Problem sheet 2

Exercise 1. *Consider the situation of Lemma 2.12 and check that*

1. $\lambda_1^2, \ldots, \lambda_d^2$ *are the Eigenvalues of* $A^t A$
2. $|\lambda_1|, \ldots, |\lambda_d|$ *are the singular values of* A
3. $\det A = \lambda_1 \cdots \lambda_d$
4. $|A|^2 = \sum_{i=1}^d \lambda_i^2$ *and* $\|A\| = \lambda_d$
5. $A \in SO(d)$ *if and only if* $\{\lambda_1, \ldots, \lambda_d\} = \{1\}$.

Exercise 2. *Let* $A, B \in \mathbb{R}^{d\times d}$. *Show that:*

(i) $(\det A)\mathbf{Id} = A^t \mathrm{cof} A$ *(Cramer's rule),*

(ii) $\det(\mathrm{cof} A) = (\det A)^{d-1}$

(iii) $\mathrm{cof} A = (\det A) A^{-t}$, *if* A *is invertible,*

(iv) $\mathrm{cof} A = A$, *if* $A \in SO(d)$,

(v) $\mathrm{cof}(AB) = (\mathrm{cof} A)(\mathrm{cof} B)$,

(vi) $\mathrm{cof}(A^t) = (\mathrm{cof} A)^t$,

(vii) $\mathrm{cof}(A^{-1}) = (\mathrm{cof} A)^{-1}$ *if* A *is invertible.*

Exercise 3. *Let* Ω *be a domain,* $\varphi : \Omega \to \mathbb{R}^d$ *a* C^1*-deformation, and* $U \subset\subset \Omega$ *a* C^1*-domain. Then* $U^\varphi := \varphi(U)$ *is a* C^1*-domain. Moreover, if* $\nu(x)$ *denotes the outer normal to* ∂U *at* $x \in \partial U$, *then* $x^\varphi := \varphi(x) \in \partial\varphi(U)$ *and*

$$\nu^\varphi(x^\varphi) := \frac{(\mathrm{cof} D\varphi(x))\nu(x)}{|(\mathrm{cof} D\varphi(x))\nu(x)|},$$

is the outer normal to $\partial\varphi(U)$ *at* x^φ.

Exercise 4. *Let* $A \in C^\infty(\mathbb{R}^d; \mathbb{R}^{d\times d})$ *and* $f \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$. *Show that*

$$\mathrm{div}(A^t f) = (\mathrm{div}\, A) \cdot f + A \cdot Df.$$

Exercise 5. *Let* $\Omega \subset \mathbb{R}^d$ *be a* C^1*-domain and consider a (countable) covering of* $\partial\Omega$ *by open balls* $B_j \subset \mathbb{R}^d$, *a subordinate partition of unity* $\zeta_i \subset C_c^\infty(B_j)$ *with* $0 \le \zeta_j \le 1$ *and* $\sum_j \zeta_j = 1$ *on* $\partial\Omega$, *and parametrizations* (ϕ_j, V_j) *with* $B_j \subset \phi_j(V_j)$. *Convince yourself (briefly) that these objects exist and recall that* $f \in L^1(\partial\Omega)$ *if and only if the functions* $V_j \mapsto f(\phi_j(x))$ *are measurable and*

$$\int_{\partial\Omega} |f|\, d\mathcal{H}^{d-1} = \sum_j \int_{V_j} |f \circ \phi_j| \zeta_j \circ \phi_j \sqrt{\det D\phi_j^t D\phi_j}\, dx < \infty.$$

Show that there exists $\varepsilon > 0$ *and a sequence* $(f_k) \in C_c^1(U_\varepsilon)$ *where* $U_\varepsilon := \{x \in \mathbb{R}^d : \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$ *denotes a small neighbourhood of* $\partial\Omega$, *s.t.*

$$\lim_{k\to\infty} \|f - f_k\|_{L^1(\partial\Omega)} = 0.$$

Discuss and sketch the approximation argument in the proof of Theorem 2.24 (b).