MKMECH: Problem sheet 3

Exercise 1. *The* **Fundamental Theorem of Calculus** *says: Let* $\Omega \subset \mathbb{R}^d$ *open,* $f \in L^1_{loc}(\Omega)$. *Then*

$$\left[\forall \eta \in C_c^\infty(\Omega) \, : \, \int_\Omega f\eta = 0\right] \qquad \Rightarrow \qquad f = 0 \text{ fast sicher in } \Omega.$$

- *Give an elementary proof in the special case* $f \in C(\Omega)$.
- *Prove the statement in the case* $f \in L^q(\Omega)$, $1 \leq q \leq \infty$, *with help of the Theorem of Riesz.*
- *Let* $\Gamma := \partial B(0;1) \cap \{x \cdot e_1 > 0\}$ *and* $f \in C(\partial B(0;1))$. *Show that: If*

$$\int_{\partial B(0;1)} f\eta \, d\mathcal{H}^{d-1} = 0 \text{ für alle } \eta \in C^\infty(\overline{B(0;1)}) \text{ mit } \eta = 0 \text{ in } \partial B(0;1) \setminus \Gamma,$$

then $f = 0$ *on* Γ.

Definition 1.1 (Principal, normal, shear, and planar stresses)**.** *Let* $T = T^\varphi(x^\varphi)$ *denote the Cauchy strain tensor in* x^φ. *The Eigenvectors of* T *are called* **principial stress directions** *and the associated Eigenvalues* τ_i *are called* **principal stresses**. *For a direction* $\nu \in S^{d-1}$ *the scalar field* $T_N := T \cdot (\nu \otimes \nu) = T\nu \cdot \nu \in \mathbb{R}$ *is called the* **normal stress** *in direction* ν *and the vector field* $T_S := T\nu - T_N\nu \in \mathbb{R}^d$ *is called* **shear stress** *associated with* ν.

Exercise 2.

- *Fix* $\nu \in S^{d-1}$ *and let* T_N *and* T_S *denote the normal stress and shear stress in direction* ν, *respectively. Show that* $|T\nu|^2 = T_N^2 + |T_S|^2$.
- *Let* $d = 3$ *and let* $\tau_1 \geq \tau_2 \geq \tau_3$ *denote the principal stresses of* T. *Then,* τ_1 *(resp.* τ_3*) is given by the maximal (resp. minimal) normal stress* $\max_{\nu \in S^2} T \cdot (\nu \otimes \nu)$ *(resp.* $\min_{\nu \in S^2} T \cdot (\nu \otimes \nu)$*). Determine the direction* ν *that maximes the normal stress.*

Exercise 3. *For an elastic material the following are equivalent:*

(i) *the material is objective,*

(ii) $\hat{T}^D(RF) = R\hat{T}^D(F)R^t$ *for all* $R \in SO(d)$ *and* $F \in GL_+(d)$

(iii) $\hat{T}(RF) = R\hat{T}(F)$ *for all* $R \in SO(d)$ *and* $F \in GL_+(d)$

(iv) $\hat{\Sigma}(RF) = \Sigma(F)$ *for all* $R \in SO(d)$ *and* $F \in GL_+(d)$

(v) *there exists a map* $\tilde{\Sigma} : \mathbb{R}^{d\times d}_{sym} \cap GL_+(d) \to \mathbb{R}^{d\times d}_{sym}$ *such that* $\tilde{\Sigma}(F^tF) = \hat{\Sigma}(F)$ *for all* $F \in GL_+(d)$.

Exercise 2. Let $d = 3$ and $A \in \mathbb{R}^{3\times 3}_{sym}$. Let $\lambda_1, \lambda_2, \lambda_3$ denote the eigenvalues of A. Prove the following relations for the principal invariants of A:

$$\begin{aligned} I_1(A) &= \operatorname{trace}(A) = \lambda_1 + \lambda_2 + \lambda_3 \\ I_2(A) &= \operatorname{trace}\operatorname{cof}(A) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 \\ I_3(A) &= \det A = \lambda_1\lambda_2\lambda_3. \end{aligned}$$

Exercise 4. *Let* $d \geq 3$. *Show that: An elastic material is isotropic and frame indifferent, if and only if there exist functions* $\gamma_0, \ldots, \gamma_{d-1} : \mathbb{R}^d \to \mathbb{R}$ *such that*

$$\hat{\Sigma}(F) = \sum_{k=0}^{d-1} \gamma_k(\mathcal{I}(C))C^k$$

where $\hat{\Sigma}$ *denotes the response function for the second Piola-Kirchhoff stress tensor and* $C = F^tF$ *the right Cauchy-Green strain tensor.*
Hint: The Cayley-Hamilton theorem states that any matrix A *is the root of it's characteristic polynomial, i.e.* $\chi_A(A) = 0$ *or equivalently*

$$(-A)^d + \sigma_{k=1}^d I_k(A)(-A)^{d-k}.$$