TECHNISCHE UNIVERSITÄT DRESDEN

Stochastic Homogenization

I) *Introduction and Motivation*

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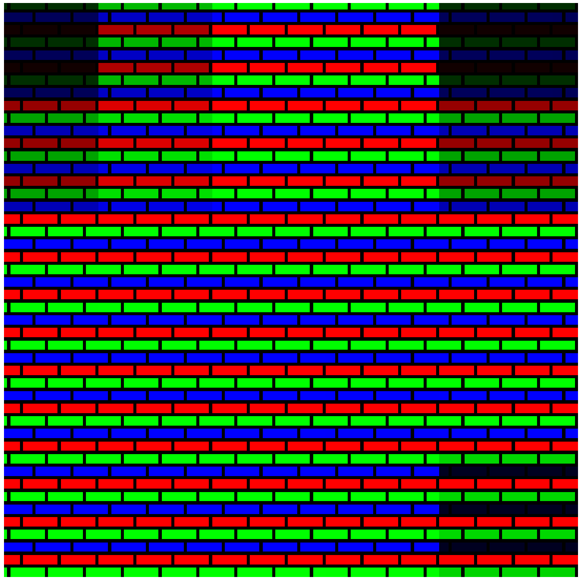
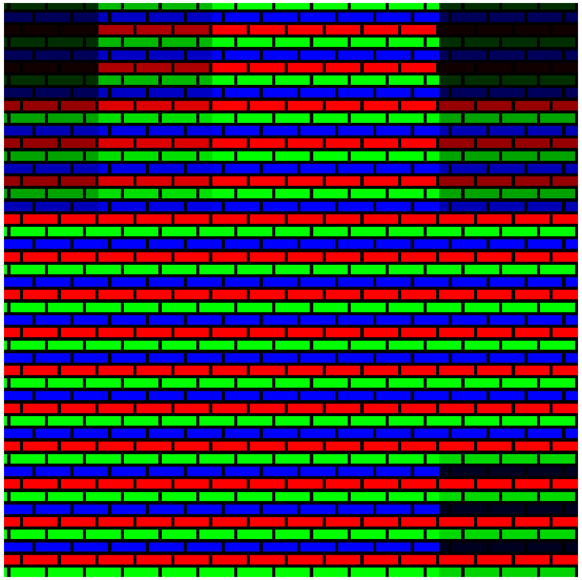
GSIS International Winter School 2017
Tohoku University

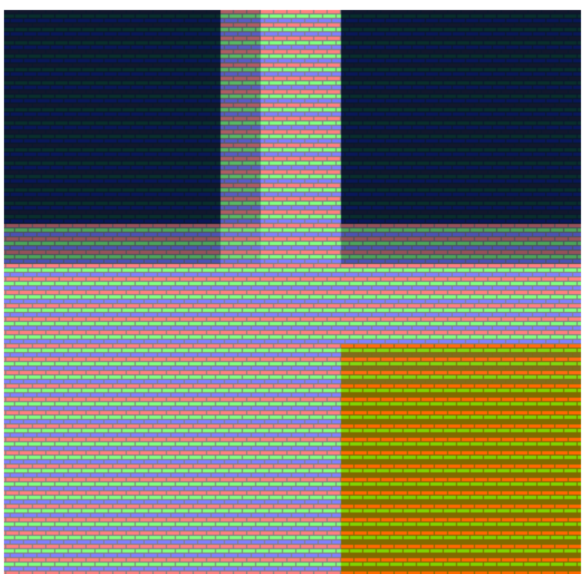
DRESDEN concept
Exzellenz aus Wissenschaft und Kultur

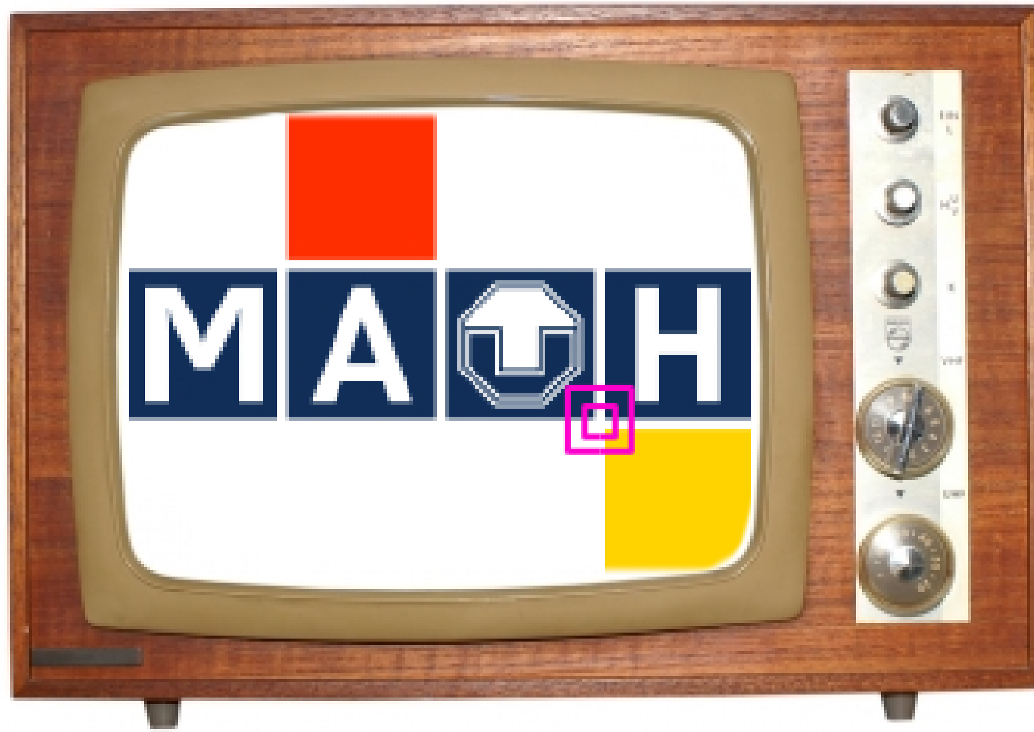


microscale



mesoscale

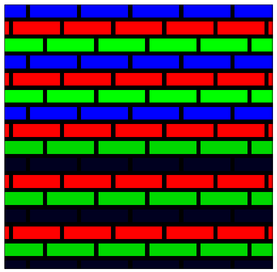




macroscale

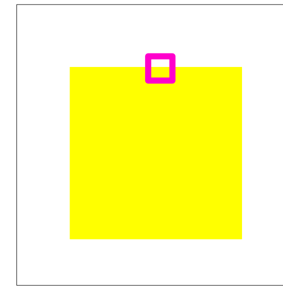
Microscale versus macroscale

- Different models on **different scales**

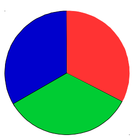


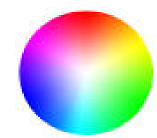
$f(\text{pixel}) \mapsto \text{intensity}$

yellow square
on a white
background

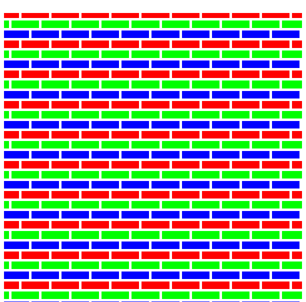


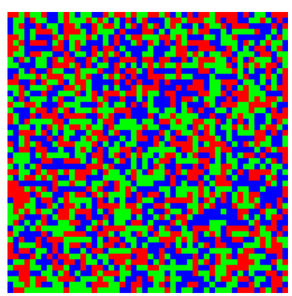
- **Emergence** of effective properties

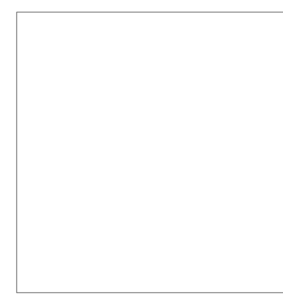




- Loss of microstructural details through „averaging“

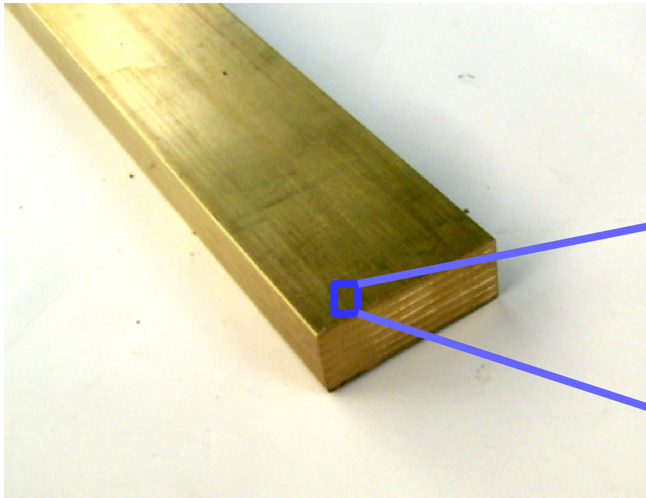




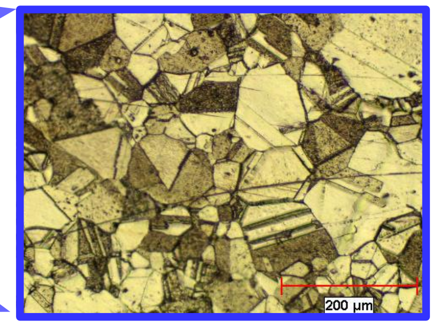


Microstructures are everywhere

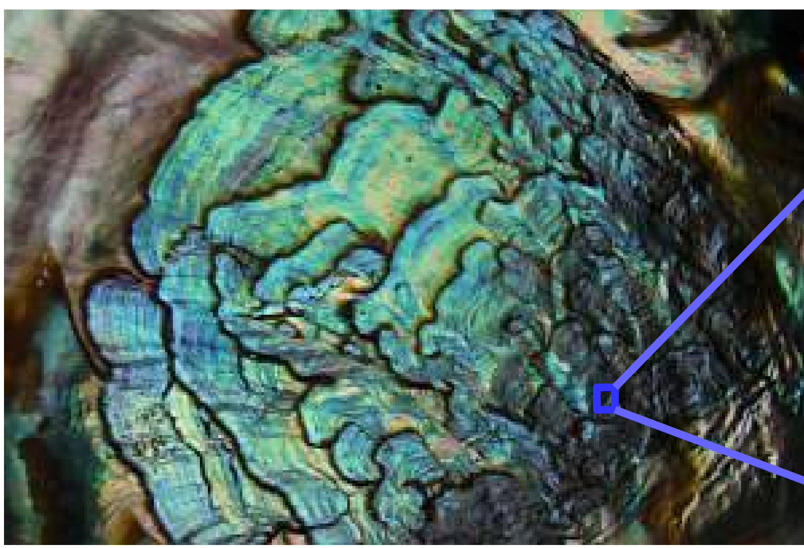
Microstructures are everywhere



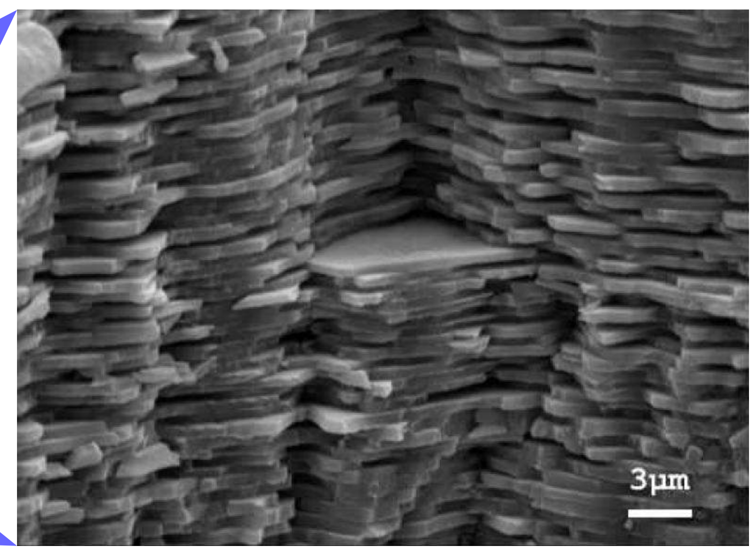
brass (a metal alloy)



grain structure
@CLEMEX.COM

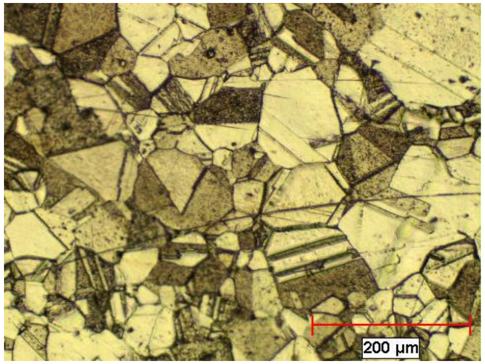


nautilus shell (nacre)
@Zawischa ITP Hannover

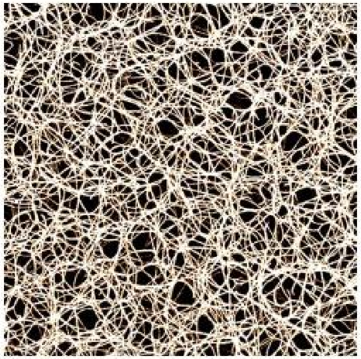


@F.Heinemann (Wikipedia)

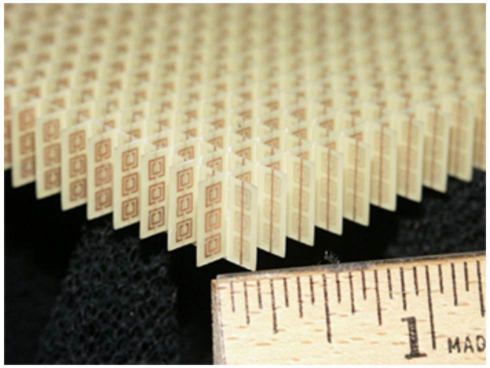
Microstructures are everywhere



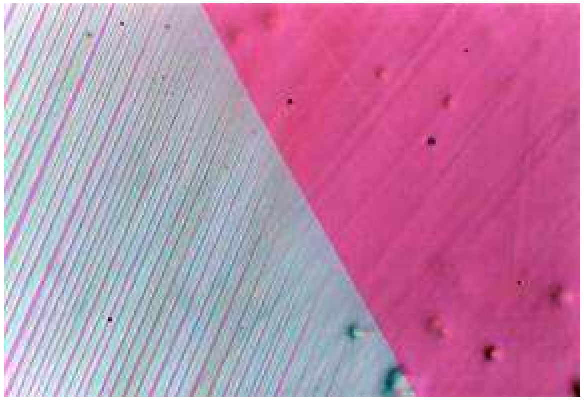
metal alloy
@CLEMEX.COM



rubber
@wikipedia.com



meta material
@wikipedia.com



shape memory alloy
@ Chu & James

Microstructures dominate the effective behavior of many materials

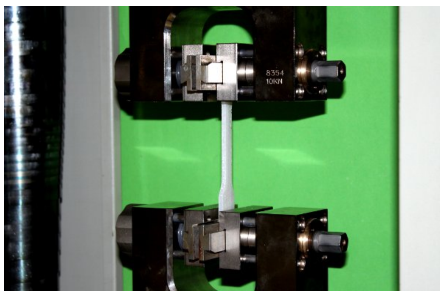
Origin of microstructures?

Connection between microstructures and macroscopic properties?

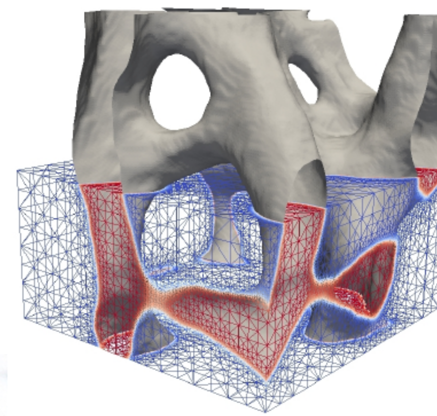
Origin of microstructures?

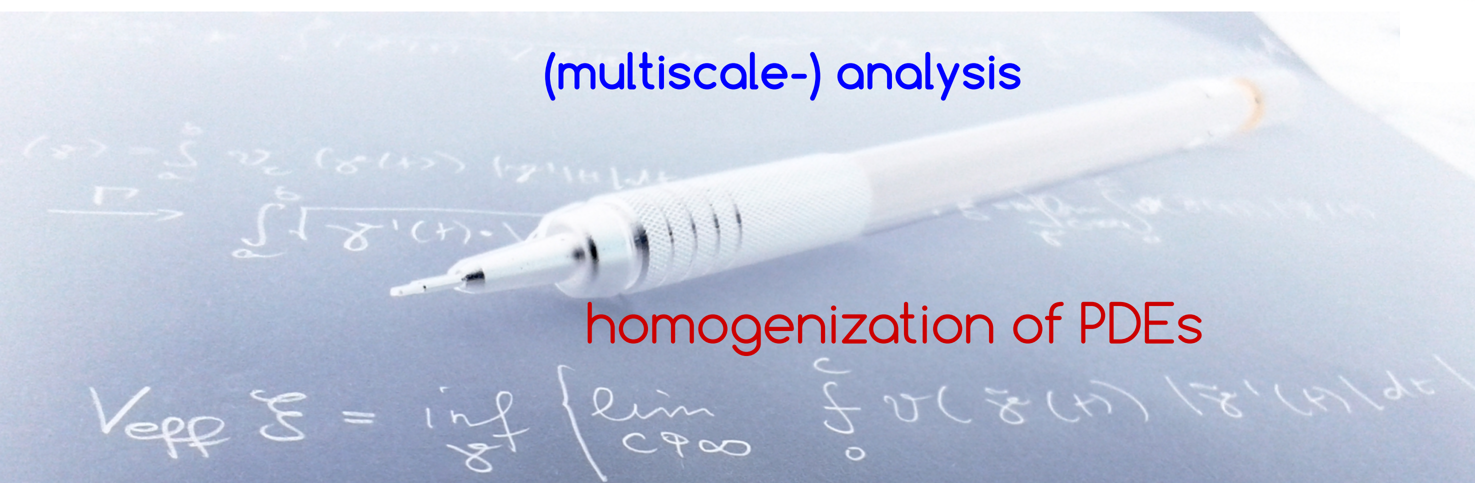
Connection between microstructures and macroscopic properties?

experiment



simulation





(multiscale-) analysis

homogenization of PDEs

... many links within mathematics

probability theory

modelling

PDEs

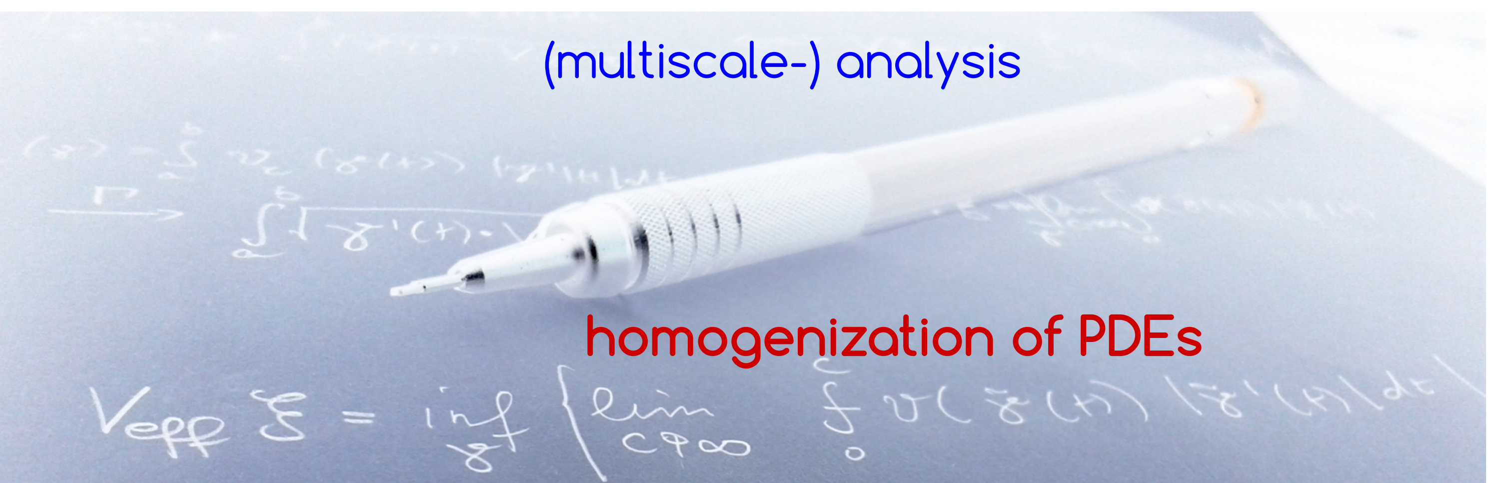
numerical analysis

simulation

calculus of variations

(multiscale-) analysis

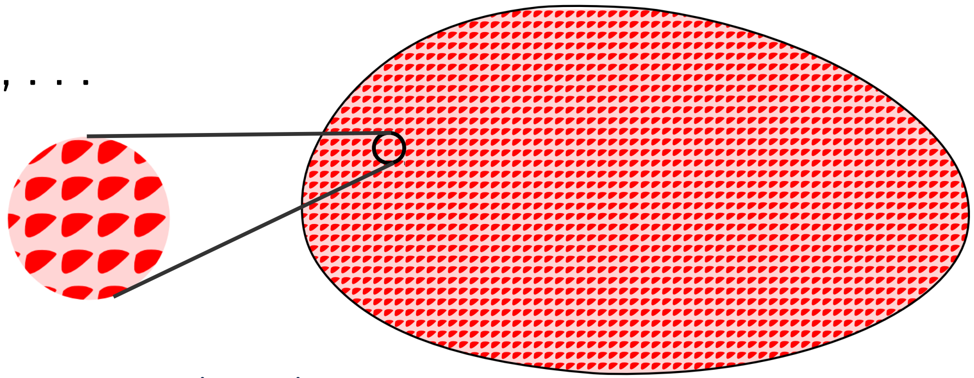
homogenization of PDEs



Homogenization

Homogenization

- composite with phases $M_1, M_2, \ldots$
- periodic microstructure

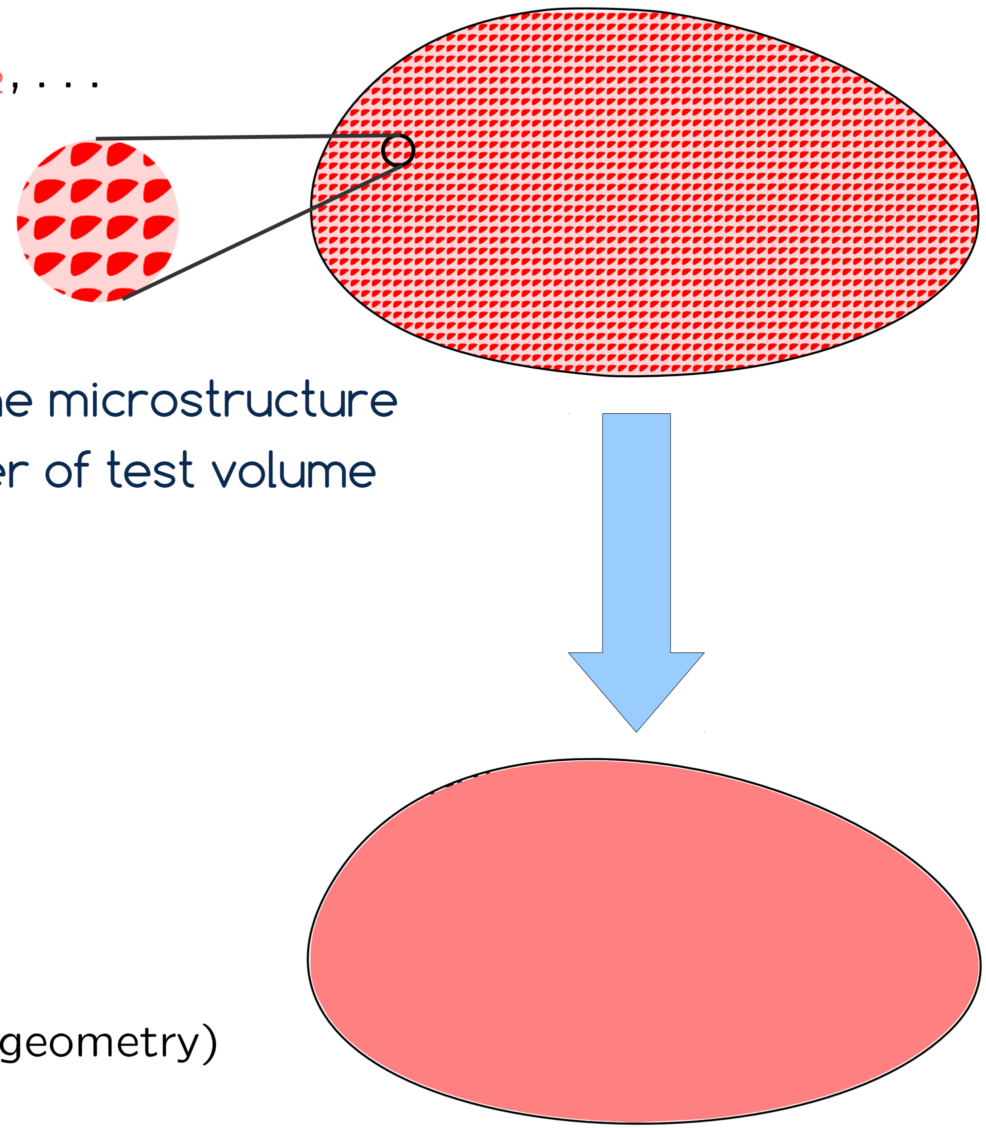


two length scales

- microscale $\ell =$ period of the microstructure
- macroscale $L =$ e.g. diameter of test volume

Homogenization

- composite with phases $M_1, M_2, \ldots$
- (periodic) microstructure



two length scales

- microscale $\ell =$ period of the microstructure
- macroscale $L =$ e.g. diameter of test volume

Paradigm of homogenization:

If $\boxed{\varepsilon = \frac{\ell}{L} \ll 1}$

(1) composite $(M_1, M_2, \ldots)$

$\rightsquigarrow$ effective medium M_{eff}

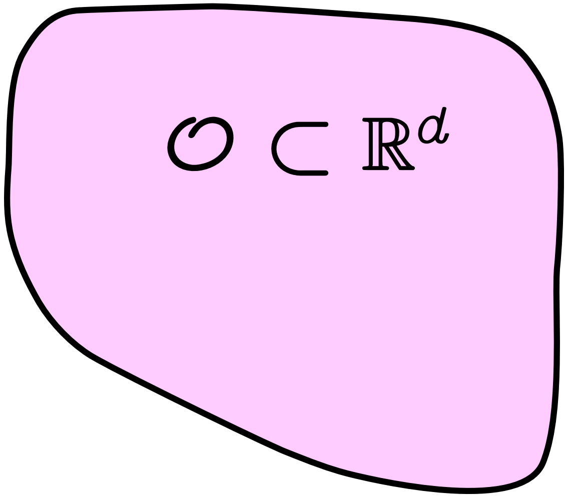
(2) $M_{\text{eff}} = \mathtt{formula}(M_1, M_2, \ldots; \mathtt{geometry})$

$$\boxed{\text{analyze asymptotics } \varepsilon = \frac{\text{microscale}}{\text{macroscale}} \downarrow 0}$$

PDE-modell for a microstructured material

PDE-modell for a microstructured material

$$
\begin{aligned}
-\nabla \cdot (a(x)\nabla u(x)) &= f(x) \qquad x \in \mathcal{O} \\
u(x) &= 0 \qquad\quad\;\; x \in \partial\mathcal{O}
\end{aligned}
$$



- $\mathcal{O}$ – domain occupied by material
- $a(x)$ – material coefficient at x
- u – physical quantity (e.g. temperature)

PDE-modell for a microstructured material

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\begin{aligned}
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u(x) &= 0 && x \in \partial\mathcal{O}
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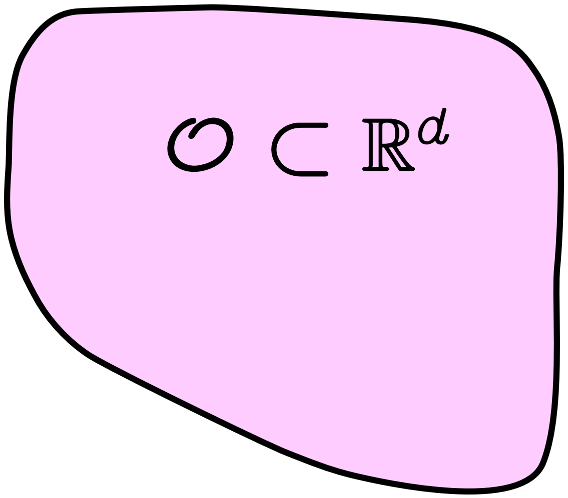


- $\mathcal{O}$ – domain occupied by material
- $a(x)$ – material coefficient at x
- u – physical quantity (e.g. temperature)

homogeneous material

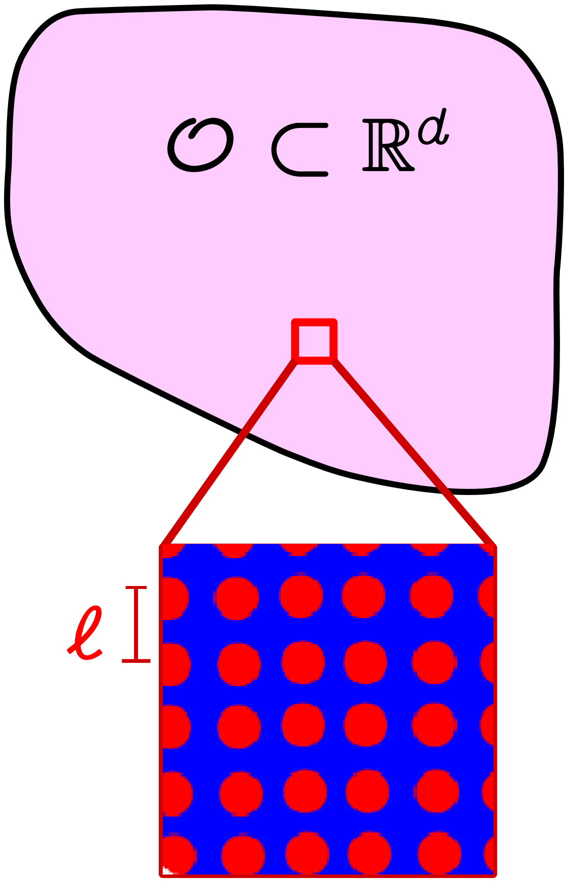
- $a(x)$ is independent of x

heterogeneous material

- $a(x)$ varies in x

PDE-modell for a microstructured material

$$-\nabla \cdot (a(x)\nabla u(x)) = f(x) \qquad x \in \mathcal{O}$$
$$u(x) = 0 \qquad x \in \partial\mathcal{O}$$



- $\mathcal{O}$ – domain occupied by material
- $a(x)$ – material coefficient at x
- u – physical quantity (e.g. temperature)

microstructured material

- a varies on scale ℓ
- f varies on scale $L \lesssim \mathrm{diam}(\mathcal{O})$

microstructure: $\varepsilon := \dfrac{\ell}{L} \ll 1$

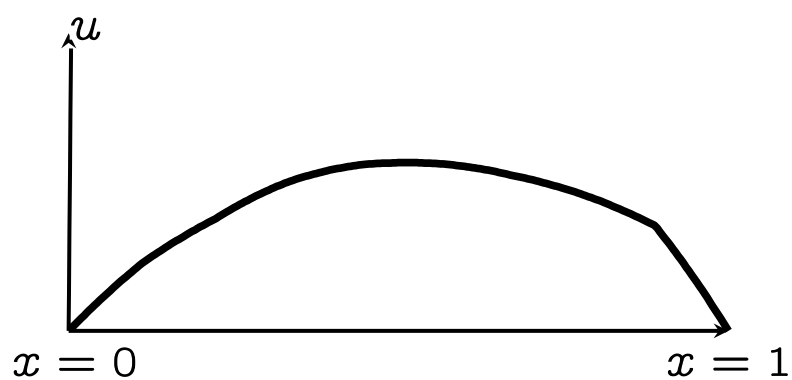
a numerical investigation

a numerical investigation

one-dimensional heat conducting rod

$$
\begin{aligned}
-\partial_x(a(x)\partial_x u(x)) &= 1 \qquad x \in (0,1) \\
u(0) = u(1) &= 0
\end{aligned}
$$

- $u(x)$ – temperature
- $a(x)$ – thermal conductivity
- $a(x)\partial_x u(x)$ – heat flux
- homogeneously heated



(1) How does u depend on a ?
(2) What happens if a varies on scale $\ell \ll 1$?

a numerical investigation

one-dimensional heat conducting rod

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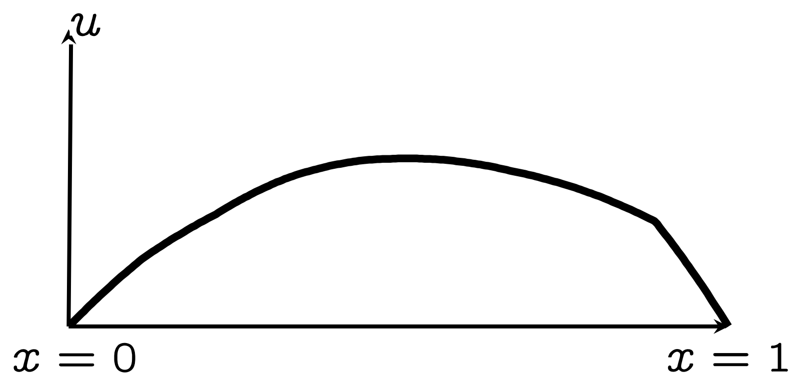

(1) How does u depend on a ?
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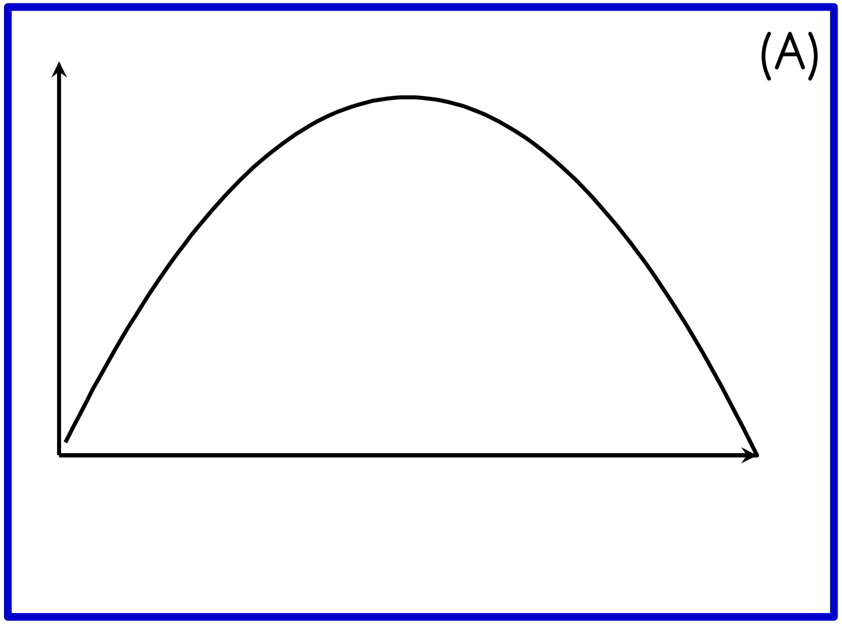
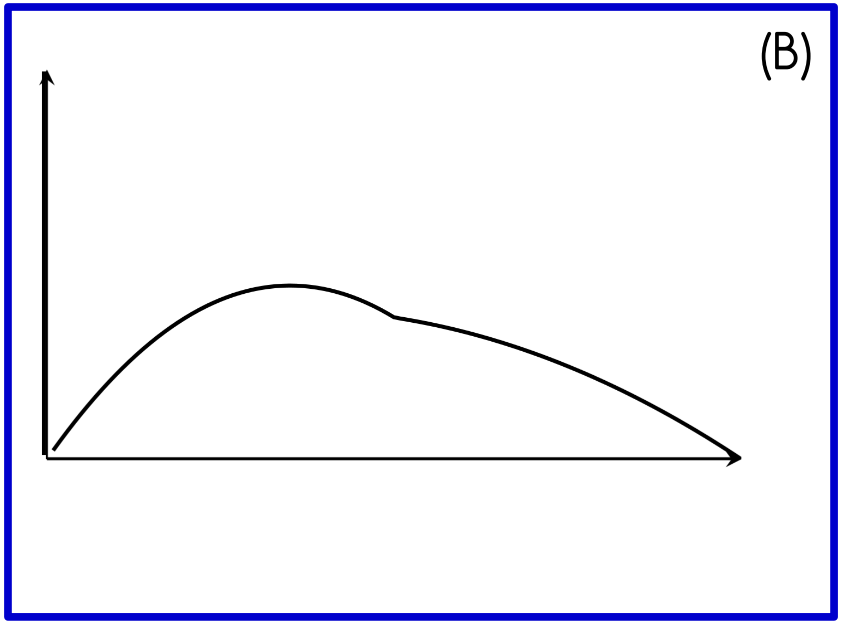
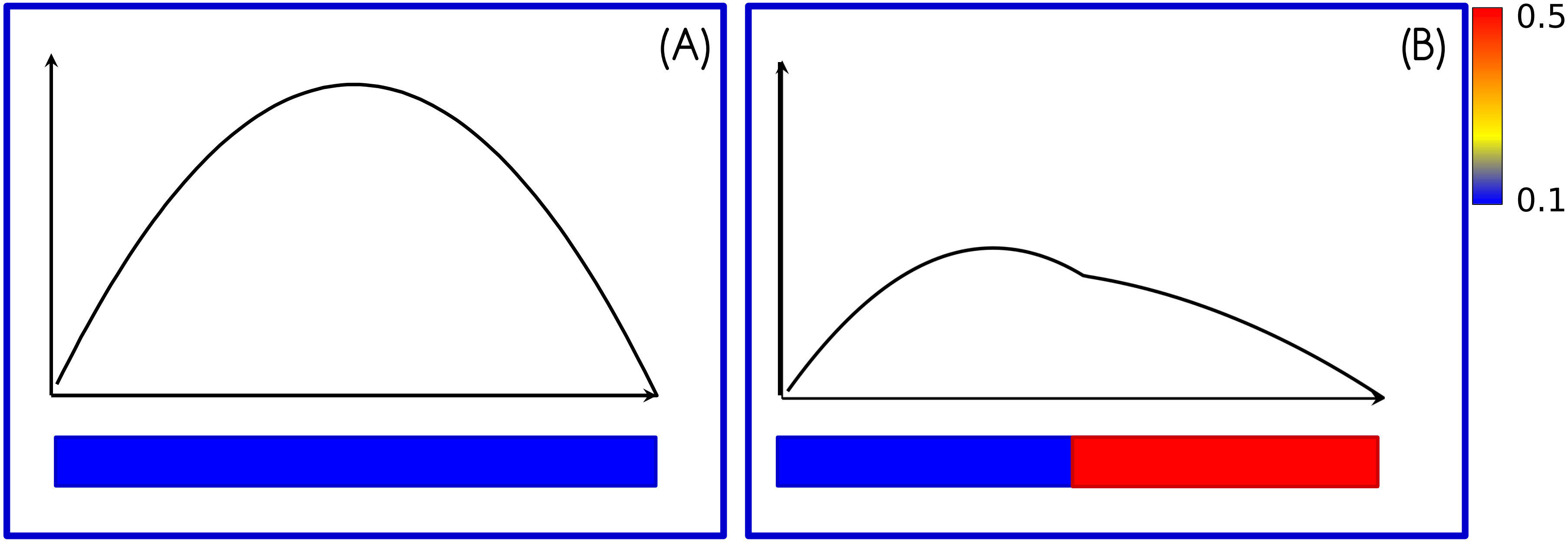
$$-\partial_x(a\partial_x u) = 1 \text{ in } (0,1), \qquad u(0) = u(1) = 0$$





Which plot corresponds to a homogeneous material ?

$$-\partial_x(a\partial_x u) = 1 \text{ in } (0,1), \qquad u(0) = u(1) = 0$$



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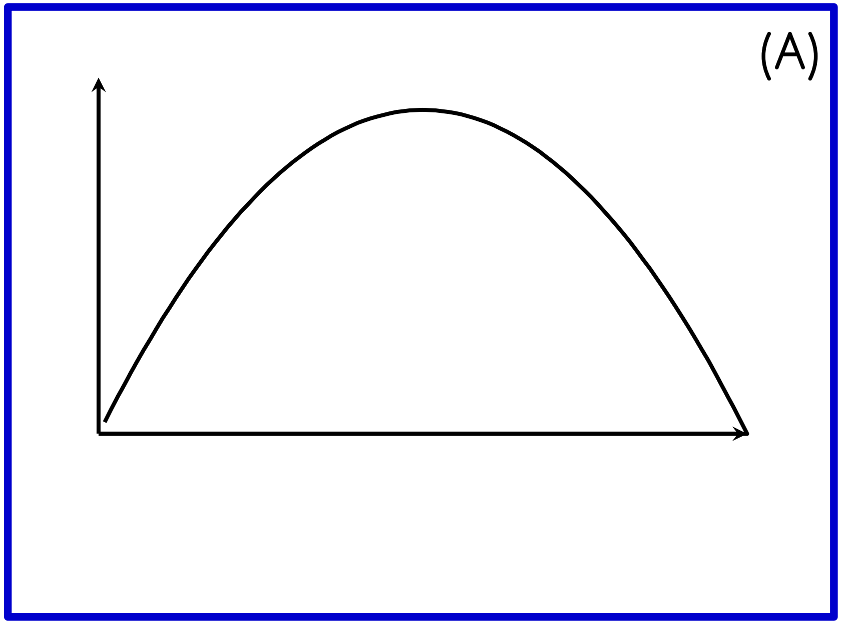
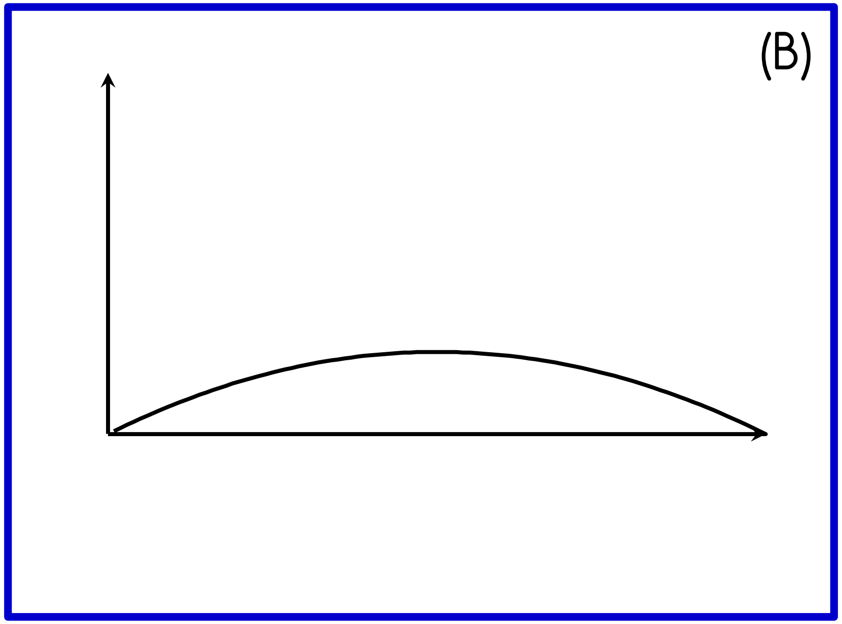
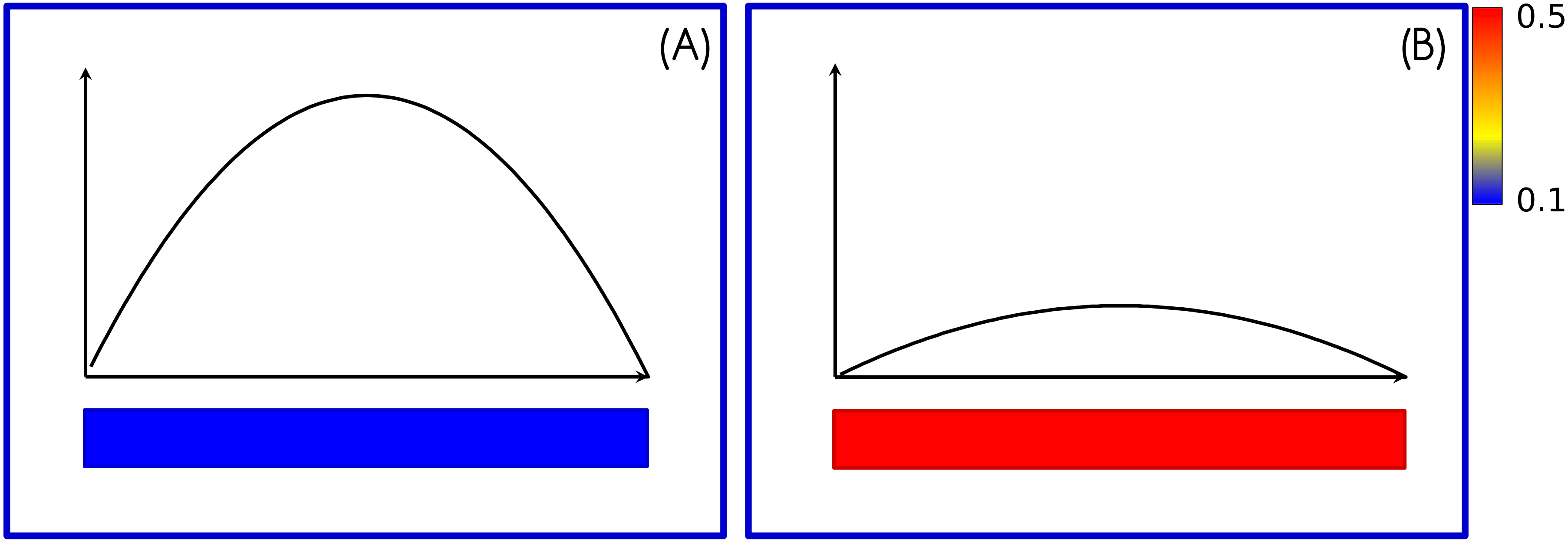




In which plot the conductivity is larger?

$$-\partial_x(a\partial_x u) = 1 \text{ in } (0,1), \qquad u(0) = u(1) = 0$$



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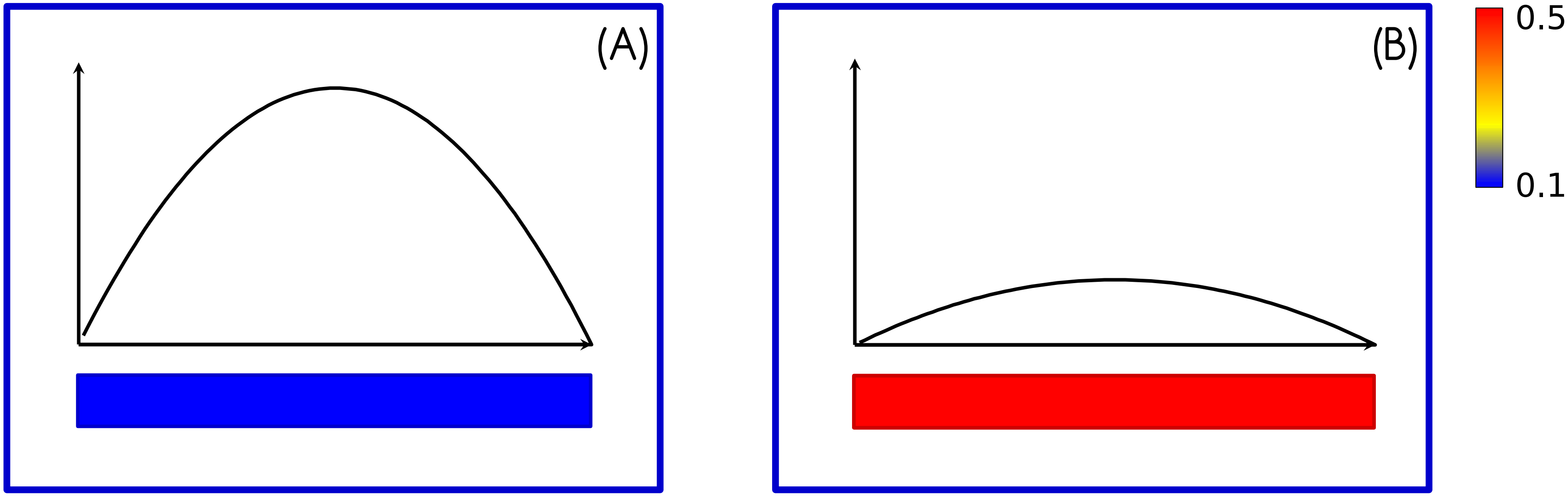
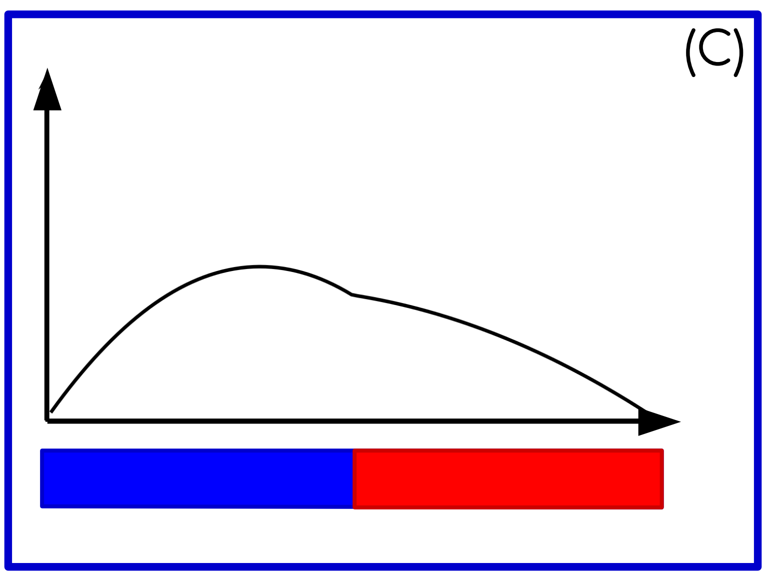




Observation:

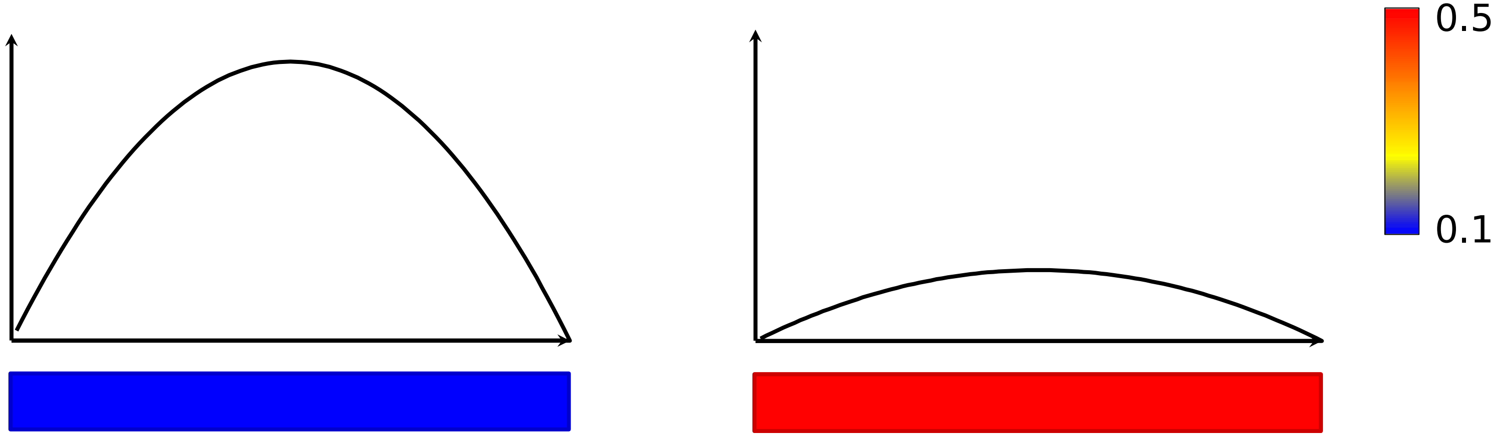
(1) material is homogeneous
$\Leftrightarrow$ u is a parabola

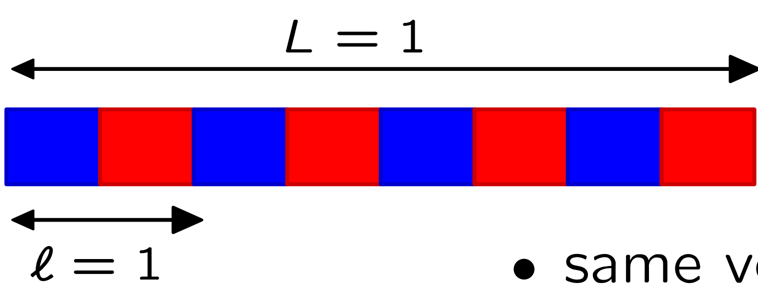
(2) in the homogeneous case:

$$u_{\max} = \frac{1}{8a}$$

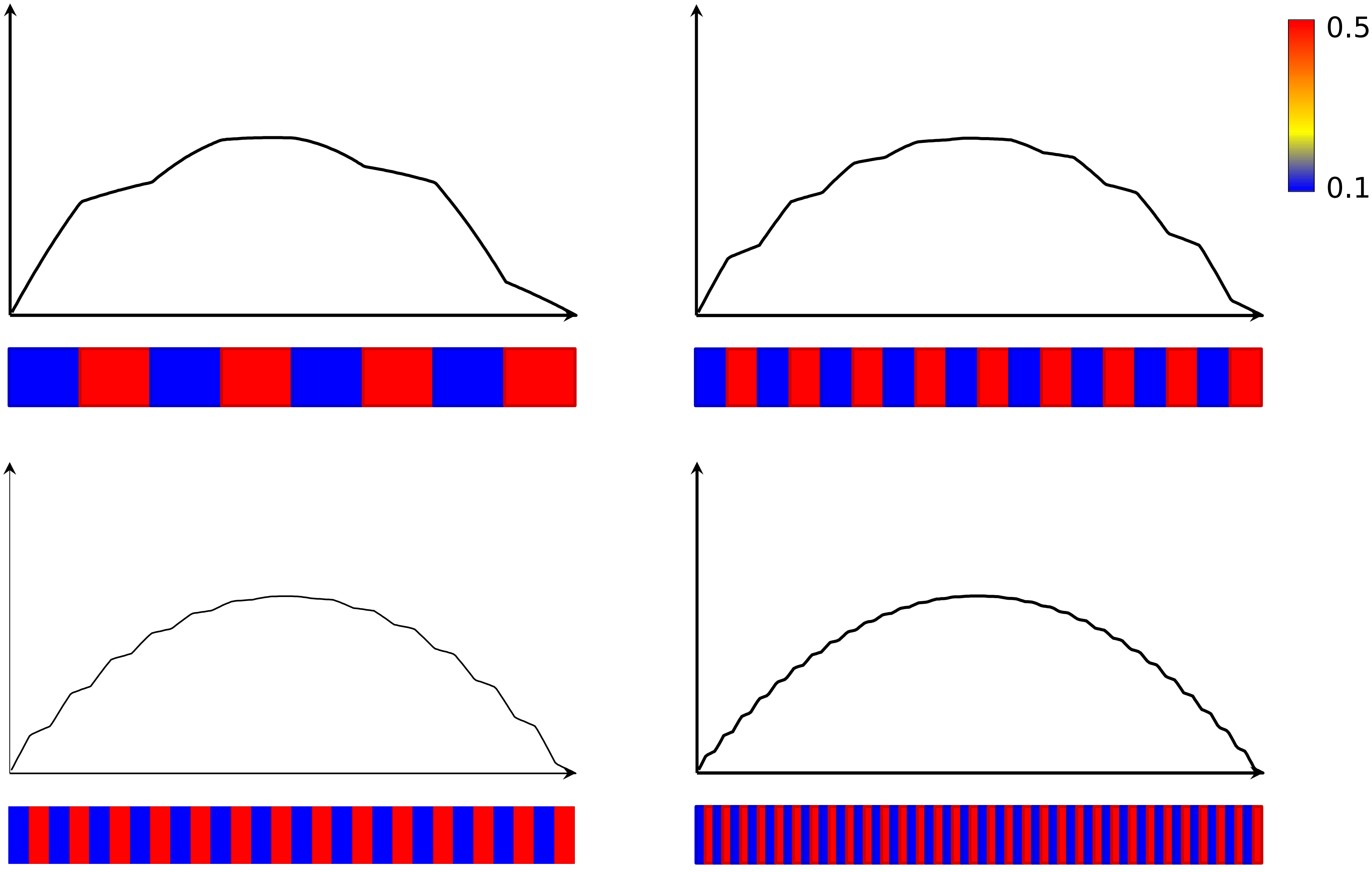
$$-\partial_x(a\partial_x u) = 1 \text{ in } (0,1), \qquad u(0) = u(1) = 0$$

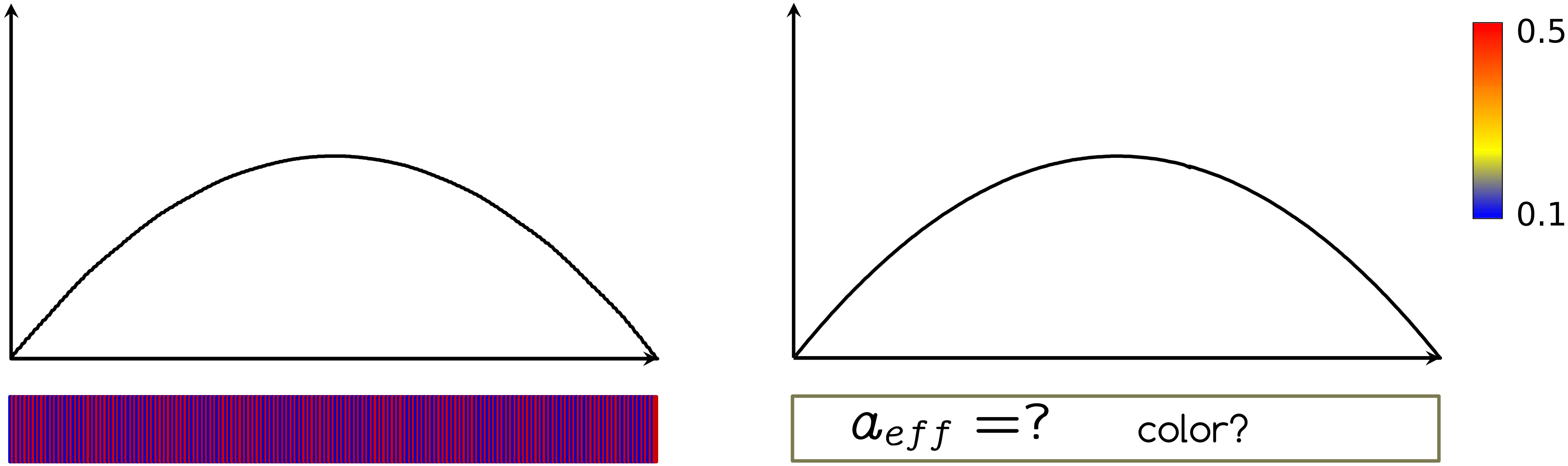
microstructured, periodic, two-phase material



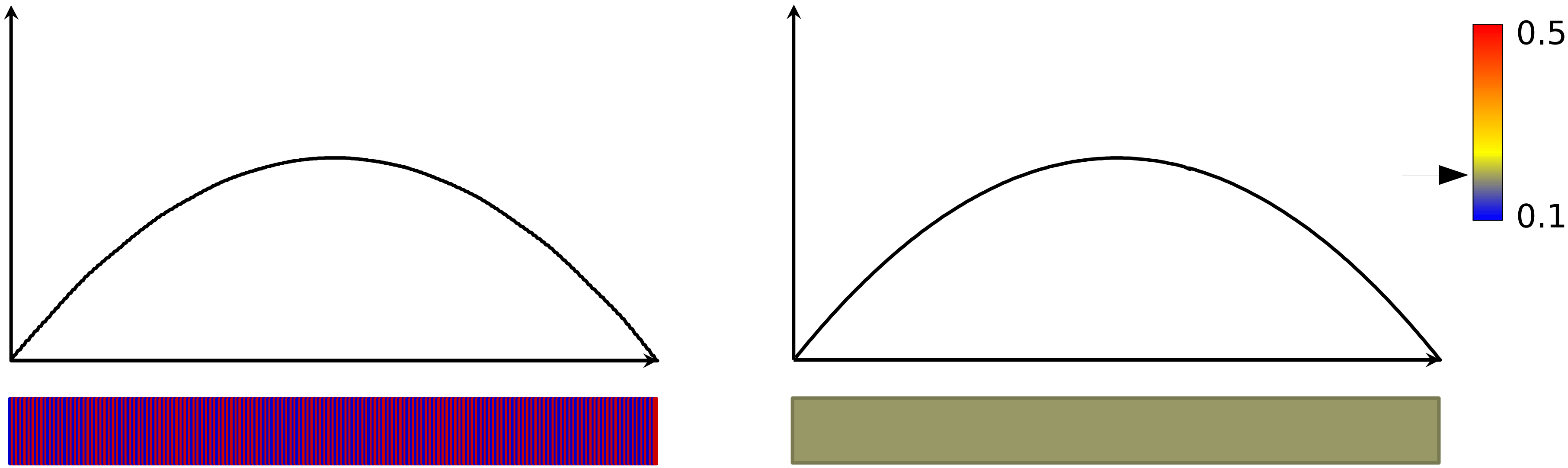


- same volume fraction of both phases
- behavior for $\ell \ll 1$?


0.5
0.1

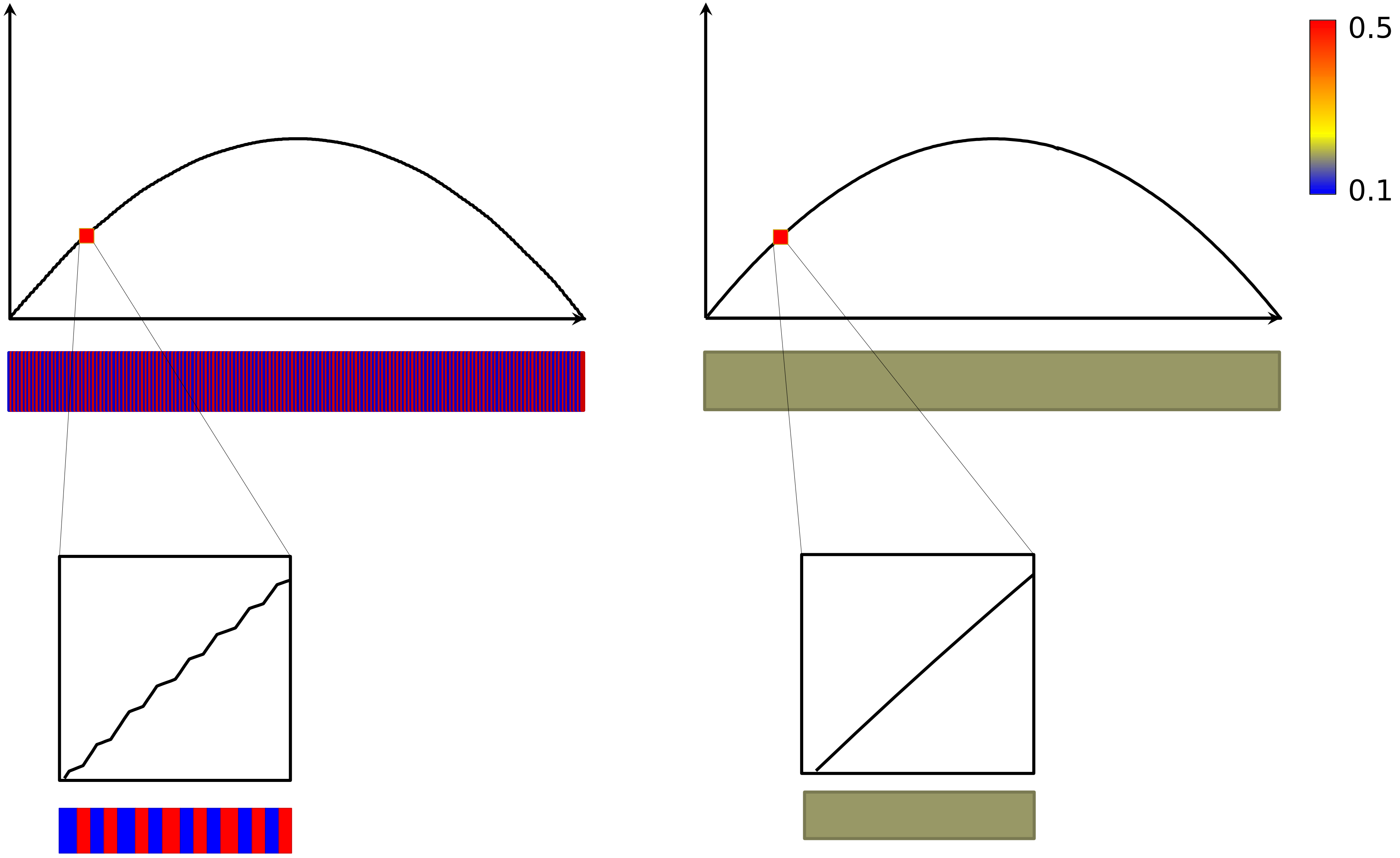


a_{eff} =?
color?
0.5
0.1



a is not a simple average!

Why does the left picture not contradict our rule
parabola $\Leftrightarrow$ homogeneous material ?


0.5
0.1

How to phrase this in the language of mathematics?

How to phrase this in the language of mathematics?

$$\begin{aligned} -\nabla \cdot a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon &= f && \text{in } \mathcal{O} \\ u_\varepsilon &= 0 && \text{auf } \partial\mathcal{O} \end{aligned}$$

with $a(\cdot)$ uniformly elliptic and periodic.

Theorem (Homogenization Gold Standard)

$\exists a_{hom} \in \mathbb{R}^{d\times d}$ uniformly elliptic, s.t. as $\varepsilon \downarrow 0$:

$$u_\varepsilon \rightharpoonup u_0 \qquad \text{weakly in } H^1(\mathcal{O})$$

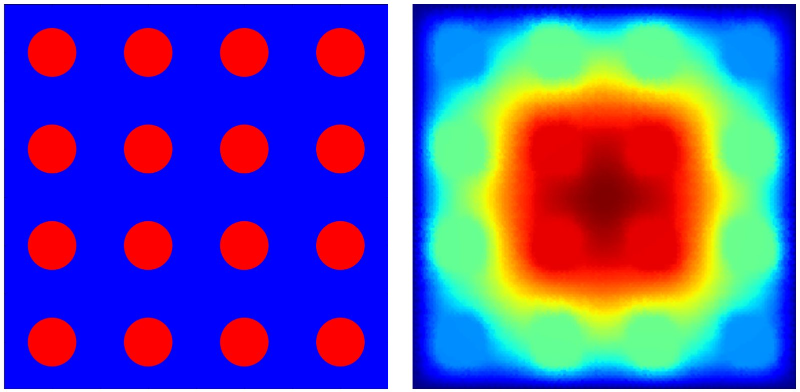
where u_0 is the unique weak solution to

$$\begin{aligned} -\nabla \cdot a_{hom}\nabla u_0 &= f && \text{in } \mathcal{O} \\ u_0 &= 0 && \text{auf } \partial\mathcal{O} \end{aligned}$$

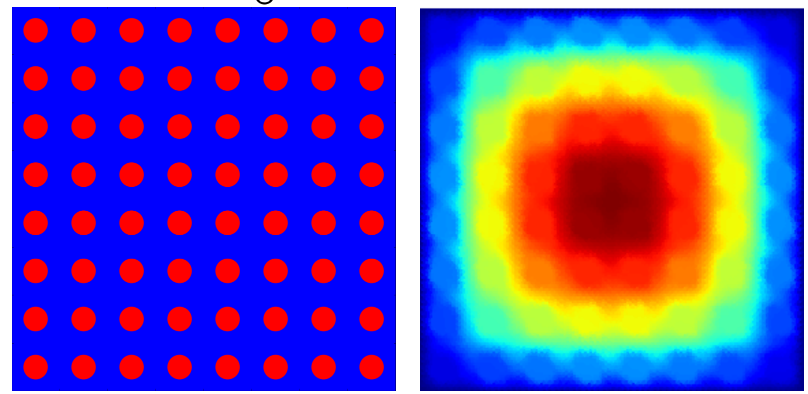
[Spagnolo, Bensoussan, Lions, Tartar, Papanicolaou & Varadhan, Kozlov, ...]

numerical illustration (two dimensional problem)

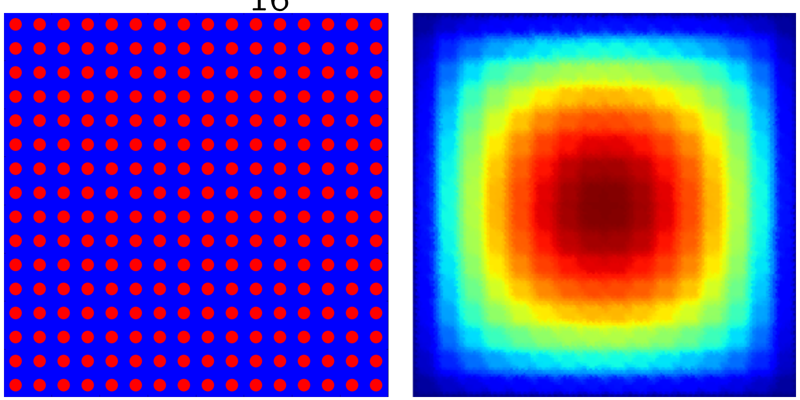
$\varepsilon = \frac{1}{4}$



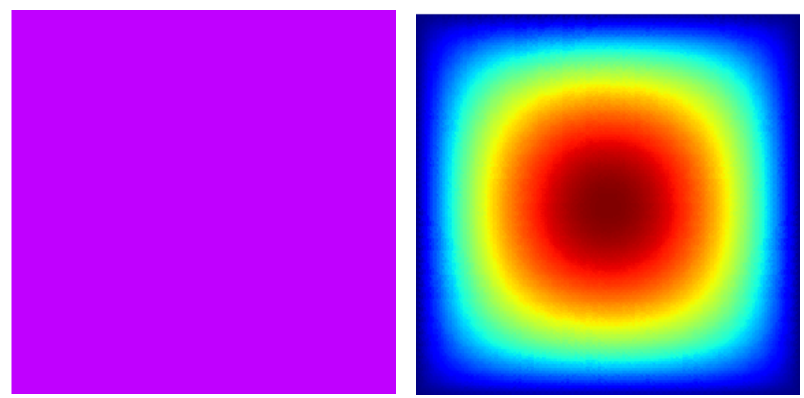
$\varepsilon = \frac{1}{8}$



$\varepsilon = \frac{1}{16}$



$\varepsilon \to 0$



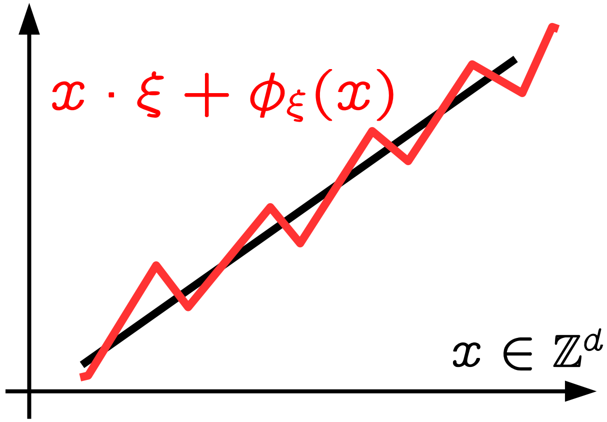
[Andreas Kunze]

formula for a_{hom}

$$\forall \xi \in \mathbb{R}^d : \qquad a_{\mathrm{hom}}\xi = \fint_{\square} a(y)(\nabla\phi_\xi(y) + \xi)\, dy.$$

corrector equation

$$-\nabla \cdot (a(\nabla\phi_\xi + \xi)) = 0 \quad \text{on } \mathbb{T}^d$$

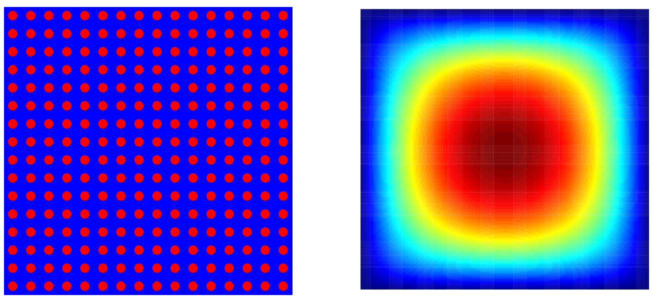


two-scale expansion

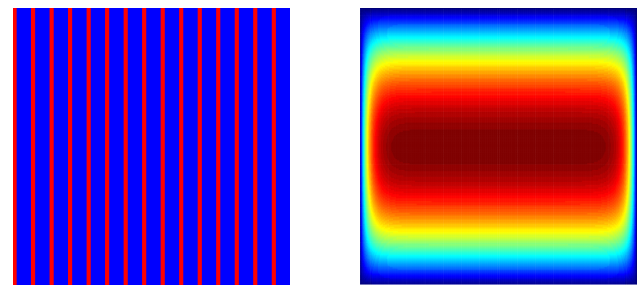
$$u_\varepsilon(x) = u_0(x) + \varepsilon \sum_{j=1}^{d} \phi_j(\tfrac{x}{\varepsilon})\partial_j u_0(x) + \text{h.o.t}$$

Homogenization formula yields
not a simple average

2 phase material



isotropic



anisotropic

(Simple) implications of the homogenization formula

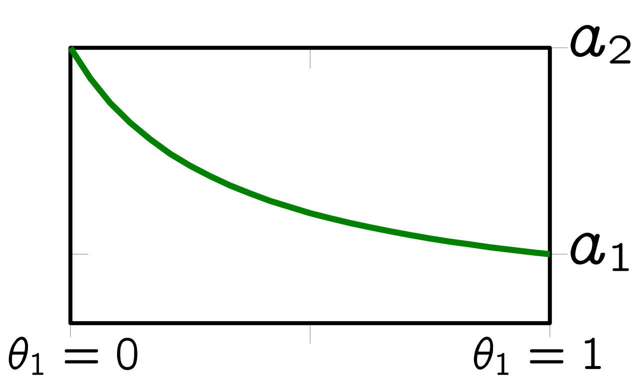
in dimension $d = 1$

$$a_{hom} = \frac{1}{\int_{(0,1)} \frac{1}{a(y)}\, dy}$$

(harmonic mean)

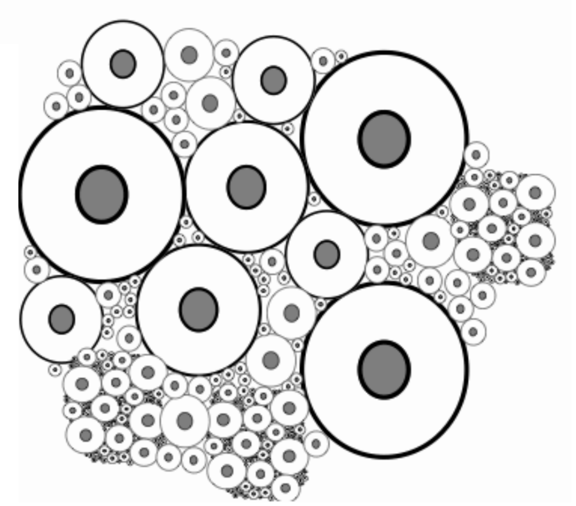
special case: 2-phasen material

$$a_{hom} = \frac{1}{\frac{\theta_1}{a_1} + \frac{1-\theta_1}{a_2}}$$



Hashin-Shtrikman Bounds

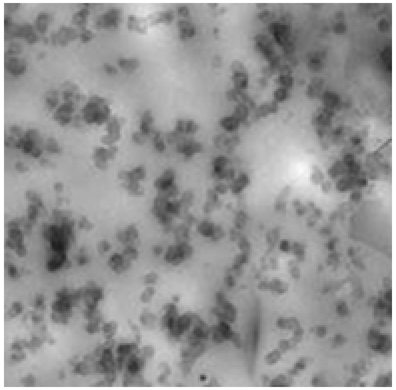
$d \geq 2$



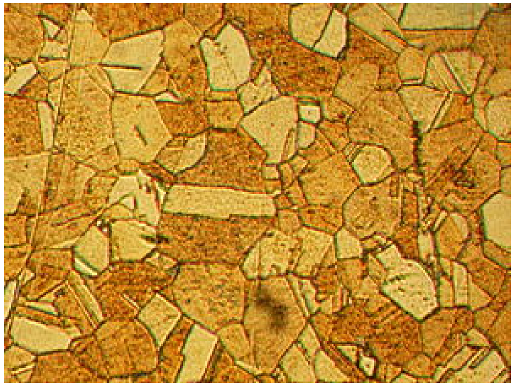
Beyond periodicity

Beyond periodicity

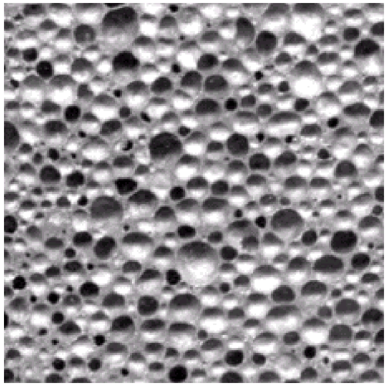
random media



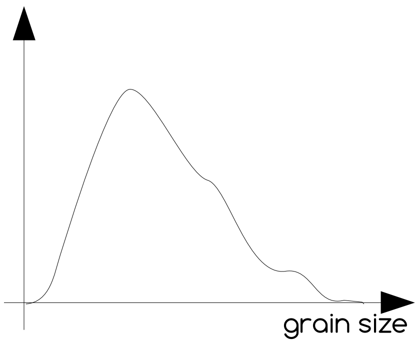
rubberwith carbon black
[Heinrich et al. '14]



brass
[Wikipedia]

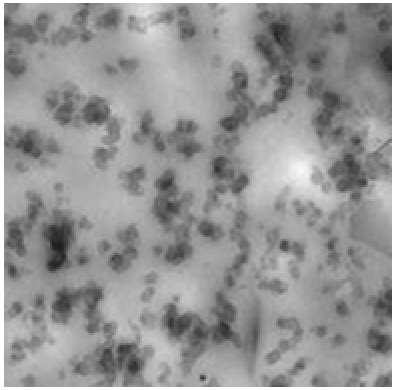


metallfoam
[Wikipedia]

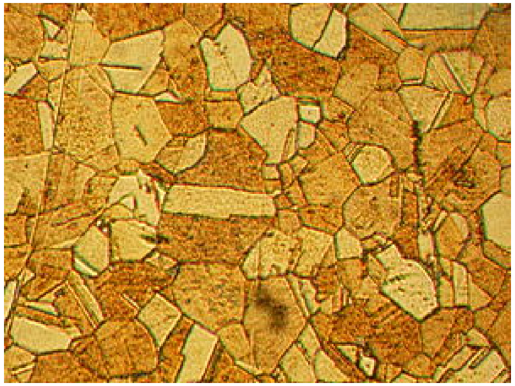


Beyond periodicity

random media



rubberwith carbon black
[Heinrich et al. '14]



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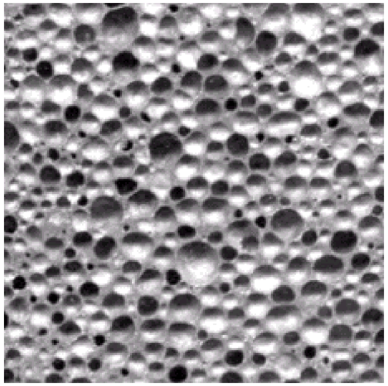

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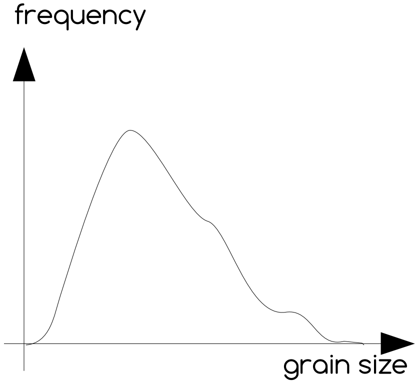


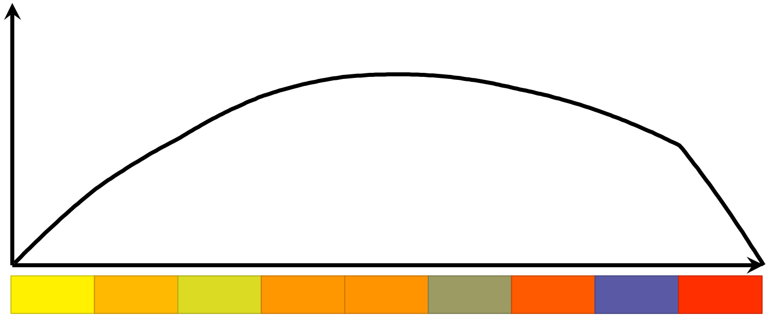
mathematical description

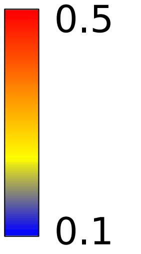
$$-\nabla \cdot (a(x)\nabla u(x)) \;=\; f(x)$$

random matrix field

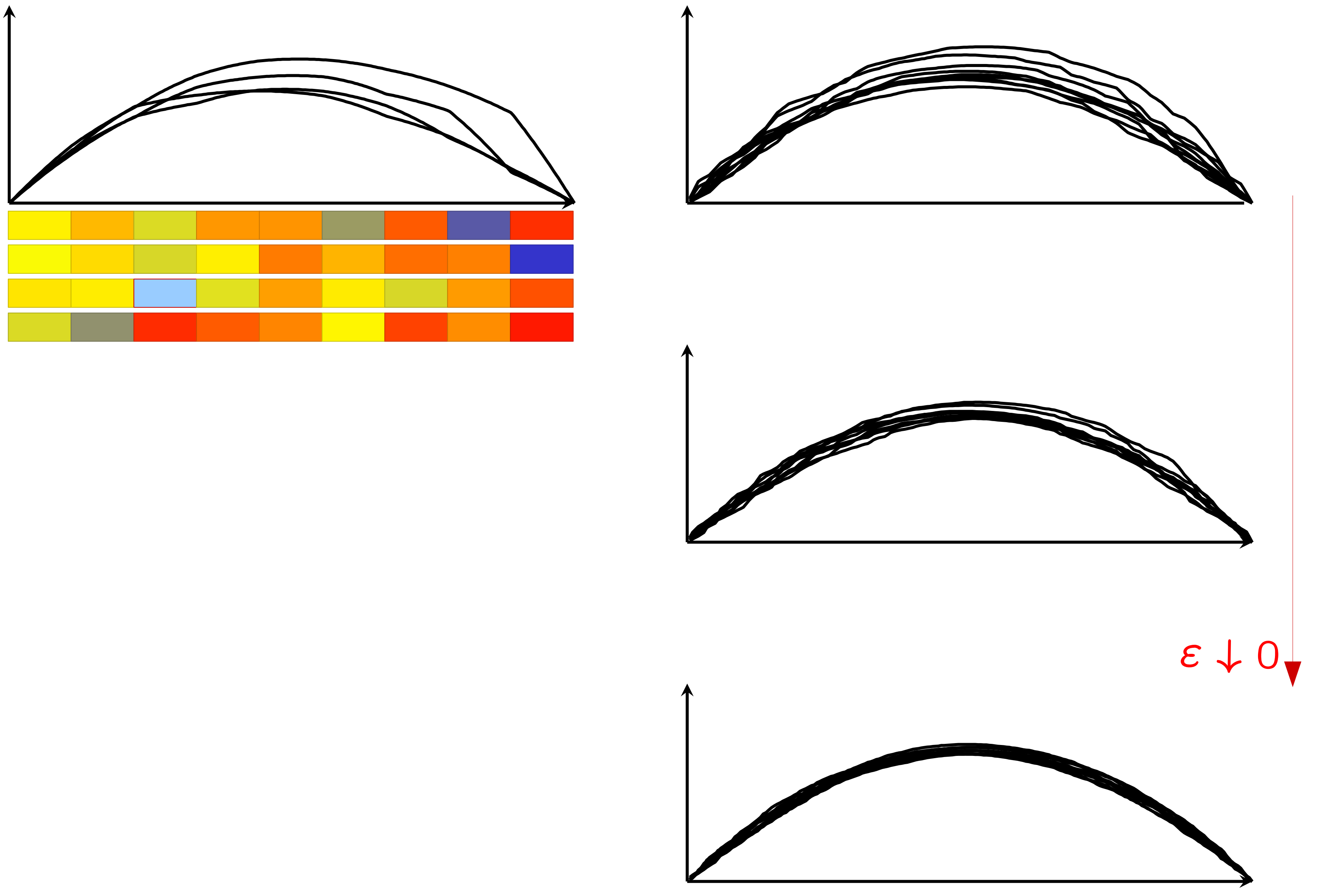
$$\boxed{\begin{aligned} \Omega &= \text{configuration space} \\ &= \{a(\cdot)\} \\ \langle\cdot\rangle &= \text{statistics on } \Omega \\ &= \text{probability measure on} \end{aligned}}$$

a numerical investigation

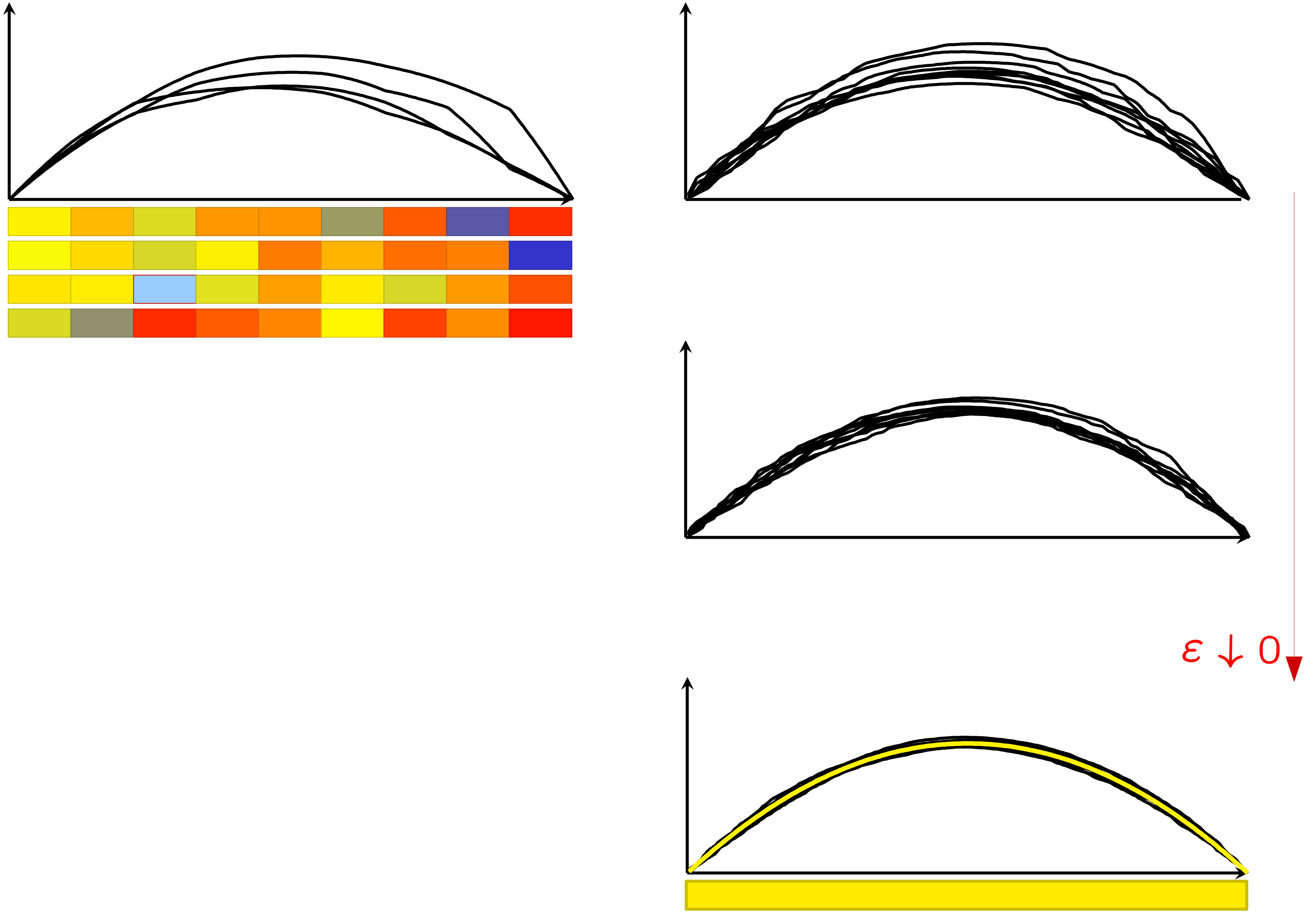




$$-\partial_x(a\partial_x u) = 1 \text{ in } (0,1), \qquad u(0) = u(1) = 0$$



$$-\partial_x(a\partial_x u) = 1 \text{ in } (0,1), \qquad u(0) = u(1) = 0$$


ε ↓ 0

Stochastic homogenization

$$\begin{aligned} -\nabla \cdot a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon &= f && \text{in } \mathcal{O} \\ u_\varepsilon &= 0 && \text{auf } \partial\mathcal{O} \end{aligned}$$

with $a(\cdot)$ uniformly elliptic, stationary and ergodic random field.

Theorem (stochastic homogenization)

$\exists a_{hom} \in \mathbb{R}^{d\times d}$ uniformly elliptic and deterministic, s.t.

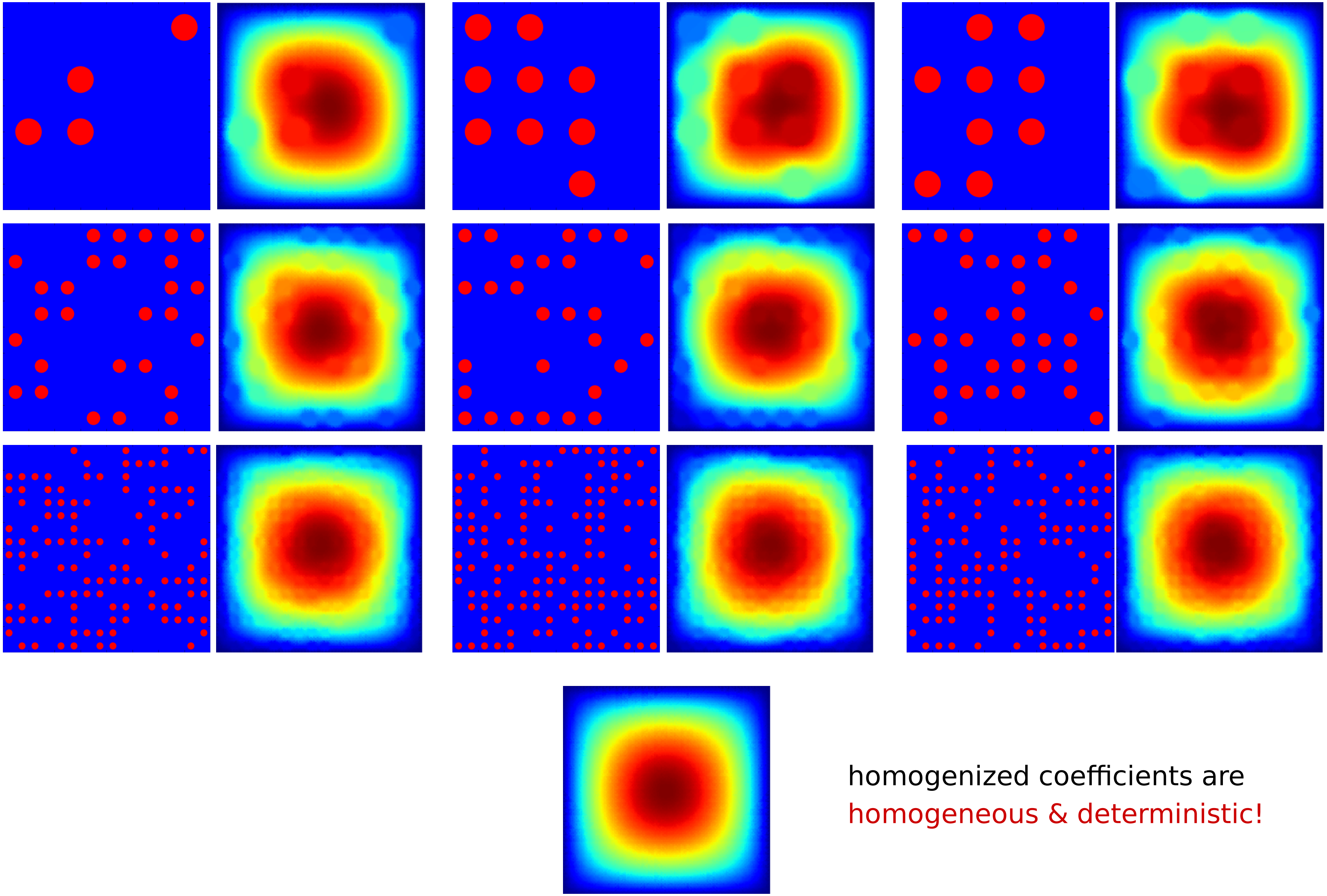
$$u_\varepsilon \rightharpoonup u_0 \qquad \text{weakly in } H^1(\mathcal{O}) \ \langle\cdot\rangle\text{-a.s.}$$

where u_0 is the unique weak solution to

$$\begin{aligned} -\nabla \cdot a_{hom}\nabla u_0 &= f && \text{in } \mathcal{O} \\ u_0 &= 0 && \text{auf } \partial\mathcal{O} \end{aligned}$$

[Papanicolaou & Varadhan '79, Kozlov '79]

numerical illustration (two dimensional problem)



homogenized coefficients are
homogeneous & deterministic!

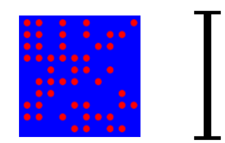
„Infinities“ in the stochastic case

invokes a corrector problem on infinite, high-dimensional space

$$a_{\mathrm{hom}} e_i = \underbrace{\lim_{L\uparrow\infty} \langle ⨍_{\Box_L}}_{\text{average over infinitely large box}} a(\nabla \phi_i + e_i) \rangle$$

...require approximation:

$$a_{\mathrm{hom},L}\, e_i = ⨍_{\Box_L} a(\nabla \phi_{L,i} + e_i)$$



L

random error: $a_{\mathrm{hom},L}$ fluctuates around mean $\langle a_{\mathrm{hom},L} \rangle$

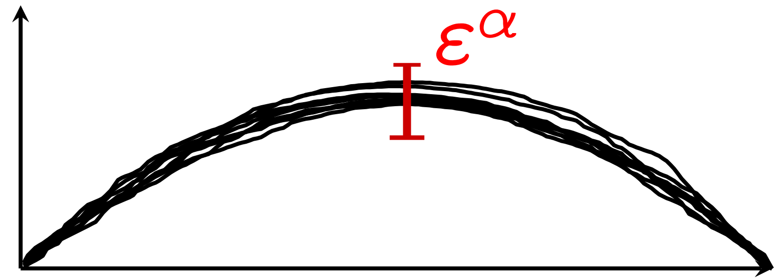
systematic error: $\langle a_{\mathrm{hom},L} \rangle \neq a_{\mathrm{hom}}$

Quantitative homogenization

Quantitative homogenization

homogenization error

$u_\varepsilon \rightharpoonup u_0$ rate ?



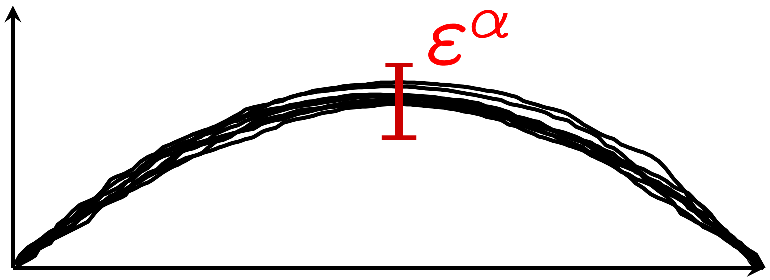
two-scale expansion

$$u_\varepsilon = u_0 + \varepsilon \phi_j(\tfrac{\cdot}{\varepsilon}) \partial_j u_0 + z_\varepsilon \qquad \|z_\varepsilon\|_{H^1} \leq C\varepsilon^\alpha, \qquad \alpha = ?$$

Quantitative homogenization

homogenization error

$$u_\varepsilon \rightharpoonup u_0 \qquad \text{rate ?}$$

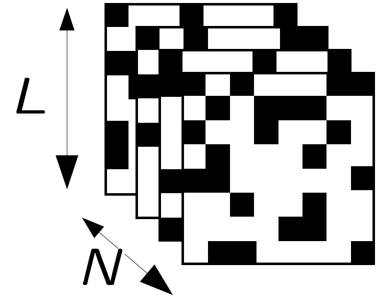


two-scale expansion

$$u_\varepsilon = u_0 + \varepsilon \phi_j(\tfrac{\cdot}{\varepsilon}) \partial_j u_0 + z_\varepsilon \qquad \|z_\varepsilon\|_{H^1} \le C\varepsilon^\alpha, \qquad \alpha = ?$$

quantitative approximation of a_{hom}

$$\|a_{\mathrm{hom}} - a_{\mathrm{hom},L}\| \le C f(L, N)$$



Programme of the winter school

- Introduction: the one-dimensional case
  - homogenization theorem
  - harmonic mean formula
  - two-scale expansion

- homogenization of elliptic equations in the periodic case
  - Tartar's method of oscillation test functions
  - the notion of the periodic corrector

- homogenization of elliptic equations in the stochastic case
  - description of random materials (stationarity and ergodicity)
  - the notion of the stochastic corrector (sublinearity)

- two-scale expansion: error representation via correctors

- quantitative stochastic homogenization
  - discrete setting
  - quantification of ergodicity via spectral gap
  - corrector bounds via semigroup decay