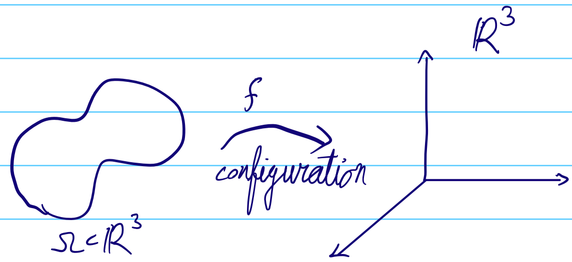
Introduction & motivation

Standard hyperelasticity



Elastic energy: $E(f) = \int_\Omega W(Df_x)\, dx$

- $W : \mathbb{R}^{3\times 3} \to [0,\infty)$, $\{W=0\} = SO(3)$ (single well structure)
- $W(RA) = W(A) \quad \forall A \in \mathbb{R}^{3\times 3},\ R \in SO(3)$ (frame indifference)
- $W(A) \geq c\, \mathrm{dist}^2(A, SO(3))$ (coercivity)

$\vdots$

Isotropy group of W $\quad G_W = \{Q \mid W(AQ) = W(A)\ \forall A\} < SO(3)$

Isotropic material $\quad G_W = SO(3)$

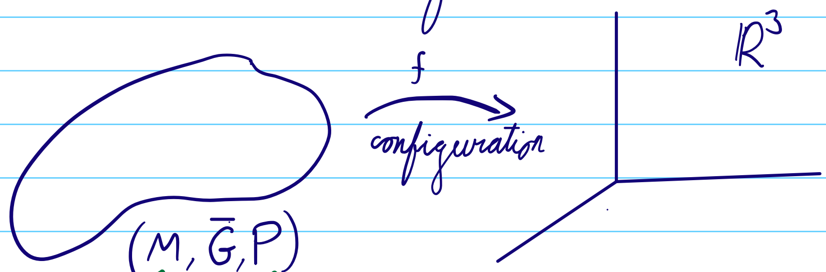
Example: $W(A) = \mathrm{dist}^2(A, SO(3))$

The well assumption $\{W=0\} = SO(3)$ implies the existence of zero energy (reference) configurations: $f(x) = Rx + b$.

Main focus of this course:

Problems in which no reference configurations exist

I Non-Euclidean elasticity



$(M, \bar{G}, P)$

3 dim. man. (single chart)

Riem. metric (reference)

Plastic strain
$P : M \to T^*M \otimes \mathbb{R}^3$
($\mathbb{R}^3$-valued oriented one form)

Volume form of $\bar{G}$: $|\det \bar{G}|^{\frac{1}{2}} dx$

<u>Elastic energy</u>: $E_M(f) = \int_M W(Df_x \circ P_x^{-1})\, dV_{\bar{G}}$, $E_M(f) = E_{M,P}(f)$; P_x^{-1} "implant map"

Compatibility of P & $\bar{G}$: $\bar{G}_x(v,w) = \langle P_x v, P_x w\rangle$, hence $P \rightsquigarrow \bar{G}$.

I.e. $P_x \in SO(\bar{G}_x) := \{A : T_xM \to \mathbb{R}^3 \mid A \text{ is orientation-preserving isometry}\}$

<u>Example</u>: In a chart, $\sqrt{\bar{G}}$ is a compatible strain.

If P, P' are two plastic strains compatible with $\bar{G}$, then $P'_x = Q_x P_x$ for some $Q_x \in SO(3)$.

$$\forall x \quad Q_x \in G_W \quad \Longrightarrow \quad E_{M,P} = E_{M,P'}$$

In particular, if W is <u>isotropic</u>, then

$$E_M(f) = \int W(Df_x \circ \bar{G}_x^{-\frac{1}{2}})\, |\det \bar{G}_x|^{\frac{1}{2}} dx$$

<u>Corollary</u>: $E_M(f) = 0 \iff Df_x \in SO(\bar{G}_x)$ a.e.

<u>Pf</u>: $W(Df_x \circ P_x^{-1}) = 0 \iff Df_x \circ P_x^{-1} \in SO(3)$

$$\iff Df_x \in SO(3) P_x = SO(\bar{G}_x)$$

Laplace Beltrami
$\frac{1}{|\bar G|^{\frac12}}\partial_j\left(\bar G^{jk}|\bar G|^{\frac12}\partial_k\right)$

Prop: $Df_x \in SO(\bar G_x)$ a.e. $\Rightarrow \Delta_{\bar G} f = 0$.
In particular, f is smooth.
This implies that $f: M\to\mathbb{R}^3$ is an isometric imm., hence $\bar G$ is flat – $\mathcal{R}^{\bar G}\equiv 0$.

Pf: $A\in SO(\bar G_x) \iff \det_{\bar G} A = 1$, $\operatorname{cof}_{\bar G} A = A$

$\det_{\bar G} A := |\bar G|^{-\frac12}\det A$
$\operatorname{cof}_{\bar G} A = |\bar G|^{-\frac12}\operatorname{cof} A\cdot \bar G$

As in Euclidean case, $\operatorname{div}_{\bar G}(\operatorname{cof}_{\bar G} Df)^{\#} = 0 \quad \forall f$

$A^{\#} = A\,\bar G^{-1}$
$\operatorname{div}_{\bar G} V = |\bar G|^{-\frac12}\partial_j\left(|\bar G|^{\frac12}V^j\right)$

Under our assumption $\operatorname{cof}_{\bar G} Df = Df$, hence
$\Delta_{\bar G} f = \operatorname{div}_{\bar G}(Df)^{\#} = 0$. (can also be written using a weak for.)

The flatness follows from Riemann's theorem.

Thm (LP'11, KMS'19): $\inf E_M = 0 \Rightarrow \mathcal{R}^{\bar G}\equiv 0$. The converse holds if M is simply-connected.

In KMS'19 also for non-Eucl. target

These works show a stronger claim: $\inf E_M = 0 \iff \exists f\in C^\infty(M;\mathbb{R}^n)$ iso. imm.

Pf: $\Rightarrow$ We will deduce it from a following estimate.
$\Leftarrow$ flatness + simply-conn. $\rightsquigarrow \exists f: M\to\mathbb{R}^3$ iso. imm. $\Rightarrow E_M(f) = 0$.

So for M simply-conn. we get $\inf E_M > 0 \iff \mathcal{R}^{\bar G}\not\equiv 0$. Can we get more quantitative?
Let's "localize" the question:

Thm (MS'19, KM'21): Let $p\in M$, then $\inf E_{B_r(p)} = r^4|\mathcal{R}^{\bar G}(p)|^2 + o(r^4)$.

some norm, ind. of p for isotropic W.

Pf: For simplicity, choose $W(A) = \operatorname{dist}^2(A, SO(n))$.
We can choose normal coordinates on $B_r(p)$ (for small enough $r>0$), and then

$$B_r(p) \rightsquigarrow B_r(0)\subset\mathbb{R}^n,\quad q\longmapsto x \qquad \bar G(x)_{ij} = \delta_{ij} + \underbrace{\tfrac13 \mathcal{R}_{kijl}(p)x^k x^l}_{T(x)} + O(|x|^3)$$

hence $\bar G^{-\frac12}(x)^j_i = \delta^j_i - \frac16 R_{kijl}(p)x^kx^l + O(|x|^3)$
and thus, for the map $\iota(x) = x$ we have

$$E_{B_r(0)}(\iota) = \fint_{B_r(0)} \operatorname{dist}^2\left(I - \tfrac16 R_{kijl}(p)x^kx^l + O(|x|^3)\right)\cdot\left(1+O(|x|^2)\right)dx \sim r^4$$

our energy scaling

<u>Lower bnd:</u> Assume $E_{B_r(p)}(f_r) \le Cr^4$. Then, since $|\bar G^{-\frac12} - I| \le C|x|^2$,

$$\operatorname{dist}(f_r, SO(n)) \le \operatorname{dist}(f_r \circ \bar G^{-\frac12}, SO(n)) + C|f_r||x|^2$$

$$\le C\left(\operatorname{dist}(f_r \circ \bar G^{-\frac12}, SO(n)) + |x|^2\right)$$

hence $\displaystyle \fint_{B_r(0)} \operatorname{dist}^2(f_r, SO(n))\,dx = O(r^4)$.

From FJM we have $R \in SO(n)$ s.t. $\displaystyle \fint_{B_r(0)} |Df_r - R|^2 \le Cr^4$.

$\exists c_r$ s.t. $\bar f_r = R^T f_r - c_r$ sat. $\displaystyle \fint |\bar f_r - x|^2 + |D\bar f_r - I|^2 \le Cr^4$

Define now $y = \frac{x}{r} \in B_1(0)$, $v_r(y) = \bar f_r(ry) - ry$, then

$\frac{1}{r^3} Dv_r \longrightarrow Df$ in $L^2(B_1(0), \mathbb{R}^n)$

$$D\bar f_r(ry) \circ \bar G^{-\frac12}(ry) = I + \underbrace{\frac{1}{r^3} Dv_r(y)}_{\xrightarrow{L^2} Df} + \underbrace{\frac{\bar G^{-\frac12}(ry) - I}{r^2}}_{\xrightarrow{L^\infty} -T(y)} + \underbrace{\frac{1}{r^3} Dv_r(y)\left(\bar G^{-\frac12}(ry) - I\right)}_{\xrightarrow{L^2} 0}$$

Hence:

$$\liminf \frac{1}{r^4} E_{B_r(p)}(f_r) = \liminf \frac{1}{r^4} E_{B_r(p)}(\bar f_r)$$

$$= \liminf \frac{1}{r^4} \fint_{\substack{B_r(0)\\ B_1(0)}} \operatorname{dist}^2\left(Df_r(\overset{ry}{x}) \circ \bar G^{-\frac12}(\overset{ry}{x})\right)\left(1 + O(\overset{r^2}{|x|^2})\right) \overset{dy}{dx}$$

$\operatorname{dist}(I + A, SO(n)) = |\operatorname{sym} A| + O(|A|^2)$

$$\ge \int_{B_1(0)} |\operatorname{sym} Df - T|^2\,dy \ge \min_{f \in W^{1,2}} \int_{B_1(0)} |\operatorname{sym} Df - T|^2\,dy =: I(\mathcal{R}(p))$$

Now, $\mathcal{R} \mapsto (I(\mathcal{R}))^{\frac12}$ is a norm:

homogeneity + triangle ineq. are easy.

Positivity follows from Saint-Venant compatibility condition:

$$T = \operatorname{sym} Df \iff \underbrace{\partial_{ij}T_{kl} + \partial_{kl}T_{ij} - \partial_{ik}T_{jl} - \partial_{jl}T_{ik}}_{R_{iklj}(P)} = 0$$

<u>Upper bound</u>: Let f be a min. of $\int_{B_1(0)} |\operatorname{sym} Df - T|^2 \, dy$

and define $f_r(x) = x + r^3 f(x/r)$.

The above calculation yields $\frac{1}{r^4} E_{B_r(P)}(f_r) \longrightarrow I(R)$ $\boxed{}$

<u>Comment</u>: If we take another isotropic W, we get $\int Q(\operatorname{sym} Df - T)$,

where $Q(A) = \frac{1}{2} D_I^2 W(A, A)$ is positive def. on sym. matrices.

If W is not isotropic, then, writing $P_x = \underset{SO(n)}{\underset{\cap}{Q_x}} G^{-\frac{1}{2}}$, we get

$$\int_{B_1(0)} Q(Q_p^T(\operatorname{sym} Df - T)Q_p) \, dy$$