Defects in solids

We saw inf $E_M = 0 \Rightarrow R^{\bar{G}} = 0$.
The converse is true only for simply-connected domains:
Topology is also an obstruction!

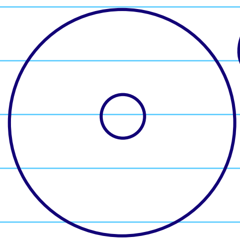
Focus on locally-flat ($R^{\bar{G}} \equiv 0$), multiply-connected bodies.

Volterra cut-and-weld constructions - Show picture

$R^{\bar{G}} \equiv 0 \Rightarrow M$ is locally isometrically embeddable in $\mathbb{R}^n$

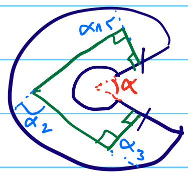
$\Rightarrow$ Notions s.a. straight lines, geodesic curv. etc. are as in $\mathbb{R}^n$.
(everything that is measured locally)
In part., locally path-ind. par. transport (identifying vectors in different tangent spaces)

Defects in 2D



$(M, \bar{G})$, $\bar{G}$ locally flat.

Disclination

- 

cannot be isom. imm. in $\mathbb{R}^2$:
The sum of angles in any triangle $= \pi + \alpha > \pi$
(preserved under isometry)

Gauss-Bonnet: for any triangle in a simply-connected surface

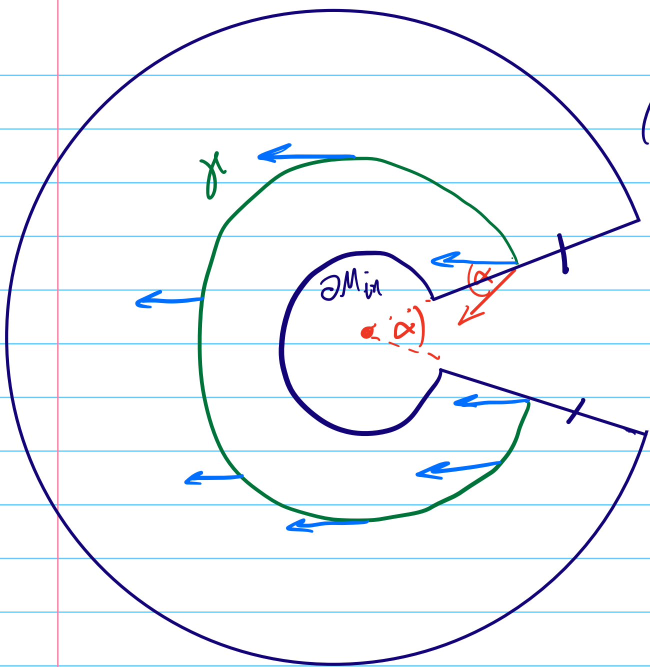
$$\iint K_{\bar{G}} = 2\pi - \sum \alpha_i - \int \kappa_g \, ds$$

in our case: $\quad 2\pi - \underbrace{(2\pi - \alpha)} - \underbrace{0} \quad = \alpha$

if time permits

hence we can think of the "hole" having total curvature α.
Given $R^{\bar{G}} \equiv 0$, we can measure this defect by calculating $2\pi - \int \kappa_g \, ds$
on any smooth closed curve around the core

Disclination = curvature charge



$(M, \bar{G})$ cannot be iso. imm. in $\mathbb{R}^2$:

If we par. trans. a vector along γ (homotopic to ∂M_{in}) we get a rotation of α!

If we do it in a closed curve in $\mathbb{R}^2$ we get zero.

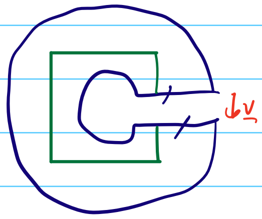
$Hol_\gamma = R_\alpha$

<u>Cor</u>: No parallel vector fields!

<u>Comment</u>: In a general manifold $(M, \bar{G})$, one can show that the holonomy of an infinitesimally small curve converges to the Riem. curv. at the point.

Disclination = curvature charge!

(edge) Dislocation

- 

  No curvature charge (Gauss-Bonnet is satisfied, $\mathrm{Hol}_\gamma = I \ \forall \gamma$)

  Cannot be isometrically immersed in $\mathbb{R}^2$:
  In the rectangle, opposite edges are not equal.

More generally, any simple, oriented closed curve γ around the core satisfies

$$\oint \Gamma^{\gamma(0)}_{\gamma(t)}\, \dot\gamma(t)\,dt = b(\gamma(0)) \in T_{\gamma(0)}M$$

Par. trans. along γ

Burgers vector (par. vector field)

The Burgers vector is the charge of the dislocation.

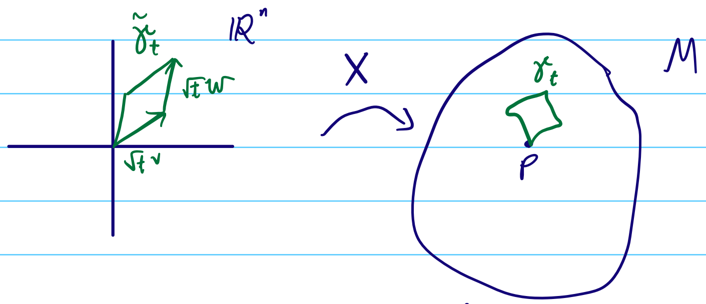
If time permits

In Riem. geo., we can define many connections, $\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$.

The torsion of the connection is $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$

$$T_{ij} = (\Gamma^k_{ij} - \Gamma^k_{ji})\,\partial_k$$

Thm: Given a connection ∇, and let $X: U \subset M \to \mathbb{R}^n$ be a coordinate chart, with $X(p) = 0$, and let $v, w \in T_pM$, $\tilde v = dX(v)$, $\tilde w = dX(w)$

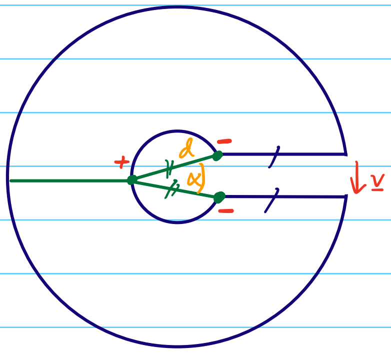


Then, $T_p(v,w) = \dfrac{d}{dt}\Big|_{t=0} \oint \underbrace{\Gamma^p_{\gamma_t(s)}}_{\text{par. trans. along } \gamma} \dot\gamma_t(s)\,ds$.

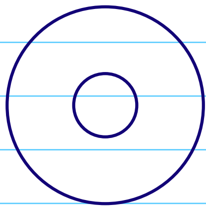
Dislocation = torsion charge.

Remark: If the core contains a disclination, then the Burgers vector is path-dependent, i.e., not well-defined.

"Edge dislocation = Curvature dipole"



$|\underline{v}| = 2d\sin\frac{\alpha}{2}$



$\mathrm{Hol}_\gamma = 0$ $\qquad b = 0$

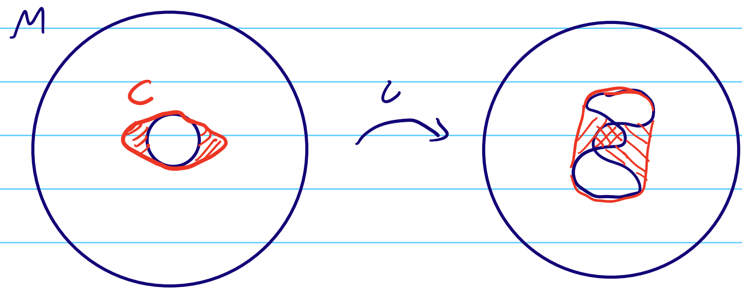
- $(M, \bar{\mathfrak{G}})$, $\bar{\mathfrak{G}}$ with no curvature and torsion charges.

  Can we embed it iso. in $\mathbb{R}^2$?

KM'23

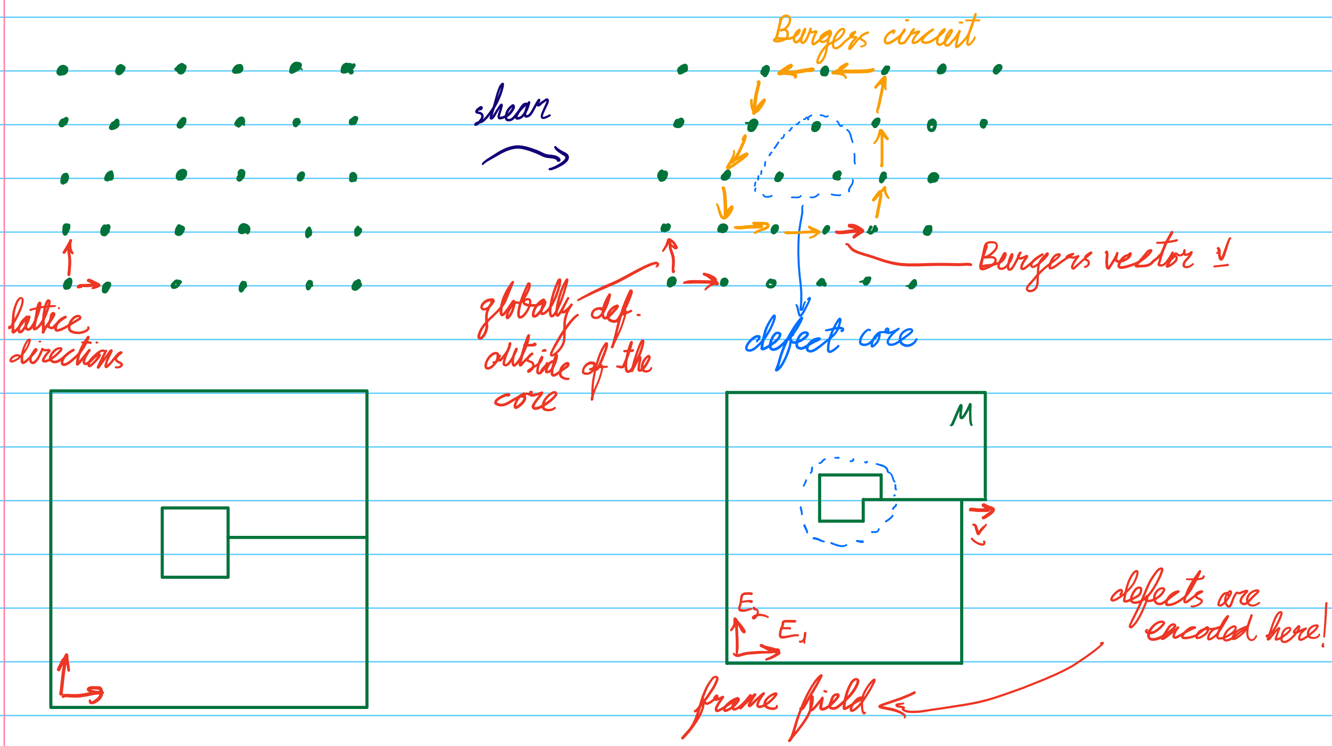
Thm (KMS'15): Let $\partial M_{in} \subset M$ be the inner boundary, and let $C = \mathrm{conv}(\partial M_{in})$ its geodesic convex hull.
Then, under the above assumptions, $(M \setminus C, \bar{\mathfrak{G}})$ can be iso. emb. in $\mathbb{R}^2$.

Ex:



$\boxed{\text{From now on — focus on dislocations}}$ (following mostly Kupferman-M. '23)

Dislocation in lattices:



$\{E_1, E_2\}$ frame-field $\rightsquigarrow E : \mathbb{R}^2 \to TM$ frame $\rightsquigarrow P : TM \to \mathbb{R}^2$ dual frame

Dislocations in $(M, \bar{G}, P)$:

① $dP = 0 \implies R^{\bar{G}} \equiv 0$, no curvature charge in core

Pf: $P = df$ locally, hence $\bar{G}(X,Y) = \langle df(X), df(Y) \rangle$ locally $\Rightarrow R^{\bar{G}} \equiv 0$.

Again, locally, $\nabla^{\bar{G}} P = \nabla^{f^*\delta} df = df(\nabla^{\delta} \mathrm{Id}) = 0$ hence P is a global, parallel section of $SO(\bar{G})$. $\implies$ no holonomy $\implies$ no dislocation.

② $\int_\gamma P = \int_0^1 P_{\gamma(t)}(\dot{\gamma}(t))\,dt = \underline{v}$ Burgers vector. $P(\underline{b}) = \underline{v}$.
$\hookrightarrow$ curve homotopic to inner bdry.

Def: A two-dim. body w. an edge dislocation is $(M, \bar{G}, P)$, $M \overset{\text{diff.}}{\cong} \mathbb{R}^2 \setminus B_1$,
P satisfies ①, ②,
∂M has winding number 1. $\left(\frac{1}{2\pi} \int_\gamma \kappa(s)\,ds = 1\right)$

Ex: $M = \{(r,\varphi) \mid r \geq |\underline{v}|\}$

- $\hat{P}_{\underline{v}} = I + \frac{\underline{v}}{2\pi} d\varphi \quad \rightsquigarrow \bar{G} = \begin{pmatrix} 1 & \frac{1}{\pi}(v_1 \cos\varphi + v_2 \sin\varphi) \\ & \left(r + \frac{1}{2\pi}(-v_1 \sin\varphi + v_2 \cos\varphi)\right)^2 \end{pmatrix}$

- $\bar{G} = \begin{pmatrix} 1 & 0 \\ & \left(r + \frac{|\underline{v}|}{\pi} \cos(\varphi - \varphi_0)\right)^2 \end{pmatrix}$

- $\bar{G} = \exp\left(\frac{|\underline{v}| \cos(\varphi - \varphi_0)}{\pi r}\right) \begin{pmatrix} 1 & 0 \\ & r^2 \end{pmatrix}$

Q: Are they isometric?

Thm (Kupferman-M.): If $(M, \bar{G}, P)$ is a complete body with an edge dislocation, then $(M \setminus \mathrm{conv}(\partial M_{in}), \bar{G}, P)$ is obtained by a Volterra construction.

Cor: $\underline{v}$ uniquely determines a body w. edge disloc. up to the shape of the core.

Energy scaling of a single dislocation

Consider $M_{\underline{v}}^R = \{(r,\varphi) \mid r \in [|v|, R]\}$, $\hat{P}_{\underline{v}} = I + \frac{\underline{v}}{2\pi} d\varphi$

$\hat{g}_{\underline{v}}$ induced by $\hat{P}_{\underline{v}}$.

Lemma: $\frac{1}{c}\delta < \hat{g}_v < c\delta$, $\frac{1}{c} < \left|\frac{dV_{\hat{g}}}{dx}\right| < c$, c ind. of $\underline{v}$

Prop: Let $\delta \in (0,1)$ s.t. $\delta R \in [|v|, R)$. Then

$$\inf_{H^1(M_{\underline{v}};\mathbb{R}^2)} \int_{M_{\underline{v}}^R \setminus M_{\underline{v}}^{\delta R}} \operatorname{dist}^2(Df, SO(\hat{g}_v))\, dV_{\hat{g}_v} \sim |\underline{v}|^2 \log \frac{1}{\delta}$$

In particular, for $\delta R = |\underline{v}|$ we get

$$\inf_{H^1(M_{\underline{v}};\mathbb{R}^2)} \int_{M_{\underline{v}}^R} \operatorname{dist}^2(Df, SO(\hat{g}_v))\, dV_{\hat{g}_v} \sim |\underline{v}|^2 \log \frac{R}{|\underline{v}|}$$

Upper bnd: Choose $\underset{M_{\underline{v}}}{f(r,\varphi)} = \underset{\mathbb{R}^2}{(r,\varphi)}$, and then

$$\operatorname{dist}(I, SO(\hat{g}_v)) \leq |I - \hat{P}_{\underline{v}}| = \frac{|v|}{2\pi}\frac{1}{r}$$

(we're lying here a bit since dist & $|\cdot|$ are w.r.t. $\hat{g}_v$

this calculation is w.r.t. Euclidean norm, but they are

uniformly equivalent, ind. of $\underline{v}$)

Thus, $\inf \int \ldots \lesssim \int_\delta^R |\underline{v}|^2 \frac{1}{r^2} r\, dr = |\underline{v}|^2 \log \frac{1}{\delta}$.

Lower bnd:

FJM for half annuli: $\Omega = \{(r,\varphi) \mid r \in (R_1, R_2), \varphi \in (0,\pi)\}$, $\frac{R_2}{R_1} \geq \frac{3}{2}$.

$\exists C > 0$ (ind. of R_1, R_2) s.t. $\forall f \in H^1(\Omega; \mathbb{R}^2)$ $\exists U \in SO(2)$ s.t.

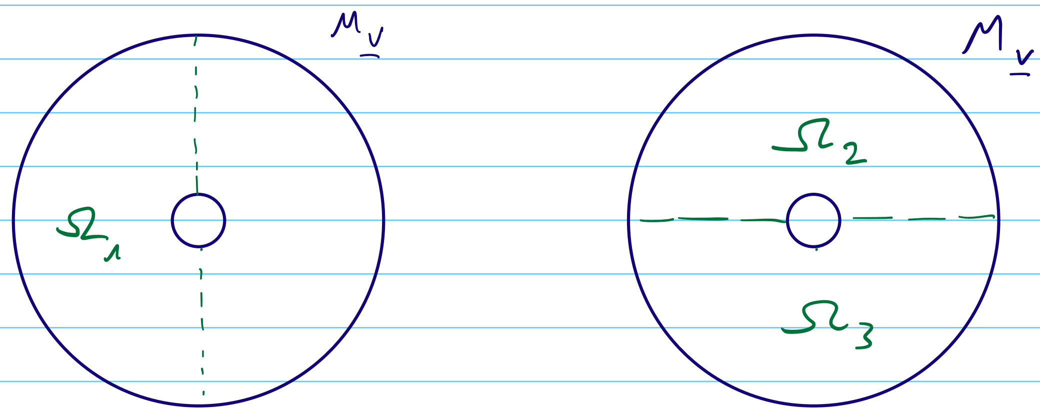
$$\int_\Omega |Df - U|^2 dx \leq C \int_\Omega \operatorname{dist}^2(Df, SO(2))\, dx$$

FJM for $\hat{M}_{\underline{v}}$: Assume $R > 10|\underline{v}|$, $\delta > 0$ s.t. $\delta R \in [|\underline{v}|, \frac{2}{3}R)$

$\exists C > 0$ ind. of $|\underline{v}|, R, \delta$ s.t. $\forall f \in H^1(\hat{M}_{\underline{v}}^R; \mathbb{R}^2)$, $\exists U \in SO(2)$ s.t.

$$\int_{\hat{M}_{\underline{v}}^R \setminus \hat{M}_{\underline{v}}^\delta} |Df - U\hat{P}_{\underline{v}}|^2 dV_{\hat{g}_v} \leq C \int_{\hat{M}_{\underline{v}}^R \setminus \hat{M}_{\underline{v}}^\delta} \operatorname{dist}^2(Df, SO(\hat{g}_v))\, dV_{\hat{g}_v}$$

Pf:



Ω_i simply connected flat manifold, hence

$$(\Omega_i, \hat{g}_v, \hat{P}_{\underline{v}}) \xrightarrow[\text{(isometry)}]{\iota} (\Omega, \delta, P_0)$$

$\Omega \subseteq \mathbb{R}^2$ uni. biLip. equiv. to half annulus

$P_0 \in SO(2)$ const. since $dP_0 = 0$

$f \rightsquigarrow \iota \circ f$

$\downarrow$ FJM

$$|Df - U\hat{P}_{\underline{v}}|^2_{g_v} \leq C \cdots \longleftarrow |D(\iota \circ f) - UP_0|^2 \leq C \cdots$$

Pf of lower bnd:

$$\int_{M_{\underline{v}}^R \setminus M_{\underline{v}}^\delta} \mathrm{dist}^2(Df, SO(\mathring{g}_v))\, dV_{\hat{g}_{\underline{v}}} \gtrsim \int |Df - U\hat{P}_{\underline{v}}|^2_{\hat{g}_v}\, dV_{\hat{g}}$$

$$\approx \int_{\delta R}^{R} \left(\int_{\{r=s\}} |Df - U\hat{P}_v|^2 d\varphi \right) s\, ds$$

$\left|\frac{\partial\varphi}{r}\right|_\delta = 1$

$$\gtrsim \int_{\delta R}^{R} \left(\int_{\{r=s\}} \left| Df\left(\tfrac{\partial\varphi}{s}\right) - U\hat{P}_v\left(\tfrac{\partial\varphi}{s}\right)\right|^2 d\varphi \right) s\, ds$$

Jensen

$$\geq \int_{\delta R}^{R} \frac{1}{2\pi s} \left| \int_{\{r=s\}} Df(\partial_\varphi) - U\tilde{P}_v(\partial_\varphi)\, d\varphi \right|^2 ds$$

$$= \int_{\delta R}^{R} \frac{1}{2\pi s} \left| \int_{\{r=s\}} Df - U\hat{P}_v \right|^2 ds$$

$$= |\underline{v}| \int_{\delta R}^{R} \frac{ds}{2\pi s} = \frac{|v|^2}{2\pi} \log\frac{1}{\delta}$$