left|\left|\left|u - P^N_{\hat{N},\hat{N}} u^N\right|\right|\right|_\varepsilon &\le \left|\left|\left|u - P^N_{\hat{N},\hat{N}} u\right|\right|\right|_\varepsilon + \left|\left|\left|P^N_{\hat{N},\hat{N}} u - P^N_{\hat{N},\hat{N}} u^N\right|\right|\right|_\varepsilon \\ &\le CN^{-2} \ln^2 N + C \left|\left|\left|I^N_{\hat{N},\hat{N}} u - u^N\right|\right|\right|_\varepsilon \end{aligned}$$

*using (5) and (8). Thus, if a supercloseness property holds we have superconvergence.*

# 3 Numerical results

As numerical example let us consider

$$\begin{aligned} -\varepsilon \Delta u - (2 + x) u_x - (3 + y^3) u_y + u &= f \quad \text{in } \Omega = (0,1)^2 \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

with $f$ such that

$$u(x, y) = \cos(x\pi/2)[1 - \exp(-2x/\varepsilon)](1 - y)^3[1 - \exp(-3y/\varepsilon)].$$

| $\widehat{N}$ | $N$ | $\left\|\left\|\left\| u - u^N_{\widehat{N},\widehat{N}} \right\|\right\|\right\|_\varepsilon$ | | $\left\|\left\|\left\| u^N_{\widehat{N},\widehat{N}} - I^N_{\widehat{N},\widehat{N}} u \right\|\right\|\right\|_\varepsilon$ | | $\left\|\left\|\left\| u - P^N_{\widehat{N},\widehat{N}} u^N_{\widehat{N},\widehat{N}} \right\|\right\|\right\|_\varepsilon$ | | $\left\|\left\|\left\| u - P_{N,N} u^N_{\widehat{N},\widehat{N}} \right\|\right\|\right\|_\varepsilon$ | |
|---|---|---|---|---|---|---|---|---|---|
| 4 | 16 | 2.951e-1 | 0.73 | 1.155e-1 | 1.48 | 1.827e-1 | 1.41 | 1.434e-1 | 1.43 |
| 8 | 64 | 1.070e-1 | 0.78 | 1.485e-2 | 1.38 | 2.598e-2 | 1.47 | 1.964e-2 | 1.29 |
| 12 | 144 | 5.673e-2 | 0.81 | 4.862e-3 | 1.40 | 7.909e-3 | 1.50 | 6.887e-3 | 1.27 |
| 16 | 256 | 3.556e-2 | 0.83 | 2.178e-3 | 1.37 | 3.336e-3 | 1.50 | 3.308e-3 | 1.27 |
| 20 | 400 | 2.457e-2 | 0.84 | 1.181e-3 | 1.42 | 1.709e-3 | 1.52 | 1.873e-3 | 1.31 |
| 28 | 784 | 1.393e-2 | 0.86 | 4.537e-4 | 1.39 | 6.138e-4 | 1.49 | 7.755e-4 | 1.32 |
| 40 | 1600 | 7.552e-3 | 0.87 | 1.684e-4 | 1.36 | 2.116e-4 | 1.45 | 3.022e-4 | 1.33 |
| 56 | 3136 | 4.203e-3 | 0.88 | 6.751e-5 | 1.33 | 7.974e-5 | 1.40 | 1.231e-4 | 1.34 |
| 80 | 6400 | 2.242e-3 | 0.89 | 2.615e-5 | 1.27 | 2.937e-5 | 1.80 | 4.718e-5 | 1.34 |
| 112 | 12544 | 1.231e-3 | 0.90 | 1.112e-5 | 1.28 | 8.734e-6 | 1.30 | 1.916e-5 | 1.35 |
| 144 | 20736 | 7.846e-4 | | 5.840e-6 | | 4.547e-6 | | 9.730e-6 | |

Table 1: Convergence, supercloseness and postprocessing of the combination method, $\varepsilon = 1e\text{-}8$

In our numerical simulations the perturbation parameter is fixed at $\varepsilon = 1e\text{-}8$. All calculations are carried out in MATLAB, using biCGstab as solver for the linear systems with an incomplete LU-decomposition.
Table 1 shows the errors of the combination method solution. In Column 2 convergence of order $\mathcal{O}\left(N^{-1}\ln N\right)$ can be observed as predicted in [1]. The following column shows the supercloseness error between the numerical solution and the two-scale interpolant. This error is of higher order than the convergence error. In column 4 the errors of the postprocessed solution, using the introduced two-scale postprocessing operator are shown. Clearly, we have superconvergence. As comparison the last column lists the errors for the piecewise biquadratic postprocessing operator on the fine mesh. We see, that the introduced operator gives better results than the standard one.

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