Technische Universität Dresden
Bereich Mathematik und Naturwissenschaften

Stochastic unfolding and homogenization of evolutionary gradient systems

Dissertation
zur Erlangung des akademischen Grades

Doctor rerum naturalium
(Dr. rer. nat.)

vorgelegt dem
Bereich Mathematik und Naturwissenschaften
der Technischen Universität Dresden

von
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Tag der Einreichung: 05.04.2019

Die Dissertation wurde in der Zeit von Oktober 2015 bis April 2019 im Institut für Wissenschaftliches Rechnen angefertigt.

Abstract

The mathematical theory of homogenization deals with the rigorous derivation of effective models from partial differential equations with rapidly-oscillating coefficients. In this thesis we deal with modeling and homogenization of random heterogeneous media. Namely, we obtain stochastic homogenization results for certain evolutionary gradient systems. In particular, we derive continuum effective models from discrete networks consisting of elasto-plastic springs with random coefficients in the setting of evolutionary rate-independent systems. Also, we treat a discrete counterpart of gradient plasticity. The second type of problems that we consider are gradient flows. Specifically, we study continuum L^2-type gradient flows driven by λ-convex energy functionals. In stochastic homogenization the derived deterministic effective equations are typically hardly-accessible for standard numerical methods. For this reason, we study approximation schemes for the effective equations that we obtain, which are well-suited for numerical analysis. For the sake of a simple treatment of these problems, we introduce a general procedure for stochastic homogenization – the stochastic unfolding method. This method presents a stochastic counterpart of the well-established periodic unfolding procedure which is well-suited for homogenization of media with periodic microstructure. The stochastic unfolding method is convenient for the treatment of equations driven by integral functionals with random integrands. The advantage of this strategy in regard to other methods in homogenization is its simplicity and the elementary analysis that mostly relies on basic functional analysis concepts, which makes it an easily accessible method for a wide audience. In particular, we develop this strategy in the setting that is suited for problems involving discrete-to-continuum transition as well as for equations defined on a continuum physical space. We believe that the stochastic unfolding method may also be useful for problems outside of the scope of this work.

Acknowledgments

First of all, I would like to express my sincere gratitude to my supervisor Prof. Stefan Neukamm for his constant support, motivation and invaluable guidance during my PhD studies. I greatly appreciate that he was always patient and ready to discuss any question that I had. I really enjoyed working with him and I am looking forward to our future discussions.

I am greatly indebted to Prof. Alexander Mielke for kindly accepting to be the reviewer of this thesis and for his useful comments during our meetings.

I wish to thank the people with whom I had the chance to work on joint projects. Special thanks go to Dr. Martin Heida for our interesting discussions and for his hospitality during my stays at the Weierstrass Institute in Berlin. I am very grateful to Prof. Goro Akagi for sharing his knowledge about evolutionary equations with me, for his useful suggestions and for the kind hospitality during my stay at the Tohoku University. I thank PD Dr. Marcus Waurick for introducing me to the interesting field of abstract operator equations and for his hospitality during my stay at the University of Strathclyde.

I gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – project number 405009441, and in the context of TU Dresden's Institutional Strategy "The Synergetic University".

Also, I really enjoyed the atmosphere at the TU Dresden. In particular, I wish to thank all my friends and colleagues at the Institute of Scientific Computing. Many thanks go to Dr. Mathias Schäffner for the many discussions during his stay in Dresden. I thank Dr. Carina Geldhauser for reading parts of the manuscript and for her useful comments. I am also grateful to Prof. Oliver Sander and Prof. Ralph Chill for useful discussions.

Moreover, I wish to thank all my friends, with a special mention to Selena Ilić who was always there to lift my spirits.

Last but not the least, I would like to thank my sister Dora and my parents Ljubica and Zoltan for being my greatest support during my studies and in life in general. Also, immense thanks to Dora for proofreading the entire manuscript of the thesis.

Mario Varga, April 2019

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Introduction

Modeling of heterogeneous materials plays a significant role in many aspects of contemporary science. For example, in mechanics the analysis of granular media, cellular and composite materials, and truss-like structures requires the development of multiscale models. Typically, such models give rise to boundary value problems or evolutionary problems in the form of partial differential equations with coefficients that feature rapid spatial oscillations. In particular, if the oscillations appear on a very small scale, say $\varepsilon \ll 1$, an efficient numerical treatment is inaccessible. Therefore, the derivation of effective (homogeneous) macroscopic models is vital for practical purposes. The mathematical theory of homogenization deals with the rigorous justification of such effective models by means of asymptotic analysis in the limit $\varepsilon \to 0$.

In this work we study random heterogeneous media. In particular, the purpose of this thesis is twofold. On the one hand, we consider specific evolutionary equations and obtain stochastic homogenization results for them. On the other hand, to allow a simple treatment of these equations, we develop a general strategy for stochastic homogenization – the stochastic unfolding method. We view this method as a general and easily accessible technique for modeling and homogenization of random media that presents an extension of the well-established periodic unfolding procedure to the random setting. In this respect, one of the two main achievements of this study is the development and detailed analysis of the stochastic unfolding method. The second main achievement are the stochastic homogenization results that we derive for certain evolutionary gradient systems. In particular, we obtain stochastic homogenization and discrete-to-continuum transition results for discrete versions of elasto-plasticity and gradient plasticity in the setting of evolutionary rate-independent systems. Also, we derive a stochastic homogenization result for a continuum L^2-type gradient flow given in terms of a λ-convex energy functional. We also consider approximation schemes for the obtained effective equations and prove their convergence with the help of the stochastic unfolding method.

Early contributions in the theory of homogenization originate from the 60s and 70s, e.g., in [Hil63] Hill considered elastic composite materials, [BLP11] is an early standard reference, and in [Tar77, MT97] Tartar and Murat developed the notion of *H-convergence*, we also refer to the works by Zhikov et al. [ZKON79, JKO12]. Variational problems were considered by Marcellini [Mar78], Spagnolo [Spa76] via *G-convergence*, and De Giorgi and Franzoni using *Γ-convergence* [DGF75]. In the 80s and later, homogenization was intensively studied for a wide range of problems including non-convex integral functionals (e.g., Müller [Mül87, GMT93] and Braides [Bra85]), or the topic of effective flow through porous media (e.g., see Hornung et al. [ADH90, HJ91, HJM94, Hor12] and Allaire [All89]). Most results in homogenization theory discuss problems with periodic microstructure, for which specific analytic tools for homogenization of linear (or monotone) operators are developed, including the notions of two-scale convergence and periodic unfolding [Ngu89, All92, CDG02]. In recent times considerable interest in applied mathematics emerged in understanding random heterogeneous materials, i.e., materials whose properties on a small length-scale are described only on a statistical level (see [Tor13, OS07]). Although the first results in stochastic homogenization were already obtained in the 70s and 80s for linear elliptic equations and convex functionals, see [PV81, Koz79, DMM85, DMM86], the theory in this setting is still less developed than in the periodic case and it is the object of various recent studies, e.g., regarding error estimates and regularity properties (see [GO11, GO12, GNO15, GNO14a, GNO14b, AS16, AKM17]), or modeling of random materials [ZP06, ACG11, CR17, HPV17, Hei17, HN17, BSS17].

The notion of two-scale convergence was introduced by Nguetseng [Ngu89] and it is further investigated by Allaire [All92] (see also [LNW02]). This notion grants a very convenient approach to periodic homogenization since two-scale limits capture information about the oscillatory behavior of rapidly-oscillating sequences, and it has been applied to a great variety of problems (see [LNW02, Section 8] for a list of some of the many references). In [ADH90] the so-called *dilation technique (operator)* is used for the study of flow through periodic porous media and similar techniques have been employed for other specific problems, e.g., in [BLM96, Len97, AC98, Gri96]. Stemming from this strategy, in [CDG02] the *periodic unfolding method* is introduced as a systematic approach to periodic homogenization (for further investigations see [CDG08, Vis04, MT07]). In particular, a linear isometric operator – the *periodic unfolding operator* – is introduced. By means of a local "blow-up", this operator transforms equations with rapidly oscillating coefficients to "unfolded" problems with mildly-varying coefficients. Also, it turns out that two-scale limits are equivalently characterized as weak limits of unfolded sequences in an extended space. In recent years this method has been applied to many multiscale problems, e.g., see [CDDA04, Gri04, MT07, Neu10, MRT14, Pta15, CGM15, LR18, PP17, HK17].

In the stochastic setting, the notion of two-scale convergence is generalized in [BMW94] (see also [AW98, SW11a]) and in [ZP06] (see also [Fag08, Hei11]). Yet, as far as the author knows, the concept of unfolding has not been investigated earlier in this case. We extend the idea of the periodic unfolding procedure to the stochastic case. Namely, we introduce a linear isometric operator, the *stochastic unfolding operator*, that enjoys many similarities to the periodic unfolding operator. Also, as in the periodic case, stochastic two-scale convergence in the mean from [BMW94] might be equivalently characterized as weak convergence of unfolded sequences. In this respect, we develop a general procedure for stochastic homogenization, which allows us to systematically extend periodic homogenization results obtained by unfolding to their stochastic counterparts, and to investigate new issues arising from the randomness of the equations, e.g., practical approximations for effective systems. Despite the many similarities to periodic unfolding, some difficulties arise in the stochastic case that have to be carefully treated. In this work, the stochastic unfolding operator is introduced for two settings. Namely, first we consider a notion suited for discrete-to-continuum transition problems (such as partial *difference* equations); second, we investigate the theory for problems given on a continuum physical space (such as partial *differential* equations).

The objective of our applications are evolutionary equations of the form

$$D\mathcal{R}_\varepsilon(\dot{y}(t)) + D_y\mathcal{E}_\varepsilon(t, y(t)) = 0, \tag{1}$$

where $y : [0,T] \to Y$ is the solution and Y is a Hilbert space. Above, $\mathcal{R}_\varepsilon : Y \to \mathbb{R}$ is a convex *dissipation* functional, $\mathcal{E}_\varepsilon : [0,T] \times Y \to \mathbb{R} \cup \{+\infty\}$ is an *energy* functional, and D and D_y denote suitable notions of derivative (or subderivative). Systems of this form are referred to as *evolutionary gradient systems* (*EGS*). In the case that $\mathcal{R}_\varepsilon$ is positively homogeneous of degree 1, e.g., $\mathcal{R}_\varepsilon(v) =$

$\|v\|$, systems of this type fall under the category of *evolutionary rate-independent systems* (*ERIS*) (see [MR15]). If $\mathcal{R}_\varepsilon$ has a quadratic structure, e.g., $\mathcal{R}_\varepsilon(v,v) = \langle rv, v\rangle$ with $r \in \mathrm{Lin}(Y)$ being positive-definite, equation (1) corresponds to a usual Hilbert space gradient flow. Asymptotic analysis of sequences of EGS (above we might see $\varepsilon \to 0$ as a parameter) is important not only for homogenization purposes, but also for a great variety of other problems, e.g., dimension reduction problems or derivation of sharp-interface limits. For this reason, in the last decades novel general strategies for the treatment of sequences of abstract EGS were developed (see [Mie16] and the references therein). In this setting, periodic homogenization results via unfolding are obtained, e.g., for elasto-plasticity [MT07], gradient plasticity [Han11], reaction-diffusion systems [MRT14, Rei15], Cahn-Hilliard equations [LR18].

In this thesis we consider stochastic homogenization for two specific cases of EGS. First, we treat a discrete version of elasto-plasticity and gradient plasticity with random and oscillating coefficients. In this case, we deal with an ERIS, where $\mathcal{R}_\varepsilon$ (positively 1-homogeneous and convex) and $\mathcal{E}_\varepsilon$ (convex) are integral functionals with random and rapidly oscillating integrands (ε denotes the small scale of the oscillations). Second, we investigate an L^2-type gradient flow where $\mathcal{R}_\varepsilon$ (quadratic) and $\mathcal{E}_\varepsilon$ (λ-convex) are as well random and rapidly oscillating integral functionals given on a continuum physical space. Based on standard abstract strategies combined with stochastic unfolding, we obtain homogenization results for these systems. Also, a peculiarity in stochastic homogenization is that most often a direct computation of the effective properties of limit systems is inaccessible by usual numerical methods and for this reason approximation algorithms are developed. A standard method for approaching such problems is the so-called *representative volume element method*, see [Owh03, BP04, EGMN14] and references therein. We present approximation schemes for the above effective systems based on this method and prove their convergence using the stochastic unfolding procedure.

References. This thesis is mostly based on the papers: [NV18] written by Stefan Neukamm and the author, and [HNV18, HNV19] written by Martin Heida, Stefan Neukamm and the author. Moreover, the thesis contains additional original work by the author. We present a detailed declaration on this matter in the summaries at the beginning of Parts II and III.

Outline. We present a brief reading guide for this thesis:

- Part I collects briefly some key results and methods in the theories of homogenization and evolutionary gradient systems. This part might be skipped by the reader familiar with these fields. In particular, in Sections 1 and 2 we present an introduction to periodic and stochastic homogenization and to two-scale methods for homogenization. These two sections serve us to put the proposed stochastic unfolding method into context. We recall existence results and discuss strategies for asymptotic analysis of evolutionary rate-independent systems and gradient flows in Sections 3 and 4. The applications that we consider later are phrased in the settings of these two sections.

- In Part II we develop the stochastic unfolding method. We start this part with a short summary of the main results. In Section 5 we define the stochastic unfolding operator in a discrete setting and examine its main properties. Section 6 is the continuum counterpart to Section 5 where an unfolding operator suited for problems defined on a continuum physical space is considered. In Section 7 we explain the stochastic unfolding procedure on a simple example of convex minimization and we briefly discuss some additional topics such as approximation schemes for effective systems and implications of stochastic unfolding to periodic homogenization.

- In Part III we apply the stochastic unfolding method to stochastic homogenization of some evolutionary gradient systems. We start this part with a short summary of the main achievements. In Section 8 we treat discrete models of elasto-plasticity and gradient plasticity in the setting of evolutionary rate-independent systems. We obtain homogenization and discrete-to-continuum transition results, and we discuss approximation algorithms for the corresponding effective systems. Section 9 is devoted to a stochastic homogenization result for an L^2-type gradient flow which is driven by a λ-convex energy functional. Also, we consider an approximation scheme for the homogenized system in a simplified case of an Allen-Cahn type equation.

Notation

- (ε) denotes a sequence of positive real numbers that converges to 0, most often we only write ε and similarly we write u_ε for a sequence of functions instead of $(u_\varepsilon)_\varepsilon$.
- d is a natural number denoting the dimension of the Euclidean space $\mathbb{R}^d$ and $\{e_i\}_{i=1,\dots,d}$ denotes the canonical basis.
- All vector spaces considered in this thesis are real vector spaces.
- If Y is a topological space, we denote by $\mathcal{B}(Y)$ its Borel σ-algebra. If $Y = \mathbb{R}^d$, the Lebesgue σ-algebra is denoted by $\mathcal{L}(\mathbb{R}^d)$.
- If $(Y, \mathcal{F})$ and $(X, \mathcal{G})$ are measurable spaces, the product σ-algebra is denoted by $\mathcal{F} \otimes \mathcal{G}$. For measurable mappings $f : Y \to X$ we frequently use the expression $(\mathcal{F}, \mathcal{G})$-measurable. In the case that $X = \mathbb{R}$ and $\mathcal{G} = \mathcal{B}(\mathbb{R})$ we only write $\mathcal{F}$-measurable.
- We frequently use the notation $⨍_S \cdot d\mu(s)$ for the averaged integral $\frac{1}{\mu(S)} \int_S \cdot d\mu(s)$.
- If Y is a Banach space, its dual space is denoted by Y^*, and the duality pairing for $\xi \in Y^*$ and $y \in Y$ is denoted by $\langle \xi, y \rangle_{Y^*,Y}$. The norm is denoted by $\|\cdot\|_Y$. If Y is a Hilbert space, the scalar product of $y_1, y_2 \in Y$ is denoted by $\langle y_1, y_2 \rangle_Y$. If the context is clear, we occasionally drop the index "Y".
- We use the letter c to denote a positive constant that is independent of the quantities of importance in the particular arguments that we consider and it may vary from line to line.
- We use the notation $\Box = [0,1)^d$ for the unit cell of periodicity and $\Box_\# = \mathbb{R}^d / \mathbb{Z}^d$ for the unit torus.
- We use the following shorthands: a.e. = almost everywhere; a.a. = almost all; l.s.c. = lower semi-continuous; PDE = partial differential equation(s); EGS = evolutionary gradient system(s); ERIS = evolutionary rate-independent system(s).

Part I

Preliminaries

1 Introduction to homogenization

In this section, we collect some basic results and notions from the theory of periodic and stochastic homogenization considering the example of an elliptic PDE with rapidly oscillating coefficients. For detailed studies we refer to the standard textbooks [BLP11, CD00, JKO12].

Periodic homogenization

Let $\varepsilon > 0$, $Q \subset \mathbb{R}^d$ be open and bounded, and we use the notation $\square = [0,1)^d$ for the reference cell of periodicity. We consider a coefficient field $A \in L^\infty(\mathbb{R}^d; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and we assume that there exists $c > 0$ such that $A(x)F \cdot F \geq c|F|^2$ for a.a. $x \in \mathbb{R}^d$ and all $F \in \mathbb{R}^d$. Moreover, we assume that A is $\square$-periodic, this means that $A(\cdot + k) = A(\cdot)$ for any $k \in \mathbb{Z}^d$. For $f \in L^2(Q)$, we consider the following equation

$$\begin{aligned} -\mathrm{div}\left(A\left(\frac{\cdot}{\varepsilon}\right)\nabla u_\varepsilon\right) &= f \quad \text{in } Q, \\ u_\varepsilon &= 0 \quad \text{on } \partial Q. \end{aligned} \tag{1.1}$$

The unique weak solution of the above equation is denoted by $u_\varepsilon \in H^1_0(Q)$. A classical result, which can be found in [BLP11], states that as $\varepsilon \to 0$,

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } H^1(Q),$$

where $u \in H^1_0(Q)$ is the unique weak solution to the *homogenized* (or *effective*) equation

$$\begin{aligned} -\mathrm{div}\left(A_{\mathrm{hom}}\nabla u\right) &= f \quad \text{in } Q, \\ u &= 0 \quad \text{on } \partial Q. \end{aligned} \tag{1.2}$$

Above, $A_{\mathrm{hom}} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ (a constant matrix) is given by the formula, for $i, j \in \{1, ..., d\}$,

$$A_{\mathrm{hom}} e_i \cdot e_j = \int_\square A(x)(e_i + \nabla\varphi_i(x)) \cdot e_j dx,$$

where $\varphi_i \in H^1_{\mathrm{per}}(\square) := \left\{\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^d) : \varphi \text{ is } \square\text{-periodic}\right\}$ is known as the *periodic corrector* and it satisfies the following equation in a distributional sense

$$-\,\mathrm{div}\left(A\left(e_i + \nabla\varphi_i\right)\right) = 0 \quad \text{in } \mathbb{R}^d. \tag{1.3}$$

This equation admits a unique solution in $H^1_{\mathrm{per}}(\square)/\mathbb{R}$.

The main difficulty in the limit passage $\varepsilon \to 0$ in the weak formulation of (1.1) is the expression

$$\int_Q \nabla u_\varepsilon(x) \cdot A\left(\frac{x}{\varepsilon}\right)\nabla\varphi(x)dx$$

that is a scalar product of two weakly convergent sequences. A classical resolution of this issue is based on the so-called *div-curl lemma* and *Tartar's method of oscillating test functions* from [MT97]. An alternative approach to this problem is granted by the notions of two-scale convergence [Ngu89, All92] and the periodic unfolding method [CDG02]. In Section 2 we briefly describe the ideas of the latter methods.

Stochastic homogenization

In stochastic homogenization, the coefficients of a PDE are assumed to be random, which means that we only possess statistical information about the constitutive laws underlying the modeled physical process. In particular, in order to describe the coefficients, we consider a probability space $(\Omega, \mathcal{F}, P)$ and a random field $A : \Omega \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ that is a $\mathcal{F} \otimes \mathcal{L}(\mathbb{R}^d)$-measurable mapping. The elliptic PDE we consider features the rescaled random field $A(\omega, \frac{x}{\varepsilon})$ as a coefficient field.

In their seminal work [PV81], Papanicolaou and Varadhan introduced the following very convenient functional analytic setting for the description of stochastic homogenization problems. Let $(\Omega, \mathcal{F}, P)$ denote a complete and separable probability space and let $\tau = \{\tau_x\}_{x\in\mathbb{R}^d}$ denote a family of invertible measurable mappings $\tau_x : \Omega \to \Omega$ such that:

(i) (*Group property*). $\tau_0 = Id$ and $\tau_{x+y} = \tau_x \circ \tau_y$ for all $x, y \in \mathbb{R}^d$.

(ii) (*Measure preservation*). $P(\tau_x E) = P(E)$ for all $E \in \mathcal{F}$ and $x \in \mathbb{R}^d$.

(iii) (*Measurability*). $(\omega, x) \mapsto \tau_x\omega$ is $\big(\mathcal{F} \otimes \mathcal{L}(\mathbb{R}^d), \mathcal{F}\big)$-measurable.

We use the notation $\langle\cdot\rangle$ for the mathematical expectation, i.e., $\langle\cdot\rangle = \int_\Omega \cdot dP$. We say that the probability space $(\Omega, \mathcal{F}, P, \tau)$ is *ergodic* (shorter $\langle\cdot\rangle$ is ergodic) if the following implication holds:

$$E \subset \mathcal{F} \text{ and } \tau_x E = E \text{ for all } x \in \mathbb{R}^d \quad \Rightarrow \quad P(E) \in \{0, 1\}.$$

Sets that satisfy the antecedent of the above implication are called *shift-invariant* sets. We say that a random field is *stationary* if it admits the form $(\omega, x) \mapsto A(\tau_x\omega)$ where $A : \Omega \to \mathbb{R}^{d\times d}$ is a random variable, i.e., it is an $\mathcal{F}$-measurable mapping. We remark that the description of stationary coefficients of PDE using an abstract dynamical system τ might seem unusual at first sight, however, "shifts" of this form appear naturally in modeling of random media that we explain on examples in Sections 5.1 and 6.1 (see Examples 5.10 and 6.6).

Let $\varepsilon > 0$ and $Q \subset \mathbb{R}^d$ be open and bounded. We consider $A \in L^\infty(\Omega; \mathbb{R}^{d\times d}_{\text{sym}})$ and we assume that there exists $c > 0$ such that $A(\omega)F \cdot F \geq c|F|^2$ for P-a.a. $\omega \in \Omega$ and all $F \in \mathbb{R}^d$. For $f \in L^2(Q)$, we consider the following equation

$$\begin{aligned} -\text{div}(A(\tau_{\frac{x}{\varepsilon}}\omega)\nabla u_\varepsilon) &= f \quad \text{in } \Omega \times Q, \\ u_\varepsilon &= 0 \quad \text{on } \Omega \times \partial Q. \end{aligned} \tag{1.4}$$

Precisely, the weak formulation of the above equation reads: Find $u_\varepsilon \in L^2(\Omega; H^1_0(Q))$ such that

$$\left\langle \int_Q A(\tau_{\frac{x}{\varepsilon}}\omega)\nabla u_\varepsilon(\omega, x) \cdot \nabla\varphi(\omega, x) dx \right\rangle = \left\langle \int_Q f(x)\varphi(\omega, x) dx \right\rangle \quad \text{for all } \varphi \in L^2(\Omega; H^1_0(Q)).$$

The Riesz representation theorem implies the existence of a unique weak solution. Assuming ergodicity and stationarity for the coefficients the above equation homogenizes to a deterministic limit. Namely, in [PV81] it is shown that u_ε, the weak solution to (1.4), satisfies

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(\Omega; H_0^1(Q)), \quad u_\varepsilon \to u \quad \text{strongly in } L^2(\Omega; L^2(Q)) \quad \text{as } \varepsilon \to 0, \tag{1.5}$$

where $u \in H_0^1(Q)$ is the unique weak solution to the (deterministic) *homogenized* equation

$$\begin{aligned} -\operatorname{div}(A_{\text{hom}} \nabla u) &= f \quad \text{in } Q, \\ u &= 0 \quad \text{on } \partial Q. \end{aligned} \tag{1.6}$$

Also, $A_{\text{hom}} \in \mathbb{R}^{d \times d}_{\text{sym}}$ and it is given by the formula

$$A_{\text{hom}} e_i \cdot e_j = \langle A(\omega)(e_i + \chi_i(\omega)) \cdot e_j \rangle, \tag{1.7}$$

where $\chi_i \in L^2_{\text{pot}}(\Omega) := \overline{\{D\varphi : \varphi \in \operatorname{dom}(D) \subset L^2(\Omega)\}} \subset L^2(\Omega)^d$ is known as the *stochastic corrector* and it is the unique solution to the following *corrector* equation

$$\langle A(e_i + \chi_i) \cdot \widetilde{\chi} \rangle = 0 \quad \text{for all } \widetilde{\chi} \in L^2_{\text{pot}}(\Omega). \tag{1.8}$$

Above, $D : \operatorname{dom}(D) \subset L^2(\Omega) \to L^2(\Omega)^d$ is the so-called *stochastic gradient* and it is given by $D\varphi = (D_1\varphi, ..., D_d\varphi)$ where D_i is the infinitesimal generator of the strongly continuous group of operators $\left\{U_{he_i} : L^2(\Omega) \to L^2(\Omega) : U_{he_i}\varphi(\cdot) = \varphi(\tau_{he_i}\cdot)\right\}_{h\in\mathbb{R}}$. The space $L^2_{\text{pot}}(\Omega)$ is the stochastic counterpart of the space $H^1_{\text{per}}(\Box)/\mathbb{R}$ from the periodic setting. An essential difference between the periodic and stochastic setting is that in the former the range of the gradient operator ∇ is already a closed subspace of $L^2(\Box)^d$ due to the Poincaré inequality, whereas in the stochastic setting the image of the stochastic gradient D is not necessarily closed since in general we do not have a Poincaré inequality at our disposal.

The homogenization result from [PV81] is based on a suitable approximation for equation (1.8) and Tartar's method of oscillating test functions – for the quantitative treatment of this problem, we refer to [GNO15]. An alternative to this procedure bases on stochastic two-scale convergence that we briefly recall in the following section.

Remark 1.1. *We remark that the above results extend to a nonergodic system with the difference that the homogenized coefficient may still depend on ω, i.e., $A_{\text{hom}} : \Omega \to \mathbb{R}^{d\times d}$.*

Remark 1.2 (Mean and quenched formulations)**.** *For P-a.a. realizations $\omega \in \Omega$, we might consider the (uniquely solvable) deterministic equation parametrized by ω:*

$$\begin{aligned} -\operatorname{div}\left(A(\tau_{\frac{x}{\varepsilon}}\omega)\nabla u_\varepsilon\right) &= f \quad \text{in } Q, \\ u_\varepsilon &= 0 \quad \text{on } \partial Q. \end{aligned} \tag{1.9}$$

The solution of the above equation defines a mapping $\Omega \ni \omega \mapsto u_\varepsilon(\omega) \in H^1_0(Q)$ that can be identified as an element of $L^2(\Omega; H^1_0(Q))$ (see Section 7.3). In this regard, we may test (1.9) by $\varphi(\omega)$, where $\varphi \in L^2(\Omega; H^1_0(Q))$ is arbitrary, and integrate it over Ω, to obtain that u_ε is a solution to (1.4). Using that both equations admit unique solutions we can identify them and this means that (1.4) and (1.9) are two formulations of the same problem. We refer to the former as mean formulation *and to the latter as quenched (or pointwise P-a.e.) formulation. Typically, the consideration of mean formulations for PDE with random coefficients leads to convergence in the $L^2(\Omega; L^2(Q))$ topology (mean convergence) as in [PV81]. On the other hand, the consideration of quenched formulations leads to a convergence in a different topology: for P-a.a. ω, $u_\varepsilon(\omega) \to u$ in $L^2(Q)$ (quenched convergence). This can be done at the expense of using a pointwise ergodic theorem, e.g., Birkhoff's ergodic theorem or subadditive ergodic theorem, and we refer to the proof for elliptic homogenization in this case to [Osa83, Neu17]. In the elliptic case, the dominated convergence theorem, using the usual a priori estimates and the fact that f is fixed and deterministic, yields that quenched implies mean convergence, whereas the other implication holds only by extracting a subsequence. However, for more general problems, the former implication might not be clear. In this work we mostly focus on problems in the mean formulation.*

Remark 1.3 (Computation of A_{hom})**.** *Another difference between the periodic and stochastic setting is the numerical treatment of the corrector equation. In particular, in the periodic case the corrector equation is an elliptic equation with periodic boundary conditions that can be computed using standard finite difference or finite element approximations. On the other hand, equation (1.8) is defined in terms of stochastic gradients on a probability space, which might be an infinite-dimensional space. This fact makes the computation of the stochastic corrector (resp. A_{hom}) inconvenient for standard numerical methods. For this reason, approximations for the corrector equation are required, see Section 7.2 where we explain the standard periodization method for the resolution of this issue and provide references.*

2 Two-scale convergence and periodic unfolding

In this section we provide a short overview of some standard methods for homogenization such as two-scale convergence and periodic unfolding, for detailed studies we refer to the works listed in the text below. We consider the specific L^2-Hilbert space case, however, most of the results translate to the L^p-setting for $p \in (1, \infty)$.

Two-scale convergence

The notion of two-scale convergence is a well-established method for periodic homogenization (see [Ngu89, All92, LNW02]). Let $Q \subset \mathbb{R}^d$ be open and $\square = [0,1)^d$. We say that a sequence $u_\varepsilon \in L^2(Q)$ two-scale converges to $u \in L^2(Q \times \square)$ ($u_\varepsilon \xrightharpoonup{2} u$) if

$$\int_Q u_\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right) dx \to \int_Q \int_\square u(x,y)\varphi(x,y)dydx \quad \text{as } \varepsilon \to 0 \tag{2.1}$$

for all $\varphi \in L^2(Q; C_{\text{per}}(\square))$, where $C_{\text{per}}(\square)$ denotes the space of continuous and $\square$-periodic functions. This is an intermediate notion between weak and strong convergence in $L^2(Q)$. Namely, it holds that if $u_\varepsilon \to u$ in $L^2(Q)$, then $u_\varepsilon \xrightharpoonup{2} u$; and if $u_\varepsilon \xrightharpoonup{2} u$, then $u_\varepsilon \rightharpoonup \int_\square u(\cdot, y)dy$ weakly in $L^2(Q)$. The advantage of this notion can be seen, e.g., if we test equation (1.1) with a test function φ_ε of the form $\varphi_\varepsilon(x) = \varphi(x, \frac{x}{\varepsilon})$ where $\varphi \in C_c^1(Q; C_{\text{per}}(\Box))$. The troublesome term in the weak formulation of the elliptic PDE boils down to

$$\int_Q A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon(x) \cdot \nabla \varphi_\varepsilon(x) dx = \int_Q \nabla u_\varepsilon(x) \cdot A\left(\frac{x}{\varepsilon}\right) \nabla \varphi_\varepsilon(x) dx.$$

On the right-hand side we might consider the expression $A\left(\frac{x}{\varepsilon}\right) \nabla \varphi_\varepsilon(x)$ as a test function in the definition (2.1) (the continuity assumption for test functions in (2.1) might be relaxed). This means that, in essence, it is sufficient to understand the two-scale limit of ∇u_ε to obtain a homogenization result. In particular, for this, the following compactness statements are helpful:

- ([All92, Theorem 1.2]). If u_ε is a bounded sequence in $L^2(Q)$, then there exist $u \in L^2(Q \times \Box)$ and a (not relabeled) subsequence such that $u_\varepsilon \overset{2}{\rightharpoonup} u$.
- ([All92, Proposition 1.14]). If u_ε is a bounded sequence in $H^1(Q)$, then there exist $u \in H^1(Q)$, $\varphi \in L^2(Q; H^1_{\text{per}}(\Box))$ and a (not relabeled) subsequence such that $u_\varepsilon \overset{2}{\rightharpoonup} u$ and $\nabla u_\varepsilon \overset{2}{\rightharpoonup} \nabla u + \nabla_y \varphi$. Here, ∇_y denotes the gradient w.r.t. the $y \in \Box$ variable.

Using these observations and by a careful choice of the test functions, in the limit $\varepsilon \to 0$, equation (1.1) reduces to the problem: Find $(u, \varphi) \in H^1_0(Q) \times L^2(Q; H^1_{\text{per}}(\Box))$ such that

$$\int_Q \int_\Box A(y)(\nabla u(x) + \nabla_y \varphi(x,y)) \cdot (\nabla \psi_1(x) + \nabla_y \psi_2(x,y))\, dy dx = \int_Q f(x) \psi_1(x) dx, \tag{2.2}$$

for all $(\psi_1, \psi_2) \in H^1_0(Q) \times L^2(Q; H^1_{\text{per}}(\Box))$. We may fix the average of φ, say $\fint_\Box \varphi(\cdot, y) dy = 0$, to ensure that the above system has a unique solution. By reason of the emergence of the new *corrector* variable φ, this system is commonly called the *two-scale* effective problem. It is equivalent to the formulation (1.2)-(1.3) by means of the formula $\varphi(x,y) = \sum_{i=1}^d \nabla_i u(x) \varphi_i(y)$, where φ_i is the usual periodic corrector given by (1.3).

Periodic unfolding

In the following we present some basic facts about the periodic unfolding method and for detailed studies we refer to [CDG02, Vis04, Vis06, MT07, CDG08, CDG18]. We follow the presentation in [MT07].

Let $Q \subset \mathbb{R}^d$ be open and bounded with $|\partial Q| = 0$ and by $\Box_\#$ we denote the unit torus, i.e., $\Box_\# = \mathbb{R}^d / \mathbb{Z}^d$. We tacitly identify elements from $\Box$ by their corresponding equivalence classes in $\Box_\#$, and we identify functions defined on Q with their extension by 0 to the whole of $\mathbb{R}^d$. A central object in this procedure is the *periodic unfolding operator*

$$\begin{aligned} &\mathcal{T}_\varepsilon : L^2(Q) \to L^2(\mathbb{R}^d \times \Box_\#), && \text{(linear isometry)} \\ &\mathcal{T}_\varepsilon u(x,y) = u\left(\lfloor x \rfloor_\varepsilon + \varepsilon y\right), && (2.3) \end{aligned}$$

where $\lfloor x \rfloor_\varepsilon \in \varepsilon \mathbb{Z}^d$ is defined by $x - \lfloor x \rfloor_\varepsilon \in \varepsilon \Box$. This operator is a linear isometry and $\mathcal{T}_\varepsilon u$ is supported in an ε-neighborhood of $Q \times \Box_\#$. Also, a suitable left-inverse of this operator may be defined, the so-called *folding operator* $\mathcal{F}_\varepsilon : L^2(\mathbb{R}^d \times \square_\#) \to L^2(Q)$, which is a linear contraction and satisfies

$$\mathcal{F}_\varepsilon \circ \mathcal{T}_\varepsilon = Id \quad \text{in } L^2(Q). \tag{folding operator}$$

As a result of the isometry property of $\mathcal{T}_\varepsilon$, the following implication holds (up to extraction of a subsequence): Let u_ε be a sequence in $L^2(Q)$, then

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{L^2(Q)} < \infty \quad \Rightarrow \quad \mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(\mathbb{R}^d \times \square_\#). \tag{weak compactness}$$

Also, it turns out that the above notion of convergence is equivalent to two-scale convergence: Let u_ε be a sequence in $L^2(Q)$ and $u \in L^2(Q \times \square_\#)$, then

$$\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(\mathbb{R}^d \times \square_\#) \quad \Leftrightarrow \quad u_\varepsilon \xrightharpoonup{2} u. \tag{equivalence to $\xrightharpoonup{2}$}$$

Moreover, the following compactness statement for sequences of gradients holds (up to extraction of a subsequence): Let u_ε be a sequence in $H^1(Q)$,

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{H^1(Q)} < \infty \quad \Rightarrow \quad \exists u \in H^1(Q),\ \varphi \in L^2(Q; H^1(\square_\#)), \tag{compactness for ∇}$$

$$\mathcal{T}_\varepsilon \nabla u_\varepsilon \rightharpoonup \nabla u + \nabla_y \varphi \quad \text{weakly in } L^2(\mathbb{R}^d \times \square_\#)^d. \tag{2.4}$$

Note that, $H^1(\square_\#)$ may be identified with $H^1_{\text{per}}(\square)$. Furthermore, for any given u, φ as above, we can find a sequence $u_\varepsilon \in H^1(Q)$ such that

$$\mathcal{T}_\varepsilon u_\varepsilon \to u \quad \text{strongly in } L^2(Q), \tag{strong recovery}$$

$$\mathcal{T}_\varepsilon \nabla_x u_\varepsilon \to \nabla u + \nabla_y \varphi \quad \text{strongly in } L^2(\mathbb{R}^d \times \square_\#)^d. \tag{2.5}$$

This might be done in a systematic way with the help of the so-called *gradient folding operator* (see [MT07, Vis04]). We remark that strong convergence of unfolded sequences (that may be taken as definition of *strong* two-scale convergence) plays a crucial role in homogenization of evolutionary gradient systems since the above recovery statement together with the transformation formula (2.6) below provides Mosco-type convergence for convex integral functionals with periodic integrands (see [MT07]).

The following important transformation formulas hold: Let $u, v \in L^2(Q)^d$,

$$\int_Q A\left(\frac{x}{\varepsilon}\right) u(x) \cdot v(x) dx = \int_{\mathbb{R}^d} \int_\square A(y) \mathcal{T}_\varepsilon u(x,y) \cdot \mathcal{T}_\varepsilon v(x,y) dy dx, \tag{transformation}$$

$$\int_Q V\left(\frac{x}{\varepsilon}, u(x)\right) dx = \int_{\mathbb{R}^d} \int_\square V(y, \mathcal{T}_\varepsilon u(x,y)) dy dx, \tag{2.6}$$

where $A \in L^\infty(\square_\#)^{d\times d}$ and $V : \square_\# \times \mathbb{R}^d \to \mathbb{R}$ is a normal integrand and $V(y, 0) = 0$. The latter assumption may be dropped by a slight modification of the right-hand side above.

We remark that despite the equivalence of the convergence notions, the approach to homogenization via periodic unfolding differs from the approach using two-scale convergence. Namely, in the periodic unfolding procedure, the application of $\mathcal{T}_\varepsilon$ transforms equations with oscillating coefficients to problems with mildly-varying coefficients in an extended space, e.g., in the case of the elliptic equation (1.1) this amounts to

$$\int_{\mathbb{R}^d}\int_{\square} A(y)\mathcal{T}_\varepsilon\nabla u_\varepsilon(x,y)\cdot\mathcal{T}_\varepsilon\nabla\varphi(x,y)dydx = \int_Q f(x)\varphi(x)dx,$$

where we used the first formula in (2.6). Note that using (2.4) and by choosing convenient test functions by (2.5), the left-hand side boils down to a scalar product of a weakly and strongly convergent sequences. In this regard, we may pass to the limit and obtain the two-scale effective problem (2.2).

Remark 2.1. *For the treatment of integral functionals, the periodic unfolding operator is specially well-suited. In particular, by (2.6) the validity of the following "liminf inequality"*

$$\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup u \quad \textit{weakly} \quad \Rightarrow \quad \liminf_{\varepsilon\to 0}\int_Q V\left(\frac{x}{\varepsilon}, u_\varepsilon(x)\right)dx \geq \int_{\mathbb{R}^d}\int_{\square} V(y,u(x,y))dydx$$

boils down to weak l.s.c. of the functional $\int_{\mathbb{R}^d}\int_\square V(y,\cdot)dydx$ in the extended space $L^2(\mathbb{R}^d\times\square_\#)^d$. Typically, in applications u_ε is replaced by a sequence of gradients ∇u_ε.

Remark 2.2. *The above described unfolding strategy extends to the treatment of problems which involve singular domains, e.g., domains with holes [CDD+12] (see also [DY12, CD15, CD16]) or slender domains [Neu10]. In these cases the use of an unfolding operator is specially convenient since it also captures the geometric properties of the environment.*

Stochastic two-scale convergence

Two-scale convergence has been extended to the stochastic setting in [BMW94] (see also [AW98, SW11a]) and in [ZP06] (see also [Fag08, Hei11]). In the following we describe the notion of *stochastic two-scale in the mean* from [BMW94].

Let $(\Omega,\mathcal{F},P,\tau)$ be an ergodic probability space as in Section 1 and $Q\subset\mathbb{R}^d$ be open. Up to slight modifications, the below results hold even in a nonergodic setting. For a sequence u_ε in $L^2(\Omega\times Q)$ we say that it stochastically two-scale converges in the mean to $u\in L^2(\Omega\times Q)$ ($u_\varepsilon \xrightharpoonup{2} u$) if

$$\left\langle\int_Q u_\varepsilon(\omega,x)\varphi(\tau_{\frac{x}{\varepsilon}}\omega,x)dx\right\rangle \to \left\langle\int_Q u(\omega,x)\varphi(\omega,x)dx\right\rangle \quad \text{as } \varepsilon\to 0, \tag{2.7}$$

for all $\varphi\in L^2(\Omega\times Q)$ such that $(\omega,x)\mapsto\varphi(\tau_{\frac{x}{\varepsilon}}\omega,x)$ defines a measurable mapping. As pointed out in [AW98], the latter measurability assumption may be dropped by extracting a suitable representative for the test function φ.

Similarly to its periodic counterpart, stochastic two-scale convergence in the mean admits the compactness statements:

- ([BMW94, Theorem 3.4]). If u_ε is a bounded sequence in $L^2(\Omega\times Q)$, then there exist $u\in L^2(\Omega\times Q)$ and a (not relabeled) subsequence such that $u_\varepsilon\xrightharpoonup{2} u$.
- ([BMW94, Theorem 3.7]). If u_ε is a bounded sequence in $L^2(\Omega;H^1(Q))$, then there exist $u\in H^1(Q)$, $\chi\in L^2(\Omega;L^2_{\text{pot}}(\Omega))$ and a (not relabeled) subsequence such that $u_\varepsilon\xrightharpoonup{2} u$ and $\nabla u_\varepsilon\xrightharpoonup{2}\nabla u+\chi$.

Similarly as in the periodic case, elliptic stochastic homogenization (cf. equation (1.4)) may be obtained using the analogous two-scale convergence strategy, i.e., in

$$\left\langle \int_Q \nabla u_\varepsilon(\omega, x) \cdot A(\tau_{\frac{x}{\varepsilon}}\omega)\nabla\varphi_\varepsilon(\omega, x)dx \right\rangle$$

the expression $A(\tau_{\frac{x}{\varepsilon}}\omega)\nabla\varphi_\varepsilon(\omega, x)$ is considered as a test function in the definition of two-scale convergence (with a suitable choice of φ_ε). In this regard, in the limit $\varepsilon \to 0$, the following *two-scale* effective problem is obtained: Find $(u, \chi) \in H^1_0(Q) \times L^2(Q; L^2_{\mathrm{pot}}(\Omega))$ such that

$$\left\langle \int_Q A(\omega)\left(\nabla u(x) + \chi(\omega, x)\right) \cdot \left(\nabla\psi_1(x) + \psi_2(\omega, x)\right)dx \right\rangle = \int_Q f(x)\psi_1(x)dx,$$

for all $(\psi_1, \psi_2) \in H^1_0(Q) \times L^2(Q; L^2_{\mathrm{pot}}(\Omega))$. This system is equivalent to the classical effective problem (1.6)-(1.7) through the relation $\chi(\omega, x) = \sum_{i=1}^d \nabla_i u(x)\chi_i(\omega)$, where χ_i is the standard stochastic corrector characterized by (1.8).

Remark 2.3 (Quenched two-scale convergence)**.** *A different extension of two-scale convergence to the stochastic setting is introduced in [ZP06] (see also [MP07, Fag08, Hei11]). In this framework, instead of demanding convergence for the quantities integrated over Ω in (2.7), the convergence $\int_Q u_\varepsilon(\omega_0, x)\varphi(\tau_{\frac{x}{\varepsilon}}\omega_0, x)dx \to \left\langle \int_Q u(\omega, x)\varphi(\omega, x)dx \right\rangle$ for P-a.a. $\omega_0 \in \Omega$ is required. For this reason, we refer to this notion as quenched stochastic two-scale convergence*[1]*. Compactness statements in this setting rely on the use of Birkhoff's individual ergodic theorem and they provide quenched convergence results for homogenization problems. In particular, this notion is also well-suited for the treatment of problems which involve random measures. We briefly discuss the relation of quenched and mean stochastic two-scale convergence in Section 7.3.3.*

3 Evolutionary rate-independent systems

In this section we briefly recall some basic facts about evolutionary rate-independent systems (*ERIS*) in the Hilbert space setting featuring quadratic energy functionals. For a detailed treatment and a more general theory, we refer to [MT04, Mie05, MR15]. Also, we summarize a standard principle for asymptotic analysis of sequences of ERIS.

Basic notions and existence

We consider a separable Hilbert space Y (*state space*) and its dual space is denoted by Y^*. The dynamics of rate-independent systems is driven by an external loading $l \in C^1([0, T], Y^*)$, where

[1] We add the word "quenched" to the originally used phrase *stochastic two-scale convergence* to emphasize the distinction from the notion of stochastic two-scale convergence in the mean.

$T > 0$ denotes a fixed time horizon. We consider an *energy* functional $\mathcal{E} : [0,T] \times Y \to \mathbb{R}$ of the form

$$\mathcal{E}(t,y) = \frac{1}{2} \langle Ay, y\rangle_{Y^*,Y} - \langle l(t), y\rangle_{Y^*,Y}, \tag{3.1}$$

where

$$\begin{aligned} &A \in \mathrm{Lin}\,(Y, Y^*) \ \text{ is symmetric,} \\ &\text{and there exists } c > 0 \text{ such that } \langle Ay, y\rangle_{Y^*,Y} \geq c \,\|y\|_Y^2 \text{ for all } y \in Y. \end{aligned} \tag{3.2}$$

Also, we consider a *dissipation* functional $\mathcal{R} : Y \to [0, +\infty]$, which is

$$\begin{gathered} \text{convex, l.s.c. and positively homogeneous of degree 1, i.e.,} \\ \mathcal{R}(\alpha v) = \alpha \mathcal{R}(v) \text{ for all } \alpha > 0 \text{ and } v \in Y, \text{ and } \mathcal{R}(0) = 0. \end{gathered} \tag{3.3}$$

The evolution of the system is described by a state variable $y : [0,T] \to Y$. We consider the following couple of inequalities

$$\mathcal{E}(t, y(t)) + \int_0^t \mathcal{R}(\dot{y}(s))ds = \mathcal{E}(0, y(0)) - \int_0^t \left\langle \dot{l}(s), y(s) \right\rangle_{Y^*,Y} ds, \tag{E}$$

$$\mathcal{E}(t, y(t)) \leq \mathcal{E}(t, \widetilde{y}) + \mathcal{R}(\widetilde{y} - y(t)) \quad \text{for all } \widetilde{y} \in Y, \tag{S}$$

together with an initial condition $y(0) = y^0$, where $y^0 \in Y$ is given. (E) is known as *global energy balance* equality and (S) is called *global stability condition* for the solution y. We precise the notion of solution in the following definition:

Definition 3.1 (Energetic solution)**.** *We say that $y \in W^{1,1}([0,T]; Y)$ is an energetic solution to the ERIS $(Y, \mathcal{E}, \mathcal{R})$ with initial datum $y^0 \in Y$ if for all $t \in [0,T]$, $y(t)$ satisfies* (E)-(S) *and $y(0) = y^0$.*

It is customary to equivalently restate condition (S) as

$$y(t) \in S(t) := \{y \in Y : \ \mathcal{E}(t,y) \leq \mathcal{E}(t,\widetilde{y}) + \mathcal{R}\,(\widetilde{y} - y) \quad \text{for all } \widetilde{y} \in Y\},$$

and we refer to $S(t)$ as the *set of stable states* at time $t \in [0,T]$.

Remark 3.2. *We remark that for more general systems, e.g., if $\mathcal{E}$ is replaced by a nonconvex functional, it is reasonable to consider energetic solutions which allow jumps in time. However, we focus only on uniformly convex problems and therefore we include the continuity assumption already in the definition of the solution.*

Remark 3.3 (Equivalent formulations)**.** *Note that by differentiating* (E) *in t and using the chain-rule $\frac{d}{dt}\mathcal{E}(t,y(t)) = -\left\langle \dot{l}(t), y(t) \right\rangle_{Y^*,Y} + \langle D_y\mathcal{E}(t,y(t)), \dot{y}(t)\rangle_{Y^*,Y}$, it follows that for an energetic solution y, we have*

$$\langle D_y\mathcal{E}(t,y(t)), \dot{y}(t)\rangle + \mathcal{R}(\dot{y}(t)) = 0 \quad \textit{for a.a. } t \in [0,T]. \tag{$\mathrm{E_{loc}}$}$$

In fact, the opposite implication also holds: If $y \in W^{1,1}(0,T;Y)$ satisfies ($\mathrm{E_{loc}}$)*, then an integration over $(0,t)$ yields* (E)*. Moreover, it can be easily seen (using the quadratic structure of the energy) that* (S) *is equivalent to the condition*

$$\langle D_y\mathcal{E}(t,y(t)), \widetilde{y}\rangle_{Y^*,Y} + \mathcal{R}(\widetilde{y}) \geq 0 \quad \textit{for all } \widetilde{y} \in Y. \tag{$\mathrm{S_{loc}}$}$$

Furthermore, using the positive 1-homogeneity of $\mathcal{R}$ (see (A.2) in Example A.2), it follows that a function $y \in W^{1,1}(0,T;Y)$ satisfies ($\mathrm{E}_{\mathrm{loc}}$)-($\mathrm{S}_{\mathrm{loc}}$) *if and only if the following differential inclusion holds*

$$0 \in D_y\mathcal{E}(t, y(t)) + \partial\mathcal{R}(\dot{y}(t)) \quad \textit{for a.a. } t \in [0,T], \tag{DI}$$

where $\partial\mathcal{R}$ is the convex subdifferential of $\mathcal{R}$. In summary, we have the following equivalence of formulations

$$\text{(E)-(S)} \quad \Leftrightarrow \quad (\mathrm{E}_{\mathrm{loc}})\text{-}(\mathrm{S}_{\mathrm{loc}}) \quad \Leftrightarrow \quad \text{(DI)}.$$

Remark 3.4 (Rate-independence)**.** *Systems of this type are called rate-independent since time-reparametrizations of the input loading l lead simply to time-reparametrizations of the corresponding solution. For example, let $y(\cdot)$ denote an energetic solution corresponding to the input $\big(l(\cdot), y^0\big)$, then for $\alpha > 0$ the solution corresponding to $\big(l(\alpha\cdot), y^0\big)$ is given by $y(\alpha\cdot)$. Indeed, this can be directly seen from the* (DI) *formulation using the fact that $\partial\mathcal{R}(\alpha v) = \partial\mathcal{R}(v)$ for any $v \in Y$, which is a simple consequence of positive 1-homogeneity and convexity of $\mathcal{R}$.*

The existence of Hilbert space rate-independent systems with quadratic energies might be shown using the theory of maximal monotone operators, employing regularization techniques such as the *Yosida approximation*. On the other hand, existence may be obtained using time-discrete approximation schemes, which are also beneficial for the numerical treatment of such problems. For these proofs we refer to [Mie05, MT04].

Theorem 3.5 (Existence and uniqueness, [Mie05, Theorem 2.1])**.** *Let $\mathcal{E}$ satisfy (3.1)-(3.2) and $\mathcal{R}$ satisfy (3.3), $l \in C^1([0,T], Y^*)$ and $y^0 \in S(0)$. Then, there exists a unique energetic solution to the ERIS $(Y, \mathcal{E}, \mathcal{R})$ with initial datum y^0. Moreover, $y \in C^{\mathrm{Lip}}([0,T], Y)$ and*

$$\|y(t) - y(s)\|_Y \le \frac{\lambda}{c}|t - s| \quad \textit{for all } t, s \in [0,T], \tag{3.4}$$

where $\lambda = \|\dot{l}(t)\|_{L^\infty(0,T;Y^*)}$.

The estimate (3.4) is essential for the asymptotic treatment of sequences of ERIS, in particular for the homogenization problems that we consider in later sections and therefore we recall the argument that leads to it:

Proof of (3.4) ([Mie05, Theorem 3.4]). Let $0 < s < t \le T$. Using the positive-definiteness of A, we obtain

$$\begin{aligned} \frac{c}{2}\|y(t) - y(s)\|_Y^2 &\le \frac{1}{2}\langle A\,(y(t) - y(s))\,, y(t) - y(s)\rangle_{Y^*,Y} \\ &= \mathcal{E}(s, y(t)) - \mathcal{E}(s, y(s)) - \langle Ay(s) - l(s), y(t) - y(s)\rangle_{Y^*,Y}\,. \end{aligned} \tag{3.5}$$

According to Remark 3.3, y satisfies ($\mathrm{S}_{\mathrm{loc}}$) and therefore (setting $\widetilde{y} = y(t) - y(s)$) the expression on the right-hand side of (3.5) may be bounded from above by

$$\begin{aligned}\mathcal{E}(s,y(t))-\mathcal{E}(s,y(s))+\mathcal{R}(y(t)-y(s)) &\leq \mathcal{E}(s,y(t))-\mathcal{E}(s,y(s))+\int_s^t \mathcal{R}(\dot{y}(\tau))d\tau \\ &\leq \mathcal{E}(s,y(t))-\mathcal{E}(t,y(t))-\int_s^t \left\langle \dot{l}(\tau),y(\tau)\right\rangle_{Y^*,Y} d\tau \\ &= \langle l(t)-l(s),y(t)\rangle_{Y^*,Y}-\int_s^t \left\langle \dot{l}(\tau),y(\tau)\right\rangle_{Y^*,Y} d\tau \\ &= \int_s^t \left\langle \dot{l}(\tau),y(t)-y(\tau)\right\rangle_{Y^*,Y} d\tau \\ &\leq \lambda \int_s^t \|y(t)-y(\tau)\|_Y \, d\tau,\end{aligned} \tag{3.6}$$

where the first inequality is obtained using Jensen's inequality and by convexity and positive 1-homogeneity of $\mathcal{R}$, the second inequality follows by (E). For fixed $t>0$, we set $f(s)=\|y(t)-y(s)\|_Y$ and in this respect (3.5) and (3.6) imply that $f^2(s)\leq \frac{2\lambda}{c}\int_s^t f(\tau)d\tau$. This implies that $-\frac{d}{ds}\int_s^t f(\tau)d\tau = f(s)\leq \left(\frac{2\lambda}{c}\int_s^t f(\tau)d\tau\right)^{\frac{1}{2}}$, that in turn results in $-\frac{d}{ds}\left(\int_s^t f(\tau)d\tau\right)^{\frac{1}{2}}\leq \left(\frac{\lambda}{2c}\right)^{\frac{1}{2}}$. An integration over (s,t) yields $\int_s^t f(\tau)d\tau \leq \frac{\lambda}{2c}(t-s)^2$ and therefore $f^2(s)\leq \frac{\lambda^2}{c^2}(t-s)^2$, which implies the claim. □

Asymptotic analysis

In the following we briefly summarize a general approach for asymptotic analysis of sequences of ERIS established in [MRS08]. This procedure is developed for systems with very general properties. In the simplified Hilbert space setting with quadratic energies, the theory is specially convenient (see [Mie16, Section 5.3]) and it has been applied to a variety of problems, e.g., periodic homogenization of elasto-plasticity [MT07] and gradient plasticity [Han11], dimension reduction in elasto-plasticity [LM11].

We consider a sequence of ERIS $(Y,\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$, where Y is a Hilbert space, the energy functional has the following form

$$\mathcal{E}_\varepsilon(t,y)=\frac{1}{2}\langle A_\varepsilon y,y\rangle_{Y^*,Y}-\langle l_\varepsilon(t),y\rangle_{Y^*,Y}, \tag{3.7}$$

where for each $\varepsilon>0$, A_ε satisfies (3.2) (with c uniform in ε) and $\mathcal{R}_\varepsilon$ satisfies (3.3). The goal is to show that the system $(Y,\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ converges as $\varepsilon\to 0$ to an ERIS $(Y,\mathcal{E}_0,\mathcal{R}_0)$, the limit system having the same quadratic structure, in the sense of *well-prepared E-convergence*:

$$\begin{aligned}&\text{If } y_\varepsilon(0)\text{“}\to\text{”}y(0), \quad \mathcal{E}_\varepsilon(0,y_\varepsilon(0))\to\mathcal{E}_0(0,y(0)),\\ &\text{then } y_\varepsilon(t)\text{“}\to\text{”}y_0(t), \quad \mathcal{E}_\varepsilon(t,y_\varepsilon(t))\to\mathcal{E}_0(t,y(t)) \quad \text{for all } t\in(0,T],\end{aligned}$$

where y_ε, y are energetic solutions to the ERIS $(Y,\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon),(Y,\mathcal{E}_0,\mathcal{R}_0)$, respectively. Note that we write “ $\to$ ” to denote a convergence notion which depends on the specific problem considered. In particular, it is customary to consider weak (or strong) convergence in the space Y, however, for our purposes we use a nonstandard notion of stochastic (cross-)two-scale convergence (see Section 8). We state the below strategy in terms of sequences of systems defined on a common state space

Y. However, it might be simply extended to the case with variable state spaces Y_ε if, e.g., each Y_ε embeds into a fixed space Y. Such a modification will be considered in Section 8 where we consider homogenization and discrete-to-continuum transition problems.

The general principle for showing such evolutionary convergence statements consist of three common steps that we briefly describe in the following (see [Mie16, Section 5.3]).

General principle: Let y_ε be the energetic solution to the ERIS $(Y, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$.
Step 1. Compactness: Find a function $y \in W^{1,1}(0,T;Y)$ and a (not relabeled) subsequence such that $y_\varepsilon(t)$"$\to$"$y(t)$ for all $t \in [0,T]$.
A typical starting point for this problem is the a priori estimate (3.4) (in the case $\limsup_{\varepsilon\to 0} \|\dot{l}_\varepsilon\| < \infty$) that might lead to an Arzelà-Ascoli type argument. Also, an important ingredient in this case is the pointwise relative compactness property of the sequence y_ε (w.r.t. " $\to$ "). E.g., if " $\to$ " is weak convergence in Y, this property is satisfied by boundedness of $y_\varepsilon(t)$ in Y by the Banach-Alaoglu theorem.

Step 2. Stability: Show that y is stable, i.e., $y(t) \in S(t)$.
A routine approach to this issue is the construction of so-called *joint recovery sequences*, i.e., if for an arbitrary $\widetilde{y} \in Y$ we can find a sequence $\widetilde{y}_\varepsilon \in Y$ such that

$$\limsup_{\varepsilon\to 0} \left(\mathcal{E}_\varepsilon(t, \widetilde{y}_\varepsilon) - \mathcal{E}_\varepsilon(t, y_\varepsilon(t)) + \mathcal{R}_\varepsilon(\widetilde{y}_\varepsilon - y_\varepsilon(t))\right) \le \mathcal{E}(t, \widetilde{y}) - \mathcal{E}(t, y(t)) + \mathcal{R}(\widetilde{y} - y(t)),$$

then $y(t) \in S(t)$ since the left-hand side is nonnegative by the stability of $y_\varepsilon(t)$. In the specific case of (3.7), the quadratic structure of the energy may be exploited (*quadratic trick*):

$$\begin{aligned}
& \mathcal{E}_\varepsilon(t, \widetilde{y}_\varepsilon) - \mathcal{E}_\varepsilon(t, y_\varepsilon(t)) + \mathcal{R}_\varepsilon(\widetilde{y}_\varepsilon - y_\varepsilon(t)) \\
= \; & \frac{1}{2} \left\langle A_\varepsilon\left(\widetilde{y}_\varepsilon - y_\varepsilon(t)\right), \widetilde{y}_\varepsilon + y_\varepsilon(t)\right\rangle_{Y^*,Y} - \langle l_\varepsilon(t), \widetilde{y}_\varepsilon - y_\varepsilon(t)\rangle_{Y^*,Y} + \mathcal{R}_\varepsilon\left(\widetilde{y}_\varepsilon - y_\varepsilon(t)\right).
\end{aligned}$$

In this regard, the construction for $\widetilde{y}_\varepsilon$ (resp. $\widetilde{y}_\varepsilon - y_\varepsilon(t)$) should allow us to jointly pass to the limit in all three terms on the right-hand side. This is attainable, e.g., in the case that "$\to$" is weak convergence in Y and if $\mathcal{E}_\varepsilon(t,\cdot) \overset{M}{\to} \mathcal{E}_0(t,\cdot)$ and $\mathcal{R}_\varepsilon \overset{C}{\to} \mathcal{R}_0$, see [Mie16, Theorem 5.7]. Here $\overset{M}{\to}$ denotes Mosco convergence, i.e., for a sequence of functionals $\mathcal{I}_\varepsilon : Y \to \mathbb{R}$ we say that it Mosco converges to $\mathcal{I}_0 : Y \to \mathbb{R}$ if the following two conditions hold:

(i) (*Liminf inequality.*) For any $y_\varepsilon \rightharpoonup y$ weakly in Y, it follows $\liminf_{\varepsilon\to 0} \mathcal{I}_\varepsilon(y_\varepsilon) \ge \mathcal{I}_0(y)$.

(ii) (*Recovery sequence.*) For any $y \in Y$, there exists $y_\varepsilon \in Y$ such that $y_\varepsilon \to y$ strongly in Y and $\limsup_{\varepsilon\to 0} \mathcal{I}_\varepsilon(y_\varepsilon) \le \mathcal{I}_0(y)$.

Also, $\overset{C}{\to}$ means convergence along strongly convergent sequences, i.e., if $y_\varepsilon \to y$ strongly in Y, then $\lim_{\varepsilon\to 0} \mathcal{R}_\varepsilon(y_\varepsilon) = \mathcal{R}_0(y)$.

Step 3. Energy balance: Show that y satisfies the energy balance equality (E) *of* $(Y, \mathcal{E}_0, \mathcal{R}_0)$.
We remark that by Proposition 3.6 below, it is sufficient to show the "$\le$" part of (E), if Step 2 is already completed. This is obtained by passing to a liminf on the left-hand side and to the limsup on the right-hand side of

$$\mathcal{E}_\varepsilon(t, y_\varepsilon(t)) + \int_0^t \mathcal{R}_\varepsilon(\dot{y}_\varepsilon(s))ds = \mathcal{E}_\varepsilon(0, y_\varepsilon(0)) - \int_0^t \left\langle \dot{l}_\varepsilon(s), y_\varepsilon(s)\right\rangle_{Y^*,Y} ds.$$

In the case that "$\to$" is weak convergence, upon assuming strong convergence for $\dot{l}_\varepsilon$, the right-hand side converges by assumption. The liminf inequality for the left-hand side may be obtained if, e.g., $\mathcal{E}_\varepsilon(t,\cdot) \xrightharpoonup{\Gamma} \mathcal{E}_0(t,\cdot)$ and $\mathcal{R}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{R}_0$. Here, $\xrightharpoonup{\Gamma}$ denotes Γ-convergence that is defined analogously as Mosco convergence with the difference that in (ii) the recovery sequence y_ε is required to converge only weakly in Y.

For convenience of the reader, we recall the proof of the following result which can be found in [MM05].

Proposition 3.6 ([MM05, Theorem 4.4])**.** *Let the assumptions of Theorem 3.5 be satisfied. Let $y \in W^{1,1}([0,T];Y)$ satisfy* (S) *for all $t \in [0,T]$. Then, for all $t \in [0,T]$,*

$$\mathcal{E}(t,y(t)) + \int_0^t \mathcal{R}(\dot{y}(s))ds \geq \mathcal{E}(0,y(0)) - \int_0^t \left\langle \dot{l}(s), y(s) \right\rangle_{Y^*,Y} ds.$$

Proof. We consider $\{t_i\}_{i\in\{1,\dots,n\}}$, a partition of the interval $[0,t]$, with $t_1 = 0$ and $t_n = t$. We have

$$\begin{aligned}
& \mathcal{E}(t,y(t)) + \int_0^t \mathcal{R}(\dot{y}(s))ds - \mathcal{E}(0,y(0)) \\
= \; & \mathcal{E}(t,y(t)) + \sum_{i=2}^n \int_{t_{i-1}}^{t_i} \mathcal{R}(\dot{y}(s))ds - \mathcal{E}(0,y(0)) \\
\geq \; & \mathcal{E}(t,y(t)) + \sum_{i=2}^n \mathcal{R}(y(t_i) - y(t_{i-1})) - \mathcal{E}(0,y(0)) \\
\geq \; & -\langle l(t_n), y(t_n)\rangle_{Y^*,Y} + \langle l(t_1), y(t_1)\rangle_{Y^*,Y} + \sum_{i=2}^n -\langle l(t_{i-1}), y(t_{i-1})\rangle_{Y^*,Y} + \langle l(t_{i-1}), y(t_i)\rangle_{Y^*,Y} \\
= \; & -\sum_{i=2}^n \int_{t_{i-1}}^{t_i} \left\langle \dot{l}(s), y(t_i) \right\rangle_{Y^*,Y} ds = -\int_0^t \left\langle \dot{l}(s), \pi_n y(s) \right\rangle_{Y^*,Y} ds,
\end{aligned}$$

where the first inequality follows by convexity and 1-homogeneity of $\mathcal{R}$ (Jensen's inequality) and the second inequality is obtained by (S). Also, $\pi_n y$ denotes the left-continuous piecewise constant interpolation of y w.r.t. $\{t_i\}$, which satisfies $\pi_n y(s) \to y(s)$ as $n \to \infty$. Using the dominated convergence theorem, we conclude the proof. $\square$

4 Gradient flows

In this section we recall the basic framework for gradient flows on Hilbert spaces and for detailed studies we refer to the textbooks [Bré73, Zei13, Rou13, AGS08]. In particular, we consider the *evolutionary variational inequality* (*EVI*) formulation (see, e.g., the lecture notes [DS10, Clé09]). Also, we recall some standard approaches for asymptotic analysis of sequences of gradient flows and describe the strategy that we apply later.

Basic notions and existence

Let Y be a separable Hilbert space and we denote its dual space with Y^*. We consider a quadratic *dissipation* potential $\mathcal{R} : Y \to [0, \infty)$ of the form

$$\begin{aligned} &\mathcal{R}(v) = \frac{1}{2} \langle rv, v \rangle_{Y^*,Y}, \text{ where } r \in \mathrm{Lin}(Y, Y^*) \text{ is symmetric,} \\ &\text{and there exists } c > 0 \text{ such that } \frac{1}{c} \|v\|_Y^2 \leq \mathcal{R}(v) \leq c \|v\|_Y^2 \text{ for all } v \in Y. \end{aligned} \tag{4.1}$$

Also, we consider an *energy* functional $\mathcal{E} : Y \to \mathbb{R} \cup \{\infty\}$ which is, for given $\lambda \in \mathbb{R}$,

$$\begin{aligned} &\text{l.s.c., proper, } \lambda\text{-convex, i.e., the mapping } y \mapsto \mathcal{E}(y) - \lambda \mathcal{R}(y) \text{ is convex,} \\ &\text{and coercive, i.e., there exists } c > 0 \text{ such that } \mathcal{E}(y) \geq \frac{1}{c} \|y\|_Y - c \text{ for all } y \in Y. \end{aligned} \tag{4.2}$$

We consider a fixed time horizon $T > 0$. The evolution of the gradient flow is described by a state variable $y : [0, T] \to Y$ and it is determined by the inequality

$$\frac{d}{dt}\mathcal{R}(y(t) - \widetilde{y}) = \langle r\dot{y}(t), y(t) - \widetilde{y} \rangle_{Y^*,Y} \leq \mathcal{E}(\widetilde{y}) - \mathcal{E}(y(t)) - \lambda \mathcal{R}(y(t) - \widetilde{y}) \quad \text{for all } \widetilde{y} \in Y. \tag{EVI}$$

We refer to the above as an evolutionary variational inequality (see [AGS08], in the case $\lambda = 0$ this corresponds to the weak formulation of gradient flows considered by Bénilan [Bén72]). We specify the notion of solution as follows:

Definition 4.1 (EVI solution)**.** *We say that $y : [0, T] \to Y$ is an EVI solution to the gradient flow $(Y, \mathcal{E}, \mathcal{R})$ with initial datum $y^0 \in \overline{\mathrm{dom}(\mathcal{E})}$ if:*

(i) y is continuous on $[0, T]$, locally absolutely continuous on $(0, T]$ (this means that y is absolutely continuous on $[a, b]$ for any $0 < a < b \leq T$);

(ii) $y(0) = y^0$, $y(t) \in \mathrm{dom}(\mathcal{E})$ for all $t \in (0, T]$, $y(t)$ satisfies (EVI) *for a.a. $t \in (0, T]$.*

Remark 4.2 (Equivalent formulations)**.** *We remark that if y is an EVI solution to $(Y, \mathcal{E}, \mathcal{R})$, then it holds*

$$\text{for a.a. } t \in (0, T], \quad y(t) \in \mathrm{dom}(\partial_F \mathcal{E}), \quad 0 \in D\mathcal{R}(\dot{y}(t)) + \partial_F \mathcal{E}(y(t)). \tag{4.3}$$

*Above ∂_F denotes the Frechét subdifferential, which in this λ-convex case has the form $\partial_F \mathcal{E}(y) = \Big\{\xi \in Y^* : \mathcal{E}(y) \leq \mathcal{E}(\widetilde{y}) + \langle \xi, y - \widetilde{y} \rangle_{Y^*,Y} - \lambda \mathcal{R}(y - \widetilde{y}) \text{ for all } \widetilde{y} \in Y\Big\}$ (see, e.g., [Kru03] for the general case). In fact, (4.3) and (*EVI*) are equivalent. In many instances, it is convenient to consider an integrated version of* (EVI)*. In particular, if y is an EVI solution to $(Y, \mathcal{E}, \mathcal{R})$, we may multiply* (EVI) *with $e^{\lambda t}$ and integrate it over the interval (s, t) for $0 < s \leq t \leq T$, in order to obtain*

$$e^{\lambda t}\mathcal{R}(y(t) - \widetilde{y}) - e^{\lambda s}\mathcal{R}(y(s) - \widetilde{y}) \leq \int_s^t e^{\lambda \tau} \left(\mathcal{E}(\widetilde{y}) - \mathcal{E}(y(\tau))\right) d\tau \quad \text{for all } \widetilde{y} \in \mathrm{dom}(\mathcal{E}). \tag{IEVI}$$

Above, we use that $\frac{d}{dt}\left(e^{\lambda t}\mathcal{R}(y(t) - \widetilde{y})\right) = e^{\lambda t}\frac{d}{dt}\mathcal{R}(y(t) - \widetilde{y}) + \lambda e^{\lambda t}\mathcal{R}(y(t) - \widetilde{y})$. In fact, the formulations (EVI) *and* (IEVI) *are equivalent (see [AGS08, Remark 4.0.5b], [Clé09, Proposition 2.1]).* (IEVI) *is very convenient since it features neither the derivatives of the functionals nor derivatives of the solution.*

Existence results for such Hilbert space gradient flows follow using standard regularization techniques (Yosida approximation) from the theory of maximal monotone operators with Lipschitz perturbations and for the proof of the following theorem we refer to [Bré73, Chapter 3] (see also [Clé09, Theorem 3.2] and [Bar10, Section 4]). An alternative approach to existence is based on time-discrete approximation schemes (*De Giorgi's minimizing movement scheme*) (see, e.g., [AGS08, RS06]).

Theorem 4.3 (Existence and uniqueness). *Let $\mathcal{R}$ satisfy (4.1) and $\mathcal{E}$ satisfy (4.2). For $y^0 \in \mathrm{dom}(\mathcal{E})$, there exists a unique EVI solution $y \in H^1(0,T;Y)$ to the gradient flow $(Y,\mathcal{E},\mathcal{R})$ with initial datum y^0. Moreover, it holds*

$$\begin{aligned} &\mathcal{E}(y(t)) \le \mathcal{E}(y(s)) \quad \textit{for all } 0 \le s \le t \le T, \\ &\int_0^t \mathcal{R}(\dot{y}(\tau))d\tau \le \mathcal{E}(y^0) - \mathcal{E}(y(t)) \quad \textit{for all } 0 < t \le T. \end{aligned} \tag{4.4}$$

The a priori estimates (4.4) play a key role in the applications that we consider in later sections and therefore we recall the argument that leads to them. By Remark 4.2, y satisfies (4.3) and testing this inclusion with $\dot{y}(t)$, we obtain

$$\frac{d}{dt}\mathcal{E}(y(t)) = -\left\langle D\mathcal{R}(\dot{y}(t)), \dot{y}(t)\right\rangle_{Y^*,Y} \le 0, \tag{4.5}$$

where we use the chain rule the λ-convex functional $\mathcal{E}$ (see, e.g., [RS06]). By the quadratic structure $\mathcal{R}$, we have $\langle D\mathcal{R}(\dot{y}), \dot{y}\rangle_{Y^*,Y} = 2\mathcal{R}(\dot{y})$ for any $\dot{y} \in Y$. Consequently, an integration of (4.5) over (s,t) yields $\int_s^t 2\mathcal{R}\left(\dot{y}(\tau)\right) d\tau = \mathcal{E}(y(s)) - \mathcal{E}(y(t))$ for all $0 \le s \le t \le T$. This implies (4.4).

Asymptotic analysis

In the following we consider a sequence of gradient flows $(Y,\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$, and $(Y,\mathcal{E}_0,\mathcal{R}_0)$ (satisfying the assumptions of Theorem 4.3, λ being uniform in ε). The objective is to conduct an asymptotic analysis for the limit $\varepsilon \to 0$ and to show *well-prepared E-convergence* of $(Y,\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ to $(Y,\mathcal{E}_0,\mathcal{R}_0)$ in the following sense:

$$\begin{aligned} &\text{If } y_\varepsilon(0) \in \mathrm{dom}(\mathcal{E}_\varepsilon), \quad y_\varepsilon(0)\text{“}\to\text{”}y(0), \quad \mathcal{E}_\varepsilon(y_\varepsilon(0)) \to \mathcal{E}_0(y(0)), \\ &\text{then } y_\varepsilon(t)\text{“}\to\text{”}y(t), \quad \mathcal{E}_\varepsilon(y_\varepsilon(t)) \to \mathcal{E}_0(y(t)) \quad \text{for all } t \in (0,T], \end{aligned}$$

where y_ε and y are EVI solutions to $(Y,\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ and $(Y,\mathcal{E}_0,\mathcal{R}_0)$, respectively. To keep the discussion simple, we assume that "$\to$" is strong convergence in Y, but keeping in mind that in our applications we use the notion of two-scale convergence. We also remark that some of the below methods do not require convergence of the initial energy as an assumption and in that case the convergence notion is called *E-convergence*.

The general principle for proving such convergence statements consists of similar steps as in Section 3: First a compactness statement for the sequence of solutions is needed and second, based on that, we are required to pass to the limit $\varepsilon \to 0$ in a suitable formulation of the gradient system. We refer to [Mie16] for a comprehensive overview of general approaches for such problems. In particular, an early contribution in this field is due to Attouch [Att78, Att84], where problems with fixed dissipation potential $\mathcal{R}_\varepsilon = \mathcal{R}$ and convex energy functionals $\mathcal{E}_\varepsilon$ are considered. This approach is based on the fact that for convex energies the Fréchet and convex subdifferentials coincide and therefore the EVI of $(Y, \mathcal{E}_\varepsilon, \mathcal{R})$ boils down to (using the Fenchel equivalence Lemma A.1 and an integration over $(0,t)$)

$$\mathcal{R}(y_\varepsilon(t)) + \int_0^t \mathcal{E}_\varepsilon(y_\varepsilon(s)) + \mathcal{E}_\varepsilon^*(-D\mathcal{R}(\dot{y}_\varepsilon(s)))ds = \mathcal{R}(y_\varepsilon(0)), \tag{4.6}$$

where $\mathcal{E}_\varepsilon^*$ denotes the convex conjugate of $\mathcal{E}_\varepsilon$ (see Section A.1). In this setting, e.g., Mosco convergence $\mathcal{E}_\varepsilon \xrightarrow{M} \mathcal{E}_0$ is sufficient to conclude well-prepared E-convergence. Novel strategies have been developed in [SS04, Ser11] and [MRS13], which allow the treatment of very general problems with varying (nonquadratic convex) dissipation potentials $\mathcal{R}_\varepsilon$ and possibly nonconvex energy functionals $\mathcal{E}_\varepsilon$. They are based on *De Giorgi's* $(\mathcal{R}, \mathcal{R}^*)$ formulation (see, e.g., [Mie16, Introduction]). Also, based on the (IEVI) formulation, in [DS10] a method for sequences with λ-convex energies is proposed (see also [Mie15]). In [Ste08], the *Brezis-Ekeland-Nayroles principle* is utilized for the development of a procedure for E-convergence for convex dissipation and energy functionals.

Typically, the methods that allow the treatment of nonconvex energy functionals rely on the assumption that the energy "sublevels" $Y_c = \{y \in Y : \mathcal{E}_\varepsilon(y) \leq c,\ \forall \varepsilon\}$ are compact in Y (or a similar strong-type compactness assumption). We remark that in our applications in Section 9 we consider problems with nonconvex (λ-convex) energies with a lack of such strong compactness property, we have merely weak-type compactness at our disposal. For this reason, we consider a modified strategy that we briefly describe in the following and we refer to Section 9 for a detailed particular application.

First, we note that by a change of variables $y_\varepsilon(t) \rightsquigarrow e^{\lambda t} y_\varepsilon(t)$ the (EVI) of $(Y, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ reduces to a formulation given in terms of a modified convex energy[1]. In particular, we consider a new time-dependent energy functional $\widetilde{\mathcal{E}}_\varepsilon : [0,T] \times Y \to \mathbb{R} \cup \{\infty\}$,

$$\widetilde{\mathcal{E}}_\varepsilon(t,u) = e^{2\lambda t} \mathcal{E}_\varepsilon(e^{-\lambda t} u) - \lambda \mathcal{R}_\varepsilon(u),$$

where $\widetilde{\mathcal{E}}_\varepsilon(t,\cdot)$ is convex for each t (see Lemma 4.5 (i)). Moreover, if we introduce a new variable $u_\varepsilon(t) := e^{\lambda t} y_\varepsilon(t)$, it follows that the considered EVI boils down to (cf. Lemma 4.5 (ii))

$$\mathcal{R}_\varepsilon(u_\varepsilon(t)) + \int_0^t \widetilde{\mathcal{E}}_\varepsilon(s, u_\varepsilon(s)) + \widetilde{\mathcal{E}}_\varepsilon^*(s, -D\mathcal{R}_\varepsilon(\dot{u}_\varepsilon(s)))ds = \mathcal{R}_\varepsilon(u_\varepsilon(0)). \tag{4.7}$$

This formulation is similar to (4.6) with the difference that the energy functionals are time dependent and that the dissipation functionals depend on ε. We extend the strategy from [Att78, Att84] to this setting and in this way we avoid the use of strong compactness statements. In particular, it might be obtained that if $\mathcal{E}_\varepsilon \xrightarrow{M} \mathcal{E}_0$, $\mathcal{R}_\varepsilon \xrightarrow{C} \mathcal{R}_0$ and $\mathcal{R}_\varepsilon \xrightharpoonup{\Gamma} \mathcal{R}_0$, then well-prepared E-convergence for the corresponding gradient flows follows. However, in our applications we deal with modified assumptions for the functionals, for the precise analysis see Section 9. The passage to the limit is obtained similarly as for ERIS in Section 3, by first deriving a priori bounds that provide weak convergence for the solution. Second, we pass to the liminf on the left-hand side of (4.7) and limsup on the right-hand side, to obtain

$$\begin{aligned} &\mathcal{R}_0(u(t)) + \int_0^t \widetilde{\mathcal{E}}_0(s, u(s)) + \widetilde{\mathcal{E}}_0^*(s, -D\mathcal{R}_0(\dot{u}(s)))ds \leq \mathcal{R}_0(u(0)) \\ \Leftrightarrow \quad &\int_0^t \widetilde{\mathcal{E}}_0(s, u(s)) + \widetilde{\mathcal{E}}_0^*(s, -D\mathcal{R}_0(\dot{u}(s))) + \langle D\mathcal{R}_0(\dot{u}(s)), u(s) \rangle_{Y^*,Y}\, ds \leq 0. \end{aligned} \tag{4.8}$$

[1]We learned this in a private communication from Goro Akagi.

Note that by the Fenchel-Young inequality the integrand on the left-hand side is nonnegative and therefore for a.a. t it holds

$$\widetilde{\mathcal{E}}_0(t, u(t)) + \widetilde{\mathcal{E}}_0^*(t, -D\mathcal{R}_0(\dot{u}(t))) + \langle D\mathcal{R}_0(\dot{u}(t)), u(t)\rangle_{Y^*,Y} = 0. \tag{4.9}$$

This equality is in turn equivalent to the EVI formulation of the system $(Y, \mathcal{E}_0, \mathcal{R}_0)$ via the transformation $y(t) = e^{-\lambda t}u(t)$ (see Lemma 4.5). The equivalence of (4.8) and (4.9) is a version of the well-known *Brezis-Ekeland-Nayroles principle* (see [BE76, Nay76, Ste08]).

Remark 4.4. *We remark that this strategy strongly relies on the quadratic structure of the dissipation functional, however, it is convenient for the treatment of λ-convex energies and relies merely on weak convergence arguments.*

Lemma 4.5 (Convex reduction)**.** *Let $\mathcal{R}$ satisfy (4.1) and $\mathcal{E}$ satisfy (4.2). For $(t, u) \in [0, T] \times Y$, we define*

$$\widetilde{\mathcal{E}}(t, u) = e^{2\lambda t}\mathcal{E}(e^{-\lambda t}u) - \lambda\mathcal{R}(u).$$

(i) $\widetilde{\mathcal{E}} : [0, T] \times Y \to \mathbb{R} \cup \{\infty\}$ is a convex normal integrand (see Definition A.3).

(ii) Let $y : [0, T] \to Y$ be continuous on $[0, T]$ and locally absolutely continuous on $(0, T]$. Then $y(t)$ satisfies (EVI) *for a.a. $t \in [0, T]$ if and only if $u(t) := e^{\lambda t}y(t)$ satisfies*

$$\frac{d}{dt}\mathcal{R}(u(t)) + \widetilde{\mathcal{E}}(t, u(t)) + \widetilde{\mathcal{E}}^*(t, -D\mathcal{R}(\dot{u}(t))) = 0 \quad \textit{for a.a. } t \in [0, T], \tag{4.10}$$

where $\widetilde{\mathcal{E}}^*(t, \cdot)$ denotes the convex conjugate of $\widetilde{\mathcal{E}}(t, \cdot)$.

Proof. (i) For fixed t, convexity of $\widetilde{\mathcal{E}}(t, \cdot)$ follows from λ-convexity of $\mathcal{E}$. Also, $\widetilde{\mathcal{E}}(t, \cdot)$ is proper and l.s.c. Indeed, this follows by continuity of $\mathcal{R}$ and by the fact that $\mathcal{E}$ is proper and l.s.c. In the following we show that $\widetilde{\mathcal{E}}$ is $\mathcal{L}(0, T) \otimes \mathcal{B}(Y)$-measurable that implies the claim of (i). First, we note that $-\lambda\mathcal{R}$ is $\mathcal{B}(Y)$-measurable since it is continuous, therefore it is sufficient to show that the mapping $(t, u) \mapsto e^{2\lambda t}\mathcal{E}(e^{-\lambda t}u)$ is $\mathcal{L}(0, T) \otimes \mathcal{B}(Y)$-measurable. We note that $\mathcal{E}(e^{-\lambda t}u)$ is the composition of the continuous mapping $(t, u) \mapsto e^{-\lambda t}u$ (thus $(\mathcal{B}(0, T) \otimes \mathcal{B}(Y), \mathcal{B}(Y))$-measurable) and the l.s.c. functional $\mathcal{E}$ (thus $\mathcal{B}(Y)$-measurable). As a result of this, it is $\mathcal{B}(0, T) \otimes \mathcal{B}(Y)$-measurable. Finally, $e^{2\lambda t}\mathcal{E}(e^{-\lambda t}u)$ is a product of a continuous and a measurable functional and therefore it is $\mathcal{L}(0, T) \otimes \mathcal{B}(Y)$-measurable.

(ii) A simple rearrangement of the terms in (EVI) yields

$$\langle r\left(\dot{y}(t) + \lambda y(t)\right), y(t) - \widetilde{y}\rangle_{Y^*,Y} + \mathcal{E}(y(t)) - \lambda\mathcal{R}(y(t)) \leq \mathcal{E}(\widetilde{y}) - \lambda\mathcal{R}(\widetilde{y}) \quad \text{for all } \widetilde{y} \in Y.$$

We multiply the above inequality with $e^{2\lambda t}$ and set $\widetilde{y} = e^{-\lambda t}\widetilde{y}$ to obtain

$$\begin{aligned} &\left\langle r\left(e^{\lambda t}\dot{y}(t) + \lambda e^{\lambda t}y(t)\right), e^{\lambda t}y(t) - \widetilde{y}\right\rangle_{Y^*,Y} + e^{2\lambda t}\mathcal{E}(e^{-\lambda t}e^{\lambda t}y(t)) - \lambda\mathcal{R}(e^{\lambda t}y(t)) \\ \leq\ & e^{2\lambda t}\mathcal{E}(e^{-\lambda t}\widetilde{y}) - \lambda\mathcal{R}(\widetilde{y}) \quad \text{for all } \widetilde{y} \in Y. \end{aligned}$$

Using the new objects $u(t) = e^{\lambda t}y(t)$ and $\widetilde{\mathcal{E}}$, the above inequality reads

$$\langle r\dot{u}(t), u(t) - \widetilde{y}\rangle_{Y^*,Y} + \widetilde{\mathcal{E}}(t, u(t)) \leq \widetilde{\mathcal{E}}(t, \widetilde{y}) \quad \text{for all } \widetilde{y} \in Y,$$

where we used that $\dot{u}(t) = e^{\lambda t}\dot{y}(t) + \lambda e^{\lambda t}y(t)$. Since $\widetilde{\mathcal{E}}(t, \cdot)$ is convex for each t, the Fenchel equivalence (Lemma A.1) implies that u satisfies (4.10), where we use the quadratic structure of $\mathcal{R}$ in the form $\frac{d}{dt}\mathcal{R}(u(t)) = \langle D\mathcal{R}(u(t)), \dot{u}(t)\rangle_{Y^*,Y} = \langle D\mathcal{R}(\dot{u}(t)), u(t)\rangle_{Y^*,Y}$. We remark that all of the above implications are, in fact, equivalences and therefore the claim of the lemma follows. $\square$

Part II

Stochastic unfolding

Summary of main results

In this part we discuss the stochastic unfolding method, in particular, we define the stochastic unfolding operator and examine its main properties. In the end, we explain the stochastic unfolding procedure on a simple example and provide some general remarks. In the following we briefly summarize our main findings indicating the analogies with periodic unfolding, cf. Section 2.

Sections 5 and 6: In these sections we develop the unfolding approach alongside in a discrete-to-continuum and continuum settings, respectively. For $\varepsilon > 0$ and $p \in (1, \infty)$, we consider a Banach space Y_ε which is, in the following, either $L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ or $L^p(\Omega \times Q)$ with $Q \subset \mathbb{R}^d$ open – in the former case we deal with *discrete* unfolding in Section 5 and in the latter with *continuum* unfolding in Section 6. Here, $(\Omega, \mathcal{F}, P, \tau)$ is a suitable probability space equipped with a discrete or continuum dynamical system τ, see Sections 5.1 and 6.1. We define the *stochastic unfolding operator*

$$\mathcal{T}_\varepsilon : Y_\varepsilon \to L^p(\Omega \times \mathbb{R}^d), \qquad \text{(linear isometry)}$$

that is a linear isometry. In the continuum case it is even an isomorphism. Also, we define a suitable *stochastic folding* operator $\mathcal{F}_\varepsilon : L^p(\Omega \times \mathbb{R}^d) \to Y_\varepsilon$ which is a contraction and

$$\mathcal{F}_\varepsilon \circ \mathcal{T}_\varepsilon = Id \quad \text{on } Y_\varepsilon. \qquad \text{(folding operator)}$$

Since $\mathcal{T}_\varepsilon$ is an isometry, we obtain the following compactness statement, which holds up to extraction of a subsequence: Let $u_\varepsilon \in Y_\varepsilon$ be a sequence, then

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{Y_\varepsilon} < \infty \quad \Rightarrow \quad \exists u \in L^p(\Omega\times\mathbb{R}^d),\ \mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup u \text{ weakly in } L^p(\Omega\times\mathbb{R}^d). \quad \text{(weak compactness)} \tag{4.11}$$

In the continuum case, for bounded sequences, the above convergence notion is equivalent to stochastic two-scale convergence in the mean from [BMW94]: Let $u_\varepsilon \in L^p(\Omega \times \mathbb{R}^d)$ be a bounded sequence and $u \in L^p(\Omega \times \mathbb{R}^d)$, then

$$\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega \times \mathbb{R}^d) \quad \Leftrightarrow \quad u_\varepsilon \overset{2}{\rightharpoonup} u. \qquad \text{(equivalence to } \overset{2}{\rightharpoonup}\text{)}$$

In the discrete setting weak convergence of $\mathcal{T}_\varepsilon u_\varepsilon$ is equivalent to a discrete-to-continuum version of stochastic two-scale convergence in the mean. In this respect, the compactness statement (4.11) in the continuum L^2-case may be obtained by referring to the proof in [BMW94]. However, the proof based on the isometry property of $\mathcal{T}_\varepsilon$ is simpler. For the reason of the above equivalence, we use the shorthand notation $u_\varepsilon \overset{2}{\rightharpoonup} u$ (resp. $u_\varepsilon \overset{2}{\to} u$) for the convergence of the unfolded sequence $\mathcal{T}_\varepsilon u_\varepsilon \to u$ weakly (resp. strongly) in $L^p(\Omega \times \mathbb{R}^d)$.

Moreover, we obtain the following compactness statement, that holds up to extraction of a subsequence: Let $u_\varepsilon \in Y_\varepsilon$ be a sequence, then

$$\limsup_{\varepsilon\to 0} \left(\|u_\varepsilon\|_{Y_\varepsilon} + \|\nabla^\varepsilon u_\varepsilon\|_{Y_\varepsilon^d}\right) < \infty \quad \Rightarrow \quad \exists u \in L^p_{\text{inv}}(\Omega) \otimes W^{1,p}(\mathbb{R}^d),\ \chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(\mathbb{R}^d), \tag{4.12}$$

$$u_\varepsilon \overset{2}{\rightharpoonup} u, \quad \nabla^\varepsilon u_\varepsilon \overset{2}{\rightharpoonup} \nabla u + \chi, \qquad \text{(compactness for } \nabla^\varepsilon\text{)}$$

where ∇^ε is the usual gradient in the physical space variable in the continuum case, and in the discrete case it is a suitable difference quotient. We remark that in the continuum L^2-case this result is proved in the setting of stochastic two-scale convergence in the mean in [BMW94] and for the L^p-case in the setting of coupled periodic and stochastic two-scale convergence in [SW11b]. The proofs that we present are similar to [BMW94], but they are based on stochastic unfolding and apply as well to the discrete setting.

We construct suitable strong recovery sequences for given functions u and χ as in (4.12). Namely, we find a sequence $u_\varepsilon \in Y_\varepsilon$ such that

$$u_\varepsilon \xrightarrow{2} u, \quad \nabla^\varepsilon u_\varepsilon \xrightarrow{2} \nabla u + \chi. \tag{strong recovery}$$

Also, similarly as in the periodic setting, we obtain convenient transformation formulas for integral functionals. In particular, in the continuum case for $Q \subset \mathbb{R}^d$ open and bounded, we have for all $u \in L^p(\Omega \times Q)^m$,

$$\int_\Omega \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, u(\omega, x))dxdP(\omega) = \int_\Omega \int_Q V(\omega, x, \mathcal{T}_\varepsilon u(\omega, x))dxdP(\omega), \tag{transformation}$$

for a normal integrand $V : \Omega \times Q \times \mathbb{R}^m \to \mathbb{R}$ with standard growth conditions. The analogous formula holds in the discrete case.

Sections 5 and 6 present analogous results and we discuss some specific additional properties in each section separately. In particular, in the former we consider limits of scaled gradients and limits of symmetrized gradients, whereas in the latter we consider l.s.c. results for integral functionals with a dependence on a "slow" variable. Nevertheless, all these statements extend from one to the other settings.

Section 7: In this section we explain the stochastic unfolding method on a simple model example and provide some general remarks regarding this method. Namely, in Section 7.1 we consider stochastic homogenization of convex minimization problems that is based on the above described properties of the stochastic unfolding operator and follows the concept of the periodic unfolding method.

In Section 7.2 we explain that in stochastic homogenization the typical formulas that describe the effective coefficients are inaccessible for standard numerical methods. A classical procedure that resolves this issue is the so-called periodization method which provides an easily-accessible approximation scheme for the considered effective coefficients. We discuss this method on the example of elliptic equations.

Differential equations with random coefficients admit two formulations – the mean and quenched formulations, cf. Remark 1.2. Namely, in the former, a realization of a random medium $\omega \in \Omega$ is considered as a variable and the weak forms of the equations feature an integration over Ω, whereas in the latter, $\omega \in \Omega$ is considered as a parameter in a deterministic parametrized problem. In Section 7.3 we argue that in a typical situation, for equations with unique solutions, the mean and quenched formulations are equivalent. We also explain that typically homogenization of any of the two formulations yields the same effective equation. Finally, we briefly compare the notions of stochastic two-scale convergence in the mean and quenched stochastic two-scale convergence.

The random setting that we consider includes as a specific example the framework for periodic homogenization. In this regard, the stochastic unfolding method provides also a procedure for periodic homogenization of discrete-to-continuum and continuum problems. However, we remark that this procedure does not match the standard periodic unfolding method explained in Section 2. In Section 7.4 we briefly discuss the consequences of the stochastic unfolding method to periodic homogenization.

References: The results of Section 5 have already been published in the joint publication with Stefan Neukamm in [NV18], and the content of Sections 6, 7.1, 7.3.2 and 7.3.3 are mainly based on a recent, joint preprint with Martin Heida and Stefan Neukamm [HNV18]. Sections 7.2, 7.3.1 and 7.4 are new.

5 Discrete unfolding

In the following section, we study stochastic unfolding suited for the treatment of discrete-to-continuum problems. In particular, we present the general framework for discrete random modeling of spring networks, introduce the unfolding operator and derive its most important properties.

5.1 General framework

In the first part of this section, we define the basic objects for the description of discrete functions and their (symmetric) gradients. The second part is devoted to the standard framework for the description of stochastic homogenization problems in the discrete setting.

5.1.1 Discrete calculus

Let $p,q \in (1,\infty)$ be dual exponents of integrability, i.e., $\frac{1}{p}+\frac{1}{q}=1$. $\{e_i\}_{i=1,...,d}$ denotes the standard basis of $\mathbb{R}^d$. For $\varepsilon>0$, we denote the Banach space of p-summable functions by $L^p(\varepsilon\mathbb{Z}^d)=\Big\{u:\varepsilon\mathbb{Z}^d\to\mathbb{R}:\left(\varepsilon^d\sum_{x\in\varepsilon\mathbb{Z}^d}|u(x)|^p\right)^{\frac{1}{p}}<\infty\Big\}$. To keep a neat notation, it is convenient to view $L^p(\varepsilon\mathbb{Z}^d)$ as the L^p-space of p-integrable functions on the measure space $\left(\varepsilon\mathbb{Z}^d,2^{\varepsilon\mathbb{Z}^d},m_\varepsilon\right)$ with $m_\varepsilon=\varepsilon^d\sum_{x\in\varepsilon\mathbb{Z}^d}\delta_x$, that is a rescaled counting measure. In particular, we use the notation $\int_{\varepsilon\mathbb{Z}^d}u(x)dm_\varepsilon(x)=\varepsilon^d\sum_{x\in\varepsilon\mathbb{Z}^d}u(x)$.

Discrete gradient. For $u:\varepsilon\mathbb{Z}^d\to\mathbb{R}$ and $g=(g_1,...,g_d):\varepsilon\mathbb{Z}^d\to\mathbb{R}^d$, we set

$$\nabla_i^\varepsilon u(x)=\frac{u(x+\varepsilon e_i)-u(x)}{\varepsilon},\qquad \nabla_i^{\varepsilon,*}u(x)=\frac{u(x-\varepsilon e_i)-u(x)}{\varepsilon},$$

$$\nabla^\varepsilon u(x)=(\nabla_1^\varepsilon u(x),...,\nabla_d^\varepsilon u(x)),\qquad \nabla^{\varepsilon,*}g(x)=\sum_{i=1}^d\nabla_i^{\varepsilon,*}g_i(x),$$

and we call ∇^ε *discrete gradient* and $\nabla^{\varepsilon,*}$ *(negative) discrete divergence* (in analogy with the usual differential operators ∇ and $-\mathrm{div}$). For $u\in L^p(\varepsilon\mathbb{Z}^d)$, $g\in L^q(\varepsilon\mathbb{Z}^d)^d$, we have the discrete integration by parts formula

$$\int_{\varepsilon\mathbb{Z}^d} \nabla^\varepsilon u(x) \cdot g(x) dm_\varepsilon(x) = \int_{\varepsilon\mathbb{Z}^d} u(x) \nabla^{\varepsilon,*} g(x) dm_\varepsilon(x).$$

Definition 5.1 (Weak and strong convergence)**.** *Consider $u \in L^p(\mathbb{R}^d)$ and a sequence $u_\varepsilon \in L^p(\varepsilon\mathbb{Z}^d)$. We say that:*

- *u_ε weakly converges to u (denoted by $u_\varepsilon \rightharpoonup u$ in $L^p(\mathbb{R}^d)$) if*

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{L^p(\varepsilon\mathbb{Z}^d)} < \infty \text{ and}$$
$$\lim_{\varepsilon\to 0} \int_{\varepsilon\mathbb{Z}^d} u_\varepsilon(x)\eta(x) dm_\varepsilon(x) = \int_{\mathbb{R}^d} u(x)\eta(x) dx \quad \text{for all } \eta \in C_c^\infty(\mathbb{R}^d).$$

- *u_ε strongly converges to u (denoted by $u_\varepsilon \to u$ in $L^p(\mathbb{R}^d)$) if*

$$u_\varepsilon \rightharpoonup u \quad \text{in } L^p(\mathbb{R}^d) \quad \text{and} \quad \lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^p(\varepsilon\mathbb{Z}^d)} = \|u\|_{L^p(\mathbb{R}^d)}.$$

It is convenient to consider piecewise-constant and piecewise-affine interpolations of functions in $L^p(\varepsilon\mathbb{Z}^d)$.

Definition 5.2 (Interpolations and discretization)**.** *(i) For $u : \varepsilon\mathbb{Z}^d \to \mathbb{R}$, its piecewise-constant interpolation $\overline{u} : \mathbb{R}^d \to \mathbb{R}$ (subordinate to $\varepsilon\mathbb{Z}^d$) is given by $\overline{u}(x) = \sum_{y\in\mathbb{Z}^d} \mathbf{1}_{y+\Box}\left(\frac{x}{\varepsilon}\right) u(\lfloor x \rfloor_\varepsilon)$, where $\Box = [0,1)^d$ is the unit box and $\lfloor x \rfloor_\varepsilon \in \varepsilon\mathbb{Z}^d$ is defined by $x - \lfloor x \rfloor_\varepsilon \in \varepsilon\Box$.*

(ii) Consider a triangulation of $\mathbb{R}^d$ into d-simplices with nodes in $\varepsilon\mathbb{Z}^d$ (e.g., Freudenthal's triangulation). For $u : \varepsilon\mathbb{Z}^d \to \mathbb{R}$, we denote its piecewise-affine interpolation w.r.t. the triangulation by $\widehat{u} : \mathbb{R}^d \to \mathbb{R}$.

(iii) The $\varepsilon\mathbb{Z}^d$-discretization $\pi_\varepsilon : L^1_{\text{loc}}(\mathbb{R}^d) \to \mathbb{R}^{\varepsilon\mathbb{Z}^d}$ is defined as

$$(\pi_\varepsilon u)(x) = \fint_{x+\varepsilon\Box} u(y) dy.$$

Remark 5.3. *Note that $\overline{(\cdot)} : L^p(\varepsilon\mathbb{Z}^d) \to L^p(\mathbb{R}^d)$, $u \mapsto \overline{u}$ defines a linear isometry. Also, $\pi_\varepsilon : L^p(\mathbb{R}^d) \to L^p(\varepsilon\mathbb{Z}^d)$ is linear and bounded with $\|\pi_\varepsilon\|_{L^p(\mathbb{R}^d)\to L^p(\varepsilon\mathbb{Z}^d)} \leq 1$. Furthermore, $\pi_\varepsilon \circ \overline{(\cdot)} = Id$ on $L^p(\varepsilon\mathbb{Z}^d)$, and we define $\overline{\pi}_\varepsilon := \overline{(\cdot)} \circ \pi_\varepsilon$, which is a contractive projection, mapping to the subspace of piecewise-constant functions (subordinate to $\varepsilon\mathbb{Z}^d$) in $L^p(\mathbb{R}^d)$.*

Lemma 5.4. *Let $p \in (1,\infty)$. Let $u_\varepsilon \in L^p(\varepsilon\mathbb{Z}^d)$ and $u \in L^p(\mathbb{R}^d)$. The following claims are equivalent:*

(i) $u_\varepsilon \rightharpoonup (\to) u$ in $L^p(\mathbb{R}^d)$.

(ii) $\overline{u}_\varepsilon \to u$ weakly (strongly) in $L^p(\mathbb{R}^d)$.

(iii) $\widehat{u}_\varepsilon \to u$ weakly (strongly) in $L^p(\mathbb{R}^d)$.

The proof of this lemma is an uncomplicated exercise, and therefore we present only (i) $\Rightarrow$ (ii), the rest of the implications follow similarly. Let $u_\varepsilon \rightharpoonup u$ in $L^p(\mathbb{R}^d)$. By the isometry property of $\overline{(\cdot)}$, it follows that $\overline{u}_\varepsilon$ is bounded in $L^p(\mathbb{R}^d)$. For any $\eta \in C_c^\infty(\mathbb{R}^d)$, it holds

$$\Big|\int_{\mathbb{R}^d} \overline{u}_\varepsilon \eta - u\eta dx\Big| \leq \Big|\int_{\mathbb{R}^d} \overline{u}_\varepsilon \eta - \overline{u}_\varepsilon \overline{\eta}_\varepsilon dx\Big| + \Big|\int_{\varepsilon\mathbb{Z}^d} u_\varepsilon \eta dm_\varepsilon - \int_{\mathbb{R}^d} u\eta dx\Big|,$$

where $\eta_\varepsilon : \varepsilon\mathbb{Z}^d \to \mathbb{R}$ is given by $\eta_\varepsilon(x) = \eta(x)$. The second term on the right-hand side vanishes in the limit $\varepsilon \to 0$. Using Hölder's inequality, the first term may be bounded by $c\,\|\eta - \overline{\eta}_\varepsilon\|_{L^q(\mathbb{R}^d)}$ that also tends to zero in the limit $\varepsilon \to 0$ by the smoothness of η. This implies that $\overline{u}_\varepsilon \rightharpoonup u$ weakly. Also, if strong convergence in (i) holds, using the isometry property of $\overline{(\cdot)}$ we conclude that $\overline{u}_\varepsilon$ converges also strongly.

The applications we consider involve problems with Dirichlet boundary conditions, and therefore the following subspace of $L^p(\varepsilon\mathbb{Z}^d)$ is convenient: For $Q \subset \mathbb{R}^d$, we set

$$L^p_0(Q \cap \varepsilon\mathbb{Z}^d) = \Big\{u \in L^p(\varepsilon\mathbb{Z}^d) : u = 0 \text{ in } \varepsilon\mathbb{Z}^d \setminus \Big(Q \cap \varepsilon\mathbb{Z}^d\Big)\Big\}.$$

Periodic lattice graphs and symmetrized gradients

Let $E_0 = \{b_1, ..., b_k\} \subset \mathbb{Z}^d \setminus \{0\}$ be an *edge generating set* and throughout this work we always assume that E_0 includes $\{e_i\}_{i=1,\dots,d}$. We consider the rescaled periodic lattice graph $(\varepsilon\mathbb{Z}^d, \varepsilon E)$, where the set of edges is given by $E = \big\{[x, x + b_i] : x \in \mathbb{Z}^d, i = 1, ..., k\big\}$. For $u : \varepsilon\mathbb{Z}^d \to \mathbb{R}^d$ the difference quotient along the edge generated by b_i is

$$\partial_i^\varepsilon u : \varepsilon\mathbb{Z}^d \to \mathbb{R}^d, \quad \partial_i^\varepsilon u(x) = \frac{u(x + \varepsilon b_i) - u(x)}{\varepsilon |b_i|}.$$

Note that for each b_i, there exists $B_i : \mathbb{Z}^d \to \mathbb{R}^d$ such that $\sum_{y\in\mathbb{Z}^d} B_i(y) = \frac{b_i}{|b_i|}$ and

$$\partial_i^\varepsilon u(x) = \sum_{y\in\mathbb{Z}^d} \nabla^\varepsilon u(x - \varepsilon y) B_i(y). \tag{5.1}$$

Here, ∇^ε is defined componentwise. Note that B_i are not uniquely determined, however, we consider one such fixed choice corresponding to a path between 0 and b_i, cf. Figure 5.1. For $u : \varepsilon\mathbb{Z}^d \to \mathbb{R}^d$, we define the *discrete symmetrized gradient* as $\nabla_s^\varepsilon u : \varepsilon\mathbb{Z}^d \to \mathbb{R}^k$,

$$\nabla_s^\varepsilon u(x) = \left(\frac{b_1}{|b_1|} \cdot \partial_1^\varepsilon u(x), ..., \frac{b_k}{|b_k|} \cdot \partial_k^\varepsilon u(x)\right).$$

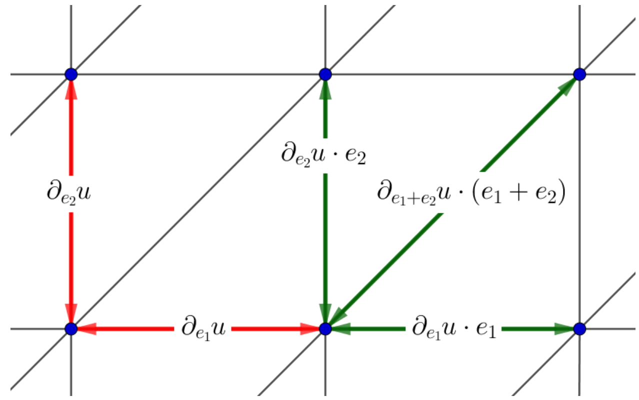
The discrete symmetrized gradient occurs naturally in a discrete version of linearized elasticity (see the introduction to Section 8) as a replacement for the linearized strain tensor "$\frac{1}{2}(\nabla u^T + \nabla u)$" from the continuum theory of elasticity.

If $\varepsilon = 1$, for the sake of simplicity, we write m, ∇u and $\nabla_s u$ instead of m_1, $\nabla^1 u$ and $\nabla_s^1 u$, respectively.

Definition 5.5 (Korn lattice graph)**.** *We say that $(\mathbb{Z}^d, E)$ is a Korn lattice graph if there exists $c(d, p) > 0$ such that for all $p \in (1, \infty)$, we have*

$$\int_{\mathbb{Z}^d} |\nabla u(x)|^p dm(x) \leq c(d, p) \int_{\mathbb{Z}^d} |\nabla_s u(x)|^p dm(x) \quad \textit{for all compactly supported } u : \mathbb{Z}^d \to \mathbb{R}^d. \tag{5.2}$$

Figure 5.1: An example of a periodic Korn lattice graph $\left(\mathbb{Z}^2, E\right)$ with the edge generating set $E_0 = (e_1, e_2, e_1 + e_2)$. The difference quotients corresponding to the usual discrete gradient are denoted by red and to the symmetrized gradient by green. In (5.1), we may consider, e.g., $B_{e_1+e_2}(0) = \frac{e_1}{|e_1+e_2|}$, $B_{e_1+e_2}(-e_1) = \frac{e_2}{|e_1+e_2|}$ and otherwise $B_{e_1+e_2} = 0$.



We remark that rescaling does not affect the the constant in the above Korn inequality. In particular, if $\left(\mathbb{Z}^d, E\right)$ is a Korn lattice, then for any $\varepsilon > 0$, it holds

$$\int_{\varepsilon\mathbb{Z}^d} |\nabla^\varepsilon u(x)|^p dm_\varepsilon(x) \leq c(d,p) \int_{\varepsilon\mathbb{Z}^d} |\nabla_s^\varepsilon u(x)|^p dm_\varepsilon(x) \quad \text{for all compactly supported } u : \varepsilon\mathbb{Z}^d \to \mathbb{R}^d. \tag{5.3}$$

Let $Q \subset \mathbb{R}^d$ be open. For $u \in W^{1,p}(Q)^d$, we define the continuum (nonstandard) *symmetrized gradient* $\nabla_s u \in L^p(Q)^k$ by

$$\left(\nabla_s u\right)_i = \frac{b_i}{|b_i|} \cdot \nabla u \frac{b_i}{|b_i|}, \quad i \in \{1, ..., k\}\,.$$

Here, we slightly abuse the notation by reusing "∇_s" for the continuum symmetrization. We do this to keep the notation simple and since the continuum symmetrization emerges in the limit $\varepsilon \to 0$ from its discrete counterpart ∇_s^ε, cf. Lemma 5.18. We hope that this does not lead to confusion and we always try to make it clear from the context which operator we use.

Remark 5.6 (Continuum Korn inequality)**.** *The existence of a Korn lattice implies a continuum version of Korn's inequality. Namely, let $Q \subset \mathbb{R}^d$ be open and bounded, $p \in (1, \infty)$, and $\left(\mathbb{Z}^d, E\right)$ be a Korn lattice graph. Then, there exists $c(d,p) > 0$ such that*

$$\int_Q |\nabla u(x)|^p dx \leq c(d,p) \int_Q |\nabla_s u(x)|^p dx \quad \textit{for all } u \in W_0^{1,p}(Q)^d. \tag{5.4}$$

This can be obtained by first approximating $u \in W_0^{1,p}(Q)^d$ by a sequence of smooth functions $u_\delta \in C_c^\infty(Q)^d$. Second, we may apply the discrete Korn inequality (5.3) to $\pi_\varepsilon u_\delta$. Finally, by passing to the limits $\varepsilon \to 0$ and $\delta \to 0$ (cf. Lemma 5.18 in the following section), (5.4) follows. Note that (5.2) also implies another, stochastic version of Korn's inequality, see Lemma 5.12 below.

We present a simple example of a Korn lattice graph, see also Figure 5.1.

Example 5.7 (Korn lattice graph)**.** *We assume that $\left\{\sum_{i=1}^d \delta_i e_i : \delta \in \{0,1\}^d \setminus 0\right\} \subset E_0$ where E_0 is the edge generating set of $\left(\mathbb{Z}^d, E\right)$. Then,*

$$\left(\mathbb{Z}^d, E\right) \textit{ is a Korn lattice graph.} \tag{5.5}$$

Using a piecewise affine interpolation w.r.t. a suitable triangulation of $\mathbb{R}^d$, the proof of this statement reduces to the usual continuum Korn inequality. In particular, in [BB06] this strategy has been applied to prove discrete Korn inequalities for three-dimensional, not necessarily periodic, networks which satisfy the so-called triangulization property. In our example, we might also consider a different edge generating set consisting of the edges of all d-simplices of some triangulation of the unit box $[0,1]^d$ (the above E_0 corresponds to Freudenthal's triangulation). For the convenience of the reader we present the proof of (5.5), which applies as well to higher dimensional lattices:

Proof of (5.5). First, we recall Freudenthal's triangulation of $\mathbb{R}^d$, see, e.g., [Moo92]. In this triangulation, the unit box $[0,1]^d$ is divided into d-simplices which are periodically extended to $\mathbb{R}^d$. Each of the simplices in $[0,1]^d$ is obtained as a convex envelope of the nodes that represent a path between 0 and 1^d, e.g., if $d = 2$ there are two simplices in $[0,1]^2$: conv $\{0, e_1, e_1 + e_2\}$ and conv $\{0, e_2, e_1 + e_2\}$. We consider a compactly supported $u : \mathbb{Z}^d \to \mathbb{R}^d$. Let $\hat{u}$ denote the piecewise-affine interpolation of u w.r.t. Freudenthal's triangulation. By construction, for each $i \in \{1, ..., d\}$, there exists a simplex $S_i \subset [0,1]^d$ such that $\int_{x+\square} |\nabla \hat{u}(y)|^p dy \geq |S_i||\partial_i u(x)|^p$ for any $x \in \mathbb{Z}^d$. Thus we obtain

$$\int_{\mathbb{Z}^d} |\nabla u(x)|^p dm(x) \leq c(d) \sum_{x\in\mathbb{Z}^d} \int_{x+\square} |\nabla \hat{u}(y)|^p dy = c(d) \int_{\mathbb{R}^d} |\nabla \hat{u}(x)|^p dx. \tag{5.6}$$

On the other hand, for a fixed simplex $S \subset [0,1]^d$ and fixed $i, j \in \{1, ..., d\}$, we have

$$\int_{x+S} |\partial_j \hat{u}_i(y) + \partial_i \hat{u}_j(y)|^p dy = |S||u_i(x + y_0 + e_j) - u_i(x + y_0) + u_j(x + y_1 + e_i) - u_j(x + y_1)|^p, \tag{5.7}$$

for some y_0, y_1 which are nodes of $[0,1]^d$. This follows since $\nabla \hat{u}$ is constant on $x + S$ and the partial derivatives of $\hat{u}$ match the partial differences of u along suitable edges of $x+S$. We continue the argument with the assumption $y_1 = y_0 + e_j + y$ where $y = \sum_{s=1}^k e_{i(s)}$ where the indices $i(s) \in \{1, ..., d\} \setminus \{i, j\}$ are all distinct. Otherwise, we may assume that $y_0 = y_1 + e_i + y$ and the argument follows analogously. We set $x_0 := x + y_0$ and the right-hand side of (5.7) may be bounded by

$$\begin{aligned}
&\frac{1}{c(p)}|u_i(x_0 + e_j) - u_i(x_0) + u_j(x_0 + y + e_j + e_i) - u_j(x_0 + y + e_j)|^p \\
\leq\ &\big|u_i(x_0 + y + e_i + e_j) - u_i(x_0) + u_j(x_0 + y + e_i + e_j) - u_j(x_0) \\
&\qquad\qquad\qquad\qquad + \sum_{s=1}^{k} u_{i(s)}(x_0 + y + e_i + e_j) - u_{i(s)}(x_0)\big|^p \\
&+\big|u_i(x_0 + e_j + y + e_i) - u_i(x_0 + e_j) + \sum_{s=1}^{k} u_{i(s)}(x_0 + e_j + y + e_i) - u_{i(s)}(x_0 + e_j)\big|^p \\
&+\big|u_j(x_0 + y + e_j) - u_j(x_0) + \sum_{s=1}^{k} u_{i(s)}(x_0 + y + e_j) - u_{i(s)}(x_0)\big|^p \\
&+\big|\sum_{s=1}^{k} u_{i(s)}(x_0 + e_j + y) - u_{i(s)}(x_0 + e_j)\big|^p.
\end{aligned}$$

Above we added and subtracted on the left-hand side the following terms $u_i(x_0+y+e_i+e_j)$, $u_j(x_0)$ and $\sum_{s=1}^k u_{i(s)}(x_0+y+e_i+e_j)+u_{i(s)}(x_0+y+e_j)+u_{i(s)}(x_0+e_j)+u_{i(s)}(x_0)$. The above inequality follows by a careful sorting of the terms and by the triangle inequality. Using the assumption on the edge generating set E_0, the right-hand side above may be bounded by $2|\nabla_s u(x_0)|^p + 2|\nabla_s u(x_0+e_j)|^p$. As a result of this and (5.7), it follows that there exists $c(d,p) > 0$ such that

$$\int_{\mathbb{R}^d} |\nabla \hat{u}(x)^T + \nabla \hat{u}(x)|^p dx \leq c(d,p) \int_{\mathbb{Z}^d} |\nabla_s u(x)|^p dm(x).$$

If we combine the above inequality with (5.6) and with the usual Korn inequality, we obtain

$$\int_{\mathbb{Z}^d} |\nabla u(x)|^p dm(x) \leq c(d,p) \int_{\mathbb{Z}^d} |\nabla_s u(x)|^p dm(x).$$

□

5.1.2 Stochastic calculus in a discrete setting

Throughout entire Section 5 we assume the following assumption to be satisfied.

Assumption 5.8. *Let $(\Omega, \mathcal{F}, P)$ be a complete and separable probability space, which is equipped with a discrete dynamical system $\tau = \{\tau_x : \Omega \to \Omega\}_{x \in \mathbb{Z}^d}$ such that:*

(i) (Measurability.) $\tau_x : \Omega \to \Omega$ is invertible and measurable for all $x \in \mathbb{Z}^d$.

(ii) (Group property.) $\tau_0 = Id$ and $\tau_{x+y} = \tau_x \circ \tau_y$ for all $x, y \in \mathbb{Z}^d$.

(iii) (Measure preservation.) $P(\tau_x E) = P(E)$ for all $E \in \mathcal{F}$ and $x \in \mathbb{Z}^d$.

We write $\langle \cdot \rangle = \int_\Omega \cdot dP(\omega)$ for the expectation and $L^p(\Omega)$ for the usual Banach space of p-integrable random variables. Above, the separability of the measure space means that $\mathcal{F}$ is the completion (w.r.t. P) of some countably generated σ-algebra. This assumption implies that $L^p(\Omega)$ is separable for $p \in [1, \infty)$ (see [Bré11, Theorem 4.13]). We say that $(\Omega, \mathcal{F}, P, \tau)$ is *ergodic* ($\langle \cdot \rangle$ is ergodic), if the following implication holds:

$$E \in \mathcal{F} \text{ and it is shift invariant, i.e., } \tau_x E = E \text{ for all } x \in \mathbb{Z}^d \quad \Rightarrow \quad P(E) \in \{0, 1\}.$$

Remark 5.9 (Ergodic theorems, see, e.g., [Tem72], [DVJ07, Section 12.2],[AK81, Theorem 2.4])**.** *Let $\langle \cdot \rangle$ be ergodic and $Q \subset \mathbb{R}^d$ be open, bounded and convex. Let $p \geq 1$ and $L \geq 1$. A multiparameter version of von Neumann's mean ergodic theorem states that if $\varphi \in L^p(\Omega)$, then*

$$⨍_{LQ \cap \mathbb{Z}^d} \varphi(\tau_x \cdot) dm(x) = \frac{1}{|LQ|} \sum_{x \in LQ \cap \mathbb{Z}^d} \varphi(\tau_x \cdot) \to \langle \varphi \rangle \quad \textit{strongly in } L^p(\Omega) \quad \textit{as } L \to \infty.$$

Moreover, a multiparameter version of Birkhoff's ergodic theorem implies that the above convergence holds for P-a.a. $\omega \in \Omega$.

We present a standard example of a random medium – independent and identically distributed (*i.i.d.*) random fields:

Example 5.10 (I.i.d. random fields). *Let $\Omega_0 \subset \mathbb{R}^{d\times d}$ and we equip it with a countably generated σ-algebra $\mathcal{F}_0$ (e.g., the Borel sets if it is open) and with a probability measure P_0. We define $(\Omega, \mathcal{F}, P)$ as the $\mathbb{Z}^d$-fold product of $(\Omega_0, \mathcal{F}_0, P_0)$, i.e.,*

$$\Omega := \Omega_0^{\mathbb{Z}^d}, \quad \mathcal{F} = \overline{\otimes_{\mathbb{Z}^d} \mathcal{F}_0}, \quad P = \otimes_{\mathbb{Z}^d} P_0.$$

Note that a configuration $\omega \in \Omega$ can be seen as a function $\omega : \mathbb{Z}^d \to \Omega_0$. By construction, for $x \in \mathbb{Z}^d$, the coordinate projections $\pi_x : \Omega \to \Omega_0$, $\pi_x(\omega) = \omega(x)$ are i.i.d. random variables (distributed according to P_0). We define a shift $\tau_x : \Omega \to \Omega$ for $x \in \mathbb{Z}^d$ as

$$\tau_x \omega(\cdot) := \omega(\cdot + x).$$

It follows that $(\Omega, \mathcal{F}, P, \tau)$ satisfies Assumption 5.8 and defines an ergodic probability space. Usually, we consider random coefficients of difference equations in the form $A(\tau_{\frac{x}{\varepsilon}}\omega)$ for a measurable $A : \Omega \to \mathbb{R}^{d\times d}$. For the above specific Ω, the choice $A(\omega) = \omega(0)$ provides coefficient fields of the form $x \mapsto \omega(\frac{x}{\varepsilon})$.

Stochastic gradient. For $\varphi : \Omega \to \mathbb{R}$ and $\psi = (\psi_1, ..., \psi_d) : \Omega \to \mathbb{R}^d$ measurable, we introduce the *stochastic gradient* D and *(negative) stochastic divergence* D^*:

$$D_i\varphi(\omega) = \varphi(\tau_{e_i}\omega) - \varphi(\omega), \qquad D_i^*\varphi(\omega) = \varphi(\tau_{-e_i}\omega) - \varphi(\omega),$$

$$D\varphi(\omega) = (D_1\varphi(\omega), ..., D_d\varphi(\omega)), \qquad D^*\psi(\omega) = \sum_{i=1}^{d} D_i^*\psi_i(\omega).$$

Remark 5.11. *Let $p \in (1, \infty)$ and $q = \frac{p}{p-1}$. $D : L^p(\Omega) \to L^p(\Omega)^d$ and $D^* : L^p(\Omega)^d \to L^p(\Omega)$ are linear and bounded operators. Furthermore, for any $\varphi \in L^p(\Omega)$ and $\psi \in L^q(\Omega)^d$ the integration by parts formula*

$$\langle D\varphi \cdot \psi \rangle = \langle \phi D^*\psi \rangle$$

holds. Hence, D (defined on $L^p(\Omega)$) and D^* (defined on $L^q(\Omega)^d$) are adjoint operators.

We denote the set of *shift-invariant functions* in $L^p(\Omega)$ by

$$L^p_{\text{inv}}(\Omega) := \left\{ \varphi \in L^p(\Omega) : \varphi(\tau_x \cdot) = \varphi(\cdot) \quad \text{for all } x \in \mathbb{Z}^d \right\},$$

and we note that $L^p_{\text{inv}}(\Omega) \simeq \mathbb{R}$ if and only if $\langle\cdot\rangle$ is ergodic. We denote by $P_{\text{inv}} : L^p(\Omega) \to L^p_{\text{inv}}(\Omega)$ the conditional expectation w.r.t. the σ-algebra generated by the family of shift invariant sets $\{E \in \mathcal{F} : \tau_x E = E \text{ for every } x \in \mathbb{Z}^d\}$. P_{inv} is a contractive projection and in the ergodic case we simply have $P_{\text{inv}}\varphi = \langle\varphi\rangle$.
It is easily verified that $L^p_{\text{inv}}(\Omega) = \ker D$ and by standard arguments (see [Bré11, Section 2.6]) we have the orthogonality relations

$$L^p_{\text{pot}}(\Omega) := \overline{\text{ran} D}^{L^p(\Omega)^d} = (\ker D^*)^\perp, \qquad \ker D = (\text{ran} D^*)^\perp, \tag{5.8}$$

where $\ker(\cdot)$ denotes the kernel and $\text{ran}(\cdot)$ the range of an operator. The above relations hold in the sense of L^p-L^q duality (we identify the dual $L^q(\Omega)^*$ with $L^p(\Omega)$). Namely, above $D : L^p(\Omega) \to L^p(\Omega)^d$ and $D^* : L^q(\Omega)^d \to L^q(\Omega)$, and the orthogonal of a set $A \subset L^q(\Omega)$ is given by

$$A^\perp = \left\{ \varphi \in L^q(\Omega)^* : \langle \varphi, \psi \rangle_{(L^q)^*, L^q} = 0 \text{ for all } \psi \in A \right\}.$$

Random fields. In the following, measurable functions defined on $\Omega \times \varepsilon\mathbb{Z}^d$ or on $\Omega \times Q$, with $Q \subset \mathbb{R}^d$ open, are called random fields. We mainly consider the space of p-integrable random fields $L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ (resp. $L^p(\Omega \times Q)$) with $p \in (1, \infty)$. We frequently use the following notation: If $X \subset L^p(\Omega)$ and $Y \subset L^p(\varepsilon\mathbb{Z}^d)$ (resp. $Y \subset L^p(Q)$) are closed subspaces, then we denote by $X \otimes Y$ the closure of

$$X \overset{a}{\otimes} Y := \left\{ \sum_{i=1}^{n} \varphi_i \eta_i : \varphi_i \in X, \eta_i \in Y, n \in \mathbb{N} \right\}$$

in $L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ (resp. $L^p(\Omega \times Q)$). In this regard, we tacitly identify linear and bounded operators on X (or Y) by their obvious extension to $X \otimes Y$. Note that in the case $X = L^p(\Omega)$ and $Y = L^p(\varepsilon\mathbb{Z}^d)$, we have $X \otimes Y = L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ (and analogously for $L^p(\Omega \times Q)$). Up to isometric isomorphisms, we may identify $L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ with the Bochner spaces $L^p(\Omega; L^p(\varepsilon\mathbb{Z}^d))$ and $L^p(\varepsilon\mathbb{Z}^d; L^p(\Omega))$ (and analogously for $L^p(\Omega \times Q)$). Slightly abusing the notation, for closed subspaces $X \subset L^p(\Omega)$ and $Y \subset W^{1,p}(Q)$, we denote by $X \otimes Y$ the closure of

$$X \overset{a}{\otimes} Y := \left\{ \sum_{i=1}^{n} \varphi_i \eta_i : \varphi_i \in X, \eta_i \in Y, n \in \mathbb{N} \right\}$$

in $L^p(\Omega; W^{1,p}(Q))$. In this regard, we may identify $u \in L^p(\Omega) \otimes W^{1,p}(Q)$ with the pair $(u, \nabla u) \in L^p(\Omega \times Q)^{1+d}$. This notation might seem unusual, however it is very convenient for keeping track of the various subspaces of $L^p(\Omega \times Q)$ that we deal with.

Symmetrized random fields

We consider a periodic lattice graph $\left(\mathbb{Z}^d, E\right)$ with its edge generating set E_0 as in Section 5.1.1. We consider the corresponding set $\left\{B_i : \mathbb{Z}^d \to \mathbb{R}^d\right\}_{i \in \{1,\dots,k\}}$ from (5.1). For a random field $F : \Omega \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$, we define its *symmetrization* by $F_s : \Omega \times \mathbb{R}^d \to \mathbb{R}^k$,

$$(F_s)_i(\omega, x) = \frac{b_i}{|b_i|} \cdot \sum_{y \in \mathbb{Z}^d} F(\tau_{-y}\omega, x) B_i(y), \quad i \in \{1, ..., k\}. \tag{5.9}$$

Note that F_s is a measurable mapping and, if $p \in (1, \infty)$, the symmetrization defines a linear bounded operator $(\cdot)_s : L^p(\Omega \times \mathbb{R}^d)^{d \times d} \to L^p(\Omega \times \mathbb{R}^d)^k$. We remark that if F is deterministic, i.e., it does not depend on ω, then

$$(F_s)_i(x) = \frac{b_i}{|b_i|} \cdot F(x) \frac{b_i}{|b_i|}.$$

In particular, for $u \in W^{1,p}(\mathbb{R}^d)^d$, $(\nabla u)_s$ matches the (nonstandard) symmetrized gradient $\nabla_s u$. We also remark that in the case that F is a stochastic gradient, i.e., $F = \chi$ where $\chi \in L^p_{\text{pot}}(\Omega)^d$, χ_s does not depend on the choice of $\{B_i\}$ but only on E_0. This might be shown by a direct computation for functions of the form $D\varphi$ and, for general $\chi \in L^p_{\text{pot}}(\Omega)^d$, by an approximation argument $D\varphi_n \to \chi$.

We present a stochastic Korn inequality that will be useful in the applications.

Lemma 5.12 (Stochastic Korn inequality)**.** *Let $p \in (1, \infty)$. We assume that $(\mathbb{Z}^d, E)$ is a Korn lattice graph. Then, there exists $c(d, p) > 0$ such that*

$$\langle |\chi|^p \rangle \le c(d, p) \langle |\chi_s|^p \rangle \quad \textit{for all } \chi \in L^p_{\text{pot}}(\Omega)^d.$$

Proof. We show the inequality in the case $\chi = D\varphi$ for $\varphi \in L^p(\Omega)^d$. For general $\chi \in L^p_{\text{pot}}(\Omega)^d$, it is obtained by an approximation argument. We denote by $\widetilde{\varphi} : \Omega \times \mathbb{Z}^d \to \mathbb{R}^d$ the stationary extension of φ, i.e., $\widetilde{\varphi}(\omega, x) = \varphi(\tau_x \omega)$. Let $R > 0$ and $K > 0$ be such that $K > \sup\{|b| : b \in E_0\}$. Let η_R be a cut-off function given by $\eta_R : \mathbb{Z}^d \to \mathbb{R}$ with $\eta_R = 1$ in $B_{R+K} \cap \mathbb{Z}^d$ and $\eta_R = 0$ otherwise ($B_R \subset \mathbb{R}^d$ is a ball of radius R with center in 0). Using the properties of η_R, the discrete Korn inequality (5.2) and the shorthand $|B_R| := m(B_R \cap \mathbb{Z}^d)$, we obtain

$$\begin{aligned}
\left\langle \fint_{B_R \cap \mathbb{Z}^d} |D\varphi(\tau_x\omega)|^p dm(x) \right\rangle &\leq \left\langle \frac{1}{|B_R|} \int_{\mathbb{Z}^d} |\nabla(\widetilde{\varphi}(\omega, x)\eta_R(x))|^p dm(x) \right\rangle \\
&\leq \left\langle \frac{c}{|B_R|} \int_{\mathbb{Z}^d} |\nabla_s(\widetilde{\varphi}(\omega, x)\eta_R(x))|^p dm(x) \right\rangle \\
&= \left\langle c \fint_{B_R \cap \mathbb{Z}^d} |\nabla_s(\widetilde{\varphi}(\omega, x))|^p dm(x) \right\rangle \\
&\quad + \left\langle \frac{c}{|B_R|} \int_{(B_{R+2K} \setminus B_R) \cap \mathbb{Z}^d} |\nabla_s(\widetilde{\varphi}(\omega, x)\eta_R(x))|^p dm(x) \right\rangle .
\end{aligned}$$

By invariance of P w.r.t. τ, the left-hand side of the above inequality equals $\langle |D\varphi|^p \rangle$ for any R. Moreover, the first term on the right-hand side equals $c\,\langle |D_s\varphi|^p \rangle$, which is obtained by a direct computation and using again the invariance of P w.r.t. τ. Therefore, it is sufficient to show that the second term vanishes in the limit $R \to \infty$. To obtain that, we estimate, for P-a.a. $\omega \in \Omega$,

$$\begin{aligned}
& \frac{1}{|B_R|} \int_{(B_{R+2K} \setminus B_R) \cap \mathbb{Z}^d} |\nabla_s(\widetilde{\varphi}(\omega, x)\eta_R(x))|^p dm(x) \qquad (5.10) \\
\leq \; & \frac{c}{|B_R|} \int_{(B_{R+2K} \setminus B_{R-K}) \cap \mathbb{Z}^d} |\widetilde{\varphi}(\omega, x)|^p |\eta_R(x)|^p dm(x) \\
\leq \; & \frac{c}{|B_R|} \int_{(B_{R+2K} \setminus B_{R-K}) \cap \mathbb{Z}^d} |\widetilde{\varphi}(\omega, x)|^p dm(x) \\
= \; & \frac{c}{|B_R|} \int_{B_{R+2K} \cap \mathbb{Z}^d} |\widetilde{\varphi}(\omega, x)|^p dm(x) - \frac{c}{|B_R|} \int_{B_{R-K} \cap \mathbb{Z}^d} |\widetilde{\varphi}(\omega, x)|^p dm(x) \\
= \; & \frac{c|B_{R+2K}|}{|B_R|} \fint_{B_{R+2K} \cap \mathbb{Z}^d} |\widetilde{\varphi}(\omega, x)|^p dm(x) - \frac{c|B_{R-K}|}{|B_R|} \fint_{B_{R-K} \cap \mathbb{Z}^d} |\widetilde{\varphi}(\omega, x)|^p dm(x).
\end{aligned}$$

In the first inequality above, we used the fact that the discrete symmetrized gradient ∇_s is a bounded operator. An integration of (5.10) over Ω yields

$$\begin{aligned}
& \left\langle \frac{c}{|B_R|} \int_{(B_{R+2K} \setminus B_R) \cap \mathbb{Z}^d} |\nabla_s(\widetilde{\varphi}(\omega, x)\eta_R(x))|^p dm(x) \right\rangle \\
\leq \; & \langle |\varphi|^p \rangle \left(\frac{c|B_{R+2K}|}{|B_R|} - \frac{c|B_{R-K}|}{|B_R|} \right) \to 0 \quad \text{as } R \to \infty.
\end{aligned}$$

This concludes the proof. $\square$

5.2 Stochastic unfolding: definition and properties

In this section we introduce a key object in our analysis, the stochastic unfolding operator, and we discuss its main properties.

Let $\varepsilon > 0$. For a random field $u : \Omega \times \varepsilon\mathbb{Z}^d \to \mathbb{R}$, we define its *unfolding* $\widetilde{\mathcal{T}}_\varepsilon u : \Omega \times \varepsilon\mathbb{Z}^d \to \mathbb{R}$ via

$$(\widetilde{\mathcal{T}}_\varepsilon u)(\omega, x) = u(\tau_{-\frac{x}{\varepsilon}}\omega, x). \tag{5.11}$$

Lemma 5.13 (Stochastic unfolding operator)**.** *Let $p \in (1, \infty)$. (5.11) defines an isometric isomorphism $\widetilde{\mathcal{T}}_\varepsilon : L^p(\Omega \times \varepsilon\mathbb{Z}^d) \to L^p(\Omega \times \varepsilon\mathbb{Z}^d)$. We refer to $\widetilde{\mathcal{T}}_\varepsilon$ as stochastic unfolding operator.*

Proof. First we note that $\widetilde{\mathcal{T}}_\varepsilon u$ is $\mathcal{F} \otimes 2^{\varepsilon\mathbb{Z}^d}$-measurable. Indeed, $u \in L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ is $\mathcal{F} \otimes 2^{\varepsilon\mathbb{Z}^d}$-measurable since $\mathcal{F} \otimes 2^{\varepsilon\mathbb{Z}^d}$ is complete w.r.t. $P \otimes m_\varepsilon$, that can be obtained by the completeness of $\mathcal{F}$ and the discreteness of the σ-algebra on $\varepsilon\mathbb{Z}^d$. Also, the mapping $(\omega, x) \mapsto (\tau_{-\frac{x}{\varepsilon}}\omega, x)$ is $(\mathcal{F} \otimes 2^{\varepsilon\mathbb{Z}^d}, \mathcal{F} \otimes 2^{\varepsilon\mathbb{Z}^d})$-measurable. As a result of this, since $\widetilde{\mathcal{T}}_\varepsilon u$ is the composition of the last two mappings, it is measurable.
Moreover, $\widetilde{\mathcal{T}}_\varepsilon$ is linear and we compute

$$\|\widetilde{\mathcal{T}}_\varepsilon u\|^p_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)} = \int_{\varepsilon\mathbb{Z}^d} \left\langle |u(\tau_{-\frac{x}{\varepsilon}}\omega, x)|^p \right\rangle dm_\varepsilon(x) = \int_{\varepsilon\mathbb{Z}^d} \langle |u(\omega, x)|^p \rangle \, dm_\varepsilon(x) = \|u\|^p_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)},$$

where we use Fubini's theorem and in the second equality the fact that P is invariant with respect to the action of $\tau_{-\frac{x}{\varepsilon}}$. Also, the inverse of $\widetilde{\mathcal{T}}_\varepsilon$ is given by $\widetilde{\mathcal{T}}_{-\varepsilon} u(\omega, x) = u(\tau_{\frac{x}{\varepsilon}}\omega, x)$, which satisfies analogous properties as $\widetilde{\mathcal{T}}_\varepsilon$. $\square$

In applications we consider discrete-to-continuum transition problems and therefore it is convenient to lift the unfolding operator to the continuum space $\mathbb{R}^d$, i.e., we consider

$$\mathcal{T}_\varepsilon := \overline{(\cdot)} \circ \widetilde{\mathcal{T}}_\varepsilon : L^p(\Omega \times \varepsilon\mathbb{Z}^d) \to L^p(\Omega \times \mathbb{R}^d).$$

$\mathcal{T}_\varepsilon$ is a linear (nonsurjective) isometry. We call $\mathcal{T}_\varepsilon$ also *stochastic unfolding operator*. Note that $\widetilde{\mathcal{T}}_\varepsilon$ (defined on L^p) and $\widetilde{\mathcal{T}}_{-\varepsilon}$ (defined on L^q, $q = \frac{p}{p-1}$) are adjoint operators and in this respect, if $p = 2$, $\widetilde{\mathcal{T}}_\varepsilon$ is a unitary transformation.

As in the periodic unfolding method, a central notion of convergence in the stochastic case is the convergence of unfolded sequences. In particular, usually we consider a sequence $u_\varepsilon \in L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ and we primarily deal with the convergence of the sequence $\mathcal{T}_\varepsilon u_\varepsilon$ either in the weak or strong topology of $L^p(\Omega \times \mathbb{R}^d)$.

Remark 5.14 (Comparison to two-scale convergence in the mean)**.** *Let $p \in (1, \infty)$. Note that the adaptation of the notion of two-scale convergence in the mean from [BMW94, AW98] to the discrete setting reads: $u_\varepsilon \in L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ stochastically two-scale converges in the mean to $u \in L^p(\Omega \times \mathbb{R}^d)$ if*

$$\limsup_{\varepsilon\to 0} \left\langle \int_{\varepsilon\mathbb{Z}^d} |u_\varepsilon(\omega, x)|^p dm_\varepsilon(x) \right\rangle < \infty, \tag{5.12}$$

$$\lim_{\varepsilon\to 0} \left\langle \int_{\varepsilon\mathbb{Z}^d} u_\varepsilon(\omega, x)\varphi(\tau_{\frac{x}{\varepsilon}}\omega)\eta(x) dm_\varepsilon(x) \right\rangle = \left\langle \int_{\mathbb{R}^d} u(\omega, x)\varphi(\omega)\eta(x) dx \right\rangle \tag{5.13}$$

for all $\varphi \in L^q(\Omega)$ and all $\eta \in C_c^\infty(\mathbb{R}^d)$. For a sequence $u_\varepsilon \in L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ that satisfies (5.12), the following equivalence holds

$$(5.13) \quad \Leftrightarrow \quad \mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega \times \mathbb{R}^d).$$

This follows using the invariance of P w.r.t. τ and Lemma 5.4.

For the reason of the above equivalence and to keep the notation uncluttered, we use the notation

$$u_\varepsilon \overset{2}{\rightharpoonup} (\overset{2}{\to}) u \quad :\Leftrightarrow \quad \mathcal{T}_\varepsilon u_\varepsilon \to u \quad \text{weakly (strongly) in } L^p(\Omega \times \mathbb{R}^d).$$

We also call this notion of convergence *weak* (*strong*) stochastic two-scale convergence in the mean, however, we remark that by this we always mean convergence of the unfolded sequences.

Remark 5.15 (Relation to usual notions of convergence)**.** *We remark that even for a constant function $u \in L^p(\Omega) \otimes C(\mathbb{R}^d)$, in general, it does not hold $u \overset{2}{\rightharpoonup} u$, cf. Remark 5.9. However, if $u_\varepsilon \in L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ and $u \in L^p(\mathbb{R}^d)$, the following equivalence holds*

$$u_\varepsilon \overset{2}{\to} u \quad \Leftrightarrow \quad \overline{u}_\varepsilon \to u \quad \textit{strongly in } L^p(\Omega \times \mathbb{R}^d).$$

The following lemma follows directly from the isometry property of $\mathcal{T}_\varepsilon$ and the usual properties of weak convergence in $L^p(\Omega \times \mathbb{R}^d)$.

Lemma 5.16 (Basic properties)**.** *Let $p \in (1, \infty)$ and $q = \frac{p}{p-1}$. We consider sequences $u_\varepsilon \in L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ and $v_\varepsilon \in L^q(\Omega \times \varepsilon\mathbb{Z}^d)$.*

(i) If $u_\varepsilon \overset{2}{\rightharpoonup} u$ in $L^p(\Omega \times \mathbb{R}^d)$, then

$$\sup_\varepsilon \|u_\varepsilon\|_{L^p(\Omega \times \varepsilon\mathbb{Z}^d)} < \infty \quad \textit{and} \quad \|u\|_{L^p(\Omega \times \mathbb{R}^d)} \leq \liminf_{\varepsilon \to 0} \|u_\varepsilon\|_{L^p(\Omega \times \varepsilon\mathbb{Z}^d)}.$$

(ii) If $\limsup_{\varepsilon \to 0} \|u_\varepsilon\|_{L^p(\Omega \times \varepsilon\mathbb{Z}^d)} < \infty$, then there exist $u \in L^p(\Omega \times \mathbb{R}^d)$ and a subsequence ε' such that $u_{\varepsilon'} \overset{2}{\rightharpoonup} u$ in $L^p(\Omega \times \mathbb{R}^d)$.

(iii) $u_\varepsilon \overset{2}{\to} u$ in $L^p(\Omega \times \mathbb{R}^d)$ if and only if $\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^p(\Omega \times \varepsilon\mathbb{Z}^d)} = \|u\|_{L^p(\Omega \times \mathbb{R}^d)}$ and $u_\varepsilon \overset{2}{\rightharpoonup} u$ in $L^p(\Omega \times \mathbb{R}^d)$.

(iv) If $u_\varepsilon \overset{2}{\to} u$ in $L^p(\Omega \times \mathbb{R}^d)$ and $v_\varepsilon \overset{2}{\rightharpoonup} v$ in $L^q(\Omega \times \mathbb{R}^d)$, then

$$\lim_{\varepsilon \to 0} \left\langle \int_{\varepsilon\mathbb{Z}^d} u_\varepsilon(\omega, x) v_\varepsilon(\omega, x) dm_\varepsilon(x) \right\rangle = \left\langle \int_{\mathbb{R}^d} u(\omega, x) v(\omega, x) dx \right\rangle.$$

As in the periodic setting, a suitable "inverse" of the unfolding operator $\mathcal{T}_\varepsilon$ is given by the linear operator

$$\mathcal{F}_\varepsilon : L^p(\Omega \times \mathbb{R}^d) \to L^p(\Omega \times \varepsilon\mathbb{Z}^d), \quad \mathcal{F}_\varepsilon = \widetilde{\mathcal{T}}_\varepsilon^{-1} \circ \pi_\varepsilon.$$

In analogy to the periodic case, we refer to $\mathcal{F}_\varepsilon$ as the *stochastic folding operator*. Note that $\mathcal{F}_\varepsilon : L^p(\Omega \times \mathbb{R}^d) \to L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ is exactly the adjoint of $\mathcal{T}_\varepsilon : L^q(\Omega \times \varepsilon\mathbb{Z}^d) \to L^q(\Omega \times \mathbb{R}^d)$.

Lemma 5.17 (Properties of folding)**.** *Let $p \in (1, \infty)$. $\mathcal{F}_\varepsilon$ is a linear bounded operator and it satisfies:*

(i) $\|\mathcal{F}_\varepsilon u\|_{L^p(\Omega \times \varepsilon\mathbb{Z}^d)} \leq \|u\|_{L^p(\Omega \times \mathbb{R}^d)}$ for all $u \in L^p(\Omega \times \mathbb{R}^d)$.

(ii) $\mathcal{F}_\varepsilon \circ \mathcal{T}_\varepsilon = Id$ on $L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ and $\mathcal{T}_\varepsilon \circ \mathcal{F}_\varepsilon = \overline{\pi}_\varepsilon$ on $L^p(\Omega \times \mathbb{R}^d)$.

(iii) $\mathcal{F}_\varepsilon u \xrightarrow{2} u$ *in* $L^p(\Omega\times\mathbb{R}^d)$ *for all* $u\in L^p(\Omega\times\mathbb{R}^d)$.

Proof. (i) follows by the isometry property of $\widetilde{\mathcal{T}}_\varepsilon$ and contraction property of π_ε. (ii) follows directly from the definition of the operators. In (iii) we may approximate u by a function of the form $u_n=\sum_{i=1}^n \varphi_i^n\eta_i^n$ where $\varphi_i^n\in L^p(\Omega)$ and $\eta_i^n\in C_c^\infty(\mathbb{R}^d)$. We have

$$\begin{aligned}\|\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon u-u\|_{L^p(\Omega\times\mathbb{R}^d)} &\le \|\overline{\pi}_\varepsilon u-\overline{\pi}_\varepsilon u_n\|_{L^p(\Omega\times\mathbb{R}^d)}+\|\overline{\pi}_\varepsilon u_n-u_n\|_{L^p(\Omega\times\mathbb{R}^d)}+\|u_n-u\|_{L^p(\Omega\times\mathbb{R}^d)}\\ &\le 2\|u_n-u\|_{L^p(\Omega\times\mathbb{R}^d)}+\|\overline{\pi}_\varepsilon u_n-u_n\|_{L^p(\Omega\times\mathbb{R}^d)}.\end{aligned}$$

Letting $\varepsilon\to 0$, the second term vanishes by the smoothness of η_i^n and subsequently letting $n\to\infty$, the claim follows. □

Limits of symmetrized gradients. We consider a periodic lattice graph $(\mathbb{Z}^d,E)$ as in Section 5.1. The following lemma characterizes two-scale limits of discrete symmetrized gradients. In the following lemma, F_s denotes the symmetrization of the random field F defined in (5.9).

Lemma 5.18 (Limits of symmetrized gradients)**.** *Let* $p\in(1,\infty)$. *We consider a sequence* $u_\varepsilon\in L^p(\Omega\times\varepsilon\mathbb{Z}^d)^d$ *and* $F\in L^p(\Omega\times\mathbb{R}^d)^{d\times d}$ *such that* $\nabla^\varepsilon u_\varepsilon \xrightharpoonup{2} F$ *in* $L^p(\Omega\times\mathbb{R}^d)^{d\times d}$. *Then*

$$\nabla_s^\varepsilon u_\varepsilon \xrightharpoonup{2} F_s \quad \text{in } L^p(\Omega\times\mathbb{R}^d)^k.$$

If we have strong two-scale convergence for $\nabla^\varepsilon u_\varepsilon$, *strong two-scale convergence for* $\nabla_s^\varepsilon u_\varepsilon$ *follows.*

Proof. For any $i\in\{1,...,k\}$, using (5.1) we compute

$$\mathcal{T}_\varepsilon(\nabla_s^\varepsilon u_\varepsilon)_i(\omega,x)=\frac{b_i}{|b_i|}\cdot\mathcal{T}_\varepsilon\sum_{y\in\mathbb{Z}^d}\nabla^\varepsilon u_\varepsilon(\omega,x-\varepsilon y)B_i(y)=\frac{b_i}{|b_i|}\cdot\sum_{y\in\mathbb{Z}^d}\mathcal{T}_\varepsilon\nabla^\varepsilon u_\varepsilon(\tau_{-y}\omega,x-\varepsilon y)B_i(y).$$

For any fixed $y\in\mathbb{Z}^d$, the function $(\omega,x)\mapsto\mathcal{T}_\varepsilon\nabla^\varepsilon u_\varepsilon(\tau_{-y}\omega,x-\varepsilon y)B_i(y)$ weakly converges to $(\omega,x)\mapsto F(\tau_{-y}\omega,x)B_i(y)$. If we assume strong two-scale convergence for the gradient, the previous quantities converge in the strong sense. Using this and the fact that $B_i(y)=0$ for all but finitely many $y\in\mathbb{Z}^d$, the claim follows. □

Integral functionals and unfolding

In the following we present continuity and l.s.c. results for integral functionals with rapidly oscillating and random integrands with respect to strong and weak two-scale convergence in the mean. These observations will be key in the stochastic unfolding procedure for variational problems in the applications. We remark that we have applications in mind where the role of u_ε below is played by a discrete symmetrized gradient.

We consider a normal integrand $V:\Omega\times\mathbb{R}^k\to\mathbb{R}$, k being a positive integer, such that $V\ge -c$ for some $c>0$. The following transformation formula will be useful: For any $u\in L^p(\Omega\times\varepsilon\mathbb{Z}^d)^k$, it holds

$$\varepsilon^d\left\langle V(\tau_{\frac{x}{\varepsilon}}\omega,u(\omega,x))\right\rangle=\varepsilon^d\left\langle V(\omega,\widetilde{\mathcal{T}}_\varepsilon u(\omega,x))\right\rangle=\left\langle\int_{x+\varepsilon\Box}V(\omega,\mathcal{T}_\varepsilon u(\omega,y))dy\right\rangle\quad\text{for all } x\in\varepsilon\mathbb{Z}^d. \tag{5.14}$$

The first equality is obtained by the invariance of P w.r.t. $\tau_{\frac{x}{\varepsilon}}$ and the second follows since $\mathcal{T}_\varepsilon u(\omega,\cdot)$ is constant on $x+\varepsilon\Box$.

Proposition 5.19. *Let $p \in (1,\infty)$ and $V : \Omega \times \mathbb{R}^k \to \mathbb{R}$ be such that $V(\omega,\cdot)$ is continuous for P-a.a. ω and $V(\cdot, F)$ is measurable for all $F \in \mathbb{R}^k$. We assume that there exists $c > 0$ such that*

$$\frac{1}{c}|F|^p - c \leq V(\omega, F) \leq c(|F|^p + 1)$$

for P-a.a. $\omega \in \Omega$ and all $F \in \mathbb{R}^k$. Let $Q, Q^{+\varepsilon} \subset \mathbb{R}^d$ be bounded domains with Lipschitz boundaries satisfying $Q \subset Q^{+\varepsilon} \subset \{x \in \mathbb{R}^d : \operatorname{dist}(x,Q) \leq c\varepsilon\}$ for some $c > 0$.

(i) If $u_\varepsilon \in L^p(\Omega \times \varepsilon\mathbb{Z}^d)^k$ and $u_\varepsilon \xrightarrow{2} u$ in $L^p(\Omega \times \mathbb{R}^d)^k$, then

$$\lim_{\varepsilon \to 0} \left\langle \int_{Q^{+\varepsilon} \cap \varepsilon\mathbb{Z}^d} V(\tau_{\frac{x}{\varepsilon}}\omega, u_\varepsilon(\omega,x)) dm_\varepsilon(x) \right\rangle = \left\langle \int_Q V(\omega, u(\omega,x)) dx \right\rangle.$$

(ii) We additionally assume that $V(\omega,\cdot)$ is convex for P-a.a. ω. If $u_\varepsilon \in L^p(\Omega \times \varepsilon\mathbb{Z}^d)^k$ and $u_\varepsilon \xrightharpoonup{2} u$ in $L^p(\Omega \times \mathbb{R}^d)^k$, then

$$\liminf_{\varepsilon \to 0} \left\langle \int_{Q^{+\varepsilon} \cap \varepsilon\mathbb{Z}^d} V(\tau_{\frac{x}{\varepsilon}}\omega, u_\varepsilon(\omega,x)) dm_\varepsilon(x) \right\rangle \geq \left\langle \int_Q V(\omega, u(\omega,x)) dx \right\rangle.$$

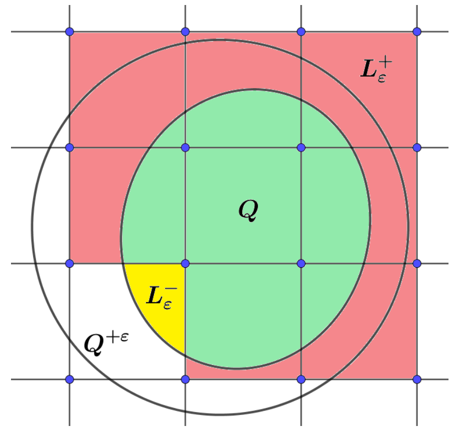
Proof. For notational convenience we set $\mathcal{E}_\varepsilon(u_\varepsilon) := \left\langle \int_{Q^{+\varepsilon} \cap \varepsilon\mathbb{Z}^d} V(\tau_{\frac{x}{\varepsilon}}\omega, u_\varepsilon(\omega,x)) dm_\varepsilon(x) \right\rangle$.
(i) By (5.14), using the shorthand $Q^\varepsilon := \cup_{x \in Q^{+\varepsilon} \cap \varepsilon\mathbb{Z}^d} (x + \varepsilon\Box)$, we obtain

$$\begin{aligned}
\mathcal{E}_\varepsilon(u_\varepsilon) &= \left\langle \int_{Q^\varepsilon} V(\omega, \mathcal{T}_\varepsilon u(\omega,x)) dx \right\rangle \\
&= \left\langle \int_Q V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega,x)) dx \right\rangle + \left\langle \int_{L_\varepsilon^+} V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega,x)) dx - \int_{L_\varepsilon^-} V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega,x)) dx \right\rangle \\
&\leq \left\langle \int_Q V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega,x)) dx \right\rangle + c(|L_\varepsilon^+| + |L_\varepsilon^-|) + c\left\langle \int_{L_\varepsilon^+} |\mathcal{T}_\varepsilon u_\varepsilon(\omega,x)|^p dx \right\rangle.
\end{aligned}$$

Above, L_ε^+ and L_ε^- are small boundary layer sets which satisfy $|L_\varepsilon^\pm| \to 0$ as $\varepsilon \to 0$ since Q has Lipschitz boundary (see Figure 5.2). Also, in the last inequality we used the growth conditions of V. The terms in the middle on the right-hand side vanish as $\varepsilon \to 0$, as well as the last term, using strong convergence of $\mathcal{T}_\varepsilon u_\varepsilon$ that implies uniform integrability of the sequence $\mathcal{T}_\varepsilon u_\varepsilon$. Moreover, using that we may extract a subsequence such that $\mathcal{T}_\varepsilon u_\varepsilon$ converges to u a.e. and by the fact that $V(\omega,\cdot)$ is continuous and has p-growth, the dominated convergence theorem ([Bog07, Theorem 2.8.8]) implies that $\limsup_{\varepsilon\to0} \mathcal{E}_\varepsilon(u_\varepsilon) \leq \limsup_{\varepsilon\to0} \left\langle \int_Q V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega,x)) dx \right\rangle = \left\langle \int_Q V(\omega, u(\omega,x)) dx \right\rangle$. Analogously, it follows that $\liminf_{\varepsilon\to0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \left\langle \int_Q V(\omega, u(\omega,x)) dx \right\rangle$.
(ii) Consider a sequence of open sets $A_k \subset\subset Q$ which satisfies $A_k \subset A_{k+1}$ and $|Q \setminus A_k| \to 0$ as $k \to \infty$. Analogously as in part (i) (by replacing Q with A_k), we obtain

$$\mathcal{E}_\varepsilon(u_\varepsilon) = \left\langle \int_{A_k} V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega,x)) dx \right\rangle + \left\langle \int_{L_\varepsilon^+} V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega,x)) dx \right\rangle - \left\langle \int_{L_\varepsilon^-} V(\omega, \mathcal{T}_\varepsilon u_\varepsilon(\omega,x)) dx \right\rangle.$$

Figure 5.2: Small boundary layers in the two-dimensional case. The inner ellipse represents Q and the outer $Q^{+\varepsilon}$, and the union of red and green squares is Q^{ε}. L_{ε}^{+} is denoted by red color and L_{ε}^{-} by yellow.



Note that if ε is small enough, L_{ε}^{-} is empty since $A_k \subset\subset Q$. Therefore, the growth conditions of V yield

$$\liminf_{\varepsilon\to 0} \mathcal{E}_{\varepsilon}(u_{\varepsilon}) \geq \liminf_{\varepsilon\to 0} \left\langle \int_{A_k} V(\omega, \mathcal{T}_{\varepsilon} u_{\varepsilon}(\omega,x))dx \right\rangle - c \lim_{\varepsilon\to 0} |L_{\varepsilon}^{+}|.$$

We have $|L_{\varepsilon}^{+}| \to |Q \setminus A_k|$. Also, since $u \mapsto \left\langle \int_{A_k} V(\omega, u(\omega,x))dx \right\rangle$ defines a convex and l.s.c. functional (arguing as in part (i)), it follows that it is weakly l.s.c. As a result of this, we obtain

$$\liminf_{\varepsilon\to 0} \mathcal{E}_{\varepsilon}(u_{\varepsilon}) \geq \left\langle \int_{A_k} V(\omega, u(\omega,x))dx \right\rangle - c|Q \setminus A_k|.$$

Finally, letting $k \to \infty$, the claim follows. □

5.3 Two-scale limits of gradients

In this section we treat two-scale limits of discrete gradients. First we present some compactness results and later we show that weak two-scale limits can be recovered in the strong two-scale sense by convenient linear constructions. Also, in the end we discuss how these results extend to functions defined in an enlarged space. To keep the exposition uncluttered, we present all the proofs in the end, in Section 5.3.1.

Compactness

The following auxiliary lemmas are useful for the proof of the main compactness statement – Proposition 5.22. The following commutator identity for measurable $u_{\varepsilon} : \Omega \times \varepsilon\mathbb{Z}^d \to \mathbb{R}$, obtained by direct computation, is practical:

$$\widetilde{\mathcal{T}}_{\varepsilon} \nabla^{\varepsilon} u_{\varepsilon} - \nabla^{\varepsilon} \widetilde{\mathcal{T}}_{\varepsilon} u_{\varepsilon} = \frac{1}{\varepsilon} D \widetilde{\mathcal{T}}_{\varepsilon} u_{\varepsilon} + (D_1 \nabla_1^{\varepsilon}, ..., D_d \nabla_d^{\varepsilon}) \widetilde{\mathcal{T}}_{\varepsilon} u_{\varepsilon}. \tag{5.15}$$

Lemma 5.20. *Let $p \in (1,\infty)$ and consider a sequence $u_{\varepsilon} \in L^p(\Omega \times \varepsilon\mathbb{Z}^d)$. Suppose that $u_{\varepsilon} \overset{2}{\rightharpoonup} u$ in $L^p(\Omega \times \mathbb{R}^d)$ and $\varepsilon \nabla^{\varepsilon} u_{\varepsilon} \overset{2}{\rightharpoonup} 0$ in $L^p(\Omega \times \mathbb{R}^d)^d$. Then $u \in L^p_{\text{inv}}(\Omega) \otimes L^p(\mathbb{R}^d)$.*
(See Section 5.3.1 for the proof.)

Lemma 5.21. *Let $p \in (1,\infty)$ and $u_\varepsilon \in L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ satisfy*

$$\limsup_{\varepsilon\to 0} \left\langle \int_{\varepsilon\mathbb{Z}^d} |u_\varepsilon(\omega,x)|^p + |\nabla^\varepsilon u_\varepsilon(\omega,x)|^p dm_\varepsilon(x) \right\rangle < \infty.$$

Then there exists $u \in L^p_{\text{inv}}(\Omega) \otimes W^{1,p}(\mathbb{R}^d)$ such that, up to a subsequence,

$$u_\varepsilon \xrightharpoonup{2} u, \quad P_{\text{inv}} u_\varepsilon \xrightharpoonup{2} u \quad \text{in } L^p(\Omega \times \mathbb{R}^d), \quad \nabla^\varepsilon P_{\text{inv}} u_\varepsilon \xrightharpoonup{2} \nabla u \quad \text{in } L^p(\Omega \times \mathbb{R}^d)^d.$$

(See Section 5.3.1 for the proof.)

Proposition 5.22 (Compactness)**.** *Let $p \in (1,\infty)$ and $\gamma \geq 0$. Let $u_\varepsilon \in L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ satisfy*

$$\limsup_{\varepsilon\to 0} \left\langle \int_{\varepsilon\mathbb{Z}^d} |u_\varepsilon(\omega,x)|^p + \varepsilon^{\gamma p} |\nabla^\varepsilon u_\varepsilon(\omega,x)|^p dm_\varepsilon(x) \right\rangle < \infty. \tag{5.16}$$

(i) If $\gamma = 0$, there exist $u \in L^p_{\text{inv}}(\Omega) \otimes W^{1,p}(\mathbb{R}^d)$ and $\chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(\mathbb{R}^d)$ such that, up to a subsequence,

$$u_\varepsilon \xrightharpoonup{2} u \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \nabla^\varepsilon u_\varepsilon \xrightharpoonup{2} \nabla u + \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.$$

(ii) If $\gamma \in (0,1)$, there exist $u \in L^p_{\text{inv}}(\Omega) \otimes L^p(\mathbb{R}^d)$ and $\chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(\mathbb{R}^d)$ such that, up to a subsequence,

$$u_\varepsilon \xrightharpoonup{2} u \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \varepsilon^\gamma \nabla^\varepsilon u_\varepsilon \xrightharpoonup{2} \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.$$

(iii) If $\gamma = 1$, there exists $u \in L^p(\Omega \times \mathbb{R}^d)$ such that, up to a subsequence,

$$u_\varepsilon \xrightharpoonup{2} u \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \varepsilon \nabla^\varepsilon u_\varepsilon \xrightharpoonup{2} Du \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.$$

(iv) If $\gamma > 1$, there exists $u \in L^p(\Omega \times \mathbb{R}^d)$ such that, up to a subsequence,

$$u_\varepsilon \xrightharpoonup{2} u \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \varepsilon^\gamma \nabla^\varepsilon u_\varepsilon \xrightharpoonup{2} 0 \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.$$

(See Section 5.3.1 for the proof.)

The statements of Proposition 5.22 can be adapted to sequences supported in a domain: Let $Q \subset \mathbb{R}^d$ be open. We denote by $W^{1,p}_0(Q)$ the closure of $C^\infty_c(Q)$ in $W^{1,p}(Q)$. Since the range of the unfolding operator consists of functions defined in $\mathbb{R}^d$, we tacitly identify functions in $L^p(Q)$ and $W^{1,p}_0(Q)$ with their trivial extension by 0 to $\mathbb{R}^d$. As a corollary of Proposition 5.22 we obtain the following:

Corollary 5.23. *Let $p \in (1,\infty)$ and $Q \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, and set $Q^{+\varepsilon} := \{x \in \mathbb{R}^d : \operatorname{dist}(x,Q) \leq c\varepsilon\}$ where $c > 0$ denotes a constant independent of $\varepsilon > 0$. Consider a sequence $u_\varepsilon \in L^p(\Omega) \otimes L^p_0(Q^{+\varepsilon} \cap \varepsilon\mathbb{Z}^d)$ satisfying (5.16). Then, in addition to the convergence statements in Proposition 5.22, the two-scale limits (from Proposition 5.22) satisfy*

- *if $\gamma = 0$, $u \in L^p_{\text{inv}}(\Omega) \otimes W^{1,p}_0(Q)$ and $\chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(Q)$;*
- *if $\gamma \in (0,1)$, $u \in L^p_{\text{inv}}(\Omega) \otimes L^p(Q)$ and $\chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(Q)$;*

- *if* $\gamma \geq 1$, $u \in L^p(\Omega) \otimes L^p(Q)$.

The proof of this corollary follows directly by the previous proposition and using the fact that $\mathcal{T}_\varepsilon u_\varepsilon(\omega,\cdot)$ and $\mathcal{T}_\varepsilon \nabla^\varepsilon u_\varepsilon(\omega,\cdot)$ are both supported in an ε-neighborhood of Q.

We remark that in Proposition 5.22 (i) and (ii) the two-scale limit u is shift-invariant and therefore in the ergodic setting it is deterministic, i.e., $u = P_{\mathrm{inv}} u = \langle u \rangle$.

Corollary 5.24. *Let $p \in (1,\infty)$, $\gamma \in [0,1)$ and $\langle\cdot\rangle$ be ergodic. Let u_ε satisfy the assumptions in Proposition 5.22. Then the claims in Proposition 5.22 (i) and (ii) hold and we have the following:*

(i) If $\gamma = 0$, then $\langle u_\varepsilon \rangle \rightharpoonup u$ in $L^p(\mathbb{R}^d)$, $\langle \nabla^\varepsilon u_\varepsilon \rangle \rightharpoonup \nabla u$ in $L^p(\mathbb{R}^d)^d$ and $u_\varepsilon - \langle u_\varepsilon \rangle \overset{2}{\rightharpoonup} 0$ in $L^p(\Omega\times\mathbb{R}^d)$.

(ii) If $\gamma \in (0,1)$, then $\langle u_\varepsilon \rangle \rightharpoonup u$ in $L^p(\mathbb{R}^d)$, $\langle \varepsilon^\gamma \nabla^\varepsilon u_\varepsilon \rangle \rightharpoonup 0$ in $L^p(\mathbb{R}^d)^d$ and $u_\varepsilon - \langle u_\varepsilon \rangle \overset{2}{\rightharpoonup} 0$ in $L^p(\Omega \times \mathbb{R}^d)$.

(iii) If $\gamma = 0$ and $Q \subset \mathbb{R}^d$ is open and bounded with Lipschitz boundary, then $\langle u_\varepsilon \rangle \to u$ strongly in $L^p(Q)$.

(iv) If $\gamma \in [0,1)$ and if, additionally, $u_\varepsilon \overset{2}{\to} u$ in $L^p(\Omega \times \mathbb{R}^d)$, then for any $\varphi \in L^\infty(\Omega)$ we have $\langle u_\varepsilon \varphi \rangle \to \langle \varphi \rangle u$ in $L^p(\mathbb{R}^d)$.

(See Section 5.3.1 for the proof.)

Recovery sequences

In the following, we show that weak two-scale accumulation points can be recovered in the strong two-scale sense.

Lemma 5.25 (Nonlinear approximation)**.** *Let $p \in (1,\infty)$. For $\chi \in L^p_{\mathrm{pot}}(\Omega) \otimes L^p(\mathbb{R}^d)$ and $\delta > 0$, there exists a sequence $g_{\delta,\varepsilon} \in L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ such that*

$$\|g_{\delta,\varepsilon}\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)} \leq \varepsilon c(\delta), \quad \limsup_{\varepsilon\to 0} \|\mathcal{T}_\varepsilon \nabla^\varepsilon g_{\delta,\varepsilon} - \chi\|_{L^p(\Omega\times\mathbb{R}^d)^d} \leq \delta, \tag{5.17}$$

where $c(\delta) > 0$.
(See Section 5.3.1 for the proof.)

Remark 5.26. *Using Attouch's diagonalization lemma [Att84, Lemma 1.15 and Corollary 1.16], we may extract a subsequence $\delta(\varepsilon) \to 0$ (as $\varepsilon \to 0$) such that $g_{\delta(\varepsilon),\varepsilon} \overset{2}{\to} 0$ and $\nabla^\varepsilon g_{\delta(\varepsilon),\varepsilon} \overset{2}{\to} 0$. On the other hand, we also prefer to consider the doubly-indexed approximation in* (5.17) *since it is a convenient building block for time-dependent recovery sequences (cf. Lemma 9.9) and it is helpful for the linear constructions from Proposition 5.27 and Corollary 5.28.*

Proposition 5.27. *Let $p \in (1,\infty)$.*

(i) Let $\gamma \in [0,1)$. For $\varepsilon > 0$ there exists a linear and bounded operator $\mathcal{G}^\gamma_\varepsilon : L^p_{\mathrm{pot}}(\Omega) \otimes L^p(\mathbb{R}^d) \to L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ such that

$$\mathcal{G}^\gamma_\varepsilon \chi \overset{2}{\to} 0 \quad \text{in } L^p(\Omega \times \mathbb{R}^d), \quad \varepsilon^\gamma \nabla^\varepsilon \mathcal{G}^\gamma_\varepsilon \chi \overset{2}{\to} \chi \quad \text{in } L^p(\Omega \times \mathbb{R}^d)^d$$

for all $\chi \in L^p_{\mathrm{pot}}(\Omega) \otimes L^p(\mathbb{R}^d)$. Moreover, the operator norm of $\mathcal{G}^\gamma_\varepsilon$ can be bounded independently of $0 < \varepsilon \leq 1$.

(ii) *Let $u \in L^p_{\mathrm{inv}}(\Omega) \otimes W^{1,p}(\mathbb{R}^d)$. We have*

$$\nabla^\varepsilon \mathcal{F}_\varepsilon u \xrightarrow{2} \nabla u \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.$$

(iii) *Let $\gamma \in (0,1)$. For $\varepsilon > 0$ there exists a linear and bounded operator $\mathcal{F}^\gamma_\varepsilon : L^p_{\mathrm{inv}}(\Omega) \otimes L^p(\mathbb{R}^d) \to L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ such that*

$$\mathcal{F}^\gamma_\varepsilon u \xrightarrow{2} u \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \varepsilon^\gamma \nabla^\varepsilon \mathcal{F}^\gamma_\varepsilon u \xrightarrow{2} 0 \text{ in } L^p(\Omega \times \mathbb{R}^d)^d$$

for all $u \in L^p_{\mathrm{inv}}(\Omega) \otimes L^p(\mathbb{R}^d)$. Moreover, the operator norm of $\mathcal{F}^\gamma_\varepsilon$ can be bounded independently of $0 < \varepsilon \leq 1$.

(iv) *Let $\gamma \geq 1$. For any $u \in L^p(\Omega \times \mathbb{R}^d)$, it holds that*

$$\varepsilon^\gamma \nabla^\varepsilon \mathcal{F}_\varepsilon u \xrightarrow{2} a_\gamma D u \text{ in } L^p(\Omega \times \mathbb{R}^d)^d,$$

where $a_\gamma = \begin{cases} 1 & \text{if } \gamma = 1, \\ 0 & \text{if } \gamma > 1. \end{cases}$

(See Section 5.3.1 for the proof.)

Corollary 5.28. *Let $p \in (1, \infty)$.*

(i) *The mapping*

$$\begin{gathered}(L^p_{\mathrm{inv}}(\Omega) \otimes W^{1,p}(\mathbb{R}^d)) \times (L^p_{\mathrm{pot}}(\Omega) \otimes L^p(\mathbb{R}^d)) \to L^p(\Omega \times \varepsilon\mathbb{Z}^d) \\ (u, \chi) \mapsto \mathcal{F}_\varepsilon u + \mathcal{G}^0_\varepsilon \chi =: u_\varepsilon(u, \chi)\end{gathered}$$

is linear and bounded, and it holds that

$$u_\varepsilon(u,\chi) \xrightarrow{2} u \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \nabla^\varepsilon u_\varepsilon(u,\chi) \xrightarrow{2} \nabla u + \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.$$

Moreover, its operator norm is bounded uniformly in $0 < \varepsilon \leq 1$.

(ii) *Let $\gamma \in (0,1)$. The mapping*

$$\begin{gathered}(L^p_{\mathrm{inv}}(\Omega) \otimes L^p(\mathbb{R}^d)) \times (L^p_{\mathrm{pot}}(\Omega) \otimes L^p(\mathbb{R}^d)) \to L^p(\Omega \times \varepsilon\mathbb{Z}^d) \\ (u, \chi) \mapsto \mathcal{F}^\gamma_\varepsilon u + \mathcal{G}^\gamma_\varepsilon \chi =: u_\varepsilon(u, \chi)\end{gathered}$$

is linear and bounded and it holds that

$$u_\varepsilon(u,\chi) \xrightarrow{2} u \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \varepsilon^\gamma \nabla^\varepsilon u_\varepsilon(u,\chi) \xrightarrow{2} \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.$$

Moreover, its operator norm is bounded uniformly in $0 < \varepsilon \leq 1$.

Let $Q \subset \mathbb{R}^d$ be open and bounded with Lipschitz boundary.

(iii) *For any $(u, \chi) \in (L^p_{\mathrm{inv}}(\Omega) \otimes W^{1,p}_0(Q)) \times (L^p_{\mathrm{pot}}(\Omega) \otimes L^p(Q))$, we can find a sequence $u_\varepsilon \in L^p(\Omega) \otimes L^p_0(Q \cap \varepsilon\mathbb{Z}^d)$ such that*

$$u_\varepsilon \xrightarrow{2} u \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \nabla^\varepsilon u_\varepsilon \xrightarrow{2} \nabla u + \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.$$

(iv) Let $\gamma \in (0,1)$. There exists a mapping

$$\begin{aligned}(L^p_{\text{inv}}(\Omega) \otimes L^p(Q)) \times (L^p_{\text{pot}}(\Omega) \otimes L^p(Q)) \to L^p(\Omega) \otimes L^p_0(Q \cap \varepsilon\mathbb{Z}^d)\\ (u,\chi) \mapsto u_\varepsilon(u,\chi),\end{aligned}$$

which is linear and bounded, and it holds that

$$u_\varepsilon(u,\chi) \xrightarrow{2} u \text{ in } L^p(\Omega \times \mathbb{R}^d), \quad \varepsilon^\gamma \nabla^\varepsilon u_\varepsilon(u,\chi) \xrightarrow{2} \chi \text{ in } L^p(\Omega \times \mathbb{R}^d)^d.$$

Moreover, its operator norm is bounded uniformly in $0 < \varepsilon \leq 1$.

(See Section 5.3.1 for the proof.)

We remark that in the case $\gamma \geq 1$, the recovery sequence for $u \in L^p(\Omega \times \mathbb{R}^d)$ is simply given by $\mathcal{F}_\varepsilon u$ and in the case of prescribed boundary data for the recovery sequence, we might consider a cut-off procedure as in (iv) above.

Remark 5.29. *Note that the construction of the recovery sequence in the whole-space cases (i) and (ii) (and if $\gamma \in (0,1)$ for a domain (iv)) is linear in the sense that the mapping $(u,\chi) \mapsto u_\varepsilon$ is linear. In contrast, the construction for a domain (iii) is nonlinear, since it relies on a cut-off procedure applied to the whole-space construction. We remark that the cut-off procedure can be avoided in certain cases: For $p = 2$, we can construct the recovery sequence, similarly as in the proof of Proposition 5.27 (i), by defining u_ε as the unique solution of $\nabla^{\varepsilon,*}\nabla^\varepsilon u_\varepsilon = \nabla^{\varepsilon,*}(\nabla^\varepsilon \mathcal{F}_\varepsilon u + \mathcal{F}_\varepsilon \chi)$ in the interior of $Q \cap \varepsilon\mathbb{Z}^d$ and with prescribed homogeneous Dirichlet boundary data. For $p \neq 2$ the same strategy applies as long as the above discrete elliptic equation satisfies maximal L^p-regularity. The latter depends on the regularity of the domain Q.*

Extension to an enlarged space

Let $p \in (1,\infty)$ and Z be a reflexive separable Banach space. We consider the Bochner space $L^p(\Omega \times \varepsilon\mathbb{Z}^d; Z)$. Functions of the following form are dense in $L^p(\Omega \times \varepsilon\mathbb{Z}^d; Z)$:

$$f_n = \sum_{i=1}^n u_i z_i, \quad u_i \in L^p(\Omega \times \varepsilon\mathbb{Z}^d),\ z_i \in Z.$$

For such f_n, we may define its unfolding by $\mathcal{T}_\varepsilon f_n = \sum_{i=1}^n (\mathcal{T}_\varepsilon u_i)\, z_i \in L^p(\Omega \times \mathbb{R}^d; Z)$. In this respect, we can extend the stochastic unfolding operator to a (not relabeled) linear isometry

$$\mathcal{T}_\varepsilon : L^p(\Omega \times \varepsilon\mathbb{Z}^d; Z) \to L^p(\Omega \times \mathbb{R}^d; Z).$$

In particular, if Z is an L^p-space, most results in the previous sections still hold for the extension. We collect some specific statements with brief (sketches of) proofs in the Hilbert space ergodic setting, which we use later in the applications.

Let $O \subset \mathbb{R}^d$ be open and bounded, and $\langle\cdot\rangle$ be ergodic. We set $Z = L^2(O)$. We identify $L^2(\Omega \times \varepsilon\mathbb{Z}^d; L^2(O))$ with $L^2(\Omega) \otimes L^2(\varepsilon\mathbb{Z}^d) \otimes L^2(O) := L^2(\Omega \times \varepsilon\mathbb{Z}^d \times O)$ and similarly as before, for suitable subspaces, we use the "$\otimes$"-notation. Let $Q \subset \mathbb{R}^d$ be open bounded with Lipschitz boundary and we set $Q^{+\varepsilon} := \{x \in \mathbb{R}^d : \text{dist}(x,Q) \leq c\varepsilon\}$ for some $c > 0$. We use the letter "x" to denote elements in Q and $\varepsilon\mathbb{Z}^d$ and the letter "y" for elements in O. In this respect, we use ∇^ε_x and ∇_x to denote the discrete gradient and the gradient w.r.t. the x-variable, respectively. We have the following compactness and recovery sequence statements:

- (Compactness). For a bounded sequence $u_\varepsilon \in L^2(\Omega) \otimes L^2_0(Q^{+\varepsilon} \cap \varepsilon\mathbb{Z}^d) \otimes L^2(O)$, there exists $u \in L^2(\Omega) \otimes L^2(Q) \otimes L^2(O)$ such that, up to a subsequence,

$$\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^2(\Omega) \otimes L^2(\mathbb{R}^d) \otimes L^2(O). \tag{5.18}$$

  This follows by the isometry property of $\mathcal{T}_\varepsilon$ and since $\mathcal{T}_\varepsilon u_\varepsilon(\omega, \cdot, y)$ is supported in an ε-neighborhood of Q.

- (Compactness for gradients). Let $u_\varepsilon \in L^2(\Omega) \otimes L^2_0(Q \cap \varepsilon\mathbb{Z}^d) \otimes L^2(O)$ be a sequence such that $(u_\varepsilon, \nabla_x^\varepsilon u_\varepsilon)$ is bounded in $\left(L^2(\Omega) \otimes L^2_0(Q^{+\varepsilon} \cap \varepsilon\mathbb{Z}^d) \otimes L^2(O)\right)^{1+d}$. Then, there exist $u \in H^1_0(Q) \otimes L^2(O)$ and $\chi \in L^2_{\mathrm{pot}}(\Omega) \otimes L^2(Q) \otimes L^2(O)$ such that, up to a subsequence,

$$(\mathcal{T}_\varepsilon u_\varepsilon, \mathcal{T}_\varepsilon \nabla_x^\varepsilon u_\varepsilon) \rightharpoonup (u, \nabla_x u + \chi) \quad \text{weakly in } (L^2(\Omega) \otimes L^2(\mathbb{R}^d) \otimes L^2(O))^{1+d}. \tag{5.19}$$

  Proof. (5.18) implies that $(\mathcal{T}_\varepsilon u_\varepsilon, \mathcal{T}_\varepsilon \nabla_x^\varepsilon u_\varepsilon) \rightharpoonup (u, v)$ weakly in $\left(L^2(\Omega) \otimes L^2(\mathbb{R}^d) \otimes L^2(O)\right)^{1+d}$, up to a subsequence. Namely, for fixed $\varphi \in L^2(\Omega)$, $\eta \in C_c^\infty(\mathbb{R}^d)$ and $\xi \in C_c^\infty(O)$, it holds, as $\varepsilon \to 0$,

$$\left\langle \int_{\mathbb{R}^d} \int_O \mathcal{T}_\varepsilon u_\varepsilon(\omega, x, y) \varphi(\omega) \eta(x) \xi(y) dy dx \right\rangle \to \left\langle \int_{\mathbb{R}^d} \int_O u(\omega, x, y) \varphi(\omega) \eta(x) \xi(y) dy dx \right\rangle,$$

  and analogously for $\mathcal{T}_\varepsilon \nabla_x^\varepsilon u_\varepsilon$. We consider the sequence $\widetilde{u}_\varepsilon(\omega, x) := \int_O u_\varepsilon(\omega, x, y)\xi(y) dy$ and Corollary 5.23 implies that $\mathcal{T}_\varepsilon \widetilde{u}_\varepsilon \rightharpoonup \widetilde{u}$ and $\mathcal{T}_\varepsilon \nabla_x^\varepsilon \widetilde{u}_\varepsilon \rightharpoonup \nabla_x \widetilde{u} + \widetilde{\chi}$ weakly in $L^2(\Omega) \otimes L^2(\mathbb{R}^d)$, where $\widetilde{u} \in L^2_{\mathrm{inv}}(\Omega) \otimes H^1_0(Q) \simeq H^1_0(Q)$ (by ergodicity) and $\widetilde{\chi} \in L^2_{\mathrm{pot}}(\Omega) \otimes L^2(Q)$. We note that $\int_O \cdot dy$ commutes with the action of the operators $\mathcal{T}_\varepsilon$ and ∇_x^ε. In this regard, we obtain that

$$\widetilde{u}(x) = \int_O u(\omega, x, y)\xi(y) dy, \quad \nabla_x \widetilde{u}(x) + \widetilde{\chi}(\omega, x) = \int_O v(\omega, x, y) \xi(y) dy.$$

  Since $\widetilde{u}$ is deterministic, we have $\int_O Du(\cdot, \cdot, y)\xi(y) dy = 0$. This holds for an arbitrary $\xi \in C_c^\infty(O)$ and therefore we conclude that u is also deterministic, i.e., $u \in L^2(\mathbb{R}^d) \otimes L^2(O)$. Similarly, it follows that $u \in H^1_0(Q) \otimes L^2(O)$ and $\chi := v - \nabla_x u \in L^2_{\mathrm{pot}}(\Omega) \otimes L^2(Q) \otimes L^2(O)$. ☐

- (Recovery sequence). For given $u \in H^1_0(Q) \otimes L^2(O)$ and $\chi \in L^2_{\mathrm{pot}}(\Omega) \otimes L^2(Q) \otimes L^2(O)$, there exists $u_\varepsilon \in L^2(\Omega) \otimes L^2_0(Q \cap \varepsilon\mathbb{Z}^d) \otimes L^2(O)$ such that

$$(\mathcal{T}_\varepsilon u_\varepsilon, \mathcal{T}_\varepsilon \nabla_x^\varepsilon u_\varepsilon) \to (u, \nabla_x u + \chi) \quad \text{strongly in } (L^2(\Omega) \otimes L^2(\mathbb{R}^d) \otimes L^2(O))^{1+d}. \tag{5.20}$$

  Proof. We may approximate u by a sequence of the form $u_n = \sum_{i=1}^n \eta_i \xi_i$ with $\eta_i \in H^1_0(Q)$ and $\xi_i \in L^2(O)$, and χ by a sequence of the form $\chi_n = \sum_{i=1}^n \widetilde{\chi}_i \xi_i$ with $\widetilde{\chi}_i \in L^2_{\mathrm{pot}}(\Omega) \otimes L^2(Q)$ and $\xi_i \in L^2(O)$. Following the analogous steps as in the proof of Corollary 5.28 (iii), we find a sequence $u_{n,\varepsilon} \in L^2(\Omega) \otimes L^2_0(Q \cap \varepsilon\mathbb{Z}^d) \otimes L^2(O)$ such that, for fixed n and as $\varepsilon \to 0$,

$$(\mathcal{T}_\varepsilon u_{n,\varepsilon}, \mathcal{T}_\varepsilon \nabla_x^\varepsilon u_{n,\varepsilon}) \to (u_n, \nabla_x u_n + \chi_n) \quad \text{strongly in } (L^2(\Omega) \otimes L^2(\mathbb{R}^d) \otimes L^2(O))^{1+d}.$$

  Finally, we can extract a diagonal sequence $n(\varepsilon) \to \infty$ as $\varepsilon \to 0$ such that $u_{n(\varepsilon),\varepsilon}$ satisfies the claim. ☐

5.3.1 Proofs

Proof of Lemma 5.20. Since $L^p_{\text{inv}}(\Omega) = (\text{ran}D^*)^\perp$, it suffices to show that

$$\left\langle \int_{\mathbb{R}^d} u(\omega, x) D_i^* \varphi(\omega)\eta(x)dx \right\rangle = 0 \tag{5.21}$$

for any $\varphi \in L^q(\Omega)$ with $q = \frac{p}{p-1}$, $\eta \in C_c^\infty(\mathbb{R}^d)$ and $i \in \{1, ..., d\}$.
We consider the sequence given by $v_\varepsilon = \mathcal{F}_\varepsilon(\varphi\eta) \in L^q(\Omega \times \varepsilon\mathbb{Z}^d)$ and by Lemma 5.17 (iii), we have $v_\varepsilon \xrightarrow{2} \varphi\eta$ in $L^q(\Omega \times \mathbb{R}^d)$. Therefore, using Lemma 5.16 (iv), we obtain

$$\varepsilon \left\langle \int_{\varepsilon\mathbb{Z}^d} u_\varepsilon(\omega, x) \nabla_i^{\varepsilon,*} v_\varepsilon(\omega, x) dm_\varepsilon(x) \right\rangle = \left\langle \int_{\varepsilon\mathbb{Z}^d} (\varepsilon\nabla_i^\varepsilon u_\varepsilon(\omega, x)) v_\varepsilon(\omega, x) dm_\varepsilon(x) \right\rangle \to 0 \quad \text{as } \varepsilon \to 0. \tag{5.22}$$

Moreover, using the definition of $\mathcal{F}_\varepsilon$, we compute

$$\begin{aligned} \varepsilon\nabla_i^{\varepsilon,*} v_\varepsilon(\omega, x) &= \varphi(\tau_{\frac{x}{\varepsilon} - e_i}\omega)\pi_\varepsilon\eta(x - \varepsilon e_i) - \varphi(\tau_{\frac{x}{\varepsilon}}\omega)\pi_\varepsilon\eta(x) \\ &= \varepsilon\varphi(\tau_{\frac{x}{\varepsilon} - e_i}\omega)\nabla_i^{\varepsilon,*}\pi_\varepsilon\eta(x) + D_i^*\varphi(\tau_{\frac{x}{\varepsilon}}\omega)\pi_\varepsilon\eta(x), \end{aligned} \tag{5.23}$$

which implies $\varepsilon\nabla_i^{\varepsilon,*} v_\varepsilon \xrightarrow{2} D_i^*\varphi\eta$ in $L^q(\Omega \times \mathbb{R}^d)$. Indeed, the first term on the right-hand side of (5.23) vanishes in the strong two-scale limit since η is compactly supported and smooth. The second term strongly two-scale converges to $D_i^*\varphi\eta$. This and Lemma 5.16 (iv) imply

$$\lim_{\varepsilon \to 0} \varepsilon \left\langle \int_{\varepsilon\mathbb{Z}^d} u_\varepsilon(\omega, x) \nabla_i^{\varepsilon,*} v_\varepsilon(\omega, x) dm_\varepsilon(x) \right\rangle = \left\langle \int_{\mathbb{R}^d} u(\omega, x) D_i^*\varphi(\omega)\eta(x)dx \right\rangle,$$

which, together with (5.22), yields (5.21). $\square$

Proof of Lemma 5.21. Step 1. We claim that $\widetilde{\mathcal{T}}_\varepsilon \circ P_{\text{inv}} = P_{\text{inv}} \circ \widetilde{\mathcal{T}}_\varepsilon = P_{\text{inv}}$. By shift invariance, we have $\widetilde{\mathcal{T}}_\varepsilon \circ P_{\text{inv}} = P_{\text{inv}}$. Hence, it suffices to prove $P_{\text{inv}} \circ \widetilde{\mathcal{T}}_\varepsilon = P_{\text{inv}}$. Let $\eta \in L^q(\varepsilon\mathbb{Z}^d)$, $\varphi \in L^q(\Omega)$ and $v_\varepsilon \in L^p(\Omega \times \varepsilon\mathbb{Z}^d)$. We have

$$\begin{aligned} \left\langle \int_{\varepsilon\mathbb{Z}^d} P_{\text{inv}}\widetilde{\mathcal{T}}_\varepsilon v_\varepsilon(\omega, x)\varphi(\omega)\eta(x) dm_\varepsilon(x) \right\rangle &= \left\langle \int_{\varepsilon\mathbb{Z}^d} v_\varepsilon(\tau_{-\frac{x}{\varepsilon}}\omega, x) P^*_{\text{inv}}\varphi(\omega)\eta(x) dm_\varepsilon(x) \right\rangle \\ &= \left\langle \int_{\varepsilon\mathbb{Z}^d} v_\varepsilon(\omega, x) P^*_{\text{inv}}\varphi(\omega)\eta(x) dm_\varepsilon(x) \right\rangle \\ &= \left\langle \int_{\varepsilon\mathbb{Z}^d} P_{\text{inv}} v_\varepsilon(\omega, x)\varphi(\omega)\eta(x) dm_\varepsilon(x) \right\rangle. \end{aligned}$$

Above, in the second equality we use the fact that $P^*_{\text{inv}} \simeq P_{\text{inv}}$ on $L^q(\Omega)$ and therefore $\widetilde{\mathcal{T}}_\varepsilon^{-1} P^*_{\text{inv}}\varphi = P^*_{\text{inv}}\varphi$. Consequently, by a density argument it follows that $P_{\text{inv}} \circ \widetilde{\mathcal{T}}_\varepsilon = P_{\text{inv}}$.
Step 2. Convergence of $P_{\text{inv}}u_\varepsilon$. Using boundedness of P_{inv} and the fact that ∇^ε and P_{inv} commute, we obtain

$$\limsup_{\varepsilon \to 0} \left\langle \int_{\varepsilon\mathbb{Z}^d} |P_{\text{inv}}u_\varepsilon(\omega, x)|^p + |\nabla^\varepsilon P_{\text{inv}}u_\varepsilon(\omega, x)|^p dm_\varepsilon(x) \right\rangle < \infty.$$

Applying Lemma 5.16 (ii) and Lemma 5.20, there exist $v \in L^p_{\text{inv}}(\Omega) \otimes L^p(\mathbb{R}^d)$ and $\widetilde{v} \in L^p(\Omega \times \mathbb{R}^d)^d$ such that

$$P_{\text{inv}}u_\varepsilon \overset{2}{\rightharpoonup} v \quad \text{in } L^p(\Omega \times \mathbb{R}^d), \quad \nabla^\varepsilon P_{\text{inv}}u_\varepsilon \overset{2}{\rightharpoonup} \widetilde{v} \quad \text{in } L^p(\Omega \times \mathbb{R}^d)^d, \tag{5.24}$$

for a (not relabeled) subsequence. Note that, additionally, it holds $\widetilde{v} \in L^p_{\text{inv}}(\Omega) \otimes L^p(\mathbb{R}^d)^d$.

Let $\varphi \in L^q(\Omega)$ and $\eta \in C_c^\infty(\mathbb{R}^d)$ and denote $v_\varepsilon = \mathcal{F}_\varepsilon(\varphi\eta)$. Since $v_\varepsilon \xrightarrow{2} \eta\varphi$ (Lemma 5.17 (iii)), for $i = 1, ..., d$, we have

$$\left\langle \int_{\varepsilon\mathbb{Z}^d} \nabla_i^\varepsilon P_{\text{inv}} u_\varepsilon(\omega, x) v_\varepsilon(\omega, x) dm_\varepsilon(x) \right\rangle \to \left\langle \int_{\mathbb{R}^d} \widetilde{v}_i(\omega, x)\varphi(\omega)\eta(x)dx \right\rangle \quad \text{as } \varepsilon \to 0. \tag{5.25}$$

On the other hand, it holds that

$$\begin{aligned}\left\langle \int_{\varepsilon\mathbb{Z}^d} \nabla_i^\varepsilon P_{\text{inv}} u_\varepsilon(\omega, x) v_\varepsilon(\omega, x) dm_\varepsilon(x) \right\rangle &= \left\langle \int_{\varepsilon\mathbb{Z}^d} P_{\text{inv}} u_\varepsilon(\omega, x)\varphi(\omega)\nabla_i^{\varepsilon,*}\pi_\varepsilon\eta(x) dm_\varepsilon(x) \right\rangle \\ &\overset{(\varepsilon\to 0)}{\to} -\left\langle \int_{\mathbb{R}^d} v(\omega, x)\varphi(\omega)\partial_i\eta(x)dx \right\rangle.\end{aligned} \tag{5.26}$$

The above convergence is obtained using that $\overline{P_{\text{inv}} u_\varepsilon} \rightharpoonup v$ weakly in $L^p(\Omega \times \mathbb{R}^d)$ (this follows from (5.24) and Step 1) and $\overline{\nabla_i^{\varepsilon,*}\pi_\varepsilon\eta} \to -\partial_i\eta$ strongly in $L^q(\mathbb{R}^d)$. The latter may be shown as follows. We have

$$\begin{aligned} & \|\overline{\nabla_i^{\varepsilon,*}\pi_\varepsilon\eta} + \partial_i\eta\|_{L^q(\mathbb{R}^d)} \\ \leq\; & \left\| \overline{\pi}_\varepsilon \left(\frac{\eta(\cdot - \varepsilon e_i) - \eta(\cdot)}{\varepsilon} \right) + \overline{\pi}_\varepsilon\partial_i\eta \right\|_{L^q(\mathbb{R}^d)} + \|\overline{\pi}_\varepsilon\partial_i\eta - \partial_i\eta\|_{L^q(\mathbb{R}^d)} \\ \leq\; & \left\| \frac{\eta(\cdot - \varepsilon e_i) - \eta(\cdot)}{\varepsilon} + \partial_i\eta \right\|_{L^q(\mathbb{R}^d)} + \|\overline{\pi}_\varepsilon\partial_i\eta - \partial_i\eta\|_{L^q(\mathbb{R}^d)}, \end{aligned}$$

where we used that $\overline{\pi}_\varepsilon$ is a contraction. Since $\eta \in C_c^\infty(\mathbb{R}^d)$, it follows by a Taylor expansion argument that both terms on the right-hand side of the above inequality vanish in the limit $\varepsilon \to 0$. Combining (5.25) and (5.26), we conclude that $v \in L^p_{\text{inv}}(\Omega) \otimes W^{1,p}(\mathbb{R}^d)$ and $\nabla v = \widetilde{v}$.

Step 3. We show that $u_\varepsilon \overset{2}{\rightharpoonup} v$ in $L^p(\Omega \times \mathbb{R}^d)$, up to another subsequence. Using Lemmas 5.16 (ii) and 5.20, we conclude that there exist another subsequence (not relabeled) and $u \in L^p_{\text{inv}}(\Omega) \otimes L^p(\mathbb{R}^d)$ such that $u_\varepsilon \overset{2}{\rightharpoonup} u$ in $L^p(\Omega \times \mathbb{R}^d)$. Since P_{inv} is a linear and bounded operator, it follows that $P_{\text{inv}}(\mathcal{T}_\varepsilon u_\varepsilon) \rightharpoonup P_{\text{inv}} u$ in $L^p(\Omega \times \mathbb{R}^d)$, and $P_{\text{inv}} u = u$ by shift invariance of u. Furthermore, by Steps 1 and 2 we have that $P_{\text{inv}}\mathcal{T}_\varepsilon u_\varepsilon = \mathcal{T}_\varepsilon P_{\text{inv}} u_\varepsilon \rightharpoonup v$ and therefore $u = v$. This completes the proof. $\square$

Proof of Proposition 5.22. (i) By Lemma 5.21 we deduce that there exists $u \in L^p_{\text{inv}}(\Omega) \otimes W^{1,p}(\mathbb{R}^d)$ such that $u_\varepsilon \overset{2}{\rightharpoonup} u$ and by boundedness of $\nabla^\varepsilon u_\varepsilon$ (Lemma 5.16 (ii)) there exists $v \in L^p(\Omega \times \mathbb{R}^d)^d$ such that $\nabla^\varepsilon u_\varepsilon \overset{2}{\rightharpoonup} v$, up to a subsequence. In order to prove the claim, it suffices to show that

$$\left\langle \int_{\mathbb{R}^d} v(\omega, x) \cdot \eta(x)\varphi(\omega)dx \right\rangle = \left\langle \int_{\mathbb{R}^d} \nabla u(\omega, x) \cdot \eta(x)\varphi(\omega)dx \right\rangle \tag{5.27}$$

for any $\varphi \in L^q(\Omega)^d$ with $D^*\varphi = 0$ and $\eta \in C_c^\infty(\mathbb{R}^d)$. Indeed, this implies that $\chi := v - \nabla u \in L^p_{\text{pot}}(\Omega) \otimes L^p(\mathbb{R}^d)$ (see (5.8)) and thus the claim of the proposition.

For the argument consider $v_\varepsilon = \mathcal{F}_\varepsilon(\eta\varphi)$, the folding acting componentwise. Since $v_\varepsilon \xrightarrow{2} \eta\varphi$ (Lemma 5.17 (iii)),

$$\left\langle \int_{\varepsilon\mathbb{Z}^d} \nabla^\varepsilon u_\varepsilon(\omega, x) \cdot v_\varepsilon(\omega, x) dm_\varepsilon(x) \right\rangle \to \left\langle \int_{\mathbb{R}^d} v(\omega, x) \cdot \eta(x)\varphi(\omega)dx \right\rangle \quad \text{as } \varepsilon \to 0.$$

On the other hand, the commutator identity (5.15) and the definition of $\mathcal{F}_\varepsilon$ yield

$$\begin{aligned}
&\left\langle \int_{\varepsilon\mathbb{Z}^d} \nabla^\varepsilon u_\varepsilon(\omega,x)\cdot v_\varepsilon(\omega,x) dm_\varepsilon(x) \right\rangle \\
= \ &\left\langle \int_{\varepsilon\mathbb{Z}^d} \left(\nabla^\varepsilon \widetilde{\mathcal{T}}_\varepsilon u_\varepsilon(\omega,x) + \frac{1}{\varepsilon} D\widetilde{\mathcal{T}}_\varepsilon u_\varepsilon(\omega,x) + (D_1\nabla_1^\varepsilon,...,D_d\nabla_d^\varepsilon)\widetilde{\mathcal{T}}_\varepsilon u_\varepsilon(\omega,x)\right)\cdot \pi_\varepsilon \eta(x)\varphi(\omega) dm_\varepsilon(x) \right\rangle.
\end{aligned}$$

Since $D^*\varphi = 0$, the contribution from the second term on the right-hand side above vanishes. After a discrete integration by parts, the right-hand side reduces to

$$\begin{aligned}
&\sum_{i=1}^d \left\langle \int_{\varepsilon\mathbb{Z}^d} \left(\widetilde{\mathcal{T}}_\varepsilon u_\varepsilon(\omega,x) + D_i \widetilde{\mathcal{T}}_\varepsilon u_\varepsilon(\omega,x)\right) \nabla_i^{\varepsilon,*}\pi_\varepsilon\eta(x)\varphi_i(\omega) dm_\varepsilon(x)\right\rangle \\
&\to -\sum_{i=1}^d \left\langle \int_{\mathbb{R}^d} \left(u(\omega,x) + D_i u(\omega,x)\right)\partial_i \eta(x)\varphi_i(\omega)dx \right\rangle \quad \text{as } \varepsilon\to 0,
\end{aligned}$$

which follows using that $u_\varepsilon \xrightharpoonup{2} u$ and that η is smooth and compactly supported (cf. Step 2 in the proof of Lemma 5.21). Since u is shift-invariant, the second term on the right-hand side vanishes. After an integration by parts, we are able to infer (5.27) and conclude the proof of part (i).

(ii) By Lemma 5.16 (ii), there exists $u\in L^p(\Omega\times\mathbb{R}^d)$ such that $u_\varepsilon \xrightharpoonup{2} u$, up to a subsequence. Since $\gamma\in(0,1)$, u_ε satisfies the assumptions in Lemma 5.20 and therefore $u\in L^p_{\text{inv}}(\Omega)\otimes L^p(\mathbb{R}^d)$. With the help of (i), we obtain that for the sequence $v_\varepsilon := \varepsilon^\gamma u_\varepsilon$, there exist $v\in L^p_{\text{inv}}(\Omega)\otimes W^{1,p}(\mathbb{R}^d)$ and $\chi\in L^p_{\text{pot}}(\Omega)\otimes L^p(\mathbb{R}^d)$ such that, up to another subsequence,

$$v_\varepsilon \xrightharpoonup{2} v \text{ in } L^p(\Omega\times\mathbb{R}^d), \quad \nabla^\varepsilon v_\varepsilon \xrightharpoonup{2} \nabla v + \chi \text{ in } L^p(\Omega\times\mathbb{R}^d)^d.$$

However, using that $u_\varepsilon \xrightharpoonup{2} u$, we conclude that $v=0$ and the claim is proven.

(iii) Lemma 5.16 (ii) implies that there exist $u\in L^p(\Omega\times\mathbb{R}^d)$ and $v\in L^p(\Omega\times\mathbb{R}^d)^d$ such that, up to a subsequence,

$$u_\varepsilon \xrightharpoonup{2} u \text{ in } L^p(\Omega\times\mathbb{R}^d), \quad \varepsilon\nabla^\varepsilon u_\varepsilon \xrightharpoonup{2} v \text{ in } L^p(\Omega\times\mathbb{R}^d)^d.$$

Following the same strategy as in Lemma 5.20 it follows that $v = Du$.

(iv) Lemma 5.16 (ii) implies that there exists $u\in L^p(\Omega\times\mathbb{R}^d)$ such that $u_\varepsilon \xrightharpoonup{2} u$ in $L^p(\Omega\times\mathbb{R}^d)$, up to a subsequence. Also, using part (iii), for the sequence $v_\varepsilon := \varepsilon^{\gamma-1}u_\varepsilon$, there exists $v\in L^p(\Omega\times\mathbb{R}^d)$ such that, up to another subsequence,

$$v_\varepsilon \xrightharpoonup{2} v, \quad \varepsilon\nabla^\varepsilon v_\varepsilon \xrightharpoonup{2} Dv.$$

The fact that $u_\varepsilon \xrightharpoonup{2} u$ implies that $v=0$ and the proof is complete. □

Proof of Corollary 5.24. (i) The claim follows directly from Lemmas 5.21 and 5.4.

(ii) Exploiting linearity and boundedness of P_{inv} and Step 1 in the proof of Lemma 5.21, we obtain that

$$\overline{\langle u_\varepsilon\rangle} = P_{\text{inv}}\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup P_{\text{inv}} u = u, \quad \overline{\langle \varepsilon^\gamma\nabla^\varepsilon u_\varepsilon\rangle} = P_{\text{inv}}\mathcal{T}_\varepsilon \varepsilon^\gamma \nabla^\varepsilon u_\varepsilon \rightharpoonup P_{\text{inv}}\chi = \langle\chi\rangle\,.$$

The above convergence, Lemma 5.4, and the fact that $\langle \chi \rangle = 0$ allow us to conclude the proof.
(iii) Lemma 5.21 implies that $\overline{\langle u_\varepsilon \rangle} \rightharpoonup u$ and $\overline{\nabla^\varepsilon \langle u_\varepsilon \rangle} \rightharpoonup \nabla u$ weakly in $L^p(\mathbb{R}^d)$. Lemma 5.4 implies that $\widehat{\langle u_\varepsilon \rangle} \rightharpoonup u$ weakly in $L^p(\mathbb{R}^d)$. Furthermore, for any $\eta \in L^q(\mathbb{R}^d)$ it holds that

$$\int_{\mathbb{R}^d} \left(\nabla \widehat{\langle u_\varepsilon \rangle}(x) - \overline{\nabla^\varepsilon \langle u_\varepsilon \rangle}(x) \right) \eta(x) dx \to 0 \text{ as } \varepsilon \to 0.$$

As a result of this, $\widehat{\langle u_\varepsilon \rangle} \rightharpoonup u$ weakly in $W^{1,p}(\mathbb{R}^d)$. Rellich's embedding theorem implies that $\widehat{\langle u_\varepsilon \rangle} \to u$ strongly in $L^p(Q)$ and using Lemma 5.4 we conclude that $\overline{\langle u_\varepsilon \rangle} \to u$ strongly in $L^p(Q)$.
(iv) We have by Jensen's inequality and boundedness of φ

$$\int_{\mathbb{R}^d} | \langle \overline{u}_\varepsilon(\omega, x)\varphi(\omega) \rangle - \langle \varphi(\omega) \rangle u(x)|^p dx \leq c \left\langle \int_{\mathbb{R}^d} |\overline{u}_\varepsilon(\omega, x) - u(x)|^p dx \right\rangle.$$

The right-hand side of the above inequality equals $\left\langle \int_{\mathbb{R}^d} |\mathcal{T}_\varepsilon u_\varepsilon(\omega, x) - u(x)|^p dx \right\rangle$ and therefore it vanishes as $\varepsilon \to 0$. □

Proof of Lemma 5.25. Let $\chi \in L^p_{\mathrm{pot}}(\Omega) \otimes L^p(\mathbb{R}^d)$ and $\delta > 0$ be fixed. By the definition of $L^p_{\mathrm{pot}}(\Omega)$, there exists $v = \sum_{j=1}^n \varphi_j \eta_j$ with $\varphi_j \in L^p(\Omega)$, $\eta_j \in C_c^\infty(\mathbb{R}^d)$ and

$$\|Dv - \chi\|_{L^p(\Omega \times \mathbb{R}^d)^d} \leq \delta.$$

We define $g_\varepsilon := \varepsilon \mathcal{F}_\varepsilon v$ and remark that $\|g_\varepsilon\|_{L^p(\Omega \times \varepsilon\mathbb{Z}^d)} \leq \varepsilon \|v\|_{L^p(\Omega \times \mathbb{R}^d)}$, which follows from the boundedness of $\mathcal{F}_\varepsilon$. This proves the first part.
Note that $\nabla_i^\varepsilon g_\varepsilon(\omega, x) = D_i \pi_\varepsilon v(\tau_{\frac{x}{\varepsilon}} \omega, x) + \varepsilon \nabla_i^\varepsilon \pi_\varepsilon v(\tau_{\frac{x}{\varepsilon}+e_i} \omega, x)$ and therefore we obtain $\mathcal{T}_\varepsilon \nabla_i^\varepsilon g_\varepsilon(\omega, x) = \overline{\pi}_\varepsilon D_i v(\omega, x) + \varepsilon \overline{\nabla_i^\varepsilon \pi_\varepsilon v}(\tau_{e_i} \omega, x)$. Hence

$$\|\mathcal{T}_\varepsilon \nabla^\varepsilon g_\varepsilon - \chi\|_{L^p(\Omega \times \mathbb{R}^d)^d} \leq \|\overline{\pi}_\varepsilon Dv - Dv\|_{L^p(\Omega \times \mathbb{R}^d)^d} + \|Dv - \chi\|_{L^p(\Omega \times \mathbb{R}^d)^d} + \varepsilon \|\nabla^\varepsilon \pi_\varepsilon v\|_{L^p(\Omega \times \varepsilon\mathbb{Z}^d)^d}.$$

The first and last terms on the right-hand side above vanish as $\varepsilon \to 0$ and therefore the claim follows. Indeed, for the first term it is sufficient to note that Dv is smooth and has compact support w.r.t. its x-variable. Also, the last term vanishes due to the boundedness of π_ε and the boundedness of difference quotients by gradients. More precisely, for $i = 1, ..., d$, it holds

$$\varepsilon^p \|\nabla_i^\varepsilon \pi_\varepsilon v\|^p_{L^p(\Omega \times \varepsilon\mathbb{Z}^d)^d} \leq c\varepsilon^p \left\langle \int_{\mathbb{R}^d} \left| \frac{v(\omega, x + \varepsilon e_i) - v(\omega, x)}{\varepsilon} \right|^p dx \right\rangle \leq c\varepsilon^p \|\nabla v\|^p_{L^p(\Omega \times \mathbb{R}^d)^d}.$$

□

Proof of Proposition 5.27. In the following proof we appeal to discrete maximal L^p-regularity for the equation (with $\lambda > 0$)

$$\lambda u + \nabla^{\varepsilon,*} \nabla^\varepsilon u = \nabla^* F + g \quad \text{in } \varepsilon\mathbb{Z}^d, \quad (\text{for some } F \in L^p(\varepsilon\mathbb{Z}^d)^d, g \in L^p(\varepsilon\mathbb{Z}^d))$$

in the form of

$$\lambda^{\frac{1}{2}} \|u\|_{L^p(\varepsilon\mathbb{Z}^d)} + \|\nabla^\varepsilon u\|_{L^p(\varepsilon\mathbb{Z}^d)^d} \leq c(d,p) \Big(\|F\|_{L^p(\varepsilon\mathbb{Z}^d)^d} + \lambda^{-\frac{1}{2}} \|g\|_{L^p(\varepsilon\mathbb{Z}^d)} \Big),$$

which is uniform in ε. For $p=2$ this is a standard a priori estimate. For $1<p<\infty$, in the continuum setting, this is a classical result (see, e.g., [Kry08, Chapter 4, Sec. 4, Theorem 2]), and follows from the Calderón-Zygmund estimate $\|\partial_{ij}u\|_{L^p(\mathbb{R}^d)} \leq C(d,p)\|\Delta u\|_{L^p(\mathbb{R}^d)}$. The estimate above follows by the same argument from the Calderón-Zygmund estimate for the discrete Laplacian on $\varepsilon\mathbb{Z}^d$, for the latter see, e.g., [GNO15, BAMN17].

(i) Let $2\gamma<\alpha<2$. For a given $\chi \in L^p_{\text{pot}}(\Omega)\otimes L^p(\mathbb{R}^d)$ we define $\mathcal{G}^\gamma_\varepsilon \chi := u_\varepsilon$ as the unique solution to the following equation in $L^p(\Omega\times\varepsilon\mathbb{Z}^d)$ (for P-a.a. $\omega\in\Omega$ that we drop from the notation)

$$\varepsilon^{-\alpha}u_\varepsilon + \nabla^{\varepsilon,*}\nabla^\varepsilon u_\varepsilon = \nabla^{\varepsilon,*}\varepsilon^{-\gamma}\mathcal{F}_\varepsilon\chi \quad \text{in } \varepsilon\mathbb{Z}^d.$$

The discrete maximal L^p-regularity theory implies that

$$\varepsilon^{-\frac{\alpha}{2}}\|u_\varepsilon\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)} + \|\nabla^\varepsilon u_\varepsilon\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)^d} \leq \varepsilon^{-\gamma}c\|\mathcal{F}_\varepsilon\chi\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)^d}.$$

As a result of this, we have $\|u_\varepsilon\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)} \leq \varepsilon^{\frac{\alpha}{2}-\gamma}c\|\chi\|_{L^p(\Omega\times\mathbb{R}^d)^d}$ and therefore $u_\varepsilon \overset{2}{\to} 0$ as $\varepsilon\to 0$. The latter also implies that $\mathcal{G}^\gamma_\varepsilon$ is a linear bounded operator with its operator norm bounded uniformly in $\varepsilon\in(0,1]$.

We consider the sequence $g_{\delta,\varepsilon}$ from Lemma 5.25 corresponding to χ. Note that $w_{\delta,\varepsilon} := u_\varepsilon - \varepsilon^{-\gamma}g_{\delta,\varepsilon}$ is the unique solution in $L^p(\Omega\times\varepsilon\mathbb{Z}^d)$ to (for P-a.a. $\omega\in\Omega$)

$$\varepsilon^{-\alpha}w_{\delta,\varepsilon} + \nabla^{\varepsilon,*}\nabla^\varepsilon w_{\delta,\varepsilon} = \nabla^{\varepsilon,*}\varepsilon^{-\gamma}(\mathcal{F}_\varepsilon\chi - \nabla^\varepsilon g_{\delta,\varepsilon}) - \varepsilon^{-\alpha-\gamma}g_{\delta,\varepsilon} \quad \text{in } \varepsilon\mathbb{Z}^d.$$

We employ again the discrete maximal L^p-regularity theory to obtain

$$\|\nabla^\varepsilon w_{\delta,\varepsilon}\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)^d} \leq c\left(\varepsilon^{-\gamma}\|\mathcal{F}_\varepsilon\chi - \nabla^\varepsilon g_{\delta,\varepsilon}\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)^d} + \varepsilon^{-\frac{\alpha}{2}-\gamma}\|g_{\delta,\varepsilon}\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)}\right).$$

Multiplication of the above inequality by ε^γ yields

$$\|\varepsilon^\gamma\nabla^\varepsilon u_\varepsilon - \nabla^\varepsilon g_{\delta,\varepsilon}\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)^d} \leq c\left(\|\mathcal{F}_\varepsilon\chi - \nabla^\varepsilon g_{\delta,\varepsilon}\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)^d} + \varepsilon^{-\frac{\alpha}{2}}\|g_{\delta,\varepsilon}\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)}\right).$$

As a result of this and with help of the isometry property of $\mathcal{T}_\varepsilon$, we obtain

$$\begin{aligned}
&\|\mathcal{T}_\varepsilon\varepsilon^\gamma\nabla^\varepsilon u_\varepsilon - \chi\|_{L^p(\Omega\times\mathbb{R}^d)^d}\\
\leq\ & c\left(\|\mathcal{F}_\varepsilon\chi - \nabla^\varepsilon g_{\delta,\varepsilon}\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)^d} + \varepsilon^{-\frac{\alpha}{2}}\|g_{\delta,\varepsilon}\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)} + \|\mathcal{T}_\varepsilon\nabla^\varepsilon g_{\delta,\varepsilon} - \chi\|_{L^p(\Omega\times\mathbb{R}^d)^d}\right).
\end{aligned}$$

Letting first $\varepsilon\to 0$ and then $\delta\to 0$, the right-hand side of the above inequality vanishes using Lemma 5.25. This completes the proof of (i).

(ii) We consider a sequence $u_\delta = \sum_{i=1}^{n(\delta)}\varphi_i^\delta\eta_i^\delta$ such that $\varphi_i^\delta\in L^p_{\text{inv}}(\Omega)$, $\eta_i^\delta\in C_c^\infty(\mathbb{R}^d)$, and

$$\|u_\delta - u\|_{L^p(\Omega\times\mathbb{R}^d)} + \|\nabla u_\delta - \nabla u\|_{L^p(\Omega\times\mathbb{R}^d)^d} \to 0 \quad \text{as } \delta\to 0.$$

Using the triangle inequality, it follows that

$$\begin{aligned}
&\|\mathcal{T}_\varepsilon\nabla^\varepsilon\mathcal{F}_\varepsilon u - \nabla u\|_{L^p(\Omega\times\mathbb{R}^d)^d} \qquad (5.28)\\
\leq\ & \|\mathcal{T}_\varepsilon\nabla^\varepsilon\mathcal{F}_\varepsilon u - \mathcal{T}_\varepsilon\nabla^\varepsilon\mathcal{F}_\varepsilon u_\delta\|_{L^p(\Omega\times\mathbb{R}^d)^d} + \|\mathcal{T}_\varepsilon\nabla^\varepsilon\mathcal{F}_\varepsilon u_\delta - \nabla u_\delta\|_{L^p(\Omega\times\mathbb{R}^d)^d} + \|\nabla u_\delta - \nabla u\|_{L^p(\Omega\times\mathbb{R}^d)^d}.
\end{aligned}$$

First, we treat the first term on the right-hand side. For $i = 1, ..., d$, by the isometry property of $\mathcal{T}_\varepsilon$ and contraction property of $\mathcal{F}_\varepsilon$, we obtain

$$\begin{aligned}\|\mathcal{T}_\varepsilon \nabla_i^\varepsilon \mathcal{F}_\varepsilon u - \mathcal{T}_\varepsilon \nabla_i^\varepsilon \mathcal{F}_\varepsilon u_\delta\|_{L^p(\Omega\times\mathbb{R}^d)}^p &\leq \left\langle \int_{\mathbb{R}^d} \left| \frac{u(\omega, x+\varepsilon e_i) - u_\delta(\omega, x+\varepsilon e_i) - u(\omega,x) + u_\delta(\omega,x)}{\varepsilon} \right|^p dx \right\rangle \\ &\leq c \left\langle \int_{\mathbb{R}^d} |\partial_i u(\omega,x) - \partial_i u_\delta(\omega,x)|^p dx \right\rangle. \end{aligned} \tag{5.29}$$

The last inequality follows using the fact that for any function $\eta \in W^{1,p}(\mathbb{R}^d)$, we have $\eta(x + \varepsilon e_i) - \eta(x) = \varepsilon \int_0^1 \partial_i \eta(x + \varepsilon t e_i) dt$ and therefore $\int_{\mathbb{R}^d} |\frac{\eta(x+\varepsilon e_i) - \eta(x)}{\varepsilon}|^p dx \leq \int_0^1 \int_{\mathbb{R}^d} |\partial_i \eta(x+\varepsilon t e_i)|^p dx dt = \int_{\mathbb{R}^d} |\partial_i \eta(x)|^p dx$.
Second, we compute, for $i = 1, ..., d$,

$$\mathcal{T}_\varepsilon \nabla_i^\varepsilon \mathcal{F}_\varepsilon u_\delta(\omega, x) = \frac{1}{\varepsilon}\left(\overline{\pi}_\varepsilon u_\delta(\tau_{e_i}\omega, x+\varepsilon e_i) - \overline{\pi}_\varepsilon u_\delta(\tau_{e_i}\omega, x)\right) + \frac{1}{\varepsilon}\left(\overline{\pi}_\varepsilon u_\delta(\tau_{e_i}\omega, x) - \overline{\pi}_\varepsilon u_\delta(\omega, x)\right). \tag{5.30}$$

The second part of the right-hand side of the above equality vanishes (for P-a.a. $\omega \in \Omega$) by shift invariance of u_δ. Furthermore, we have

$$\|\mathcal{T}_\varepsilon \nabla_i^\varepsilon \mathcal{F}_\varepsilon u_\delta - \partial_i u_\delta\|_{L^p(\Omega\times\mathbb{R}^d)} \leq \left\langle \int_{\mathbb{R}^d} \left| \frac{\overline{\pi}_\varepsilon u_\delta(\omega, x+\varepsilon e_i) - \overline{\pi}_\varepsilon u_\delta(\omega,x)}{\varepsilon} - \partial_i u_\delta(\omega,x) \right|^p dx \right\rangle^{\frac{1}{p}}.$$

For any $\delta > 0$ the last expression converges to 0 as $\varepsilon \to 0$ since u_δ is smooth in its x-variable. Finally, in (5.28) we first let $\varepsilon \to 0$ and then $\delta \to 0$ to conclude the proof.
(iii) Let $0 < \alpha < 2\gamma$. For a given $u \in L^p_{\text{inv}}(\Omega) \otimes L^p(\mathbb{R}^d)$ we define $\mathcal{F}_\varepsilon^\gamma u := u_\varepsilon$ as the unique solution to the following equation in $L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ (for P-a.a. $\omega \in \Omega$)

$$\varepsilon^{-\alpha} u_\varepsilon + \nabla^{\varepsilon,*}\nabla^\varepsilon u_\varepsilon = \varepsilon^{-\alpha}\mathcal{F}_\varepsilon u \quad \text{in } \varepsilon\mathbb{Z}^d.$$

The maximal L^p-regularity theory and boundedness of $\mathcal{F}_\varepsilon$ imply that

$$\|\nabla^\varepsilon u_\varepsilon\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)^d} \leq \varepsilon^{-\frac{\alpha}{2}} c \|u\|_{L^p(\Omega\times\mathbb{R}^d)}.$$

As a result of this and the isometry property of $\mathcal{T}_\varepsilon$, we obtain that $\varepsilon^\gamma \nabla^\varepsilon u_\varepsilon \overset{2}{\to} 0$.
We consider a sequence $u_\delta = \sum_{i=1}^{n(\delta)} \varphi_i^\delta \eta_i^\delta$ such that $\varphi_i^\delta \in L^p_{\text{inv}}(\Omega)$, $\eta_i^\delta \in C_c^\infty(\mathbb{R}^d)$, and

$$\|u_\delta - u\|_{L^p(\Omega\times\mathbb{R}^d)} \to 0 \text{ as } \delta \to 0.$$

Note that $w_{\delta,\varepsilon} := u_\varepsilon - \mathcal{F}_\varepsilon u_\delta$ is the unique solution in $L^p(\Omega \times \varepsilon\mathbb{Z}^d)$ to (for P-a.a. $\omega \in \Omega$)

$$\varepsilon^{-\alpha} w_{\delta,\varepsilon} + \nabla^{\varepsilon,*}\nabla^\varepsilon w_{\delta,\varepsilon} = \varepsilon^{-\alpha}\left(\mathcal{F}_\varepsilon u - \mathcal{F}_\varepsilon u_\delta\right) - \nabla^{\varepsilon,*}\nabla^\varepsilon \mathcal{F}_\varepsilon u_\delta \quad \text{in } \varepsilon\mathbb{Z}^d.$$

The maximal L^p-regularity theory implies that

$$\varepsilon^{-\frac{\alpha}{2}} \|w_{\delta,\varepsilon}\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)} \leq c\left(\varepsilon^{-\frac{\alpha}{2}} \|\mathcal{F}_\varepsilon u - \mathcal{F}_\varepsilon u_\delta\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)} + \|\nabla^\varepsilon \mathcal{F}_\varepsilon u_\delta\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)^d}\right).$$

We multiply the above inequality by $\varepsilon^{\frac{\alpha}{2}}$ and use boundedness of $\mathcal{F}_\varepsilon$, to obtain

$$\|u_\varepsilon - \mathcal{F}_\varepsilon u_\delta\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)} \leq c\left(\|u - u_\delta\|_{L^p(\Omega\times\mathbb{R}^d)} + \varepsilon^{\frac{\alpha}{2}} \|\nabla^\varepsilon \mathcal{F}_\varepsilon u_\delta\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)^d}\right).$$

Using the above inequality and the isometry property of $\mathcal{T}_\varepsilon$, we obtain

$$\|\mathcal{T}_\varepsilon u_\varepsilon - u\|_{L^p(\Omega\times\mathbb{R}^d)} \le c\left(\|u-u_\delta\|_{L^p(\Omega\times\mathbb{R}^d)} + \varepsilon^{\frac{\alpha}{2}}\|\nabla^\varepsilon \mathcal{F}_\varepsilon u_\delta\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)^d} + \|\mathcal{T}_\varepsilon\mathcal{F}_\varepsilon u_\delta - u_\delta\|_{L^p(\Omega\times\mathbb{R}^d)}\right).$$

Letting $\varepsilon\to 0$, the last two terms on the right-hand side of the above inequality vanish. Indeed, the middle term is bounded by $c\varepsilon^{\frac{\alpha}{2}}\|\nabla u_\delta\|_{L^p(\Omega\times\mathbb{R}^d)^d}$ (cf. part (ii) (5.29)) and the last term vanishes using Lemma 5.17 (iii). Finally, letting $\delta\to 0$ we conclude that $u_\varepsilon \overset{2}{\to} u$.
(iv) We consider a sequence $u_\delta = \sum_{i=1}^{n(\delta)}\varphi_i^\delta\eta_i^\delta$ such that $\varphi_i^\delta\in L^p(\Omega)$, $\eta_i^\delta\in C_c^\infty(\mathbb{R}^d)$, and

$$\|u_\delta - u\|_{L^p(\Omega\times\mathbb{R}^d)} \to 0 \text{ as } \delta\to 0.$$

We have

$$\begin{aligned} & \|\mathcal{T}_\varepsilon\varepsilon^\gamma\nabla^\varepsilon\mathcal{F}_\varepsilon u - a_\gamma Du\|_{L^p(\Omega\times\mathbb{R}^d)^d} \qquad (5.31)\\ \le\ & \|\mathcal{T}_\varepsilon\varepsilon^\gamma\nabla^\varepsilon\mathcal{F}_\varepsilon(u-u_\delta)\|_{L^p(\Omega\times\mathbb{R}^d)^d} + \|a_\gamma D\left(u_\delta-u\right)\|_{L^p(\Omega\times\mathbb{R}^d)^d} + \|\mathcal{T}_\varepsilon\varepsilon^\gamma\nabla^\varepsilon\mathcal{F}_\varepsilon u_\delta - a_\gamma Du_\delta\|_{L^p(\Omega\times\mathbb{R}^d)^d}. \end{aligned}$$

The first term on the right-hand side above is bounded by $\varepsilon^{\gamma-1}c\|u-u_\delta\|_{L^p(\Omega\times\mathbb{R}^d)}$, using boundedness of all of the appearing operators. We compute, as in (5.30) (part (ii)), for $i=1,...,d$

$$\mathcal{T}_\varepsilon\varepsilon^\gamma\nabla_i^\varepsilon\mathcal{F}_\varepsilon u_\delta(\omega,x) = \varepsilon^{\gamma-1}\left(\overline{\pi}_\varepsilon u_\delta(\tau_{e_i}\omega, x+\varepsilon e_i) - \overline{\pi}_\varepsilon u_\delta(\tau_{e_i}\omega,x)\right) + \varepsilon^{\gamma-1}\overline{\pi}_\varepsilon D_i u_\delta(\omega,x).$$

As a result of this, we obtain

$$\begin{aligned} & \|\mathcal{T}_\varepsilon\varepsilon^\gamma\nabla_i^\varepsilon\mathcal{F}_\varepsilon u_\delta - a_\gamma D_i u_\delta\|_{L^p(\Omega\times\mathbb{R}^d)}\\ \le\ & \varepsilon^\gamma\left\|\frac{\overline{\pi}_\varepsilon u_\delta(\cdot,\cdot+\varepsilon e_i) - \overline{\pi}_\varepsilon u_\delta(\cdot,\cdot)}{\varepsilon}\right\|_{L^p(\Omega\times\mathbb{R}^d)} + \|\varepsilon^{\gamma-1}\overline{\pi}_\varepsilon D_i u_\delta - a_\gamma D_i u_\delta\|_{L^p(\Omega\times\mathbb{R}^d)}. \end{aligned}$$

The first term on the right-hand side above is bounded by $\varepsilon^\gamma c\|\nabla u_\delta\|_{L^p(\Omega\times\mathbb{R}^d)^d}$ and therefore it vanishes in the limit $\varepsilon\to 0$. The second term vanishes as well in the limit $\varepsilon\to 0$.
Collecting the above claims and letting first $\varepsilon\to 0$ and then $\delta\to 0$ in (5.31), we conclude the proof. $\square$

Proof of Corollary 5.28. (i) and (ii) are obtained directly from Proposition 5.27 and Lemma 5.17 (iii).
(iii) For $\delta>0$ we consider a cut-off function $\eta_\delta\in C_c^\infty(\mathbb{R}^d)$ such that $0\le\eta_\delta\le 1$, $\eta_\delta=0$ in $\mathbb{R}^d\setminus Q$, $\eta_\delta = 1$ in $Q^{-\delta}:=\{x\in Q: \mathrm{dist}(x,\partial Q)\ge\delta\}$ and $|\nabla\eta_\delta|\le\frac{c}{\delta}$.
Also, by density, we can choose a sequence $u_\delta(\omega,x)=\sum_{i=1}^{n(\delta)}\varphi_i^\delta(\omega)\xi_i^\delta(x)$ such that $\varphi_i^\delta\in L^p_{\mathrm{inv}}(\Omega)$ and $\xi_i^\delta\in C_c^\infty(\mathbb{R}^d)$, $\mathrm{dist}(\mathrm{supp}(u_\delta),\partial Q)\ge\mu(\delta)>0$ (with $\mu(\delta)\to 0$ as $\delta\to 0$) and

$$u_\delta\to u \quad \text{strongly in } L^p(\Omega)\otimes W^{1,p}(\mathbb{R}^d) \quad \text{as } \delta\to 0.$$

Let $u_{\varepsilon,\delta}=\mathcal{F}_\varepsilon u_\delta + \eta_\delta\mathcal{G}_\varepsilon^0\chi$, where $\mathcal{G}_\varepsilon^0$ denotes the operator given in Proposition 5.27. We have

$$\begin{aligned} & \|u_{\varepsilon,\delta} - (\mathcal{F}_\varepsilon u + \mathcal{G}_\varepsilon^0\chi)\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)} + \|\nabla^\varepsilon(u_{\varepsilon,\delta} - (\mathcal{F}_\varepsilon u + \mathcal{G}_\varepsilon^0\chi))\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)^d} \qquad (5.32)\\ \le\ & \|\mathcal{F}_\varepsilon u_\delta - \mathcal{F}_\varepsilon u\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)} + \|\left(\eta_\delta - 1\right)\mathcal{G}_\varepsilon^0\chi\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)} + \|\nabla^\varepsilon\left(\mathcal{F}_\varepsilon u_\delta - \mathcal{F}_\varepsilon u\right)\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)^d}\\ & + \|\nabla^\varepsilon\left(\eta_\delta\mathcal{G}_\varepsilon^0\chi - \mathcal{G}_\varepsilon^0\chi\right)\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)^d}. \end{aligned}$$

Above on the right-hand side, the first term can be bounded by $\|u_\delta - u\|_{L^p(\Omega\times\mathbb{R}^d)}$ (by boundedness of $\mathcal{F}_\varepsilon$), the second term is bounded by $\|\mathcal{T}_\varepsilon\mathcal{G}^0_\varepsilon\chi\|_{L^p(\Omega\times\mathbb{R}^d\setminus Q^{-\delta})}$ (using the properties of η_δ) and the third term is bounded by $c\|\nabla u_\delta - \nabla u\|_{L^p(\Omega\times\mathbb{R}^d)^d}$ (similarly as in (5.29)). The last term is treated as follows. We take advantage of the following product rule: For $f, g : \varepsilon\mathbb{Z}^d \to \mathbb{R}$ it holds that $\nabla^\varepsilon_i\left(f(x)g(x)\right) = f(x+\varepsilon e_i)\nabla^\varepsilon_i g(x) + g(x)\nabla^\varepsilon_i f(x)$. Consequently, we obtain

$$\begin{aligned} &\|\nabla^\varepsilon\left(\eta_\delta\mathcal{G}^0_\varepsilon\chi - \mathcal{G}^0_\varepsilon\chi\right)\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)^d} \\ \leq\ &\|\left(\eta_\delta - 1\right)\nabla^\varepsilon\mathcal{G}^0_\varepsilon\chi\|_{L^p(\Omega\times\varepsilon\mathbb{Z}^d)^d} + c\sum_{i=1}^d\left\langle\int_{\varepsilon\mathbb{Z}^d}|\mathcal{G}^0_\varepsilon\chi(\omega, x+\varepsilon e_i)\nabla^\varepsilon_i\eta_\delta(x)|^p dm_\varepsilon(x)\right\rangle^{\frac{1}{p}}. \end{aligned} \tag{5.33}$$

The first term on the right-hand side of (5.33) is bounded by $\|\mathcal{T}_\varepsilon\nabla^\varepsilon\mathcal{G}^0_\varepsilon\chi\|_{L^p(\Omega\times\mathbb{R}^d\setminus Q^{-\delta})^d}$ and for small enough ε, the second term is bounded by $\frac{c}{\delta}\|\mathcal{T}_\varepsilon\mathcal{G}^0_\varepsilon\chi\|_{L^p(\Omega\times\mathbb{R}^d)}$. Note that

$$\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0}\Big(\|\mathcal{T}_\varepsilon\nabla^\varepsilon\mathcal{G}^0_\varepsilon\chi\|_{L^p(\Omega\times\mathbb{R}^d\setminus Q^{-\delta})^d} + \frac{c}{\delta}\|\mathcal{T}_\varepsilon\mathcal{G}^0_\varepsilon\chi\|_{L^p(\Omega\times\mathbb{R}^d)}\Big) = 0$$

since $\mathcal{T}_\varepsilon\mathcal{G}^0_\varepsilon\chi \to 0$, $\mathcal{T}_\varepsilon\nabla^\varepsilon\mathcal{G}^0_\varepsilon\chi \to \chi$ as $\varepsilon \to 0$ (Proposition 5.27 (i)) and $\chi(\omega,\cdot)$ vanishes outside of Q. Collecting all the above bounds for the inequality (5.32), using the isometry property of $\mathcal{T}_\varepsilon$ and with the help of part (i), we obtain that

$$\limsup_{\delta\to 0}\limsup_{\varepsilon\to 0}\Big(\|\mathcal{T}_\varepsilon u_{\varepsilon,\delta} - u\|_{L^p(\Omega\times\mathbb{R}^d)} + \|\mathcal{T}_\varepsilon\nabla^\varepsilon u_{\varepsilon,\delta} - \nabla u - \chi\|_{L^p(\Omega\times\mathbb{R}^d)^d} + g(\varepsilon,\delta)\Big) = 0,$$

where $g(\varepsilon,\delta) = \begin{cases} 0 & \text{if } \varepsilon \leq \frac{\mu(\delta)}{c(d)} \\ 1 & \text{if } \varepsilon > \frac{\mu(\delta)}{c(d)} \end{cases}$ and $c(d)$ is the diameter of $\Box$. Hence, there exists a diagonal sequence $u_\varepsilon := u_{\varepsilon,\delta(\varepsilon)}$ which satisfies the claim of the corollary.

(iv) For a given $(u,\chi) \in (L^p_{\text{inv}}(\Omega)\otimes L^p(Q)) \times (L^p_{\text{pot}}(\Omega)\otimes L^p(Q))$ we set $u_\varepsilon(u,\chi) = \eta_{\delta(\varepsilon)}\left(\mathcal{F}^\gamma_\varepsilon u + \mathcal{G}^\gamma_\varepsilon\chi\right)$, where $\eta_{\delta(\varepsilon)}$ is the cut-off function from part (iii) with $\delta(\varepsilon) = \varepsilon^{\frac{\gamma}{2}}$. This mapping defines a linear and bounded operator. For notational convenience, we write u_ε instead of $u_\varepsilon(u,\chi)$. We have

$$\begin{aligned} &\|\mathcal{T}_\varepsilon u_\varepsilon - u\|_{L^p(\Omega\times\mathbb{R}^d)} + \|\mathcal{T}_\varepsilon\varepsilon^\gamma\nabla^\varepsilon u_\varepsilon - \chi\|_{L^p(\Omega\times\mathbb{R}^d)^d} \\ \leq\ &\|\mathcal{T}_\varepsilon u_\varepsilon - \mathcal{T}_\varepsilon\left(\mathcal{F}^\gamma_\varepsilon u + \mathcal{G}^\gamma_\varepsilon\chi\right)\|_{L^p(\Omega\times\mathbb{R}^d)} + \|\mathcal{T}_\varepsilon\left(\mathcal{F}^\gamma_\varepsilon u + \mathcal{G}^\gamma_\varepsilon\chi\right) - u\|_{L^p(\Omega\times\mathbb{R}^d)} \\ &+\|\mathcal{T}_\varepsilon\varepsilon^\gamma\nabla^\varepsilon u_\varepsilon - \mathcal{T}_\varepsilon\varepsilon^\gamma\nabla^\varepsilon\left(\mathcal{F}^\gamma_\varepsilon u + \mathcal{G}^\gamma_\varepsilon\chi\right)\|_{L^p(\Omega\times\mathbb{R}^d)^d} + \|\mathcal{T}_\varepsilon\varepsilon^\gamma\nabla^\varepsilon\left(\mathcal{F}^\gamma_\varepsilon u + \mathcal{G}^\gamma_\varepsilon\chi\right) - \chi\|_{L^p(\Omega\times\mathbb{R}^d)^d}. \end{aligned}$$

The second and last terms on the right-hand side above vanish as $\varepsilon \to 0$ using the claim of part (ii). The first term is bounded by $\|\mathcal{T}_\varepsilon\left(\mathcal{F}^\gamma_\varepsilon u + \mathcal{G}^\gamma_\varepsilon\chi\right)\|_{L^p(\Omega\times\mathbb{R}^d\setminus Q^{-\delta(\varepsilon)})}$ (cf. part (iii)) and therefore it vanishes as $\varepsilon \to 0$ using the fact that $\mathcal{T}_\varepsilon\left(\mathcal{F}^\gamma_\varepsilon u + \mathcal{G}^\gamma_\varepsilon\chi\right)$ converges strongly and therefore it is uniformly integrable. For small enough ε, the third term is bounded (up to a constant) by $\|\mathcal{T}_\varepsilon\varepsilon^\gamma\nabla^\varepsilon\left(\mathcal{F}^\gamma_\varepsilon u + \mathcal{G}^\gamma_\varepsilon\chi\right)\|_{L^p(\Omega\times\mathbb{R}^d\setminus Q^{-\delta(\varepsilon)})^d} + \varepsilon^{\frac{\gamma}{2}}\|\mathcal{T}_\varepsilon\left(\mathcal{F}^\gamma_\varepsilon u + \mathcal{G}^\gamma_\varepsilon\chi\right)\|_{L^p(\Omega\times\mathbb{R}^d)}$ (cf. (5.33) in part (iii)). The last expression vanishes in the limit $\varepsilon \to 0$ using the properties of $\mathcal{F}^\gamma_\varepsilon u + \mathcal{G}^\gamma_\varepsilon\chi$. The proof is complete. $\square$

6 Continuum unfolding

In this section we study stochastic unfolding suited for the treatment of problems given on a continuum physical space. In particular, after laying the ground by presenting the general framework for modeling of continuum random media, the unfolding operator is established with its noteworthy properties.

6.1 General framework

Let $p, q \in (1, \infty)$ be dual exponents of integrability, i.e., $\frac{1}{p}+\frac{1}{q} = 1$, and $Q \subset \mathbb{R}^d$ be open. Throughout Section 6 we assume the following assumption to hold:

Assumption 6.1. *Let $(\Omega, \mathcal{F}, P)$ be a complete and separable probability space. Let $\tau = \{\tau_x\}_{x\in\mathbb{R}^d}$ denote a group of invertible measurable mappings $\tau_x : \Omega \to \Omega$ such that:*

(i) (Group property). $\tau_0 = Id$ and $\tau_{x+y} = \tau_x \circ \tau_y$ for all $x, y \in \mathbb{R}^d$.

(ii) (Measure preservation). $P(\tau_x E) = P(E)$ for all $E \in \mathcal{F}$ and $x \in \mathbb{R}^d$.

(iii) (Measurability). $(\omega, x) \mapsto \tau_x\omega$ is $\big(\mathcal{F} \otimes \mathcal{L}(\mathbb{R}^d), \mathcal{F}\big)$-measurable.

We denote by $\langle\cdot\rangle = \int_\Omega \cdot dP(\omega)$ the mathematical expectation. We denote by $L^p(\Omega)$ and $L^p(Q)$ the usual Banach spaces of p-integrable functions defined on $(\Omega, \mathcal{F}, P)$ and Q, respectively. Note that the separability assumption on the measure space implies that $L^p(\Omega)$ is separable. We say that $(\Omega, \mathcal{F}, P, \tau)$ is *ergodic* ($\langle\cdot\rangle$ is ergodic), if

$$\text{every shift invariant } E \in \mathcal{F} \text{ (i.e., } \tau_x E = E \text{ for all } x \in \mathbb{R}^d) \text{ satisfies } P(E) \in \{0, 1\}.$$

A measurable mapping (random field) $\widetilde{\varphi} : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is called *stationary* if it admits the form $\widetilde{\varphi}(\omega, x) = \varphi(\tau_x\omega)$ for a random variable $\varphi : \Omega \to \mathbb{R}$. In this regard, we might identify random variables with their stationary extensions, which is defined in the following lemma.

Lemma 6.2 (Stationary extension)**.** *Let $\varphi : \Omega \to \mathbb{R}$ be $\mathcal{F}$-measurable. Then $S\varphi : \Omega \times \mathbb{R}^d \to \mathbb{R}$, $S\varphi(\omega, x) := \varphi(\tau_x\omega)$ defines an $\mathcal{F} \otimes \mathcal{L}(\mathbb{R}^d)$-measurable function – called the stationary extension of φ. Moreover, if $Q \subset \mathbb{R}^d$ is open and bounded, for all $1 \le p < \infty$, the stationary extension defines a linear injection $S : L^p(\Omega) \to L^p(\Omega \times Q)$, which satisfies*

$$\|S\varphi\|_{L^p(\Omega\times Q)} = |Q|^{\frac{1}{p}} \|\varphi\|_{L^p(\Omega)}.$$

Proof. $S\varphi$ is measurable as a composition of the measurable mappings φ and $(\omega, x) \mapsto \tau_x\omega$. Also, we have $\left\langle \int_Q |S\varphi(\omega, x)|^p dx \right\rangle = \int_Q \langle|\varphi(\tau_x\omega)|^p\rangle\, dx$ and since P is invariant w.r.t. the action of τ_x, it holds $\langle|\varphi(\tau_x\omega)|^p\rangle = \langle|\varphi(\omega)|^p\rangle$ for all $x \in \mathbb{R}^d$. This implies the claim. $\square$

Remark 6.3 (Ergodic theorems, see, e.g., [DVJ07, Section 12.2], [Tem72])**.** *Let $\langle \cdot \rangle$ be ergodic and $Q \subset \mathbb{R}^d$ be open, bounded and convex. Let $p \geq 1$ and $L \geq 1$. If $\varphi \in L^p(\Omega)$, then it holds*

$$\fint_{LQ} S\varphi(\cdot, x)dx \to \langle \varphi \rangle \quad \text{in } L^p(\Omega) \quad \text{as } L \to \infty.$$

This is the statement of a multiparameter version of von Neumann's mean ergodic theorem. Moreover, the individual ergodic theorem (multiparameter version of Birkhoff's ergodic theorem) implies that for any $\varphi \in L^1(\Omega)$ for P-a.a. $\omega \in \Omega$, it holds $\fint_{LQ} S\varphi(\omega, x)dx \to \langle \varphi \rangle$ as $L \to \infty$.

Stochastic gradient. We consider the group of isometric operators $\{U_x\}_{x\in\mathbb{R}^d}$, $U_x : L^p(\Omega) \to L^p(\Omega)$ defined by $U_x\varphi(\omega) = \varphi(\tau_x\omega)$. This group is strongly continuous (see [JKO12, Section 7.1]). For $i = 1, ..., d$, we consider the one-parameter group of operators $\{U_{he_i}\}_{h\in\mathbb{R}}$ ($\{e_i\}$ being the usual basis of $\mathbb{R}^d$) and its infinitesimal generator $D_i : \mathcal{D}_i \subset L^p(\Omega) \to L^p(\Omega)$,

$$D_i\varphi = \lim_{h\to 0} \frac{U_{he_i}\varphi - \varphi}{h},$$

which we refer to as the *stochastic* derivative. D_i is a linear and closed operator and its domain $\mathcal{D}_i$ is dense in $L^p(\Omega)$. We set $W^{1,p}(\Omega) = \cap_{i=1}^d \mathcal{D}_i$ and define for $\varphi \in W^{1,p}(\Omega)$ the stochastic gradient as $D\varphi = (D_1\varphi, ..., D_d\varphi)$. In this manner, we obtain a linear, closed and densely defined operator $D : W^{1,p}(\Omega) \to L^p(\Omega)^d$, and we denote by

$$L^p_{\text{pot}}(\Omega) := \overline{\text{ran}(D)} \subset L^p(\Omega)^d$$

the closure of the range of D in $L^p(\Omega)^d$. We denote the adjoint of D by $D^* : \mathcal{D}^* \subset L^q(\Omega)^d \to L^q(\Omega)$ which is a linear, closed and densely defined operator ($\mathcal{D}^*$ is the domain of D^*). Note that $W^{1,q}(\Omega)^d \subset \mathcal{D}^*$ and for all $\varphi \in W^{1,p}(\Omega)$ and $\psi \in W^{1,q}(\Omega)$ we have the integration by parts formula, $i = 1, ..., d$,

$$\langle D_i\varphi\psi \rangle = - \langle \varphi D_i\psi \rangle,$$

and thus $D^*\psi = -\sum_{i=1}^d D_i\psi_i$ for $\psi \in W^{1,q}(\Omega)^d$. We define the subspace of shift invariant functions in $L^p(\Omega)$ by

$$L^p_{\text{inv}}(\Omega) = \left\{ \varphi \in L^p(\Omega) : U_x\varphi = \varphi \quad \text{for all } x \in \mathbb{R}^d \right\},$$

and denote by $P_{\text{inv}} : L^p(\Omega) \to L^p_{\text{inv}}(\Omega)$ the conditional expectation with respect to the σ-algebra of shift invariant sets $\{E \in \mathcal{F} : \tau_x E = E \text{ for all } x \in \mathbb{R}^d\}$. P_{inv} is a contractive projection and for $p = 2$ it coincides with the orthogonal projection onto $L^2_{\text{inv}}(\Omega)$.

Remark 6.4. *We recall some basic facts from functional analysis. Let $p \in (1, \infty)$ and $q = \frac{p}{p-1}$.*

(i) $\langle \cdot \rangle$ *is ergodic* $\Leftrightarrow L^p_{\text{inv}}(\Omega) \simeq \mathbb{R} \Leftrightarrow P_{\text{inv}} f = \langle f \rangle$.

(ii) The following orthogonality relations hold (for a proof see [Bré11, Section 2.6]): We identify the dual space $L^p(\Omega)^*$ with $L^q(\Omega)$, and define for a set $A \subset L^q(\Omega)$ its orthogonal complement $A^\perp \subset L^p(\Omega)$ as $A^\perp = \left\{ \varphi \in L^p(\Omega) : \langle \varphi, \psi \rangle_{L^p, L^q} = 0 \text{ for all } \psi \in A \right\}$. Then

$$\ker(D) = \text{ran}(D^*)^\perp, \quad L^p_{\text{pot}}(\Omega) = \overline{\text{ran}(D)} = \ker(D^*)^\perp. \tag{6.1}$$

Above, $\ker(\cdot)$ *denotes the kernel and* $\text{ran}(\cdot)$ *the range of an operator.*

Random fields. We introduce function spaces for functions defined on $\Omega \times Q$ as follows: For closed subspaces $X \subset L^p(\Omega)$ and $Y \subset L^p(Q)$, we denote by $X \otimes Y$ the closure of

$$X \overset{a}{\otimes} Y := \left\{ \sum_{i=1}^{n} \varphi_i \eta_i : \varphi_i \in X, \eta_i \in Y, n \in \mathbb{N} \right\}$$

in $L^p(\Omega \times Q)$. Note that in the case $X = L^p(\Omega)$ and $Y = L^p(Q)$, we have $X \otimes Y = L^p(\Omega \times Q)$. Up to isometric isomorphisms, we may identify $L^p(\Omega \times Q)$ with the Bochner spaces $L^p(\Omega; L^p(Q))$ and $L^p(Q; L^p(\Omega))$. Slightly abusing the notation, for closed subspaces $X \subset L^p(\Omega)$ and $Y \subset W^{1,p}(Q)$, we denote by $X \otimes Y$ the closure of

$$X \overset{a}{\otimes} Y := \left\{ \sum_{i=1}^{n} \varphi_i \eta_i : \varphi_i \in X, \eta_i \in Y, n \in \mathbb{N} \right\}$$

in $L^p(\Omega; W^{1,p}(Q))$. In this regard, we may identify $u \in L^p(\Omega) \otimes W^{1,p}(Q)$ with the pair $(u, \nabla u) \in L^p(\Omega \times Q)^{1+d}$. We mostly focus on the space $L^p(\Omega \times Q)$ and the above notation is convenient for keeping track of its various subspaces.

We conclude this section with some standard examples of random (and deterministic) media that fit in the above described framework, see [Tor13, DG17] for other standard models of random media. We note that in applications typically coefficients of equations are described by stationary random fields, e.g., they take the form $A(\tau_{\frac{x}{\varepsilon}}\omega)$ where $A : \Omega \to \mathbb{R}^{d\times d}$ is a random variable.

Example 6.5 (Periodic setting). *We take Ω to be the unit torus $\Box_\# = \mathbb{R}^d/\mathbb{Z}^d$ and equip it with the Lebesgue σ-algebra $\mathcal{L}(\Box_\#)$ and Lebesgue measure dy (in fact with the push-forward of these objects with respect to the canonical mapping $[0,1)^d \to \Box_\#$). We define a dynamical system, for $x \in \mathbb{R}^d$, by $\tau_x : \Box_\# \to \Box_\#$,*

$$\tau_x(y) = x + y \mod 1.$$

The system $(\Box_\#, \mathcal{L}(\Box_\#), dy, \tau)$ defines an ergodic probability space that satisfies Assumption 6.1. Also, for a measurable function on the torus $A : \Box_\# \to \mathbb{R}^{d\times d}$, $x \mapsto A(\tau_{\frac{x}{\varepsilon}} y)$ defines a rapidly oscillating periodic coefficient field. In this respect, periodic homogenization problems might also be treated by the below considered unfolding procedure (see Section 7.4).

Example 6.6 (Random checkerboard). *We present a continuum counterpart of the discrete i.i.d. random field from Example 5.10–a randomly-colored checkerboard. We consider a probability space $(\Omega, \mathcal{F}, P) = \left(\Omega_0^{\mathbb{Z}^d}, \overline{\otimes_{\mathbb{Z}^d}\mathcal{F}_0}, \otimes_{\mathbb{Z}^d}P_0\right)$ as in Example 5.10 which is given in terms of a base probability space $(\Omega_0, \mathcal{F}_0, P_0)$ with $\Omega_0 \subset \mathbb{R}^{d\times d}$. A realization $\omega : \mathbb{Z}^d \to \Omega_0$ may be identified with its piecewise constant interpolation $\overline{\omega} : \mathbb{R}^d \to \Omega_0$ and in this sense we identify Ω with the set of piecewise constant functions in $\mathbb{R}^d$ (subordinate to $\mathbb{Z}^d$). Each component of the matrix valued mapping $\overline{\omega} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ can be associated with a checkerboard with random i.i.d. coloring of the tiles, see left image in Figure 6.1 below. Note that this probability space is stationary merely with respect to discrete spatial shifts $\tau_x : \overline{\omega} \mapsto \overline{\omega}(\cdot + x)$ for $x \in \mathbb{Z}^d$. We may modify this construction by choosing the center of the tiles also randomly (e.g., w.r.t. the Lebesgue measure on the unit torus) in order to obtain a probability space which is stationary w.r.t. continuum spatial shifts. We detail this construction in the following. We consider the unit torus $\Box_\#$ and we define the following probability space*

$$(\Omega, \mathcal{F}, P) = \left(\Omega_0^{\mathbb{Z}^d} \otimes \Box_\#, \overline{\otimes_{\mathbb{Z}^d}\mathcal{F}_0 \otimes \mathcal{L}(\Box_\#)}, \otimes_{\mathbb{Z}^d}P_0 \otimes dy\right).$$

We see a realization $\omega \in \Omega$ as a pair (ω_1, y) where $\omega_1 : \mathbb{Z}^d \to \Omega_0$ and $y \in \Box_{\#}$. In this regard, we define a dynamical system, for $x \in \mathbb{R}^d$, $\tau_x : \Omega \to \Omega$ by

$$\tau_x \omega = (\omega_1(\cdot + \lfloor y + x \rfloor), y + x - \lfloor y + x \rfloor),$$

where $\lfloor x \rfloor \in \mathbb{Z}^d$ is the integer part of $x \in \mathbb{R}^d$. The above system defines an ergodic probability space satisfying Assumption 6.1. In this respect, we may define $A(\omega) = \omega_1(0)$ and in this case for a fixed realization $\omega = (\omega_1, y)$, $x \mapsto A(\tau_{\frac{x}{\varepsilon}}\omega) = \omega_1(\lfloor y + \frac{x}{\varepsilon} \rfloor)$ defines a piecewise constant and rapidly oscillating coefficient field, which corresponds to a checkerboard with independently colored tiles of size ε, i.e., the range of dependence is ε. On the other hand, we may define a coefficient field with help of $A(\omega) = \sum_{z \in \mathbb{Z}^d} \rho(-z)\omega_1(z)$ with $\rho \in L^1(\mathbb{Z}^d)$ (in this case we assume that Ω_0 is bounded). In this case, the associated oscillating coefficient field corresponds to a checkerboard where the coloring of the tiles is correlated, see right image in Figure 6.1.

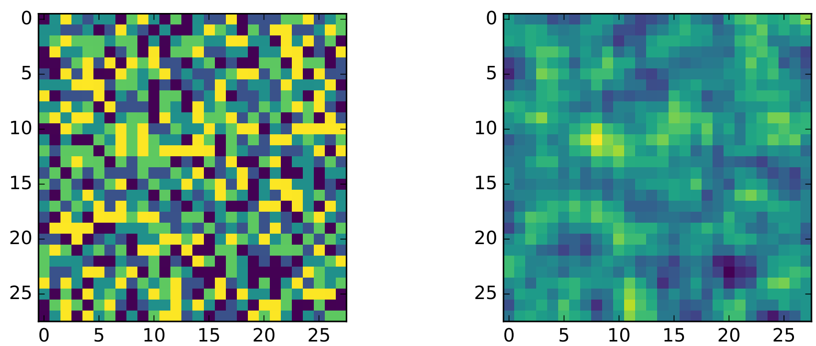


Figure 6.1: Random checkerboard. On the left side the tiles are i.i.d. colored, whereas on the right image the colors of the tiles are correlated.

6.2 Stochastic unfolding: definition and properties

In the following we introduce a key object in this study - the stochastic unfolding operator.

Lemma 6.7. *Let $\varepsilon > 0$, $1 < p < \infty$, $q = \frac{p}{p-1}$, and $Q \subset \mathbb{R}^d$ be open. There exists a unique linear isometric isomorphism*

$$\mathcal{T}_\varepsilon : L^p(\Omega \times Q) \to L^p(\Omega \times Q)$$

which satisfies

$$\text{for all } u \in L^p(\Omega) \overset{a}{\otimes} L^p(Q), \qquad (\mathcal{T}_\varepsilon u)(\omega, x) = u(\tau_{-\frac{x}{\varepsilon}}\omega, x) \qquad \text{a.e. in } \Omega \times Q.$$

Moreover, its adjoint is the unique linear isometric isomorphism $\mathcal{T}_\varepsilon^* : L^q(\Omega \times Q) \to L^q(\Omega \times Q)$ that satisfies $(\mathcal{T}_\varepsilon^* u)(\omega, x) = u(\tau_{\frac{x}{\varepsilon}}\omega, x)$ a.e. in $\Omega \times Q$ for all $u \in L^q(\Omega) \overset{a}{\otimes} L^q(Q)$.

Proof. We first define an operator $\mathcal{T}_\varepsilon : L^p(\Omega) \overset{a}{\otimes} L^p(Q) \to L^p(\Omega \times Q)$: For $u = \sum_i \varphi_i \eta_i \in L^p(\Omega) \overset{a}{\otimes} L^p(Q)$ with $\varphi_i \in L^p(\Omega)$ and $\eta_i \in L^p(Q)$, let

$$(\mathcal{T}_\varepsilon u)(\omega, x) = \sum_i (S\varphi_i)\left(\omega, -\frac{x}{\varepsilon}\right)\eta_i(x) = u(\tau_{-\frac{x}{\varepsilon}}\omega, x).$$

$\mathcal{T}_\varepsilon$ is a linear operator which is isometric by the following observation: For $u \in L^p(\Omega) \overset{a}{\otimes} L^p(Q)$,

$$\|\mathcal{T}_\varepsilon u\|^p_{L^p(\Omega \times Q)} = \int_Q \left\langle |u(\tau_{-\frac{x}{\varepsilon}}\omega, x)|^p \right\rangle dx = \int_Q \langle |u(\omega, x)|^p \rangle \, dx = \|u\|^p_{L^p(\Omega \times Q)},$$

where the first and last equality is Fubini's theorem and in the middle, for fixed a.a. $x \in Q$, we use a change of variables $\tau_{-\frac{x}{\varepsilon}}\omega \rightsquigarrow \omega$ and the P-preserving property of this transformation (Assumption (6.1) (ii)).

Since $L^p(\Omega) \overset{a}{\otimes} L^p(Q)$ is dense in $L^p(\Omega \times Q)$, $\mathcal{T}_\varepsilon$ extends to a linear isometry from $L^p(\Omega \times Q)$ to $L^p(\Omega \times Q)$. We define a linear isometry $\mathcal{T}_{-\varepsilon} : L^q(\Omega \times Q) \to L^q(\Omega \times Q)$ analogously as $\mathcal{T}_\varepsilon$, with ε replaced by $-\varepsilon$. Then for any $\varphi \in L^p(\Omega) \overset{a}{\otimes} L^p(Q)$ and $\psi \in L^q(\Omega) \overset{a}{\otimes} L^q(Q)$ we have (thanks to the measure preserving property of τ):

$$\begin{aligned}\left\langle \int_Q (\mathcal{T}_\varepsilon\varphi)\psi \, dx \right\rangle &= \int_Q \left\langle \varphi(\tau_{-\frac{x}{\varepsilon}}\omega, x)\psi(\omega, x) \right\rangle dx \\ &= \int_Q \left\langle \varphi(\omega, x)\psi(\tau_{\frac{x}{\varepsilon}}\omega, x) \right\rangle dx = \left\langle \int_Q \varphi(\mathcal{T}_{-\varepsilon}\psi) dx \right\rangle.\end{aligned}$$

Since $L^p(\Omega) \overset{a}{\otimes} L^p(Q)$ and $L^q(\Omega) \overset{a}{\otimes} L^q(Q)$ are dense in $L^p(\Omega \times Q)$ and $L^q(\Omega \times Q)$, respectively, we conclude that $\mathcal{T}_\varepsilon^* = \mathcal{T}_{-\varepsilon}$. Since $\mathcal{T}_\varepsilon^*$ is an isometry, it follows that $\mathcal{T}_\varepsilon$ is surjective (see [Bré11, Theorem 2.20]). Analogously, $\mathcal{T}_\varepsilon^*$ is also surjective. □

Definition 6.8 (Unfolding operator)**.** *The operator $\mathcal{T}_\varepsilon : L^p(\Omega\times Q) \to L^p(\Omega\times Q)$ given in Lemma 6.7 is called the stochastic unfolding operator.*

In this work a key notion of convergence is the convergence of unfolded sequence. In particular, for a sequence $u_\varepsilon \in L^p(\Omega \times Q)$, in most cases we consider the convergence of $\mathcal{T}_\varepsilon u_\varepsilon$ in either the weak or strong topology of $L^p(\Omega \times Q)$.

Remark 6.9 (A technical remark about measurability)**.** *We remark that an element $u \in L^p(\Omega\times Q)$ is an $\overline{\mathcal{F} \otimes \mathcal{L}(Q)}$-measurable function, i.e., measurable w.r.t. the $P \otimes dx$-completion of the product σ-algebra $\mathcal{F} \otimes \mathcal{L}(Q)$. On the other hand, the transformation $T_\varepsilon : (\omega, x) \mapsto (\tau_{-\frac{x}{\varepsilon}}\omega, x)$ is $(\mathcal{F} \otimes \mathcal{L}(Q), \mathcal{F} \otimes \mathcal{L}(Q))$-measurable. In this respect, a priori the composition $(u \circ T_\varepsilon)(\omega, x) = u(\tau_{-\frac{x}{\varepsilon}}\omega, x)$ does not necessarily define an $\overline{\mathcal{F} \otimes \mathcal{L}(Q)}$-measurable function. We avoid the (fruitless) discussion of such measurability issues by defining the unfolding operator on a dense subset of $L^p(\Omega \times Q)$ (where measurability is clear, cf. Lemma 6.2) and by extending it to the entire space.*

Remark 6.10 (Comparison with two-scale convergence in the mean from [BMW94])**.** *Let $p \in (1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. For a bounded sequence u_ε in $L^p(\Omega \times Q)$, we have*

$$\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup u \quad \text{weakly in } L^p(\Omega \times Q) \quad \Leftrightarrow \quad u_\varepsilon \overset{2}{\rightharpoonup} u,$$

where the convergence on the right-hand side is stochastic two-scale convergence in the mean from [BMW94], which means

$$\lim_{\varepsilon\to 0} \left\langle \int_Q u_\varepsilon(\omega, x)\varphi(\tau_{\frac{x}{\varepsilon}}\omega, x) dx \right\rangle = \left\langle \int_Q u(\omega, x)\varphi(\omega, x) dx \right\rangle, \tag{6.2}$$

for any $\varphi \in L^q(\Omega \times Q)$ that is admissible (in the sense that the transformation $(\omega, x) \mapsto \varphi(\tau_{\frac{x}{\varepsilon}}\omega, x)$ is well-defined). Indeed, with help of $\mathcal{T}_\varepsilon$, and its adjoint, we might rephrase the integral on the left-hand side in (6.2) as

$$\left\langle \int_Q u_\varepsilon(\mathcal{T}_\varepsilon^*\varphi) \, dx \right\rangle = \left\langle \int_Q (\mathcal{T}_\varepsilon u_\varepsilon)\varphi dx \right\rangle, \tag{6.3}$$

which proves the equivalence.

For the reason of the above equivalence and to keep the notation simple, we use the following notation

$$u_\varepsilon \overset{2}{\rightharpoonup} (\overset{2}{\to}) u \quad \text{in } L^p(\Omega \times Q) \quad :\Leftrightarrow \quad \mathcal{T}_\varepsilon u_\varepsilon \to u \quad \text{weakly (strongly) in } L^p(\Omega \times Q).$$

We also call this notion of convergence *weak* (*strong*) stochastic two-scale convergence in the mean, however, we remark that by this we always mean convergence of unfolded sequences.

The below lemma directly follows from the isometry property of $\mathcal{T}_\varepsilon$ and the usual properties of weak and strong convergence in $L^p(\Omega \times Q)$; therefore, we do not present its proof.

Lemma 6.11 (Basic properties). *Let $p \in (1, \infty)$, $q = \frac{p}{p-1}$ and $Q \subset \mathbb{R}^d$ be open. Consider sequences u_ε in $L^p(\Omega \times Q)$ and v_ε in $L^q(\Omega \times Q)$.*

(i) If $u_\varepsilon \overset{2}{\rightharpoonup} u$ in $L^p(\Omega \times Q)$, then $\sup_{\varepsilon \in (0,1)} \|u_\varepsilon\|_{L^p(\Omega \times Q)} < \infty$ and

$$\|u\|_{L^p(\Omega \times Q)} \leq \liminf_{\varepsilon \to 0} \|u_\varepsilon\|_{L^p(\Omega \times Q)}.$$

(ii) If $\limsup_{\varepsilon \to 0} \|u_\varepsilon\|_{L^p(\Omega \times Q)} < \infty$, then there exist a subsequence ε' and $u \in L^p(\Omega \times Q)$ such that $u_{\varepsilon'} \overset{2}{\rightharpoonup} u$ in $L^p(\Omega \times Q)$.

(iii) $u_\varepsilon \overset{2}{\to} u$ in $L^p(\Omega \times Q)$ if and only if $u_\varepsilon \overset{2}{\rightharpoonup} u$ in $L^p(\Omega \times Q)$ and $\|u_\varepsilon\|_{L^p(\Omega \times Q)} \to \|u\|_{L^p(\Omega \times Q)}$.

(iv) If $u_\varepsilon \overset{2}{\rightharpoonup} u$ in $L^p(\Omega \times Q)$ and $v_\varepsilon \overset{2}{\to} v$ in $L^q(\Omega \times Q)$, then

$$\left\langle \int_Q u_\varepsilon(\omega, x) v_\varepsilon(\omega, x) dx \right\rangle \to \left\langle \int_Q u(\omega, x) v(\omega, x) dx \right\rangle.$$

Remark 6.12. *The stochastic unfolding operator enjoys many similarities to the periodic unfolding operator, however, we would like to point out one considerable difference. Namely, in the periodic case if a sequence u_ε in $L^p(Q)$ satisfies $u_\varepsilon \to u$ strongly in $L^p(Q)$, it follows that $\mathcal{T}_\varepsilon u_\varepsilon \to u$ strongly in $L^p(Q \times \Box)$ (see, e.g., [MT07, Proposition 2.4]). On the other hand, this property does not translate to the stochastic case. In particular, even for a fixed function $u \in L^p(\Omega \times Q)$, in general it does not hold that $\mathcal{T}_\varepsilon u \rightharpoonup u$, cf. Remark 6.3. However, if $\langle \cdot \rangle$ is ergodic, using Proposition 6.14 below, it follows that for a bounded sequence u_ε in $L^p(\Omega) \otimes W^{1,p}(Q)$ such that $u_\varepsilon \rightharpoonup u$ weakly in $L^p(\Omega \times Q)$, it holds that $u_\varepsilon \overset{2}{\rightharpoonup} \langle u \rangle$. In this respect, stochastic two-scale convergence might be viewed as a weak von Neumann-type ergodic theorem for weakly convergent sequences of random fields (cf. Remark 6.3).*

Integral functionals and unfolding

For homogenization of variational problems, in particular problems driven by convex integral functionals, the following transformation and (lower semi-)continuity properties are very useful.

Proposition 6.13. *Let $p \in (1, \infty)$ and $Q \subset \mathbb{R}^d$ be open and bounded. Let $V : \Omega \times Q \times \mathbb{R}^m \to \mathbb{R}$ be such that $V(\cdot, \cdot, F)$ is $\mathcal{F} \otimes \mathcal{L}(Q)$-measurable for all $F \in \mathbb{R}^m$ and $V(\omega, x, \cdot)$ is continuous for a.a. $(\omega, x) \in \Omega \times Q$. Also, we assume that there exists $c > 0$ such that for a.a. $(\omega, x) \in \Omega \times Q$*

$$|V(\omega, x, F)| \leq c(1 + |F|^p), \quad \textit{for all } F \in \mathbb{R}^m.$$

(i) For all $u \in L^p(\Omega \times Q)^m$, we have

$$\left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, u(\omega, x))dx \right\rangle = \left\langle \int_Q V(\omega, x, \mathcal{T}_\varepsilon u(\omega, x))dx \right\rangle . \tag{6.4}$$

(ii) If $u_\varepsilon \xrightarrow{2} u$ in $L^p(\Omega \times Q)^m$, then

$$\lim_{\varepsilon \to 0} \left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, u_\varepsilon(\omega, x))dx \right\rangle = \left\langle \int_Q V(\omega, x, u(\omega, x))dx \right\rangle .$$

(iii) We additionally assume that for a.a. $(\omega, x) \in \Omega \times Q$, $V(\omega, x, \cdot)$ is convex. Then, if $u_\varepsilon \xrightharpoonup{2} u$ in $L^p(\Omega \times Q)^m$,

$$\liminf_{\varepsilon \to 0} \left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, u_\varepsilon(\omega, x))dx \right\rangle \geq \left\langle \int_Q V(\omega, x, u(\omega, x))dx \right\rangle .$$

Proof. We first note that V is a *Carathéodory integrand* in the sense of Remark A.4 (if necessary we tacitly redefine it by $V(\omega, x, \cdot) = 0$ for (ω, x) in a set of measure 0) and therefore it follows that V is a normal integrand (see Appendix A.2). For fixed $\varepsilon > 0$, the mapping $(\omega, x) \mapsto (\tau_{\frac{x}{\varepsilon}}\omega, x)$ is $(\mathcal{F} \otimes \mathcal{L}(Q), \mathcal{F} \otimes \mathcal{L}(Q))$-measurable and therefore $(\omega, x, F) \mapsto V(\tau_{\frac{x}{\varepsilon}}\omega, x, F)$ defines as well a Carathéodory and thus normal integrand. Hence, with the help of the growth condition, all the integrals in the statement of the proposition are well-defined.

(i) We first argue that it suffices to prove that

$$\left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, u(\omega, x))dx \right\rangle = \left\langle \int_Q V(\omega, x, \mathcal{T}_\varepsilon u(\omega, x))dx \right\rangle \quad \text{for all } u \in L^p(\Omega) \overset{a}{\otimes} L^p(Q)^m. \tag{6.5}$$

Indeed, for any $u \in L^p(\Omega \times Q)^m$ we can find a sequence $u_k \in L^p(\Omega) \overset{a}{\otimes} L^p(Q)^m$ such that $u_k \to u$ strongly in $L^p(\Omega \times Q)^m$, and by passing to a subsequence (not relabeled) we may additionally assume that $u_k \to u$ pointwise a.e. in $\Omega \times Q$. By continuity of V in its last variable, we thus have $V(\tau_{\frac{x}{\varepsilon}}\omega, x, u_k(\omega, x)) \to V(\tau_{\frac{x}{\varepsilon}}\omega, x, u(\omega, x))$ for a.a. $(\omega, x) \in \Omega \times Q$. Since $|V(\tau_{\frac{x}{\varepsilon}}\omega, x, u_k(\omega, x))| \leq c(1 + |u_k(\omega, x)|^p)$ a.e. in $\Omega \times Q$, the dominated convergence theorem ([Bog07, Theorem 2.8.8]) implies that $\lim_{k\to\infty} \left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, u_k(\omega, x))dx \right\rangle = \left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, u(\omega, x))dx \right\rangle$. In the same way we conclude that

$$\lim_{k \to \infty} \left\langle \int_Q V(\omega, x, \mathcal{T}_\varepsilon u_k(\omega, x))dx \right\rangle = \left\langle \int_Q V(\omega, x, \mathcal{T}_\varepsilon u(\omega, x))dx \right\rangle ,$$

and thus (6.5) extends to general $u \in L^p(\Omega \times Q)^m$.

It is left to show (6.5). Let $u \in L^p(\Omega) \overset{a}{\otimes} L^p(Q)^m$. By Fubini's theorem, the measure preserving property of τ, and by the transformation $\omega \mapsto \tau_{-\frac{x}{\varepsilon}}\omega$ in the second equality below, it follows

$$\left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, u(\omega, x))dx \right\rangle = \int_Q \left\langle V(\tau_{\frac{x}{\varepsilon}}\omega, x, u(\omega, x)) \right\rangle dx = \int_Q \left\langle V(\omega, x, u(\tau_{-\frac{x}{\varepsilon}}\omega, x)) \right\rangle dx.$$

Since $u \in L^p(\Omega) \overset{a}{\otimes} L^p(Q)$, we have $u(\tau_{-\frac{x}{\varepsilon}}\omega, x) = \mathcal{T}_\varepsilon u(\omega, x)$, and thus the proof of (i) is complete.

(ii) By part (i) we get $\left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, u_\varepsilon(\omega,x))dx \right\rangle = \left\langle \int_Q V(\omega, x, \mathcal{T}_\varepsilon u_\varepsilon(\omega,x))dx \right\rangle$. Since by assumption $\mathcal{T}_\varepsilon u_\varepsilon \to u$ strongly in $L^p(\Omega\times Q)^m$, using the growth conditions of V and the dominated convergence theorem, it follows, similarly as in part (i), that $\lim_{\varepsilon\to 0}\left\langle \int_Q V(\omega, x, \mathcal{T}_\varepsilon u_\varepsilon(\omega,x))dx \right\rangle = \left\langle \int_Q V(\omega, x, u(\omega,x))dx \right\rangle$.

(iii) We note that the functional $L^p(\Omega\times Q)^m \ni u \mapsto \left\langle \int_Q V(\omega,x,u(\omega,x))dx \right\rangle$ is convex and lower semi-continuous, therefore it is weakly lower semi-continuous (see [Bré11, Corollary 3.9]). Combining this fact with the transformation formula from (i) and the weak convergence $\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup u$ (by assumption), the claim follows. □

6.3 Two-scale limits of gradients

In the first part of this section we derive compactness results for sequences with bounded gradients. The second part is devoted to the construction of strong recovery sequences. The proofs are presented in the end, in Section 6.3.1.

Proposition 6.14 (Compactness). *Let $p \in (1,\infty)$ and $Q \subset \mathbb{R}^d$ be open. Let u_ε be a bounded sequence in $L^p(\Omega)\otimes W^{1,p}(Q)$. Then, there exist $u \in L^p_{\mathrm{inv}}(\Omega)\otimes W^{1,p}(Q)$ and $\chi \in L^p_{\mathrm{pot}}(\Omega)\otimes L^p(Q)$ such that, up to a subsequence,*

$$u_\varepsilon \overset{2}{\rightharpoonup} u \quad \text{in } L^p(\Omega\times Q), \qquad \nabla u_\varepsilon \overset{2}{\rightharpoonup} \nabla u + \chi \quad \text{in } L^p(\Omega\times Q)^d. \tag{6.6}$$

If, additionally, $\langle\cdot\rangle$ is ergodic, then $u = P_{\mathrm{inv}}u = \langle u\rangle \in W^{1,p}(Q)$ and $\langle u_\varepsilon\rangle \rightharpoonup u$ weakly in $W^{1,p}(Q)$.
(For the proof see Section 6.3.1.)

Remark 6.15. *Note that the proof of the above proposition reveals that $P_{\mathrm{inv}}u_\varepsilon \rightharpoonup u$ weakly in $L^p_{\mathrm{inv}}(\Omega)\otimes W^{1,p}(Q)$ (see Lemma 6.21). If we consider a closed subspace $X \subset W^{1,p}(Q)$ and assume that $u_\varepsilon(\omega) \in X$ P-a.e., then $P_{\mathrm{inv}}u_\varepsilon \in L^p_{\mathrm{inv}}(\Omega)\otimes X$. Therefore, it follows that $u \in L^p_{\mathrm{inv}}(\Omega)\otimes X$. This observation is useful if we consider boundary value problems, e.g., if $X = W^{1,p}_0(Q)$. We may argue similarly for closed convex subsets in $W^{1,p}(Q)$.*

Lemma 6.16 (Nonlinear recovery sequence). *Let $p, s \in (1,\infty)$ and $Q \subset \mathbb{R}^d$ be open. For $\chi \in L^p_{\mathrm{pot}}(\Omega)\otimes L^p(Q)$ and $\delta > 0$, there exists a sequence $g_{\delta,\varepsilon}(\chi) \in L^p(\Omega)\otimes W^{1,p}_0(Q)$ such that*

$$\|g_{\delta,\varepsilon}(\chi)\|_{L^s(\Omega\times Q)} \le \varepsilon c(\delta), \qquad \limsup_{\varepsilon\to 0}\|\mathcal{T}_\varepsilon \nabla g_{\delta,\varepsilon}(\chi) - \chi\|_{L^p(\Omega\times Q)^d} \le \delta, \tag{6.7}$$

where $c(\delta) > 0$ depends only on δ, χ, s and Q.
(For the proof see Section 6.3.1.)

We may extract a strongly converging diagonal sequence $g_{\delta(\varepsilon),\varepsilon}$ as in Remark 6.18, however, the doubly-indexed nonlinear approximation in (6.7) is also a useful tool for the linear construction in Proposition 6.17 and for time-dependent recovery sequences as in Lemma 9.9.

Proposition 6.17 (Gradient folding operator). *Let $p \in (1,\infty)$ and $Q \subset \mathbb{R}^d$ be open and bounded with C^1 boundary. For $\varepsilon > 0$ there exists a linear operator $\mathcal{G}_\varepsilon : L^p_{\mathrm{pot}}(\Omega)\otimes L^p(Q) \to L^p(\Omega)\otimes W^{1,p}_0(Q)$, that is uniformly bounded in ε, with the property that for any $\chi \in L^p_{\mathrm{pot}}(\Omega)\otimes L^p(Q)$, as $\varepsilon \to 0$,*

$$\mathcal{G}_\varepsilon \chi \overset{2}{\to} 0 \quad \text{in } L^p(\Omega\times Q), \qquad \nabla\mathcal{G}_\varepsilon\chi \overset{2}{\to} \chi \quad \text{in } L^p(\Omega\times Q)^d.$$

(For the proof see Section 6.3.1.)

Remark 6.18 (Strong recovery). *If $Q \subset \mathbb{R}^d$ is open, bounded and C^1, using Proposition 6.17, we obtain a mapping*

$$\left(L^p_{\mathrm{inv}}(\Omega) \otimes W^{1,p}(Q)\right) \times \left(L^p_{\mathrm{pot}}(\Omega) \otimes L^p(Q)\right) \ni (u,\chi) \mapsto u_\varepsilon(u,\chi) := u + \mathcal{G}_\varepsilon \chi \in L^p(\Omega) \otimes W^{1,p}(Q)$$

which is linear, uniformly bounded in ε and it satisfies, for all (u,χ),

$$u_\varepsilon(u,\chi) \xrightarrow{2} u \quad \text{in } L^p(\Omega \times Q), \quad \nabla u_\varepsilon(u,\chi) \xrightarrow{2} \nabla u + \chi \quad \text{in } L^p(\Omega \times Q)^d. \tag{6.8}$$

In the case that Q is merely open, we can use the nonlinear construction from Lemma 6.16. Specifically, for $(u,\chi) \in \left(L^p_{\mathrm{inv}}(\Omega) \otimes W^{1,p}(Q)\right) \times \left(L^p_{\mathrm{pot}}(\Omega) \otimes L^p(Q)\right)$ we define $u_{\delta,\varepsilon}(u,\chi) = u + g_{\delta,\varepsilon}(\chi)$ with $p = s$ from Lemma 6.16. Using Attouch's diagonalization lemma [Att84, Lemma 1.15 and Corollary 1.16], we find a sequence $u_\varepsilon(u,\chi) = u_{\delta(\varepsilon),\varepsilon}$ which satisfies (6.8). We remark that in both cases, the recovery sequence u_ε matches the boundary conditions of the function u (see constructions in Section 6.3.1).

6.3.1 Proofs

Before stating the proof of Proposition 6.14, we present some auxiliary lemmas.

Lemma 6.19. *Let $p \in (1,\infty)$ and $q = \frac{p}{p-1}$.*

(i) If $\varphi \in \left\{D^*\Psi : \Psi \in W^{1,q}(\Omega)^d\right\}^\perp$, then $\varphi \in L^p_{\mathrm{inv}}(\Omega)$.

(ii) If $\varphi \in \left\{\Psi \in W^{1,q}(\Omega)^d : D^*\Psi = 0\right\}^\perp$, then $\varphi \in L^p_{\mathrm{pot}}(\Omega)$.

Proof. (i) First, we note that

$$\varphi \in L^p_{\mathrm{inv}}(\Omega) \quad \Leftrightarrow \quad U_{he_i} U_y \varphi = U_y \varphi \quad \text{for all } y \in \mathbb{R}^d, h \in \mathbb{R}, i = 1,...,d.$$

We consider $\varphi \in \left\{D^*\Psi : \Psi \in W^{1,q}(\Omega)^d\right\}^\perp$ and we show that $\varphi \in L^p_{\mathrm{inv}}(\Omega)$ using the above equivalence. Let $\Psi = \psi e_i$ with $\psi \in W^{1,q}(\Omega)$ and $i \in \{1,...,d\}$, which implies $D^*\Psi = D^*_i \psi$. Then, by the group property we have $U_{-he_i}\psi - \psi = \int_0^h U_{-te_i} D_i^* \psi dt$ and therefore

$$\langle (U_{he_i}\varphi - \varphi)\psi \rangle = \langle \varphi(U_{-he_i}\psi - \psi) \rangle = \left\langle \varphi \int_0^h U_{-te_i} D_i^* \psi dt \right\rangle = \int_0^h \langle \varphi D_i^*(U_{-te_i}\psi) \rangle \, dt.$$

Since $U_{-te_i}\psi \in W^{1,q}(\Omega)$ for any $t \in [0,h]$, we obtain $\langle \varphi D_i^*(U_{-te_i}\psi)\rangle = \langle \varphi D^* U_{-te_i}\Psi \rangle = 0$ and thus $U_{he_i}\varphi = \varphi$. Furthermore, for any $y \in \mathbb{R}^d$, we have $\langle (U_{he_i}U_y\varphi - U_y\varphi)\psi\rangle = \langle (U_{he_i}\varphi - \varphi)U_{-y}\psi \rangle = 0$ by the same argument.

(ii) In view of $L^p_{\mathrm{pot}}(\Omega) = \ker(D^*)^\perp$ (see (6.1)), it is sufficient to prove that $\left\{\varphi \in W^{1,q}(\Omega)^d : D^*\varphi = 0\right\}$ is dense in $\ker(D^*)$. This follows by an approximation argument as in [JKO12], Section 7.2. Let $\varphi \in \ker(D^*)$ and we define for $t > 0$

$$\varphi^t(\omega) = \int_{\mathbb{R}^d} p_t(y)\varphi(\tau_y\omega)dy, \quad \text{where } p_t(y) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|y|^2}{4t}}.$$

Then the claimed density follows, since $\varphi^t \in W^{1,q}(\Omega)^d$, $D^*\varphi^t = 0$ for any $t > 0$ and $\varphi^t \to \varphi$ strongly in $L^q(\Omega)^d$ as $t \to 0$. The last statement can be seen as follows. By the continuity property of U_y, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\langle |\varphi(\tau_y\omega) - \varphi(\omega)|^q \rangle \le \varepsilon$ for any $y \in B_\delta(0)$. It follows that

$$\begin{aligned}
\langle |\varphi^t - \varphi|^q \rangle &= \left\langle \left| \int_{\mathbb{R}^d} p_t(y)\,(\varphi(\tau_y\omega) - \varphi(\omega))\,dy \right|^q \right\rangle \\
&\le \int_{\mathbb{R}^d} p_t(y)\,\langle |\varphi(\tau_y\omega) - \varphi(\omega)|^q \rangle\,dy \\
&= \int_{B_\delta} p_t(y)\,\langle |\varphi(\tau_y\omega) - \varphi(\omega)|^q \rangle\,dy + \int_{\mathbb{R}^d \setminus B_\delta} p_t(y)\,\langle |\varphi(\tau_y\omega) - \varphi(\omega)|^q \rangle\,dy.
\end{aligned}$$

The first term on the right-hand side of the above inequality is bounded by ε as well as the second term for sufficiently small $t > 0$. $\square$

Lemma 6.20. *Let $u_\varepsilon \in L^p(\Omega) \otimes W^{1,p}(Q)$ be such that $u_\varepsilon \overset{2}{\rightharpoonup} u$ in $L^p(\Omega \times Q)$ and $\varepsilon \nabla u_\varepsilon \overset{2}{\rightharpoonup} 0$ in $L^p(\Omega \times Q)^d$. Then $u \in L^p_{\text{inv}}(\Omega) \otimes L^p(Q)$.*

Proof. Consider a sequence $v_\varepsilon = \varepsilon \mathcal{T}_\varepsilon^*(\varphi\eta)$ such that $\varphi \in W^{1,q}(\Omega)$ and $\eta \in C_c^\infty(Q)$. Note that $\mathcal{T}_\varepsilon v_\varepsilon = \varepsilon\varphi\eta$ and we have, for $i = 1, ..., d$ and as $\varepsilon \to 0$,

$$\left\langle \int_Q \partial_i u_\varepsilon v_\varepsilon dx \right\rangle = \left\langle \int_Q \mathcal{T}_\varepsilon \partial_i u_\varepsilon \mathcal{T}_\varepsilon v_\varepsilon dx \right\rangle = \left\langle \int_Q \mathcal{T}_\varepsilon \partial_i u_\varepsilon \varepsilon\varphi\eta dx \right\rangle \to 0.$$

Moreover, it holds that $\partial_i v_\varepsilon = \mathcal{T}_\varepsilon^*(D_i\varphi\eta + \varepsilon\varphi\partial_i\eta)$ and therefore

$$\begin{aligned}
\left\langle \int_Q \partial_i u_\varepsilon v_\varepsilon dx \right\rangle &= -\left\langle \int_Q u_\varepsilon \partial_i v_\varepsilon dx \right\rangle = -\left\langle \int_Q u_\varepsilon \mathcal{T}_\varepsilon^*(D_i\varphi\eta + \varepsilon\varphi\partial_i\eta) dx \right\rangle \\
&= -\left\langle \int_Q \mathcal{T}_\varepsilon u_\varepsilon D_i\varphi\eta + \varepsilon \mathcal{T}_\varepsilon u_\varepsilon \varphi \partial_i\eta dx \right\rangle.
\end{aligned}$$

The last expression converges to $-\left\langle \int_Q u D_i\varphi\eta dx \right\rangle$ as $\varepsilon \to 0$. As a result of this, $\langle u(x) D_i\varphi \rangle = 0$ for almost every $x \in Q$ and therefore $u \in L^p_{\text{inv}}(\Omega) \otimes L^p(Q)$ by Lemma 6.19 (i). $\square$

Lemma 6.21. *Let u_ε be a bounded sequence in $L^p(\Omega) \otimes W^{1,p}(Q)$. Then there exists $u \in L^p_{\text{inv}}(\Omega) \otimes W^{1,p}(Q)$ such that (up to a subsequence)*

$$u_\varepsilon \overset{2}{\rightharpoonup} u \text{ in } L^p(\Omega \times Q), \quad P_{\text{inv}} u_\varepsilon \overset{2}{\rightharpoonup} u \text{ in } L^p(\Omega \times Q), \quad P_{\text{inv}} \nabla u_\varepsilon \overset{2}{\rightharpoonup} \nabla u \text{ in } L^p(\Omega \times Q)^d.$$

In particular, it holds that $P_{\text{inv}} u_\varepsilon \rightharpoonup u$ weakly in $L^p_{\text{inv}}(\Omega) \otimes W^{1,p}(Q)$.

Proof. Step 1. We show $P_{\text{inv}} \circ \mathcal{T}_\varepsilon = \mathcal{T}_\varepsilon \circ P_{\text{inv}} = P_{\text{inv}}$. The second equality holds clearly. To show that $P_{\text{inv}} \circ \mathcal{T}_\varepsilon = P_{\text{inv}}$, we consider $v \in L^p(\Omega \times Q)$, $\varphi \in L^q(\Omega)$ and $\eta \in L^q(Q)$. We have

$$\left\langle \int_Q (P_{\text{inv}} \mathcal{T}_\varepsilon v)(\varphi\eta) dx \right\rangle = \left\langle \int_Q (\mathcal{T}_\varepsilon v) P^*_{\text{inv}}(\varphi\eta) dx \right\rangle = \left\langle \int_Q v P^*_{\text{inv}}(\varphi\eta) dx \right\rangle = \left\langle \int_Q (P_{\text{inv}} v)(\varphi\eta) dx \right\rangle,$$

where we use the fact that $\mathcal{T}_\varepsilon^* P^*_{\text{inv}} = P^*_{\text{inv}}$ since the adjoint P^*_{inv} of P_{inv} satisfies $\text{ran}(P^*_{\text{inv}}) \subset L^q_{\text{inv}}(\Omega)$. The claim follows by an approximation argument since $L^q(\Omega) \overset{a}{\otimes} L^q(Q)$ is dense in $L^q(\Omega \times Q)$.

Step 2. Convergence of $P_{\text{inv}}u_\varepsilon$. P_{inv} is bounded and it commutes with ∇, and therefore

$$\limsup_{\varepsilon\to 0}\left\langle \int_Q |P_{\text{inv}}u_\varepsilon|^p + |\nabla P_{\text{inv}}u_\varepsilon|^p dx\right\rangle < \infty.$$

As a result of this and with help of Lemma 6.11 (ii) and Lemma 6.20, it follows that $P_{\text{inv}}u_\varepsilon \overset{2}{\rightharpoonup} v$ and $\nabla P_{\text{inv}}u_\varepsilon \overset{2}{\rightharpoonup} w$ (up to a subsequence), where $v\in L^p_{\text{inv}}(\Omega)\otimes L^p(Q)$ and $w\in L^p_{\text{inv}}(\Omega)\otimes L^p(Q)^d$. Let $\varphi\in W^{1,q}(\Omega)$ and $\eta\in C^\infty_c(Q)$. On the one hand, we have, as $\varepsilon\to 0$,

$$\left\langle \int_Q (\partial_i P_{\text{inv}}u_\varepsilon)\mathcal{T}^*_\varepsilon(\varphi\eta)dx\right\rangle = \left\langle \int_Q \mathcal{T}_\varepsilon(\partial_i P_{\text{inv}}u_\varepsilon)(\varphi\eta)dx\right\rangle \to \left\langle \int_Q w_i\varphi\eta dx\right\rangle.$$

On the other hand,

$$\left\langle \int_Q (\partial_i P_{\text{inv}}u_\varepsilon)\mathcal{T}^*_\varepsilon(\varphi\eta)dx\right\rangle = -\frac{1}{\varepsilon}\left\langle \int_Q (P_{\text{inv}}u_\varepsilon)(D_i\varphi\eta)dx\right\rangle - \left\langle \int_Q (P_{\text{inv}}u_\varepsilon)(\varphi\partial_i\eta)dx\right\rangle.$$

The first term on the right-hand side vanishes since $P_{\text{inv}}u_\varepsilon(\cdot,x)\in L^p_{\text{inv}}(\Omega)$ for almost every $x\in Q$ and by (6.1). The second term converges to $-\left\langle \int_Q v\varphi\partial_i\eta dx\right\rangle$ as $\varepsilon\to 0$. Consequently, we obtain $w=\nabla v$ and therefore $v\in L^p_{\text{inv}}(\Omega)\otimes W^{1,p}(Q)$. Moreover, using Step 1, we have $P_{\text{inv}}u_\varepsilon \rightharpoonup u$ weakly in $L^p_{\text{inv}}(\Omega)\otimes W^{1,p}(Q)$.

Step 3. Convergence of u_ε. Since u_ε is bounded, by Lemma 6.11 (ii) and Lemma 6.20 there exists $u\in L^p_{\text{inv}}(\Omega)\otimes L^p(Q)$ such that $u_\varepsilon \overset{2}{\rightharpoonup} u$ in $L^p(\Omega\times Q)$. Also, P_{inv} is a linear and bounded operator which, together with Step 1, implies that $P_{\text{inv}}u_\varepsilon \rightharpoonup u$. Using this, we conclude that $u=v$. □

Proof of Proposition 6.14. Lemma 6.21 implies that $u_\varepsilon \overset{2}{\rightharpoonup} u$ in $L^p(\Omega\times Q)$ (up to a subsequence), where $u\in L^p_{\text{inv}}(\Omega)\otimes W^{1,p}(Q)$. Moreover, it follows that there exists $v\in L^p(\Omega\times Q)^d$ such that $\nabla u_\varepsilon \overset{2}{\rightharpoonup} v$ in $L^p(\Omega\times Q)^d$ (up to another subsequence). We show that $\chi := v-\nabla u\in L^p_{\text{pot}}(\Omega)\otimes L^p(Q)$. Let $\varphi\in W^{1,q}(\Omega)^d$ with $D^*\varphi=0$ and $\eta\in C^\infty_c(Q)$. We have, as $\varepsilon\to 0$,

$$\left\langle \int_Q \nabla u_\varepsilon\cdot\mathcal{T}^*_\varepsilon(\varphi\eta)dx\right\rangle = \left\langle \int_Q \mathcal{T}_\varepsilon\nabla u_\varepsilon\cdot\varphi\eta dx\right\rangle \to \left\langle \int_Q v\cdot\varphi\eta dx\right\rangle. \tag{6.9}$$

On the other hand,

$$\begin{aligned}\left\langle \int_Q \nabla u_\varepsilon\cdot\mathcal{T}^*_\varepsilon(\varphi\eta)dx\right\rangle &= -\left\langle \int_Q u_\varepsilon\sum_{i=1}^d \mathcal{T}^*_\varepsilon(\frac{1}{\varepsilon}D_i\varphi\eta + \varphi_i\partial_i\eta)dx\right\rangle \\ &= \frac{1}{\varepsilon}\left\langle \int_Q (\mathcal{T}_\varepsilon u_\varepsilon)(D^*\varphi\eta)dx\right\rangle - \left\langle \int_Q (\mathcal{T}_\varepsilon u_\varepsilon)\sum_{i=1}^d \varphi_i\partial_i\eta dx\right\rangle.\end{aligned} \tag{6.10}$$

Above, the first term on the right-hand side vanishes by assumption and the second converges to $\left\langle \int_Q \nabla u\cdot\varphi\eta\right\rangle$ as $\varepsilon\to 0$. Using (6.10), (6.9) and Lemma 6.19 (ii) we complete the proof. □

Proof of Lemma 6.16. For $\chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(Q)$ and $\delta > 0$, by definition of the space $L^p_{\text{pot}}(\Omega) \otimes L^p(Q)$ and by density of $\text{ran}(D)$ in $L^p_{\text{pot}}(\Omega)$, we find $g_\delta = \sum_{i=1}^{n(\delta)} \varphi_i^\delta \eta_i^\delta$ with $\varphi_i^\delta \in W^{1,p}(\Omega)$ and $\eta_i^\delta \in C_c^\infty(Q)$ such that

$$\|\chi - Dg_\delta\|_{L^p(\Omega \times Q)^d} \leq \delta.$$

Note that we can choose φ_i^δ above so that $\varphi_i^\delta \in L^s(\Omega)$. This can be seen by a standard truncation and mollification argument (see [BMW94, Lemma 2.2] for the L^2-case) that we present here for the convenience of the reader. For a given $\varphi \in W^{1,p}(\Omega)$, by density of $L^\infty(\Omega)$ in $L^p(\Omega)$, we find a sequence $\varphi_k \in L^\infty(\Omega)$ such that $\varphi_k \to \varphi$ in $L^p(\Omega)$. For a sequence of standard mollifiers $\rho_n \in C_c^\infty(\mathbb{R}^d)$, $\rho_n \geq 0$, we define

$$\varphi_k^n = \int_{\mathbb{R}^d} \rho_n(y) U_y \varphi_k dy, \quad \varphi^n = \int_{\mathbb{R}^d} \rho_n(y) U_y \varphi dy.$$

It holds that $\varphi_k^n \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$, $D_i \varphi_k^n = \int_{\mathbb{R}^d} -\partial_i \rho_n(y) U_y \varphi_k dy$ and $D_i \varphi^n = \int_{\mathbb{R}^d} -\partial_i \rho_n(y) U_y \varphi dy = \int_{\mathbb{R}^d} \rho_n(y) U_y D_i \varphi dy$. Similarly as in the proof of Lemma 6.19 (ii), it follows that $D\varphi^n \to D\varphi$ in $L^p(\Omega)^d$ as $n \to \infty$. In the following we show that for fixed $n \in \mathbb{N}$, $D_i \varphi_k^n \to D_i \varphi^n$ in $L^p(\Omega)$ as $k \to \infty$, which yields the claim (up to extraction of a subsequence $k(n)$). We have

$$\langle |D_i \varphi_k^n - D_i \varphi^n|^p \rangle = \left\langle \Big| \int_{\mathbb{R}^d} -\partial_i \rho_n(y) \left(U_y \varphi_k - U_y \varphi \right) dy \Big|^p \right\rangle \leq c(n) \left\langle |\varphi_k - \varphi|^p \right\rangle \to 0 \quad \text{as } k \to \infty,$$

where in the last inequality we use that $\partial_i \rho_n$ is compactly supported and L^∞, and Jensen's inequality. This means that in the definition of g_δ above, we can choose $\varphi_i^\delta \in L^s(\Omega) \cap W^{1,p}(\Omega)$.

We define $g_{\delta,\varepsilon} = \varepsilon \mathcal{T}_\varepsilon^{-1} g_\delta$ and note that $g_{\delta,\varepsilon} \in L^p(\Omega) \otimes W_0^{1,p}(Q) \cap L^s(\Omega \times Q)$ and $\nabla g_{\delta,\varepsilon} = \mathcal{T}_\varepsilon^{-1} D g_\delta + \mathcal{T}_\varepsilon^{-1} \varepsilon \nabla g_\delta$. As a result of this and with help of the isometry property of $\mathcal{T}_\varepsilon^{-1}$, the claim of the lemma follows. □

Proof of Proposition 6.17. For $\chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(Q)$ we define $\mathcal{G}_\varepsilon \chi = v_\varepsilon$ as the unique weak solution in $W_0^{1,p}(Q)$ to the equation (for P-a.a. $\omega \in \Omega$)

$$-\Delta v_\varepsilon(\omega) = -\nabla \cdot (\mathcal{T}_\varepsilon^{-1} \chi(\omega)). \tag{6.11}$$

Above and further in this proof, we use the notation $u(\omega) := u(\omega, \cdot) \in L^p(Q)$ for functions $u \in L^p(\Omega \times Q)$. By the Poincaré inequality and the Calderón-Zygmund estimate, we obtain

$$\|v_\varepsilon(\omega)\|_{L^p(Q)} \leq c \|\nabla v_\varepsilon(\omega)\|_{L^p(Q)^d} \leq c \|\mathcal{T}_\varepsilon^{-1} \chi(\omega)\|_{L^p(Q)^d},$$

and therefore

$$\|v_\varepsilon\|_{L^p(\Omega \times Q)} \leq c \|\nabla v_\varepsilon\|_{L^p(\Omega \times Q)^d} \leq c \|\chi\|_{L^p(\Omega \times Q)^d}.$$

Using Lemma 6.16 with $p = s$, we find a sequence $g_{\delta,\varepsilon} \in L^p(\Omega) \otimes W_0^{1,p}(Q)$ such that

$$\|g_{\delta,\varepsilon}(\chi)\|_{L^p(\Omega \times Q)} \leq \varepsilon c(\delta), \quad \limsup_{\varepsilon \to 0} \|\mathcal{T}_\varepsilon \nabla g_{\delta,\varepsilon}(\chi) - \chi\|_{L^p(\Omega \times Q)^d} \leq \delta.$$

Note that $v_\varepsilon(\omega) - g_{\delta,\varepsilon}(\omega) \in W_0^{1,p}(Q)$ (for P-a.a. $\omega \in \Omega$) and it is the unique weak solution to

$$-\Delta (v_\varepsilon(\omega) - g_{\delta,\varepsilon}(\omega)) = -\nabla \cdot (\mathcal{T}_\varepsilon^{-1} \chi(\omega) - \nabla g_{\delta,\varepsilon}(\omega)).$$

As before, we have

$$\|v_\varepsilon - g_{\delta,\varepsilon}\|_{L^p(\Omega\times Q)} \le c\|\nabla v_\varepsilon - \nabla g_{\delta,\varepsilon}\|_{L^p(\Omega\times Q)^d} \le c\|\chi - \mathcal{T}_\varepsilon \nabla g_{\delta,\varepsilon}\|_{L^p(\Omega\times Q)^d}. \tag{6.12}$$

Therefore, using the isometry property of $\mathcal{T}_\varepsilon$, we obtain

$$\begin{aligned}\|\mathcal{T}_\varepsilon \nabla v_\varepsilon - \chi\|_{L^p(\Omega\times Q)^d} &\le \|\nabla v_\varepsilon - \nabla g_{\delta,\varepsilon}\|_{L^p(\Omega\times Q)^d} + \|\mathcal{T}_\varepsilon \nabla g_{\delta,\varepsilon} - \chi\|_{L^p(\Omega\times Q)^d} \\ &\le c\|\chi - \mathcal{T}_\varepsilon \nabla g_{\delta,\varepsilon}\|_{L^p(\Omega\times Q)^d}.\end{aligned}$$

Consequently, first letting $\varepsilon \to 0$ and then $\delta \to 0$ we obtain that $\nabla v_\varepsilon \overset{2}{\to} \chi$ in $L^p(\Omega \times Q)^d$. Furthermore, using (6.12) we obtain that $v_\varepsilon \overset{2}{\to} 0$ in $L^p(\Omega \times Q)$ which completes the proof. □

7 Unfolding method and general remarks

In this section we present some general properties of the stochastic unfolding method. To keep the discussion uncluttered, we discuss the continuum setting, however, most of the statements analogously hold in the discrete case. In Section 7.1 we explain the stochastic unfolding procedure on a standard example of convex minimization. Also, we discuss practical approximation schemes for the effective problems in Section 7.2. In Section 7.3 we shortly discuss mean and quenched formulations for equations with random coefficients. Finally, in Section 7.4 we discuss the implications of the stochastic unfolding procedure in the particular periodic setting.

7.1 Homogenization of convex minimization

In the following we consider stochastic homogenization of convex integral functionals in the continuum setting. We refer to Section 8.1 for a similar treatment of the discrete case in the setting of elasticity.

Let $(\Omega, \mathcal{F}, P, \tau)$ be a probability space that satisfies Assumption 6.1. Let $p \in (1,\infty)$ and $Q \subset \mathbb{R}^d$ be open and bounded. We consider an integrand $V : \Omega \times Q \times \mathbb{R}^d \to \mathbb{R}$ and the following set of assumptions:

(A1) $V(\cdot,\cdot,F)$ is $\mathcal{F} \otimes \mathcal{L}(Q)$-measurable for all $F \in \mathbb{R}^d$.

(A2) $V(\omega, x, \cdot)$ is convex for a.a. $(\omega, x) \in \Omega \times Q$.

(A3) There exists $c > 0$ such that

$$\frac{1}{c}|F|^p - c \le V(\omega,x,F) \le c(|F|^p + 1)$$

for a.a. $(\omega, x) \in \Omega \times Q$ and all $F \in \mathbb{R}^d$.

We consider an energy functional $\mathcal{E}_\varepsilon : L^p(\Omega) \otimes W_0^{1,p}(Q) \to \mathbb{R}$ given by

$$\mathcal{E}_\varepsilon(u) = \left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, \nabla u(\omega, x))dx \right\rangle. \tag{7.1}$$

As $\varepsilon \to 0$, we derive an effective functional $\mathcal{E}_0 : (L^p_{\text{inv}}(\Omega) \otimes W_0^{1,p}(Q)) \times \left(L^p_{\text{pot}}(\Omega) \otimes L^p(Q)\right) \to \mathbb{R}$,

$$\mathcal{E}_0(u, \chi) = \left\langle \int_Q V(\omega, x, \nabla u(\omega, x) + \chi(\omega, x))dx \right\rangle. \tag{7.2}$$

Note that this functional features a new corrector variable χ and therefore we refer to $\mathcal{E}_0$ as the *two-scale* effective energy.

Theorem 7.1 (Two-scale homogenization)**.** *Let $p \in (1, \infty)$ and $Q \subset \mathbb{R}^d$ be open and bounded. Assume (A1)-(A3).*

(i) (Compactness) Let $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)$ be such that $\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) < \infty$. There exist $(u, \chi) \in (L^p_{\text{inv}}(\Omega) \otimes W_0^{1,p}(Q)) \times \left(L^p_{\text{pot}}(\Omega) \otimes L^p(Q)\right)$ and a (not relabeled) subsequence such that

$$u_\varepsilon \overset{2}{\rightharpoonup} u \quad \text{in } L^p(\Omega \times Q), \quad \nabla u_\varepsilon \overset{2}{\rightharpoonup} \nabla u + \chi \quad \text{in } L^p(\Omega \times Q)^d. \tag{7.3}$$

(ii) (Liminf inequality) If (7.3) holds for the entire sequence, then

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}_0(u, \chi). \tag{7.4}$$

(iii) (Limsup inequality) Let $(u, \chi) \in (L^p_{\text{inv}}(\Omega) \otimes W_0^{1,p}(Q)) \times \left(L^p_{\text{pot}}(\Omega) \otimes L^p(Q)\right)$. There exists a sequence $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)$ such that

$$u_\varepsilon \overset{2}{\to} u \quad \text{in } L^p(\Omega \times Q), \quad \nabla u_\varepsilon \overset{2}{\to} \nabla u + \chi \quad \text{in } L^p(\Omega \times Q)^d, \quad \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}_0(u, \chi). \tag{7.5}$$

Proof. (i) The Poincaré inequality and (A3) imply that u_ε is bounded in $L^p(\Omega) \otimes W^{1,p}(Q)$. By Proposition 6.14 there exist $u \in L^p_{\text{inv}}(\Omega) \otimes W^{1,p}(Q)$ and $\chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(Q)$ such that (7.3) holds. Since $P_{\text{inv}} u_\varepsilon \in L^p_{\text{inv}}(\Omega) \otimes W_0^{1,p}(Q)$ and using that $P_{\text{inv}} u_\varepsilon \rightharpoonup u$ in $L^p_{\text{inv}}(\Omega) \otimes W_0^{1,p}(Q)$ (see Lemma 6.21), we conclude that $u \in L^p_{\text{inv}}(\Omega) \otimes W_0^{1,p}(Q)$ (cf. Remark 6.15).
(ii) The claim follows from Proposition 6.13 (iii).
(iii) The existence of a strongly two-scale convergent sequence $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)$ follows from Remark 6.18. Moreover, $\mathcal{E}_\varepsilon(u_\varepsilon) \to \mathcal{E}_0(u, \chi)$ follows from Proposition 6.13 (ii). □

Corollary 7.2 (Convergence for minimizers)**.** *Let the assumptions of Theorem 7.1 be satisfied. Let $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)$ be a minimizer of $\mathcal{E}_\varepsilon$. Then, there exist a (not relabeled) subsequence, $u \in L^p_{\text{inv}}(\Omega) \otimes W_0^{1,p}(Q)$ and $\chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(Q)$ such that*

$$\begin{aligned} &u_\varepsilon \overset{2}{\rightharpoonup} u \quad \text{in } L^p(\Omega \times Q), \quad \nabla u_\varepsilon \overset{2}{\rightharpoonup} \nabla u + \chi \quad \text{in } L^p(\Omega \times Q)^d, \\ &\lim_{\varepsilon\to 0} \min \mathcal{E}_\varepsilon = \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}_0(u, \chi) = \min \mathcal{E}_0. \end{aligned} \tag{7.6}$$

Proof. The statement follows by a standard argument from Γ-convergence: Since u_ε is a minimizer, we conclude that $\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u_\varepsilon) \le \limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(0) < \infty$. Hence, by Theorem 7.1 there exist $u \in L^p_{\text{inv}}(\Omega)\otimes W^{1,p}_0(Q)$ and $\chi \in L^p_{\text{pot}}(\Omega)\otimes L^p(Q)$ (and a subsequence) such that (7.6) holds and

$$\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u_\varepsilon) \ge \mathcal{E}_0(u,\chi).$$

Let $(\widetilde{u},\widetilde{\chi}) \in (L^p_{\text{inv}}(\Omega)\otimes W^{1,p}_0(Q)) \times \left(L^p_{\text{pot}}(\Omega)\otimes L^p(Q)\right)$ be arbitrary. Then by Theorem 7.1 (iii) there exists a recovery sequence v_ε such that $\mathcal{E}_\varepsilon(v_\varepsilon)\to\mathcal{E}_0(\widetilde{u},\widetilde{\chi})$, and thus

$$\mathcal{E}_0(\widetilde{u},\widetilde{\chi}) = \lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(v_\varepsilon) \ge \liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u_\varepsilon) = \liminf_{\varepsilon\to 0}\min\mathcal{E}_\varepsilon \ge \mathcal{E}_0(u,\chi).$$

This means that (u,χ) minimizes $\mathcal{E}_0$. Setting $(\widetilde{u},\widetilde{\chi}) = (u,\chi)$ yields $\mathcal{E}_\varepsilon(u_\varepsilon) = \min\mathcal{E}_\varepsilon \to \min\mathcal{E}_0 = \mathcal{E}_0(u,\chi)$. $\square$

Remark 7.3 (Convergence for the entire sequence)**.** *If $V(\omega,x,\cdot)$ is strictly convex, the minimizer of $\mathcal{E}_0$ is unique and the convergence in the above corollary holds for the entire sequence.*

Remark 7.4 (Continuous perturbations)**.** *We might consider the perturbed energy functional $\mathcal{I}_\varepsilon(\cdot) = \mathcal{E}_\varepsilon(\cdot) + \langle l_\varepsilon,\cdot\rangle_{L^q,L^p}$ with $l_\varepsilon \xrightarrow{2} l$ in $L^q(\Omega\times Q)$, where $q=\frac{p}{p-1}$. As in Corollary 7.2, minimizers of $\mathcal{I}_\varepsilon$ converge as in (7.6) to minimizers of $(u,\chi)\mapsto \mathcal{I}_0(u,\chi) := \mathcal{E}_0(u,\chi) + \langle P_{\text{inv}}l,u\rangle_{L^q,L^p}$.*

If we additionally assume that $\langle\cdot\rangle$ is ergodic, the limit functional reduces to a *single-scale* energy

$$\mathcal{E}_{\text{hom}} : W^{1,p}_0(Q)\to\mathbb{R},\quad \mathcal{E}_{\text{hom}}(u) = \int_Q V_{\text{hom}}(x,\nabla u(x))dx,$$

where the homogenized integrand V_{hom} is given by

$$V_{\text{hom}}(x,F) = \inf_{\chi\in L^p_{\text{pot}}(\Omega)} \langle V(\omega,x,F+\chi(\omega))\rangle\,,\quad \text{for } x\in\mathbb{R}^d \text{ and } F\in\mathbb{R}^d. \tag{7.7}$$

Remark 7.5 (Quadratic case)**.** *If V has a quadratic structure, V_{hom} as well admits a quadratic form. Namely, we set $V(\omega,x,F) = A(\omega)F\cdot F$ with $A\in L^\infty(\Omega,\mathbb{R}^{d\times d}_{\text{sym}})$ such that $A(\omega)F\cdot F\ge c|F|^2$ for P-a.a. ω and all $F\in\mathbb{R}^d$. Then the homogenized integrand V_{hom} boils down to $A_{\text{hom}}F\cdot F$ where $A_{\text{hom}}\in\mathbb{R}^{d\times d}_{\text{sym}}$ is given by*

$$A^{ij}_{\text{hom}} = \langle A(\omega)\,(e_i+\chi_i(\omega))\cdot e_j\rangle\,,$$

where $\chi_i\in L^2_{\text{pot}}(\Omega)$ is the solution to the usual corrector equation

$$\langle A(\omega)\,(e_i+\chi_i(\omega))\cdot\widetilde{\chi}(\omega)\rangle = 0 \quad \textit{for all } \widetilde{\chi}\in L^2_{\text{pot}}(\Omega).$$

Theorem 7.6 (Ergodic case)**.** *Let the assumptions of Theorem 7.1 be in effect and $\langle\cdot\rangle$ be ergodic.*

(i) Let $u_\varepsilon\in L^p(\Omega)\otimes W^{1,p}_0(Q)$ be such that $\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u_\varepsilon)<\infty$. There exist $u\in W^{1,p}_0(Q)$ and a (not relabeled) subsequence such that

$$u_\varepsilon \overset{2}{\rightharpoonup} u \quad \text{in } L^p(\Omega\times Q),\quad \liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u_\varepsilon)\ge\mathcal{E}_{\text{hom}}(u). \tag{7.8}$$

Moreover, $\langle u_\varepsilon\rangle \rightharpoonup u$ weakly in $W^{1,p}(Q)$.

(ii) Let $u \in W_0^{1,p}(Q)$. There exists a sequence $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)$ such that

$$u_\varepsilon \xrightarrow{2} u \quad \text{in } L^p(\Omega \times Q), \quad \langle u_\varepsilon \rangle \to u \quad \text{strongly in } W^{1,p}(Q), \quad \lim_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}_{\text{hom}}(u).$$

Proof. (i) According to Theorem 7.1 (i) and (ii) there exist $u \in W_0^{1,p}(Q)$ and $\chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(Q)$ such that u_ε satisfies (7.3)-(7.4), up to a subsequence. Hence, it holds

$$\liminf_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}_0(u, \chi) \geq \mathcal{E}_{\text{hom}}(u).$$

Moreover, the convergence for $\langle u_\varepsilon \rangle$ follows by Proposition 6.14.
(ii) We notice that it is sufficient to show that for fixed $u \in W_0^{1,p}(Q)$, there exists $\chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(Q)$ such that

$$\mathcal{E}_0(u, \chi) = \mathcal{E}_{\text{hom}}(u).$$

Indeed, this implies the claim by the application of Theorem 7.1 (iii) for (u, χ) (strong convergence for $\langle u_\varepsilon \rangle$ follows by construction in Remark 6.18).
To show the above claim we apply a measurable selection argument (see Appendix A.2). First, we define an integrand $f : Q \times L^p_{\text{pot}}(\Omega) \to \mathbb{R}$, $f(x, \chi) = \langle V(\omega, x, \nabla u(x) + \chi(\omega)) \rangle$. This integrand is finite everywhere and for fixed x, $f(x, \cdot)$ is continuous, which follows using the growth conditions of V by a standard Fatou lemma argument, cf. proof of Proposition 6.13. We fix $\chi \in L^p_{\text{pot}}(\Omega)$. The integrand V is a Carathéodory integrand as defined in Remark A.4 (if necessary, we tacitly redefine it by $V(\omega, x, \cdot) = 0$ on a set $\widetilde{\Omega} \times \widetilde{Q}$ of measure 0). As a result of this, the mapping $(\omega, x) \mapsto V(\omega, x, \nabla u(x) + \chi(\omega))$ is integrable, for which we may use the growth assumptions of V. Fubini's theorem implies that $x \mapsto \langle V(\omega, x, \nabla u(x) + \chi(\omega)) \rangle = f(x, \chi)$ is $\mathcal{L}(Q)$-measurable. The above statements imply that f is a Carathéodory integrand, which is also convex and therefore it is a convex normal integrand (Remark A.4). As a result of this, Proposition A.7 (see Remark A.8) implies

$$\mathcal{E}_{\text{hom}}(u) = \int_Q \inf_{\chi \in L^p_{\text{pot}}(\Omega)} f(x, \chi) dx = \inf_{L^p_{\text{pot}}(\Omega) \otimes L^p(Q)} \int_Q f(x, \chi(x)) dx.$$

The infimum on the right-hand side is in fact a minimum, that can be obtained by the direct method of calculus of variations using the convexity and growth assumptions of V, and therefore the claim follows. □

We consider problems with an additional strong convexity assumption and consequently obtain that the whole sequence of unique minimizers of $\mathcal{E}_\varepsilon$ converges strongly in the usual strong topology of $L^p(\Omega \times Q)$ to the unique minimizer of $\mathcal{E}_{\text{hom}}$.

(A4) For a.a. $(\omega, x) \in \Omega \times Q$, $V(\omega, x, \cdot)$ is uniformly convex with modulus $(\cdot)^p$, i.e., there exists $c > 0$ independent of ω and x, such that for all $F, G \in \mathbb{R}^d$ and $t \in [0,1]$

$$V(\omega, x, tF + (1-t)G) \leq tV(\omega, x, F) + (1-t)V(\omega, x, G) - (1-t)tc|F - G|^p.$$

Proposition 7.7 (Strong convergence)**.** *Let the assumptions of Theorem 7.6 and (A4) hold. $\mathcal{E}_\varepsilon$ and $\mathcal{E}_{\text{hom}}$ admit unique minimizers $u_\varepsilon \in L^p(\Omega) \otimes W_0^{1,p}(Q)$ and $u \in W_0^{1,p}(Q)$, respectively. We have*

$$u_\varepsilon \to u \quad \text{strongly in } L^p(\Omega \times Q), \quad \nabla u_\varepsilon \xrightarrow{2} \nabla u + \chi \quad \text{in } L^p(\Omega \times Q)^d,$$

where $\chi \in L^p_{\text{pot}}(\Omega) \otimes L^p(Q)$ is uniquely characterized by

$$V_{\text{hom}}(x, \nabla u(x)) = \langle V(\omega, x, \nabla u(x) + \chi(\omega, x)) \rangle \quad \text{for a.a. } x \in Q. \tag{7.9}$$

Proof. The uniqueness of minimizers follows by the uniform convexity assumption on the integrand V. Corollary 7.2 implies that there exists $\chi \in L^p_{\mathrm{pot}}(\Omega) \otimes L^p(Q)$ and a subsequence such that $u_\varepsilon \overset{2}{\rightharpoonup} u$, $\nabla u_\varepsilon \overset{2}{\rightharpoonup} \nabla u + \chi$ and $\lim_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}_0(u, \chi)$. Moreover, analogously as in the proof of Corollary 7.2, Theorem 7.6 implies

$$\lim_{\varepsilon \to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}_{\mathrm{hom}}(u).$$

This results in $\mathcal{E}_{\mathrm{hom}}(u) = \mathcal{E}_0(u, \chi)$ and since $V_{\mathrm{hom}}(x, \nabla u(x)) \leq \langle V(\omega, x, \nabla u(x) + \chi(\omega, x)) \rangle$, it follows that χ satisfies (7.9). For (u, χ), we find a strong two-scale recovery sequence $v_\varepsilon \in L^p(\Omega) \otimes W^{1,p}_0(Q)$ using Remark 6.18. We have

$$\begin{aligned} & \|\mathcal{T}_\varepsilon u_\varepsilon - u\|_{L^p(\Omega \times Q)} + \|\mathcal{T}_\varepsilon \nabla u_\varepsilon - \nabla u - \chi\|_{L^p(\Omega \times Q)^d} \\ \leq\ & \|\mathcal{T}_\varepsilon u_\varepsilon - \mathcal{T}_\varepsilon v_\varepsilon\|_{L^p(\Omega \times Q)} + \|\mathcal{T}_\varepsilon v_\varepsilon - u\|_{L^p(\Omega \times Q)} + \|\mathcal{T}_\varepsilon \nabla u_\varepsilon - \mathcal{T}_\varepsilon \nabla v_\varepsilon\|_{L^p(\Omega \times Q)^d} \\ & + \|\mathcal{T}_\varepsilon \nabla v_\varepsilon - \nabla u - \chi\|_{L^p(\Omega \times Q)^d} \end{aligned} \tag{7.10}$$

The second and fourth term on the right-hand side vanish in the limit $\varepsilon \to 0$. In the following we show that the third term also vanishes, which implies that the first term as well tends to zero using the Poincaré inequality. We use (A4) to obtain

$$\frac{c}{4} \|\mathcal{T}_\varepsilon \nabla u_\varepsilon - \mathcal{T}_\varepsilon \nabla v_\varepsilon\|^p_{L^p(\Omega \times Q)^d} = \frac{c}{4} \|\nabla u_\varepsilon - \nabla v_\varepsilon\|^p_{L^p(\Omega \times Q)^d} \leq \frac{1}{2} \mathcal{E}_\varepsilon(v_\varepsilon) + \frac{1}{2} \mathcal{E}_\varepsilon(u_\varepsilon) - \mathcal{E}_\varepsilon\left(\frac{1}{2}(u_\varepsilon + v_\varepsilon)\right)$$

Since u_ε is a minimizer, the right-hand side is bounded by $\frac{1}{2}\mathcal{E}_\varepsilon(v_\varepsilon) - \frac{1}{2}\mathcal{E}_\varepsilon(u_\varepsilon)$ that vanishes in the limit by the properties of u_ε and v_ε. We conclude that $u_\varepsilon \overset{2}{\to} u$ in $L^p(\Omega \times Q)$ and $\nabla u_\varepsilon \overset{2}{\to} \nabla u + \chi$ and since $\mathcal{T}_\varepsilon u = u$ it follows that $u_\varepsilon \to u$ strongly in $L^p(\Omega \times Q)$. $\square$

Remark 7.8 (Alternative assumption)**.** *We remark that in the case $p \in (1, 2)$, condition (A4) is not favorable, since even the model example $V(\omega, x, F) = |F|^p$ is not uniformly convex, see [Xu91]. An alternative approach for obtaining strong convergence, that also applies in the general case $p \in (1, \infty)$, could be based on the (weaker than (A4)) assumption of strict convexity of $V(\omega, x, \cdot)$ and the general principle developed by Visintin in [Vis84]. In particular, for a minimizer u_ε, we have the weak convergence $\mathcal{T}_\varepsilon \nabla u_\varepsilon \rightharpoonup \nabla u + \chi$, on the other hand, it also holds that $\left\langle \int_Q V(\omega, x, \mathcal{T}_\varepsilon \nabla u_\varepsilon) \right\rangle \to \left\langle \int_Q V(\omega, x, \nabla u + \chi) \right\rangle$. In this respect, [Vis84, Theorem 3] implies that $\mathcal{T}_\varepsilon \nabla u_\varepsilon \to \nabla u + \chi$ strongly in $L^p(\Omega \times Q)^d$, which also entails $u_\varepsilon \to u$ strongly in $L^p(\Omega \times Q)$ (cf. (7.10)).*

Remark 7.9 (Periodic boundary conditions)**.** *The above results are not restricted to Dirichlet boundary conditions, e.g., we may consider periodic boundary conditions that we discuss as follows. In particular, we set $Q = \square$ and we define the energy $\widetilde{\mathcal{E}}_\varepsilon : L^p(\Omega) \otimes W^{1,p}_{\mathrm{per,av}}(\square) \to \mathbb{R}$,*

$$\widetilde{\mathcal{E}}_\varepsilon(u) = \left\langle \int_\square V(\tau_{\frac{x}{\varepsilon}} \omega, x, \nabla u(\omega, x)) dx \right\rangle,$$

where $W^{1,p}_{\mathrm{per,av}}(\square) = \left\{ u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d) : \ u \text{ is } \square\text{-periodic}, \ \int_\square u(x) dx = 0 \right\}$ is the space of periodic and average-less functions, in fact, we identify $W^{1,p}_{\mathrm{per,av}}(\square)$ with a closed subspace of $W^{1,p}(\square)$. If the assumptions of Proposition 7.7 hold, then the unique minimizer u_ε of $\widetilde{\mathcal{E}}_\varepsilon$ satisfies

$$u_\varepsilon \to u \quad \text{strongly in } L^p(\Omega \times \square), \quad \nabla u_\varepsilon \overset{2}{\to} \nabla u + \chi \quad \text{in } L^p(\Omega \times \square)^d,$$

where $u \in W^{1,p}_{\mathrm{per,av}}(\square)$ is the unique minimizer of $\widetilde{\mathcal{E}}_{\mathrm{hom}} : W^{1,p}_{\mathrm{per,av}}(\square) \to \mathbb{R}$,

$$\widetilde{\mathcal{E}}_{\mathrm{hom}}(u) = \int_{\square} V_{\mathrm{hom}}(x, \nabla u(x))dx.$$

Also, $\chi \in L^p_{\mathrm{pot}}(\Omega) \otimes L^p(\square)$ is uniquely characterized by the formula

$$V_{\mathrm{hom}}(x, \nabla u(x)) = \langle V(\omega, x, \nabla u(x) + \chi(\omega, x))\rangle \quad \text{for a.a. } x \in \square.$$

These statements are obtained analogously as in Proposition 7.7 with two slight modifications that we point out here. First, the compactness statement in Theorem 7.1 (i) has to be modified. In particular, if a sequence $u_\varepsilon \in L^p(\Omega) \otimes W^{1,p}_{\mathrm{per,av}}(\square)$ satisfies $\limsup_{\varepsilon\to 0} \widetilde{\mathcal{E}}_\varepsilon(u_\varepsilon) < \infty$, then we can extract a subsequence and functions $(u, \chi) \in (L^p_{\mathrm{inv}}(\Omega) \otimes W^{1,p}_{\mathrm{per,av}}(\square)) \times \big(L^p_{\mathrm{pot}}(\Omega) \otimes L^p(\square)\big)$ such that

$$u_\varepsilon \overset{2}{\rightharpoonup} u \quad \text{in } L^p(\Omega \times \square), \qquad \nabla u_\varepsilon \overset{2}{\rightharpoonup} \nabla u + \chi \quad \text{in } L^p(\Omega \times \square)^d.$$

In particular, this follows analogously as in Theorem 7.1 (i) with the help of Proposition 6.14 and using the fact that $P_{\mathrm{inv}} u_\varepsilon \in L^p_{\mathrm{inv}}(\Omega) \otimes W^{1,p}_{\mathrm{per,av}}(\square)$. Second, for such limit (u, χ), the recovery sequence from Theorem 7.1 (iii) has to be slightly modified. In particular, we consider the construction u_ε from Remark 6.18 and we define the recovery sequence as $v_\varepsilon(\omega, x) := u_\varepsilon(\omega, x) - \int_\square u_\varepsilon(\omega, y)dy$ which is periodic and average-less by construction and satisfies the analogues of the strong convergences (7.5).

Remark 7.10 (Comparison with other methods)**.** *The treatment of integral functionals is a well-studied topic in stochastic homogenization and the results that we present above are not new, however, the proposed argumentation using stochastic unfolding presents very simple alternative proofs. In particular, previous results typically rely on the subadditive ergodic theorem (see, e.g., [DMM86, NSS17]) or on the notion of quenched stochastic two-scale convergence (see [HN17]). The analysis via unfolding is less involved than these methods since it merely relies on weak l.s.c. of convex l.s.c. functionals and weak compactness properties of "unfolded" sequences in $L^p(\Omega \times Q)$. On the other hand, we remark that the method we present yields a convergence result in the topology of $L^p(\Omega \times Q)$, that is typically weaker than results obtained using other procedures, e.g., the analysis based on the subadditive ergodic theorem (e.g., [NSS17]) yields convergence for every typical realization of the medium and it even allows to consider nonconvex functionals. We also refer to a recent study [BSS17] for an investigation of homogenization of nonconvex integral functionals by a two-scale Γ-convergence approach. Convex integral functionals were also treated in the setting of coupled periodic and stochastic homogenization in [SW11b] with the help of stochastic two-scale convergence in the mean from [BMW94]. However, despite the equivalence of convergence notions (cf. Remark 6.10), the stochastic unfolding method differs from the analysis based on stochastic two-scale convergence in the mean. In particular, stochastic unfolding is based on the transformation of "rapidly-oscillating" to "mildly-varying" problems via formula (6.4), and, in this respects, extends the idea of the periodic unfolding method to the random case (cf. Section 2).*

7.2 Representative volume element approximations

Typically, in stochastic homogenization the derived deterministic effective coefficients are described by formulas which are not easily accessible for standard numerical analysis. For example, in Section

7.1, even in the linear ergodic case from Remark 7.5, the homogenized integrand A_{hom} is defined through an equation on the probability space: Find $\chi_i \in L^2_{\text{pot}}(\Omega)$ such that

$$\langle A(e_i + \chi_i) \cdot \widetilde{\chi} \rangle = 0 \quad \text{for all } \widetilde{\chi} \in L^2_{\text{pot}}(\Omega). \tag{7.11}$$

Ω is typically an infinite-dimensional space and for this reason the standard finite element approach is inadequate. Also, (7.11) admits a PDE counterpart, in particular, the function $\varphi_i(\omega, \cdot) \in H^1_{\text{loc}}(\mathbb{R}^d)$ that satisfies the relation $\nabla\varphi_i(\omega, x) = \chi_i(\tau_x\omega)$ presents a distributional solution to the following equation, for P-a.a. ω,

$$-\operatorname{div}\left(A(\tau_x\omega)\left(e_i + \nabla\varphi_i(\omega, \cdot)\right)\right) = 0 \quad \text{in } \mathbb{R}^d. \tag{7.12}$$

For conditions which grant uniqueness of solutions to this equation, see, e.g., [Neu17, Section 2.2]. Nevertheless, (7.12) is a usual PDE, however, we need to solve it on the entire space $\mathbb{R}^d$, which is impossible.

The above described difficulties require the development of approximations for the homogenized quantity A_{hom} and respectively for (7.11). A standard approach to this problem is the so-called *representative volume element (RVE) method* , which we briefly discuss in the case of an elliptic equation in an ergodic environment. For detailed studies, we refer to [Owh03, BP04, EGMN14] and the references therein.

We consider the following equation accompanied with homogeneous Dirichlet boundary conditions:

$$-\operatorname{div}\left(A(\tau_{\frac{x}{\varepsilon}}\omega)\nabla u\right) = f, \tag{7.13}$$

where A is given as in Remark 7.5 and $f \in L^2(Q)$. The homogenized equation takes the form

$$-\operatorname{div}\left(A_{\text{hom}}\nabla u\right) = f, \tag{7.14}$$

where $A^{ij}_{\text{hom}} = \langle A\left(e_i + \chi_i\right) \cdot e_j\rangle$ and χ_i solves (7.11).

In the RVE method, in equation (7.13) the coefficient field $A(\tau_{\frac{x}{\varepsilon}}\omega)$ is replaced, e.g., by its *periodization* on a large set, say $L\Box$ where $L \gg 1$. In particular, we set $A_L(\omega, x) = A(\tau_x\omega)$ on $L\Box$ and periodically extend it to $\mathbb{R}^d$, and we consider the following equation

$$-\operatorname{div}\left(A_L\left(\omega, \frac{x}{\varepsilon}\right)\nabla u\right) = f. \tag{7.15}$$

For P-a.a. ω, this equation fits into the setting of periodic homogenization theory, which in the limit $\varepsilon \to 0$ yields the following homogenized coefficient $A_{\text{hom},L}(\omega) \in \mathbb{R}^{d\times d}$,

$$A^{ij}_{\text{hom},L}(\omega) = \fint_{L\Box} A_L(\omega, x)\left(e_i + \nabla\varphi_i(\omega, x)\right) \cdot e_j dx,$$

where $\varphi_i(\omega, \cdot) \in H^1_{\text{per}}(L\Box)$ is the periodic corrector satisfying

$$-\operatorname{div}\left(A_L(\omega, x)(e_i + \nabla\varphi_i)\right) = 0 \quad \text{in } \mathbb{R}^d. \tag{7.16}$$

In particular, $A_{\text{hom},L}$ is used as a proxy for A_{hom}. $A_{\text{hom},L}$ is still random, yet suitable for computational purposes since for P-a.a. ω, (7.16) is a standard elliptic PDE with periodic boundary conditions. In this case, the RVE method is also called the *periodization (in space) method.* Note that (7.16) may be equipped with other boundary conditions, e.g., homogeneous Dirichlet conditions, i.e.,

$$\begin{aligned} -\mathrm{div}\,(A_L(\omega,x)(e_i+\nabla\widetilde{\varphi}_i)) &= 0 \quad \text{in } L\Box,\\ \widetilde{\varphi}_i &= 0 \quad \text{on } \partial L\Box. \end{aligned} \tag{7.17}$$

This choice yields another proxy coefficient $\widetilde{A}_{\mathrm{hom},L}(\omega)\in\mathbb{R}^{d\times d}$ given by

$$\widetilde{A}^{ij}_{\mathrm{hom},L}(\omega)=\fint_{L\Box} A_L(\omega,x)\,(e_i+\nabla\widetilde{\varphi}_i(\omega,x))\cdot e_j dx. \tag{7.18}$$

We refer to [BP04] for a detailed discussion on different admissible choices for the boundary conditions. Based on usual elliptic homogenization strategies, in [Owh03, BP04] it has been shown that for P-a.a. $\omega\in\Omega$,

$$A_{\mathrm{hom},L}(\omega)\to A_{\mathrm{hom}},\quad \widetilde{A}_{\mathrm{hom},L}(\omega)\to A_{\mathrm{hom}}\quad \text{as } L\to\infty. \tag{7.19}$$

This implies pointwise P-a.e. convergence for the solutions of the corresponding elliptic equations. We present a simple alternative argument for these convergences in a different topology that is based on stochastic unfolding and avoids the use of Birkhoff's pointwise ergodic theorem, we merely use von Neumann's mean ergodic theorem. We use the following lemma in our applications in Section 9.1.1, where we consider approximations for effective Allen-Cahn type equations.

Lemma 7.11 (Convergence of approximations)**.** *Let $\langle\cdot\rangle$ be ergodic and $A\in L^\infty(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})$. We assume that there exists $c>0$ such that $A(\omega)F\cdot F\geq c|F|^2$ for P-a.a. ω and all $F\in\mathbb{R}^d$. Then:*

(i) $A_{\mathrm{hom},L}$ and $\widetilde{A}_{\mathrm{hom},L}$ are bounded sequences in $L^\infty(\Omega,\mathbb{R}^{d\times d}_{\mathrm{sym}})$ and there exists $c>0$ such that

$$A_{\mathrm{hom},L}(\omega)F\cdot F\geq c|F|^2,\quad \widetilde{A}_{\mathrm{hom},L}(\omega)F\cdot F\geq c|F|^2\quad \textit{for } P\textit{-a.a. } \omega,\ \textit{all } F\in\mathbb{R}^d \textit{ and all } L\geq 1.$$

(ii) If $L\to\infty$, then

$$A_{\mathrm{hom},L}\to A_{\mathrm{hom}}\quad \textit{strongly in } L^2(\Omega)^{d\times d},\quad \widetilde{A}_{\mathrm{hom},L}\to A_{\mathrm{hom}}\quad \textit{strongly in } L^2(\Omega)^{d\times d}. \tag{7.20}$$

Despite being weaker than (7.19), the convergence statements (7.20) already imply

$$u_L\to u\quad \text{strongly in } L^2(\Omega\times Q),$$

where $u\in H^1_0(Q)$ and $u_L\in L^2(\Omega)\otimes H^1_0(Q)$ are the solutions of the elliptic equations with coefficients A_{hom} and $A_{\mathrm{hom},L}$ (or $\widetilde{A}_{\mathrm{hom},L}$), respectively.

Proof of Lemma 7.11. (i) First, we consider $\widetilde{A}_{\mathrm{hom},L}$. We denote by $\varphi_{i,L}\in L^2(\Omega)\otimes H^1_0(L\Box)$ the unique solution to (7.17). As discussed in the following Section 7.3, $\omega\mapsto\varphi_{i,L}(\omega,\cdot)$ is indeed a measurable mapping. Standard a priori estimates for this equation imply that for P-a.a. ω,

$$\fint_{L\Box}|\frac{1}{L}\varphi_{i,L}(\omega,x)|^2+|\nabla\varphi_{i,L}(\omega,x)|^2dx\leq c,$$

where $c > 0$ is independent of ω and L. Here we use the fact that $A_L(\omega, x) = A(\tau_x \omega)$ on $L\square$. As a result of this, it follows that $\widetilde{A}_{\text{hom},L}$ is a bounded sequence in $L^\infty(\Omega)^{d\times d}$. Symmetry of $\widetilde{A}_{\text{hom},L}$ follows using symmetry of A. For an arbitrary $F \in \mathbb{R}^d$, using equation (7.17) we have

$$\begin{aligned}
\widetilde{A}_{\text{hom},L}(\omega)F \cdot F &= \fint_{L\square} A(\tau_x\omega)\left(F + \sum_{i=1}^d F_i \nabla\varphi_{i,L}(\omega,x)\right) \cdot \left(F + \sum_{i=1}^d F_i \nabla\varphi_{i,L}(\omega,x)\right) dx \\
&\geq c\fint_{L\square} |F + \sum_{i=1}^d F_i \nabla\varphi_{i,L}(\omega,x)|^2 dx \\
&\geq c|F|^2,
\end{aligned}$$

where the second line is obtained using positive-definiteness of A and the last inequality follows by Jensen's inequality and by the fact that $\nabla\varphi_{i,L}$ is average-less. The properties of $A_{\text{hom},L}$ follow analogously.

(ii) First we show that $\widetilde{A}_{\text{hom},L} \to A_{\text{hom}}$. We fix $i \in \{1, ..., d\}$ and drop it from the notation. We use the notation $\varepsilon := \frac{1}{L}$ and we set $\varphi_\varepsilon(\omega, x) := \frac{1}{L}\varphi_L(\omega, Lx)$, where φ_L is the solution to (7.17). A change of variables $x \rightsquigarrow \frac{x}{L}$ in (7.17) implies that $\varphi_\varepsilon \in L^2(\Omega) \otimes H_0^1(\square)$ is the unique solution of the minimization problem

$$\min_{\varphi \in L^2(\Omega)\otimes H_0^1(\square)} \left(\mathcal{E}_\varepsilon(\varphi) = \left\langle \int_\square A(\tau_{\frac{x}{\varepsilon}}\omega)\left(e_i + \nabla\varphi(\omega,x)\right) \cdot \left(e_i + \nabla\varphi(\omega,x)\right) dx \right\rangle\right).$$

For this $\mathcal{E}_\varepsilon$, with $p = 2$ the assumptions of Proposition 7.7 are satisfied and it follows that

$$\left\langle \int_\square |\nabla\varphi_\varepsilon - \mathcal{T}_{-\varepsilon}\chi - \nabla u|^2 dx \right\rangle = \left\langle |\mathcal{T}_\varepsilon \nabla\varphi_\varepsilon - \chi - \nabla u|^2 dx \right\rangle \to 0, \tag{7.21}$$

where (u, χ) is the unique minimizer of $(u, \chi) \mapsto \left\langle \int_\square A(\omega)\left(e_i + \nabla u + \chi\right) \cdot \left(e_i + \nabla u + \chi\right) dx \right\rangle$. Yet, it can be easily seen that the latter functional admits the minimizer $(u, \chi) = (0, \chi_i)$ where χ_i is the solution to (7.11). Furthermore, we estimate

$$\begin{aligned}
&\left\langle |\widetilde{A}_{\text{hom},L}(\omega)e_i \cdot e_j - A_{\text{hom}}e_i \cdot e_j|^2 \right\rangle \\
= \quad &\left\langle \Big| \int_\square A(\tau_{\frac{x}{\varepsilon}}\omega)(e_i + \nabla\varphi_\varepsilon) \cdot e_j dx - A_{\text{hom}}e_i \cdot e_j \Big|^2 \right\rangle \qquad (7.22) \\
\leq \quad & 2\left\langle \Big| \int_\square A(\tau_{\frac{x}{\varepsilon}}\omega)(e_i + \nabla\varphi_\varepsilon) \cdot e_j dx - \int_\square A(\tau_{\frac{x}{\varepsilon}}\omega)(e_i + \mathcal{T}_{-\varepsilon}\chi_i) \cdot e_j dx \Big|^2 \right\rangle \\
&+2\left\langle \Big| \int_\square A(\tau_{\frac{x}{\varepsilon}}\omega)(e_i + \mathcal{T}_{-\varepsilon}\chi_i) \cdot e_j dx - A_{\text{hom}}e_i \cdot e_j \Big|^2 \right\rangle. \qquad (7.23)
\end{aligned}$$

By Jensen's inequality and by boundedness of A, the first term on the right-hand side may be bounded by $\left\langle \int_\square |\nabla\varphi_\varepsilon - \mathcal{T}_{-\varepsilon}\chi_i|^2 dx \right\rangle$ that vanishes in the limit $\varepsilon \to 0$ by (7.21). Note that $\omega \mapsto A(\omega)(e_i + \chi_i(\omega)) \cdot e_j$ is an element of $L^2(\Omega)$. Consequently, von Neumann's ergodic theorem, see Remark 6.3, implies that the second term as well vanishes. This proves the claim for $\widetilde{A}_{\text{hom},L}$.

The convergence $A_{\text{hom},L} \to A_{\text{hom}}$ follows analogously as above using Remark 7.9. $\square$

Remark 7.12 (Variance reduction)**.** *We remark that the random objects $A_{\mathrm{hom},L}$ and $\widetilde{A}_{\mathrm{hom},L}$ may have a large variance, which is an issue for practical purposes. A consideration of Monte Carlo type approximations, i.e., $\frac{1}{n}\sum_{i=1}^{n} A_{\mathrm{hom},L}(\omega_i)$ with different realizations $\omega_i \in \Omega$, reduces the variance by a factor $\frac{1}{n}$. In particular, specific criteria for the choice of the considered realizations yield an even better variance reduction as it is discussed, e.g., in the studies [BCLBL12, Fis18].*

Periodization in law

An alternative to the above described method is the so-called periodization *in law* strategy. In this procedure, instead of periodizing the coefficients of the equation in the physical space variable (cf. (7.15)), their probability law is periodized. For detailed studies, we refer to [KFG$^+$03, GNO15, EGMN14, DG16] . We explain this concept on the example of a correlated checkerboard structure from Example 6.6.

We first briefly recall the setting from Example 6.6. We consider a bounded set $\Omega_0 \subset \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and we assume that there exists $c > 0$ such that $AF \cdot F \geq c|F|^2$ for all $A \in \Omega_0$ and all $F \in \mathbb{R}^d$. Also, we equip Ω_0 with a σ-algebra $\mathcal{F}_0$ and a probability measure P_0. We consider the probability space

$$(\Omega, \mathcal{F}, P) = \left(\Omega_0^{\mathbb{Z}^d} \otimes \Box_{\#}, \overline{\otimes_{\mathbb{Z}^d}\mathcal{F}_0 \otimes \mathcal{L}(\Box_{\#})}, \otimes_{\mathbb{Z}^d} P_0 \otimes dy\right), \tag{7.24}$$

equipped with a shift $\tau_x : \Omega \to \Omega$ given by $\tau_x\omega = (\omega_1(\cdot + \lfloor y + x\rfloor), y + x - \lfloor y + x\rfloor)$. Here, we see a realization $\omega \in \Omega$ as a pair (ω_1, y) where $\omega_1 : \mathbb{Z}^d \to \Omega_0$ and $y \in \Box_{\#}$. We consider an integer parameter $L > 1$ (up to slight modifications below it may be chosen to be real). We define the periodization mapping $\pi_L : \Omega \to \Omega$,

$$\pi_L\omega = (\widetilde{\pi}_L\omega_1, y),$$

where $\widetilde{\pi}_L\omega_1 : \mathbb{Z}^d \to \Omega_0$ is defined by $\widetilde{\pi}_L\omega_1(x) = \omega_1(x)$ in $L\Box \cap \mathbb{Z}^d$ and we periodically extended it to the entire $\mathbb{Z}^d$. π_L is a measurable mapping and its push-forward is the probability space $(\Omega, \mathcal{F}_L, P_L)$ with a measure P_L that concentrates on $L\Box$-periodic fields. To make the difference clear, we will denote this space by $(\Omega_L, \mathcal{F}_L, P_L)$. On this space we can define a shift τ analogously as before and in this way the probability space $(\Omega_L, \mathcal{F}_L, P_L, \tau)$ satisfies Assumption 6.1. However, this space is not ergodic since even on large distances the realizations are strongly correlated by periodicity.

In the periodization in law method, the coefficient $A(\tau_{\frac{x}{\varepsilon}}\omega)$ in equation (7.13), where ω is sampled w.r.t. P, is replaced by $A(\tau_{\frac{x}{\varepsilon}}\pi_L\omega)$. This is equivalent to the consideration of the equation

$$-\operatorname{div}\left(A(\tau_{\frac{x}{\varepsilon}}\omega)\nabla u_\varepsilon\right) = f, \quad \omega \in \Omega_L \text{ being sampled w.r.t. the new measure } P_L. \tag{7.25}$$

We remark that for P_L-a.a. $\omega \in \Omega_L$, $x \mapsto A(\tau_x\omega)$ defines an $L\Box$-periodic coefficient field. Consequently, the periodic homogenization theory yields an effective coefficient $\overline{A}_{\mathrm{hom},L}(\omega) \in \mathbb{R}^{d\times d}$, for P_L-a.a. $\omega \in \Omega_L$,

$$\overline{A}^{ij}_{\mathrm{hom},L}(\omega) = \fint_{L\Box} A(\tau_x\omega)(e_i + \nabla\overline{\varphi}_i(\omega, x) + e_i) \cdot e_j dx,$$

where $\overline{\varphi}_i(\omega, \cdot) \in H^1_{\mathrm{per}}(L\Box)$ is the periodic corrector which solves, for P_L-a.a. $\omega \in \Omega_L$,

$$-\operatorname{div}\left(A(\tau_x\omega)(e_i + \nabla\overline{\varphi}_i(\omega, \cdot))\right) = 0.$$

In particular, using the specific structure of the space $(\Omega_L, \mathcal{F}_L, P_L)$, the above definition may be equivalently rephrased as (we re-use the same notation): for P-a.a. $\omega \in \Omega$,

$$\overline{A}^{ij}_{\mathrm{hom},L}(\omega) = \fint_{L\square} A(\tau_x \pi_L \omega)(e_i + \nabla \overline{\varphi}_i(\omega, x)) \cdot e_j dx, \tag{7.26}$$

where $\overline{\varphi}(\omega, \cdot) \in H^1_{\mathrm{per}}(L\square)$ is the periodic corrector which solves, for P-a.a. $\omega \in \Omega$,

$$-\operatorname{div}\left(A(\tau_x \pi_L \omega)(e_i + \nabla \overline{\varphi}_i(\omega, \cdot))\right) = 0. \tag{7.27}$$

The coefficient $\overline{A}_{\mathrm{hom},L}(\omega)$ may be computed by usual finite element approximations and it serves as an approximation for A_{hom} for large L.

Remark 7.13 (Comparison to periodization in space)**.** *We remark that in general the periodization in space $A_L(\omega, x)$ from (7.15) and the periodization in law $A(\tau_x \pi_L \omega)$ do not coincide. However, if we deal with an i.i.d. checkerboard structure, e.g., $A(\omega) = \omega_1(0)$, then the two periodizations are equal and we have $A_{\mathrm{hom},L} = \overline{A}_{\mathrm{hom},L}$. Also, we note that periodization in law defines a coefficient field that is stationary w.r.t. shifts in the space $(\Omega_L, \mathcal{F}_L, P_L, \tau)$, whereas the periodization in space is not necessarily stationary. In this respect, we may apply the stochastic unfolding procedure to the problem (7.25) to obtain an equivalent probability space characterization of $\overline{A}_{\mathrm{hom},L}$ given by*

$$\overline{\overline{A}}^{ij}_{\mathrm{hom},L} = P_{\mathrm{inv}}\left(A(e_i + \chi_i) \cdot e_j\right),$$

where $\chi_i \in L^p_{\mathrm{pot}}(\Omega_L)$ solves the corrector equation

$$\int_{\Omega_L} A(\omega)(e_i + \chi_i(\omega)) \cdot \widetilde{\chi}(\omega) dP_L(\omega) = 0 \quad \text{for all } \widetilde{\chi} \in L^p_{\mathrm{pot}}(\Omega_L).$$

In particular, it holds that $\int_{\Omega_L} \overline{\overline{A}}_{\mathrm{hom},L}(\omega) dP_L(\omega) = \int_{\Omega_L} \overline{A}_{\mathrm{hom},L}(\omega) dP_L(\omega)$.

Similarly as before, we may also replace the boundary conditions in (7.27) with homogeneous Dirichlet conditions, i.e., we may consider for $\omega \in \Omega$,

$$\widehat{A}^{ij}_{\mathrm{hom},L}(\omega) = \fint_{L\square} A(\tau_x \pi_L \omega)(e_i + \nabla \widehat{\varphi}_i(\omega, x)) \cdot e_j dx, \tag{7.28}$$

where $\widehat{\varphi}_i(\omega, \cdot) \in H^1_0(L\square)$ is the Dirichlet corrector which solves, for P-a.a. $\omega \in \Omega$,

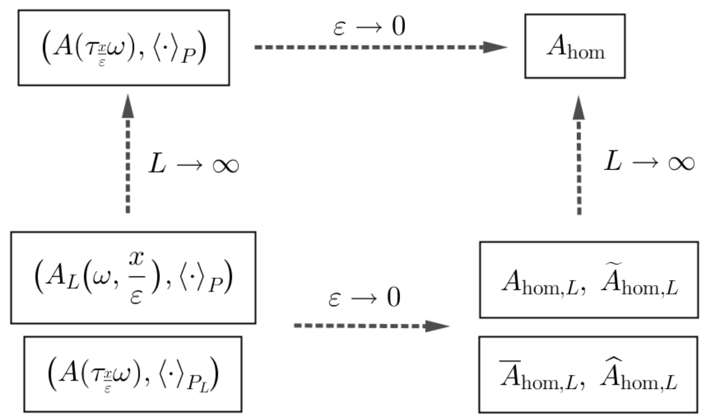
$$\begin{aligned} -\operatorname{div}\left(A(\tau_x \pi_L \omega)(e_i + \nabla \widehat{\varphi}_i(\omega, \cdot))\right) &= 0 \quad \text{in } L\square, \\ \widehat{\varphi}_i(\omega, \cdot) &= 0 \quad \text{on } \partial L\square. \end{aligned} \tag{7.29}$$

In the following, we consider a particular choice for the coefficient A and we show that both $\overline{A}_{\mathrm{hom},L}$ and $\widehat{A}_{\mathrm{hom},L}$ approximate A_{hom} in the limit $L \to \infty$. In particular, we assume that

$$A(\omega) = (\rho * \omega_1)(0) = \sum_{z \in \mathbb{Z}^d} \rho(-z) \omega_1(z), \tag{7.30}$$

where $\rho \in L^1(\mathbb{Z}^d)$ such that $\rho \neq 0$ and $\rho(z) \geq 0$ for all z. In a similar setting, in [EGMN14] it is shown that $\overline{A}_{\mathrm{hom},L}(\omega) \to A_{\mathrm{hom}}$ for P-a.a. ω as $L \to \infty$. In the following lemma we show the (weaker) strong convergence in $L^2(\Omega)^{d\times d}$ that is based on Lemma 7.11 which relies on the stochastic unfolding procedure. The proof is similar to the proof in [EGMN14] and relies on the comparison of $\overline{A}_{\mathrm{hom},L}$ (resp. $\widehat{A}_{\mathrm{hom},L}$) and $A_{\mathrm{hom},L}$ (resp. $\widetilde{A}_{\mathrm{hom},L}$).

Figure 7.1: The following diagram briefly illustrates the approximation strategies described in this section. We discuss all but the left convergence $L \to \infty$. However, for fixed ε, by similar argumentation as in Lemmas 7.11 and 7.14, it may be shown that the proxies $A_L(\omega, \frac{x}{\varepsilon})$ and $A(\tau_{\frac{x}{\varepsilon}}\omega)$ (sampled w.r.t. P_L) approximate $A(\tau_{\frac{x}{\varepsilon}}\omega)$ (sampled w.r.t. P) in the limit $L \to \infty$.



Lemma 7.14 (Convergence of approximations)**.** *Let $(\Omega, \mathcal{F}, P)$ be the probability space given in (7.24). We consider $A \in L^\infty(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ defined by (7.30). Let $\overline{A}_{\mathrm{hom},L}$ and $\widehat{A}_{\mathrm{hom},L}$ be defined by (7.26) and (7.28), respectively. It holds, as $L \to \infty$,*

$$\overline{A}_{\mathrm{hom},L} \to A_{\mathrm{hom}} \quad \textit{strongly in } L^2(\Omega)^{d\times d}, \quad \widehat{A}_{\mathrm{hom},L} \to A_{\mathrm{hom}} \quad \textit{strongly in } L^2(\Omega)^{d\times d}.$$

Proof. We show below that $\widehat{A}_{\mathrm{hom},L} \to A_{\mathrm{hom}}$ in two steps and the convergence $\overline{A}_{\mathrm{hom},L} \to A_{\mathrm{hom}}$ follows analogously. In particular, the first step is independent of the corrector equation. The second step relies on Lemma 7.11 (ii) and Meyers' estimate which holds for both the Dirichlet and periodic correctors.

Step 1. We show that for an arbitrary $q \in (1, \infty)$, it holds:

$$E_q(L) := \left\langle \fint_{L\square} |A(\tau_x \pi_L \omega) - A(\tau_x \omega)|^q dx \right\rangle \to 0 \quad \text{as } L \to \infty.$$

For $K > 0$, which we specify later, we set $\rho_K = \mathbf{1}_{K\square\cap\mathbb{Z}^d}\rho$ and $\rho_{\mathrm{rest},K} = \rho - \rho_K$. We estimate for P-a.a. $\omega = (\omega_1, y)$,

$$\begin{aligned}
&\fint_{L\square} |A(\tau_x \pi_L \omega) - A(\tau_x \omega)|^q dx \\
={}& \fint_{L\square} |\,(\rho * \widetilde{\pi}_L \omega_1)\,(\lfloor y + x \rfloor) - (\rho * \omega_1)\,(\lfloor y + x \rfloor)|^q dx \\
\leq{}& c\fint_{L\square} |\,(\rho_K * (\widetilde{\pi}_L \omega_1 - \omega_1))\,(\lfloor y + x \rfloor)|^q dx + c\fint_{L\square} |\,(\rho_{\mathrm{rest},K} * (\widetilde{\pi}_L \omega_1 - \omega_1))\,(\lfloor y + x \rfloor)|^q dx.
\end{aligned}$$

Since Ω_0 was chosen to be a bounded set in $\mathbb{R}^{d\times d}$, it follows that $\widetilde{\pi}_L\omega_1$ and ω_1 are uniformly bounded in L^∞ (independently of L and the realization ω). As a result of this, the second term above may be bounded by $c\left(\sum_{z\in\mathbb{R}^d\setminus K\square\cap\mathbb{Z}^d} |\rho(-z)|\right)^q$ that vanishes in the case that $K \to \infty$. The first term is treated as follows. We assume that $K < L$ and we compute

$$
\begin{aligned}
&\fint_{L\square} |\left(\rho_K * (\widetilde{\pi}_L\omega_1 - \omega_1)\right)(\lfloor y+x \rfloor)|^q dx \\
= \;& \frac{1}{L^d}\int_{L\square\setminus(L-K)\square} \Big| \sum_{z\in K\square\cap\mathbb{Z}^d} \rho(-z)\left(\widetilde{\pi}_L\omega_1(z+\lfloor y+x\rfloor) - \omega_1(z+\lfloor y+x\rfloor)\right)\Big|^q dx \\
&+\frac{1}{L^d}\int_{(L-K)\square} \Big| \sum_{z\in K\square\cap\mathbb{Z}^d} \rho(-z)\left(\widetilde{\pi}_L\omega_1(z+\lfloor y+x\rfloor) - \omega_1(z+\lfloor y+x\rfloor)\right)\Big|^q dx
\end{aligned}
$$

The second term is 0 since in the considered domain, $\widetilde{\pi}_L\omega_1$ and ω_1 coincide. Since $\rho \in L^1(\mathbb{Z}^d)$ and by boundedness of ω_1, the second term may be bounded by $c\frac{|L\square\setminus(L-K)\square|}{L^d}$. If we set, e.g., $K = L^{\frac{1}{2}}$, the last expression vanishes in the limit $L\to\infty$. These observations imply that for P-a.a. ω,

$$
\fint_{L\square} |A(\tau_x\pi_L\omega) - A(\tau_x\omega)|^q dx \to 0 \quad \text{as } L\to\infty. \tag{7.31}
$$

The dominated convergence theorem implies that $E_q(L)\to 0$.

Step 2. We show that $\widetilde{A}_{\mathrm{hom},L} - \widehat{A}_{\mathrm{hom},L} \to 0$ in $L^2(\Omega)^{d\times d}$ which implies $\widehat{A}_{\mathrm{hom},L}\to A_{\mathrm{hom}}$ using the triangle inequality and Lemma 7.11 (ii).

We consider $\widetilde{A}_{\mathrm{hom},L}$ from (7.18). We fix $i\in\{1,...,d\}$ that we drop from the notation and we consider the Dirichlet correctors $\widetilde{\varphi}_L$ and $\widehat{\varphi}_L$, the solutions to (7.17) and (7.28), respectively. We estimate,

$$
\begin{aligned}
&\left\langle |\widehat{A}_{\mathrm{hom},L}(\omega)e_i\cdot e_j - \widetilde{A}_{\mathrm{hom},L}(\omega)e_i\cdot e_j|^2\right\rangle \\
= \;& \left\langle \Big|\fint_{L\square} A(\tau_x\pi_L\omega)(e_i+\nabla\widehat{\varphi}_L(\omega,x))\cdot e_j - A(\tau_x\omega)\left(e_i+\nabla\widetilde{\varphi}_L(\omega,x)\right)\cdot e_j dx\Big|^2\right\rangle \\
\le \;& c\left\langle \Big|\fint_{L\square}\left(A(\tau_x\pi_L\omega)-A(\tau_x\omega)\right)e_i\cdot e_j dx\Big|^2\right\rangle + c\left\langle \Big|\fint_{L\square}\left(A(\tau_x\pi_L\omega)-A(\tau_x\omega)\right)\nabla\widehat{\varphi}_L\cdot e_j dx\Big|^2\right\rangle \\
&+c\left\langle \Big|\fint_{L\square} A(\tau_x\omega)\left(\nabla\widehat{\varphi}_L - \nabla\widetilde{\varphi}_L\right)\cdot e_j dx\Big|^2\right\rangle \\
\le \;& cE_2(L) + c\left\langle \fint_{L\square}|\nabla\widehat{\varphi}_L - \nabla\widetilde{\varphi}_L|^2 dx\right\rangle,
\end{aligned} \tag{7.32}
$$

where in the last inequality we bound the second term by $E_2(L)$ using the standard a priori estimate $\fint_{L\square}|\nabla\widehat{\varphi}_L|^2dx\le c$ for equation (7.29), which is independent of ω. In the following, we show that the last term in (7.32) is bounded by $c\left(E_2(L) + (E_q(L))^{\frac{2}{q}}\right)$ for some $q\in(1,\infty)$, which concludes the proof using Step 1.

Using the positive-definiteness of $A(\tau_x\pi_L\omega)$, we obtain

$$
\begin{aligned}
\left\langle \fint_{L\square}|\nabla\widehat{\varphi}_L - \nabla\widetilde{\varphi}_L|^2 dx\right\rangle &\le \left\langle \fint_{L\square} A(\tau_x\pi_L\omega)\left(\nabla\widehat{\varphi}_L - \nabla\widetilde{\varphi}_L\right)\cdot\left(\nabla\widehat{\varphi}_L - \nabla\widetilde{\varphi}_L\right)dx\right\rangle \\
&= \left\langle \fint_{L\square}\left(A(\tau_x\pi_L\omega)-A(\tau_x\omega)\right)e_i\cdot\left(\nabla\widetilde{\varphi}_L - \nabla\widehat{\varphi}_L\right)dx\right\rangle \\
&\quad - \left\langle \fint_{L\square}\left(A(\tau_x\pi_L\omega)-A(\tau_x\omega)\right)\nabla\widetilde{\varphi}_L\cdot\left(\nabla\widehat{\varphi}_L - \nabla\widetilde{\varphi}_L\right)dx\right\rangle,
\end{aligned}
$$

where the last equality follows by testing equations (7.17) and (7.29) with $\widehat{\varphi}_L - \widetilde{\varphi}_L$. Using Young's inequality, the above inequality yields

$$\left\langle \fint_{L\Box} |\nabla\widehat{\varphi}_L - \nabla\widetilde{\varphi}_L|^2 dx \right\rangle \leq c \left\langle \fint_{L\Box} |A(\tau_x\omega) - A(\tau_x\pi_L\omega)|^2 + |\left(A(\tau_x\omega) - A(\tau_x\pi_L\omega)\right)\nabla\widetilde{\varphi}_L|^2 dx \right\rangle.$$

The first term is $cE_2(L)$. We bound the second term with the help of Meyers' theorem. In particular, [Mey63, Theorem 1] implies that there exists some $p > 2$ such that $\langle \fint_{L\Box} |\nabla\widetilde{\varphi}_L|^p dx\rangle \leq c$. Therefore, by the application of Hölder's inequality with exponents $\left(\frac{p}{2}, q = \frac{p}{p-2}\right)$ to the second term, it may be bounded by $c(E_{2q}(L))^{\frac{1}{q}}$. This concludes the proof. $\square$

7.3 Mean vs. quenched homogenization

In this section we compare *mean* and *quenched* formulations of PDE and minimization problems with random coefficients, and we discuss the corresponding homogenization results. We explain the concepts of mean and quenched formulations on an example from Section 7 (see also Remark 1.2). In particular, we call the problem of minimization of

$$\mathcal{E} : L^p(\Omega; W_0^{1,p}(Q)) \to \mathbb{R}, \quad \mathcal{E}(u) = \int_\Omega \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, \nabla u(\omega, x)) - f(\omega, x)u(\omega, x)dxdP(\omega) \tag{7.33}$$

a mean formulation since the realization of the random medium $\omega \in \Omega$ is considered as a variable of the problem and the functional features an integral over Ω. On the other hand, in the corresponding quenched formulation we consider the following parametrized minimization problem: For P-a.a. $\omega \in \Omega$, minimize

$$\mathcal{E}^\omega : W_0^{1,p}(Q) \to \mathbb{R}, \quad \mathcal{E}^\omega(u) = \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, \nabla u(x)) - f(\omega, x)u(x)dx, \tag{7.34}$$

where $\omega \in \Omega$ is considered as a parameter in a deterministic problem. Typically, homogenization results for problems in the mean formulation yield convergence in the topology of $L^p(\Omega \times Q)$ (cf. Proposition 7.7 and the discussion in Section 1), whereas quenched homogenization implies convergence of the solution for P-a.a. $\omega \in \Omega$ in $L^p(Q)$ (cf. Remarks 1.2 and 7.10 and references therein). Also, the homogenization approaches for the two formulations are different: Quenched results usually rely on some individual ergodic theorem such as the subadditive ergodic theorem from [AK81]. In particular, in the setting of two-scale convergence the notion of *quenched* stochastic two-scale convergence is developed in [ZP06] (see also [MP07, Fag08, Hei11]) based on Birkhoff's ergodic theorem. Homogenization for problems in the mean formulation is usually based on the weaker von Neumann's ergodic theorem or on the use of stochastic two-scale convergence in the mean from [BMW94]. The stochastic unfolding method that we propose is also suited for problems in the mean formulation.

In this section we discuss the following questions regarding the comparison of the two formulations:

(i) Can we identify the minimizers of the mean and quenched functionals (7.33) and (7.34)? In the case that both $\mathcal{E}$ and $\mathcal{E}^\omega$ admit unique minimizers, it follows that for P-a.a. ω they coincide; the corresponding discussion is given in Section 7.3.1 below.

(ii) Can we identify solutions of mean and quenched formulations of evolutionary gradient systems? Similarly as for minimization problems, if uniqueness is available, we may identify the corresponding solutions. We discuss this also in Section 7.3.1.

(iii) Does homogenization in the mean and quenched formulations yield the same effective problem? We discuss this question on the example of general random functionals in Section 7.3.2.

(iv) Can we relate stochastic unfolding (resp. stochastic two-scale convergence in the mean) to the notion of quenched stochastic two-scale convergence? We discuss briefly this question in Section 7.3.3.

7.3.1 Equivalence of formulations

In order to answer questions (i) and (ii), we first consider this problem from an abstract viewpoint on general random functionals.

Throughout this section we assume that $(\Omega, \mathcal{F}, P)$ is a complete and separable probability space. We consider a separable Banach space Y with dual space Y^*. Let $p \in (1, \infty)$. We consider the following family of integral functionals parametrized by $\omega \in \Omega$:

$$\mathcal{I}^\omega : Y \to \mathbb{R} \cup \{\infty\} \qquad \text{(quenched)}$$

and the corresponding averaged (mean) functional

$$\mathcal{I} : L^p(\Omega; Y) \to \mathbb{R} \cup \{\infty\}, \quad \mathcal{I}(y) = \int_\Omega \mathcal{I}^\omega(y(\omega)) dP(\omega). \qquad \text{(mean)}$$

We assume the following:

(B1) The mapping $\Omega \times Y \to \mathbb{R} \cup \{\infty\}$, $(\omega, y) \mapsto \mathcal{I}^\omega(y)$ is a normal integrand (see Definition A.3).

(B2) There exists $c > 0$ such that for P-a.a. $\omega \in \Omega$, $\mathcal{I}^\omega(y) \geq \frac{1}{c} \|y\|_Y^p - c$ for all $y \in Y$.

(B3) There exists $y \in L^p(\Omega, Y)$ such that $\mathcal{I}(y) < \infty$.

(B4) For P-a.a. $\omega \in \Omega$, $\mathcal{I}^\omega$ is strictly convex.

In particular, (B1)-(B2) guarantee that the mean functional $\mathcal{I}$ is well-defined. Also, if we assume (B1)-(B3), Theorem A.7 implies that (see also Remark A.8)

$$\int_\Omega \inf_{y \in Y} \mathcal{I}^\omega(y) dP(\omega) = \inf_{y \in L^p(\Omega; Y)} \mathcal{I}(y). \tag{7.35}$$

If we additionally assume (B4), then the minimizers y^ω and y of, respectively, $\mathcal{I}^\omega$ and $\mathcal{I}$ are unique and we have

$$y^\omega = y(\omega, \cdot) \quad \text{for } P\text{-a.a. } \omega \in \Omega. \tag{7.36}$$

In the following, we illustrate the effects of this simple abstract argument on the specific equations that we consider later in the applications.

Convex minimization

We consider the setting from Section 7.1. Let $V : \Omega \times Q \times \mathbb{R}^d \to \mathbb{R}$ satisfy (A1)-(A3) and $V(\omega, x, \cdot)$ be strictly convex for a.a. (ω, x). We consider $f \in L^q(\Omega; L^q(Q))$ with $q = \frac{p}{p-1}$. We define $\mathcal{E}^\omega$ and $\mathcal{E}$ by (7.34) and (7.33), respectively. Under these assumptions, we have the following:

Lemma 7.15. *Let u^ω and u denote the unique minimizers of $\mathcal{E}^\omega$ and $\mathcal{E}$, respectively. Then, it holds*

$$u^\omega = u(\omega, \cdot) \quad \textit{for } P\textit{-a.a. } \omega \in \Omega.$$

Proof. We may set $Y = W_0^{1,p}(Q)$ and $\mathcal{I}^\omega = \mathcal{E}^\omega$ which satisfies (B1)-(B4). As a result of this, (7.35) and (7.36) imply the claim. □

This observation extends to the case of functionals defined on a discrete physical space which we consider in Section 8.1.

Rate-independent systems

In the following we discuss equivalence of mean and quenched formulations for ERIS (see Section 3 for the general theory).

In particular, we consider a separable Hilbert space Y. The quenched formulation of the ERIS is given in terms of the following parametrized functionals

$$\mathcal{R}^\omega : Y \to [0, \infty], \tag{dissipation}$$

$$\mathcal{E}^\omega : [0, T] \times Y \to \mathbb{R}, \quad \mathcal{E}^\omega(t, y) = \frac{1}{2} \langle \mathbb{A}^\omega y, y \rangle_{Y^*, Y} - \langle l^\omega(t), y \rangle_{Y^*, Y}. \tag{energy}$$

The mean formulation of the ERIS is given in terms of the state space $L^2(\Omega; Y)$ and the corresponding mean functionals:

$$\mathcal{R} : L^2(\Omega; Y) \to [0, \infty], \quad \mathcal{R}(\dot{y}) = \int_\Omega \mathcal{R}^\omega(\dot{y}(\omega)) dP(\omega), \tag{dissipation}$$

$$\mathcal{E} : [0, T] \times L^2(\Omega; Y) \to \mathbb{R}, \quad \mathcal{E}(t, y) = \int_\Omega \mathcal{E}^\omega(t, y(\omega)) dP(\omega). \tag{energy}$$

We assume that the mappings $(\omega, y) \mapsto \mathcal{R}^\omega(y)$ and $(\omega, y) \mapsto \mathcal{E}^\omega(t, y)$ for all $t \in [0, T]$, define convex normal integrands. This implies that the mean functionals are well-defined. We also assume that the systems $(Y, \mathcal{E}^\omega, \mathcal{R}^\omega)$ (for P-a.a. ω) and $\left(L^2(\Omega; Y), \mathcal{E}, \mathcal{R}\right)$ satisfy the assumptions of the existence and uniqueness Theorem 3.5 (such that the ellipticity ratio of $\mathbb{A}^\omega$ is independent of ω). We denote by $S^\omega(t)$ and $S(t)$ the sets of stable states of these systems. Under these assumptions, we obtain the following:

Lemma 7.16. *Let $l \in C^1([0, T], L^2(\Omega; Y))$ and $y^0 \in L^2(\Omega; Y)$ be such that $y^0(\omega) \in S^\omega(0)$ P-a.e. and $y^0 \in S(0)$. We denote by $y^\omega \in C^{\mathrm{Lip}}([0, T], Y)$ the unique energetic solution to the ERIS $(Y, \mathcal{E}^\omega, \mathcal{R}^\omega)$ with initial datum $y^0(\omega)$ and by $y \in C^{\mathrm{Lip}}([0, T], L^2(\Omega; Y))$ the unique energetic solution to the ERIS $\left(L^2(\Omega; Y), \mathcal{E}, \mathcal{R}\right)$ with initial datum y^0. Then, it holds, for all $t \in [0, T]$ and P-a.a. ω,*

$$y^\omega(t) = y(t)(\omega).$$

Proof. It is sufficient to show that for a.a. $t \in [0,T]$, the mappings $\omega \mapsto y^\omega(t)$ and $\omega \mapsto \dot{y}^\omega(t)$ are $(\mathcal{F}, \mathcal{B}(Y))$-measurable. Indeed, if this holds, we may integrate over Ω the equivalent local formulation ($\mathrm{E}_{\mathrm{loc}}$)-($\mathrm{S}_{\mathrm{loc}}$) from Remark 3.3 of $(Y, \mathcal{E}^\omega, \mathcal{R}^\omega)$, with test functions of the form $\omega \mapsto \widetilde{y}(\omega)$ in ($\mathrm{S}_{\mathrm{loc}}$). As a result of this, it follows that $(t,\omega) \mapsto y^\omega(t)$ can be identified with the unique energetic solution of the system $\big(L^2(\Omega;Y), \mathcal{E}, \mathcal{R}\big)$.

To show the above measurability, we recall the time-discrete approximation scheme for the system $(Y, \mathcal{E}^\omega, \mathcal{R}^\omega)$ from [Mie05] that is, in fact, the basis of the existence result Theorem 3.5. In particular, we consider a partition $\{t_i\}_{i=1,\dots,n}$ of the interval $[0,T]$ and we consider the time-incremental minimization algorithm:

$$
\begin{aligned}
&y_n^{t_1}(\omega) = y^0(\omega),\\
&y_n^{t_i}(\omega) \in \operatorname{Argmin}\big(\mathcal{E}^\omega(t_i, y) + \mathcal{R}^\omega(y - y^{t_{i-1}}) : \ y \in Y\big), \quad i = 2, \dots, n.
\end{aligned}
$$

If we set $p = 2$ and $\mathcal{I}^\omega(y) = \mathcal{E}^\omega(t_2, y) + \mathcal{R}^\omega(y - y^0(\omega))$, then the assumptions (B1)-(B4) are fulfilled. As a result of this, (7.35) and (7.36) imply that $\omega \mapsto y_n^{t_2}(\omega)$ is an $(\mathcal{F}, \mathcal{B}(Y))$-measurable function. Iteratively, we obtain that for each $i \in \{1, \dots, n\}$, $\omega \mapsto y_n^{t_i}(\omega)$ is measurable. We denote by $y_n(\omega) : [0,T] \to Y$ the piecewise constant interpolation of $\big\{y_n^{t_i}(\omega)\big\}$. In [Mie05, Theorems 2.1 and 4.3] it is proved that for all $t \in [0,T]$, $y_n(\omega)(t) \to y^\omega(t)$ P-a.e. as $n \to \infty$. In this respect, we conclude that for each $t \in [0,T]$, $\omega \mapsto y^\omega(t)$ is $(\mathcal{F}, \mathcal{B}(Y))$-measurable. Also, the time-derivative is, in fact, a pointwise a.e. limit of difference quotients, i.e., $\dot{y}^\omega(t) = \lim_{\delta \to 0} \frac{1}{\delta}\left(y^\omega(t+\delta) - y^\omega(t)\right)$ for a.a. t. As a result of this, $\omega \mapsto \dot{y}^\omega$ is also measurable and with that, the proof is done. $\square$

We treat systems of the above form later in the applications, cf. systems $(Y_\varepsilon, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$, $(Y_L, \mathcal{E}_L, \mathcal{R}_L)$ in Section 8.2 and $\big(Y_\varepsilon, \mathcal{E}_L^\gamma, \mathcal{R}_\varepsilon\big)$, $(Y_{\mathrm{ap}}, \mathcal{E}_L, \mathcal{R}_L)$ in Section 8.3. In particular, we always consider mean formulations. However, the above discussed equivalence with the quenched formulation is essential for practical purposes for the approximations of effective systems that we consider, cf. Remarks 8.16 and 8.24.

Gradient flows

In this section we briefly discuss the equivalence of mean and quenched formulations for gradient flows in the EVI formulation (see Section 4 for the general theory).

Specifically, we consider a separable Hilbert space Y. The quenched EVI formulation of the gradient flow is given in terms of the parametrized functionals

$$
\begin{aligned}
&\mathcal{R}^\omega : Y \to \mathbb{R}, \quad \mathcal{R}^\omega(\dot{y}) = \frac{1}{2}\left\langle r^\omega \dot{y}, \dot{y}\right\rangle_{Y^*,Y}, && \text{(dissipation)}\\
&\mathcal{E}^\omega : Y \to \mathbb{R} \cup \{\infty\}. && \text{(energy)}
\end{aligned}
$$

On the other hand, the mean formulation is given in terms of the state space $L^2(\Omega;Y)$ and the corresponding mean functionals:

$$
\begin{aligned}
&\mathcal{R} : L^2(\Omega;Y) \to \mathbb{R}, \quad \mathcal{R}(\dot{y}) = \int_\Omega \mathcal{R}^\omega(\dot{y}(\omega)) dP(\omega), && \text{(dissipation)}\\
&\mathcal{E} : L^2(\Omega;Y) \to \mathbb{R} \cup \{\infty\}, \quad \mathcal{E}(y) = \int_\Omega \mathcal{E}^\omega(y(\omega)) dP(\omega). && \text{(energy)}
\end{aligned}
$$

We assume that the mappings $(\omega, y) \mapsto \mathcal{R}^\omega(y)$ and $(\omega, y) \mapsto \mathcal{E}^\omega(y)$ are normal integrands. This implies that the mean functionals are well-defined. Also, we assume that the gradient flows $(Y, \mathcal{E}^\omega, \mathcal{R}^\omega)$ (for P-a.a. ω) and $\big(L^2(\Omega; Y), \mathcal{E}, \mathcal{R}\big)$ satisfy the assumptions of the existence and uniqueness Theorem 4.3. We remark that the energy $\mathcal{E}^\omega$ is not necessarily convex, but we assume that it is λ-convex with $\lambda \in \mathbb{R}$ independent of ω. Also, we assume that for each $\omega \in \Omega$, there exists $\delta^* > 0$ such that $\mathcal{E}^\omega + \frac{1}{\delta^*}\mathcal{R}^\omega$ has compact sublevels. The latter assumption is used in [RS06, Theorem 2] to prove the convergence of the approximation scheme (7.37) below. Under these assumptions, we obtain the following:

Lemma 7.17. *Let $y^0 \in L^2(\Omega; Y)$ be such that $y^0(\omega) \in \mathrm{dom}(\mathcal{E}^\omega)$ and $y^0 \in \mathrm{dom}(\mathcal{E})$. We denote by $y^\omega \in H^1(0, T; Y)$ the unique EVI solution to the gradient flow $(Y, \mathcal{E}^\omega, \mathcal{R}^\omega)$ with initial datum $y^0(\omega)$ and by $y \in H^1(0, T; L^2(\Omega; Y))$ the unique EVI solution to the gradient flow $\big(L^2(\Omega; Y), \mathcal{E}, \mathcal{R}\big)$ with initial datum y^0. Then, it holds, for all $t \in [0, T]$ and P-a.a. ω,*

$$y^\omega(t) = y(t)(\omega).$$

Sketch of proof. This claim can be shown similarly as Lemma 7.16 by proving measurability of the solution y^ω w.r.t. ω and by an integration of the formulation (EVI) or (IEVI) over Ω. The measurability may be also obtained using a time-discrete approximation for the system $(Y, \mathcal{E}^\omega, \mathcal{R}^\omega)$ that is presented, e.g., in [RS06]. We briefly recall this scheme and we refer to [RS06, Theorem 2] for its convergence. In particular, an equidistant partition $\{t_i\}_{i=1,\dots,n}$ of the interval $[0, T]$ with $\delta = t_2 - t_1$ is considered. The following time-incremental minimization scheme (known as De Giorgi's minimizing movement scheme) provides an approximation for y^ω:

$$\begin{aligned} & y_n^{t_1}(\omega) = y^0(\omega), \\ & y_n^{t_i}(\omega) \in \mathrm{Argmin}\left(\mathcal{E}^\omega(y) + \frac{1}{\delta}\mathcal{R}^\omega(y) - \frac{1}{\delta}\left\langle r^\omega y_n^{t_{i-1}}(\omega), y\right\rangle_{Y^*,Y} :\ y \in Y\right), \quad i \in \{2, \dots, n\}. \end{aligned} \tag{7.37}$$

Note that even if $\mathcal{E}^\omega$ is nonconvex, for small enough δ, $\mathcal{E}^\omega + \frac{1}{\delta}\mathcal{R}^\omega$ is strictly convex. As a result, (7.37) fits into the abstract setting in (7.35) and by a similar argumentation as in Lemma 7.16 measurability of $\omega \mapsto y^\omega$ follows. □

We consider systems of the above form in Section 9, in particular, see systems $(Y, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ in Section 9.1 and $(Y, \mathcal{E}_L, \mathcal{R}_L)$ in Section 9.1.1 (cf. Remark 9.7).

7.3.2 Equality of effective models

In this section we show that for sequences of general random functionals both mean and quenched stochastic homogenization, if both are possible, yield the same effective functional.

Let $p \in (1, \infty)$ and $Q \subset \mathbb{R}^d$ be open. Consider $\{\mathcal{E}_\varepsilon^\omega : L^p(Q) \to \mathbb{R} \cup \{\infty\}\}_{\omega \in \Omega}$, a family of random functionals. We consider the mean counterpart of this sequence of functionals $\mathcal{E}_\varepsilon : L^p(\Omega; L^p(Q)) \to \mathbb{R} \cup \{\infty\}$,

$$\mathcal{E}_\varepsilon(u) = \int_\Omega \mathcal{E}_\varepsilon^\omega(u(\omega)) dP(\omega).$$

We assume the following assumptions: There exist $\Omega' \subset \Omega$ with $P(\Omega') = 1$, $c > 0$, and $\psi \in L^1(\Omega)$ such that:

(C1) The mapping $\Omega \times L^p(Q) \ni (\omega, u) \mapsto \mathcal{E}_\varepsilon^\omega(u)$ defines a convex normal integrand and for all $\omega \in \Omega'$, $\inf_u \mathcal{E}_\varepsilon^\omega(u) \leq \psi(\omega)$.

(C2) It holds that $\mathrm{dom}(\mathcal{E}_\varepsilon^\omega) = X \subset W^{1,p}(Q)$ where X is convex, closed and compactly embedded in $L^p(Q)$, and for all $\omega \in \Omega'$, $\mathcal{E}_\varepsilon^\omega(u) \geq \frac{1}{c}\|u\|^p_{W^{1,p}(Q)} - c$ for all $u \in X$.

Assumptions (C1)-(C2) imply that $\mathcal{E}_\varepsilon$ is well-defined. Also, we have $\mathcal{E}_\varepsilon(u) \geq \frac{1}{c}\|u\|^p_{L^p(\Omega;W^{1,p}(Q))} - c$ for all $u \in L^p(\Omega; X)$. Moreover, $\mathcal{E}_\varepsilon^\omega$ (resp. $\mathcal{E}_\varepsilon$) is equi-coercive in $L^p(Q)$ (resp. w.r.t. weak two-scale convergence in the mean).

Also, we assume that in the limit $\varepsilon \to 0$, $\mathcal{E}_\varepsilon^\omega$ and $\mathcal{E}_\varepsilon$ Γ-converge:

(C3) For P-a.a. ω, $\mathcal{E}_\varepsilon^\omega$ Γ-converges to a functional $\mathcal{E}_{\mathrm{hom}} : L^p(Q) \to \mathbb{R} \cup \{\infty\}$, i.e., it holds:

  (i) If $u_\varepsilon, u \in L^p(Q)$ and $u_\varepsilon \to u$ strongly in $L^p(Q)$, then $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon^\omega(u_\varepsilon) \geq \mathcal{E}_{\mathrm{hom}}(u)$.

  (ii) For $u \in L^p(Q)$, there exists a sequence $u_\varepsilon \in L^p(Q)$ such that $u_\varepsilon \to u$ strongly in $L^p(Q)$ and $\mathcal{E}_\varepsilon^\omega(u_\varepsilon) \to \mathcal{E}_{\mathrm{hom}}(u)$.

(C4) $\mathcal{E}_\varepsilon$ Γ-converges w.r.t. weak stochastic two-scale convergence in the mean to a functional $\widetilde{\mathcal{E}}_{\mathrm{hom}} : L^p(Q) \to \mathbb{R} \cup \{\infty\}$ in the following sense:

  (i) If $u_\varepsilon \in L^p(\Omega \times Q)$, $u \in L^p(Q)$ and $u_\varepsilon \xrightarrow{2} u$, then $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \widetilde{\mathcal{E}}_{\mathrm{hom}}(u)$.

  (ii) For $u \in L^p(Q)$, there exists $u_\varepsilon \in L^p(\Omega \times Q)$, such that $u_\varepsilon \xrightarrow{2} u$, $\mathcal{E}_\varepsilon(u_\varepsilon) \to \widetilde{\mathcal{E}}_{\mathrm{hom}}(u)$.

A specific example of this situation are integral functionals discussed in Section 7.1, see Theorem 7.1 for a mean homogenization result and Remark 7.10 for references to quenched homogenization results (see also Remark 1.2). For a generic situation, we show that the mean and quenched Γ-limits match:

Proposition 7.18. *Let $p \in (1,\infty)$, $Q \subset \mathbb{R}^d$ be open. We assume (C1)-(C4) to hold. Then,*

$$\mathcal{E}_{\mathrm{hom}} = \widetilde{\mathcal{E}}_{\mathrm{hom}}.$$

Proof. Note that using the Γ-convergences in (C3)-(C4) and the corresponding properties of $\mathcal{E}_\varepsilon^\omega$ and $\mathcal{E}_\varepsilon$, it follows that $\mathcal{E}_{\mathrm{hom}}$ and $\widetilde{\mathcal{E}}_{\mathrm{hom}}$ are convex, proper and l.s.c. functionals. To prove the claim, it is sufficient to show that $\widetilde{\mathcal{E}}^*_{\mathrm{hom}} = \mathcal{E}^*_{\mathrm{hom}}$ since $\widetilde{\mathcal{E}}^{**}_{\mathrm{hom}} = \widetilde{\mathcal{E}}_{\mathrm{hom}}$ and $\mathcal{E}^{**}_{\mathrm{hom}} = \mathcal{E}_{\mathrm{hom}}$. Here $(\cdot)^*$ denotes the Legendre-Fenchel transformation (cf. Section A.1).

In the following, we identify $L^p(Q)^*$ and $L^p(\Omega; L^p(Q))^*$, respectively, with $L^q(Q)$ and $L^q(\Omega; L^q(Q))$, where $q = \frac{p}{p-1}$. With the help of Proposition A.7, for any $f \in L^q(\Omega; L^q(Q))$, we have

$$\int_\Omega \left(\mathcal{E}_\varepsilon^\omega\right)^* (f(\omega)) dP(\omega) = \mathcal{E}_\varepsilon^*(f).$$

$\mathcal{E}^*_{\mathrm{hom}} = \widetilde{\mathcal{E}}^*_{\mathrm{hom}}$ follows by passing to the limit $\varepsilon \to 0$ above. In particular, in the limit $\varepsilon \to 0$, it holds:

(a) $\int_\Omega \left(\mathcal{E}_\varepsilon^\omega\right)^* (f) dP(\omega) \to \mathcal{E}^*_{\mathrm{hom}}(f)$ for any $f \in L^q(Q)$.

(b) $\mathcal{E}_\varepsilon^*(f) \to \widetilde{\mathcal{E}}^*_{\mathrm{hom}}(f)$ for any $f \in L^q(Q)$.

In the following we show only (a), and (b) follows similarly (cf. proof of Corollary 7.2). For an arbitrary $u \in L^p(Q)$, using the growth assumption in (C2), it follows that

$$\mathcal{E}_\varepsilon^\omega(u) - \int_Q fudx \leq \left(\frac{c}{p}+1\right)\mathcal{E}_\varepsilon^\omega(u) + \frac{c^2}{p} + \frac{1}{q}\|f\|^q_{L^q(Q)}.$$

As a result of this and using the assumption $\inf_{u\in L^p(Q)} \mathcal{E}_\varepsilon^\omega(u) \leq \psi(\omega)$, it follows that

$$\inf_{u\in L^p(Q)}\left(\mathcal{E}_\varepsilon^\omega(u) - \int_Q fudx\right) \leq \left(\frac{c}{p}+1\right)\psi(\omega) + \frac{c^2}{p} + \frac{1}{q}\|f\|^q_{L^q(Q)}.$$

Note that for P-a.a. $\omega \in \Omega$, $\psi(\omega) < \infty$, thus the above inequality and the Γ-convergence $\mathcal{E}_\varepsilon^\omega \xrightarrow{\Gamma} \mathcal{E}_{\text{hom}}$ imply that for P-a.a. $\omega \in \Omega$ (using a standard Γ-convergence argument, cf. proof of Corollary 7.2)

$$-\left(\mathcal{E}_\varepsilon^\omega\right)^*(f) = \inf_{u\in L^p(Q)}\left(\mathcal{E}_\varepsilon^\omega(u) - \int_Q fudx\right) \to \inf_{u\in L^p(Q)}\left(\mathcal{E}_{\text{hom}}(u) - \int_Q fu\right) = -\mathcal{E}^*_{\text{hom}}(f).$$

Consequently, the dominated convergence theorem and the fact that $\mathcal{E}^*_{\text{hom}}(f)$ is deterministic imply (a). $\square$

7.3.3 Mean vs. quenched two-scale limits

In this section we briefly discuss the relation of stochastic two-scale convergence in the mean (resp. stochastic unfolding) and the notion of quenched stochastic two-scale convergence from [ZP06]. We only present a formal discussion and for a detailed presentation we refer to our article [HNV18, Section 4] (see also [HNV]). The notion of quenched stochastic two-scale convergence is introduced in [ZP06] (see also [MP07, Fag08, Hei11]). This notion of convergence is developed for the treatment of random measures on $\mathbb{R}^d$, however, in the following we focus on the simplified case described by the Lebesgue measure on $\mathbb{R}^d$.

In particular, let $p \in (1,\infty)$ and $Q \subset \mathbb{R}^d$ be open and bounded. We consider an ergodic probability space $(\Omega, \mathcal{F}, P, \tau)$ that satisfies Assumption 6.1. Let $\mathcal{D}$ be a fixed countable and dense subset of $L^p(\Omega)$. We recall the definition of quenched two-scale convergence from [ZP06]: Let $\omega_0 \in \Omega$. For a sequence $u_\varepsilon \in L^p(Q)$ it is said to ω_0-two-scale converges to $u^{\omega_0} \in L^p(\Omega \times Q)$ if in the limit $\varepsilon \to 0$, it holds

$$\int_Q u_\varepsilon(x)\varphi(\tau_{\frac{x}{\varepsilon}}\omega_0)\eta(x)dx \to \int_\Omega\int_Q u^{\omega_0}(\omega,x)\varphi(\omega)\eta(x)dxdP(\omega), \tag{7.38}$$

for all $\varphi \in \mathcal{D}$ and $\eta \in C_c^\infty(Q)$. We say that u_ε quenched stochastically two-scale converges if the above convergence holds for P-a.a. realizations $\omega_0 \in \Omega$.

We remark that in this notion the two-scale limits still depend on the specific realization ω_0. However, typically in homogenization problems, e.g., if the limit equations have unique solutions, the two-scale limits of solutions turn out to be constant in ω_0. Compactness statements in this setting rely on Birkhoff's ergodic theorem and for such results we refer to [ZP06, Section 5].

In the following we will relate quenched two-scale convergence and two-scale convergence in the mean with help of Young measures. For sequences of vector valued functions the relation of weak limit points and pointwise weak accumulation points can be described with the help of *Young* *measures*, see [RS06] (cf. [Bal84]). Similarly, the connection of mean two-scale limits and quenched two-scale limits can be characterized in terms of parametrized Young measures. In particular, we say that a parametrized family of measures $\{\nu_{\omega_0}$ is a measure on $L^p(\Omega\times Q)\}_{\omega_0\in\Omega}$ is a Young measure if

$$\text{for all } \omega_0\in\Omega,\quad \nu_{\omega_0} \text{ is a probability measure on } L^p(\Omega\times Q),$$
$$\omega_0\mapsto \nu_{\omega_0}(B) \text{ is } \mathcal{F}\text{-measurable for all Borel sets B in } L^p(\Omega\times Q).$$

The following characterization of stochastic two-scale limits in the mean holds:

- ([HNV18, Theorem 4.11]). Let u_ε be a bounded sequence in $L^p(\Omega\times Q)$. Then, there exists a (not relabeled) subsequence and a Young measure $\{\nu_{\omega_0}\}_{\omega_0\in\Omega}$ such that for each $\omega_0\in\Omega$, ν_{ω_0} concentrates on the quenched ω_0-two-scale accumulation points of the sequence $u_\varepsilon(\omega_0,\cdot)$, i.e., all cluster points w.r.t. convergence (7.38). Moreover, it holds

$$u_\varepsilon \overset{2}{\rightharpoonup} u \quad \text{in } L^p(\Omega\times Q),\quad u=\int_\Omega\int_{L^p(\Omega\times Q)}\xi d\nu_{\omega_0}(\xi)dP(\omega_0),$$
$$\liminf_{\varepsilon\to 0}\|u_\varepsilon\|^p_{L^p(\Omega\times Q)}\geq \int_\Omega\int_{L^p(\Omega\times Q)}\|\xi\|^p_{L^p(\Omega\times Q)}\,d\nu_{\omega_0}(\xi)dP(\omega_0).$$

For a precise statement and the proof, we refer to [HNV18].

Remark 7.19 (Application)**.** *We may apply this theorem to lift mean homogenization result to quenched results, however, with some additional effort. In particular, using the above result and suitable pointwise a.e. liminf estimates, the weak two-scale convergence of the minimizers* $u_\varepsilon \overset{2}{\rightharpoonup} u$ *from Corollary 7.2 may be improved to: for* P*-a.a.* ω*, up to a not relabeled subsequence, it holds*

$$u_\varepsilon(\omega,\cdot)\rightharpoonup u \quad \textit{weakly in } W^{1,p}(Q).$$

For the detailed analysis, we refer to [HNV18, Theorem 4.18].

7.4 Periodic unfolding in the mean

The stochastic unfolding procedure provides as well a technique for periodic homogenization. We briefly discuss this matter and provide a comparison to the standard periodic unfolding method.

According to Example 6.5, the choice of $(\Omega,\mathcal{F},P)=(\Box_\#,\mathcal{L}(\Box_\#),dy)$ with a shift $\tau_x(y)=x+y$ mod 1 ($x\in\mathbb{R}^d$) defines a suitable probability space for the description of periodic homogenization problems in a continuum setting, e.g., if we consider PDE with coefficients of the form $A(\tau_{\frac{x}{\varepsilon}}y)$. In this regard, by Lemma 6.7, we obtain an unfolding operator $\mathcal{T}_\varepsilon: L^2(\Box_\#\times Q)\to L^2(\Box_\#\times Q)$, where $Q\subset\mathbb{R}^d$ is open and $p\in(1,\infty)$. This operator does not coincide with the standard periodic unfolding operator from [CDG02], cf. the definition (2.3) in Section 2. For this reason, we refer to the stochastic unfolding operator $\mathcal{T}_\varepsilon$ in this case as *periodic unfolding operator in the mean.* A similar method for periodic homogenization is considered in [Nes07], where two-scale convergence in combination with averaging over *phase-shifts* of the coefficients is applied to the treatment of visco-plasticity equations.

We remark that the convergence notions obtained by periodic unfolding and its mean counterpart also differ, since in the latter we consider problems in the mean formulation. In particular, a typical convergence statement obtained by periodic unfolding is $u_\varepsilon(y,\cdot) \to u(\cdot)$ in $L^p(Q)$ for all $y \in \square_\#$, whereas in the mean case we obtain $u_\varepsilon \to u$ in $L^p(\square_\# \times Q)$ (cf. Proposition 7.7). However, if the the sequence $\{y \mapsto u_\varepsilon(y,\cdot)\}_\varepsilon$ is equicontinuous, it may be shown that the mean convergence readily implies pointwise convergence in y. We elaborate on this on the example of an elliptic PDE:

Let $Q \subset \mathbb{R}^d$ be open and bounded, and $A \in L^\infty(\square_\#; \mathbb{R}^{d\times d}_{\text{sym}})$ be such that there exists $c > 0$ with $A(y)F \cdot F \geq c|F|^2$ for a.a. $y \in \square_\#$ and all $F \in \mathbb{R}^d$. For $f \in L^2(Q)$, we consider the equation

$$\begin{aligned} -\text{div}\left(A(\tau_{\frac{x}{\varepsilon}}y)\nabla u_\varepsilon\right) &= f \quad \text{in } \square_\# \times Q, \\ u_\varepsilon &= 0 \quad \text{on } \square_\# \times \partial Q. \end{aligned} \tag{7.39}$$

Using the stochastic unfolding procedure, analogously as in Proposition 7.7, it follows that $u_\varepsilon \to u$ strongly in $L^2(\square_\# \times Q)$, where $u \in H^1_0(Q)$ is the solution of the effective problem

$$\begin{aligned} -\text{div}\left(A_{\text{hom}}\nabla u\right) &= f \quad \text{in } Q, \\ u &= 0 \quad \text{on } \partial Q. \end{aligned} \tag{7.40}$$

We emphasize that the following proposition is obtained only using the stochastic unfolding strategy (i.e., convergence in the mean) and by appealing to equicontinuity of the solution, and it does not rely on any other homogenization method.

Proposition 7.20 (Pointwise convergence)**.** *Let $Q \subset \mathbb{R}^d$ be open and bounded with C^1 boundary, $f \in L^2(Q)$ and $A \in L^\infty(\square_\#; \mathbb{R}^{d\times d}_{\text{sym}})$ be such that there exists $c > 0$ with $A(y)F \cdot F \geq c|F|^2$ for a.a. $y \in \square_\#$ and all $F \in \mathbb{R}^d$. Let u_ε and u be the solutions to (7.39) and (7.40), respectively. Then:*

(i) (Equicontinuity). The family of functions $\left\{u_\varepsilon : \square_\# \to L^2(Q)\right\}_\varepsilon$ is equicontinuous.

(ii) (Convergence). For all $y \in \square_\#$, it holds $u_\varepsilon(y,\cdot) \to u(\cdot)$ strongly in $L^2(Q)$.

Proof. (i) In this proof we take advantage of Meyers' theorem ([Mey63, Theorem 1]), which states that there exist $p, r > 2$ such that the equation, for fixed $y \in \square_\#$,

$$-\text{div}\left(A(\tau_{\frac{\cdot}{\varepsilon}}y)\nabla u\right) = f_\delta \quad \text{in } Q, \quad (\text{where } f_\delta \in L^r(Q)) \tag{7.41}$$

admits a unique solution $u(y,\cdot) \in W^{1,p}_0(Q)$ and

$$\|\nabla u(y,\cdot)\|_{W^{1,p}_0(Q)} \leq c\,\|f_\delta\|_{L^r(Q)}, \tag{7.42}$$

where $c > 0$ is independent of ε, y and f_δ.

We find a sequence $f_\delta \in L^r(Q)$ such that $f_\delta \to f$ strongly in $L^2(Q)$ as $\delta \to 0$. We denote by $u_{\varepsilon,\delta} \in L^p(\square_\#; W^{1,p}_0(Q))$ the unique solution to (7.41) with this f_δ as a right-hand side. For arbitrary $y_1, y_2 \in \square_\#$, we have

$$\begin{aligned} & \|u_\varepsilon(y_1) - u_\varepsilon(y_2)\|_{H^1_0(Q)} \\ \leq\ & \|u_\varepsilon(y_1) - u_{\varepsilon,\delta}(y_1)\|_{H^1_0(Q)} + \|u_{\varepsilon,\delta}(y_1) - u_{\varepsilon,\delta}(y_2)\|_{H^1_0(Q)} + \|u_{\varepsilon,\delta}(y_2) - u_\varepsilon(y_2)\|_{H^1_0(Q)} \\ \leq\ & c\,\|f - f_\delta\|_{L^2(Q)} + \|u_{\varepsilon,\delta}(y_1) - u_{\varepsilon,\delta}(y_2)\|_{H^1_0(Q)}. \end{aligned}$$

The first and third term in the second line are both estimated by the first term in the last line by usual elliptic a priori estimates. The second term on the right-hand side above is estimated as follows

$$\|u_{\varepsilon,\delta}(y_1) - u_{\varepsilon,\delta}(y_2)\|^2_{H^1_0(Q)} \leq \int_Q \left(A(\tau_{\frac{x}{\varepsilon}} y_2) - A(\tau_{\frac{x}{\varepsilon}} y_1) \right) \nabla u_{\varepsilon,\delta}(y_2, x) \cdot (\nabla u_{\varepsilon,\delta}(y_1, x) - \nabla u_{\varepsilon,\delta}(y_2, x))\, dx,$$

where we use equation (7.41). As a result of this, Young's inequality yields

$$\begin{aligned}\|u_{\varepsilon,\delta}(y_1) - u_{\varepsilon,\delta}(y_2)\|^2_{H^1_0(Q)} &\leq c \int_Q | \left(A(\tau_{\frac{x}{\varepsilon}} y_2) - A(\tau_{\frac{x}{\varepsilon}} y_1) \right) \nabla u_{\varepsilon,\delta}(y_2, x)|^2 dx \\ &\leq c \left(\int_Q |A(\tau_{\frac{x}{\varepsilon}} y_2) - A(\tau_{\frac{x}{\varepsilon}} y_1)|^{2q} dx \right)^{\frac{1}{q}} \left(\int_Q |\nabla u_{\varepsilon,\delta}(y_2, x)|^p dx \right)^{\frac{2}{p}},\end{aligned}$$

where the second inequality follows by the application of Hölder's inequality with exponents $\left(\frac{p}{2}, q\right)$ with $q = \frac{p}{p-2} < \infty$. Collecting the above estimates and with the help of (7.42), we obtain

$$\|u_\varepsilon(y_1) - u_\varepsilon(y_2)\|^2_{H^1_0(Q)} \leq c \|f - f_\delta\|^2_{L^2(Q)} + c \|f_\delta\|^2_{L^r(Q)} \left(\int_Q |A(\tau_{\frac{x}{\varepsilon}} y_2) - A(\tau_{\frac{x}{\varepsilon}} y_1)|^{2q} dx \right)^{\frac{1}{q}}.$$

For given $\gamma > 0$ we may choose $\delta(\gamma)$ such that $c \|f - f_\delta\|^2_{L^2(Q)} \leq \frac{\gamma}{2}$. Furthermore, if we choose $|y_1 - y_2|$ small enough, the second term above is also bounded by $\frac{\gamma}{2}$ (and the equicontinuity follows). Indeed, this might be seen by the periodicity of A and the following computation

$$\begin{aligned}\int_Q |A(\tau_{\frac{x}{\varepsilon}} y_2) - A(\tau_{\frac{x}{\varepsilon}} y_1)|^{2q} dx &\leq \sum_{x \in Q^{+\varepsilon} \cap \varepsilon \mathbb{Z}^d} \int_{x + \varepsilon \Box} \left| A\left(y_2 + \frac{\widetilde{x}}{\varepsilon}\right) - A\left(y_1 + \frac{\widetilde{x}}{\varepsilon}\right) \right|^{2q} d\widetilde{x} \\ &= \sum_{x \in Q^{+\varepsilon} \cap \varepsilon \mathbb{Z}^d} \varepsilon^d \int_\Box |A(y_2 + \widetilde{x}) - A(y_1 + \widetilde{x})|^{2q} d\widetilde{x} \\ &= m_\varepsilon(Q^{+\varepsilon} \cap \varepsilon \mathbb{Z}^d) \int_\Box |A(y_2 + \widetilde{x}) - A(y_1 + \widetilde{x})|^{2q} d\widetilde{x}\end{aligned}$$

where $Q^{+\varepsilon} \subset \{x : \operatorname{dist}(x, Q) \leq c\varepsilon\}$. The second line is obtained by a change of variables $\widetilde{x} \rightsquigarrow \frac{\widetilde{x}}{\varepsilon}$ and by the periodicity of A. Note that $m_\varepsilon(Q^{+\varepsilon} \cap \varepsilon \mathbb{Z}^d) \to |Q|$ and therefore, using the fact that spatial shifts are continuous in $L^{2q}(\Box)$, choosing $|y_1 - y_2|$ small enough, the above expression may be made arbitrarily small independently of ε. This means that we obtain even more than the claim, i.e., we obtain equicontinuity w.r.t. the $H^1(Q)$-norm.

(ii) Using the stochastic unfolding procedure, as in Proposition 7.7, we conclude that $u_\varepsilon \to u$ in $L^2(\Box_\# \times Q)$. Using the equicontinuity of u_ε from (i) and the usual a priori bound $\|u_\varepsilon(y, \cdot)\|_{H^1_0(Q)} \leq c \|f\|_{L^2(Q)}$ (that holds for all $y \in \Box_\#$), the Arzelà-Ascoli theorem yields the existence of a (not relabeled) subsequence such that for all $y \in \Box_\#$,

$$u_\varepsilon(y, \cdot) \to u(\cdot) \quad \text{strongly in } L^2(Q).$$

Note that here we use the compact embedding $H^1_0(Q) \subset\subset L^2(Q)$. Also, we may dispense of the extraction of the subsequence since for any subsequence of ε, up to extraction of another subsequence, the above argument applies. $\Box$

Remark 7.21. *Part (ii) in the above theorem does not rely on the specific equation, but bases merely on strong convergence in the mean and on equicontinuity. In this respect, we may obtain pointwise convergence using the periodic unfolding method in the mean as long as these two ingredients are available.*

Remark 7.22 (Discrete case)**.** *The stochastic unfolding procedure in the discrete setting (cf. Section 5) also provides a method for periodic homogenization, well-suited for discrete-to-continuum transition problems. In particular, for $N \in \mathbb{N}$, we set $\Omega = \square_N := \mathbb{Z}^d_{/N\mathbb{Z}^d}$ the discrete N-torus equipped with a (rescaled) counting measure. We define a discrete dynamical system, for $x \in \mathbb{Z}^d$, $\tau_x y = y + x \mod N$. This choice of the probability space fits into the framework of Section 5 and consequently we obtain a method that is well-suited for the treatment of problems involving periodic and rapidly-oscillating coefficients of the form $\mathbb{Z}^d \ni x \mapsto A(\tau_{\frac{x}{\varepsilon}} y)$, where $A : \square_N \to \mathbb{R}^{d\times d}$. In this case, pointwise convergence in $y \in \square_N$ follows directly from convergence in the mean since $\square_N$ contains only finitely many elements. In particular, if $u_\varepsilon \to u$ strongly in $L^p(\square_N \times \mathbb{R}^d)$ for $u_\varepsilon \in L^p(\square_N \times \varepsilon\mathbb{Z}^d)$ and $u \in L^p(\mathbb{R}^d)$ (cf. Definition 5.1 and Remark 5.15), then for all $y \in \square_N$, $u_\varepsilon(y, \cdot) \to u(\cdot)$ strongly in $L^p(\mathbb{R}^d)$.*

Part III

Applications

Summary of main results

In this part we derive stochastic homogenization results for some evolutionary equations with the help of the stochastic unfolding procedure. In particular, Section 8 is devoted to homogenization and discrete-to-continuum transition for certain evolutionary rate-independent systems. Section 9 treats homogenization of a continuum L^2-type gradient flow driven by a λ-convex energy functional. In the following we briefly summarize our main findings.

Section 8: In this section, we examine discrete networks consisting of springs with random and oscillating coefficients. In particular, in Section 8.2 we consider an ERIS given in terms of the following energy and dissipation functionals, (state variable $y_\varepsilon = (u_\varepsilon, z_\varepsilon)$, time $t \in [0,T]$)

$$\mathcal{E}_\varepsilon(t, y_\varepsilon) = \frac{1}{2}\left\langle \int_{Q^{+\varepsilon}\cap\varepsilon\mathbb{Z}^d} A(\tau_{\frac{x}{\varepsilon}}\omega)\begin{pmatrix}\nabla_s^\varepsilon u_\varepsilon(\omega,x)\\ z_\varepsilon(\omega,x)\end{pmatrix}\cdot\begin{pmatrix}\nabla_s^\varepsilon u_\varepsilon(\omega,x)\\ z_\varepsilon(\omega,x)\end{pmatrix} - \pi_\varepsilon l(t,x)\cdot u_\varepsilon(\omega,x) dm_\varepsilon(x)\right\rangle,$$
$$\mathcal{R}_\varepsilon(\dot{y}_\varepsilon) = \left\langle \int_{Q^{+\varepsilon}\cap\varepsilon\mathbb{Z}^d} \rho(\tau_{\frac{x}{\varepsilon}}\omega, \dot{z}_\varepsilon(\omega,x)) dm_\varepsilon(x)\right\rangle$$

defined on the state space $Y_\varepsilon = \left(L^2(\Omega)\otimes L^2_0(Q\cap\varepsilon\mathbb{Z}^d)\right)^d \times \left(L^2(\Omega)\otimes L^2_0(Q^{+\varepsilon}\cap\varepsilon\mathbb{Z}^d)\right)^k$. Above, $Q, Q^{+\varepsilon}\subset\mathbb{R}^d$, $(\Omega,\mathcal{F},P,\tau)$ is an ergodic probability space given as in Section 5.1 and $t\mapsto l(t)$ is a given external loading. Systems of this type emerge in modeling of the rate-independent response of discrete networks that consist of random elasto-plastic springs of typical size $\varepsilon \ll 1$ (see Section 8). In this context, u_ε is a displacement variable and z_ε is an internal variable accounting for the plastic deformations of the springs. Under certain standard assumptions on the integrands A and ρ, we obtain a stochastic homogenization result. Particularly, in Theorem 8.12 we show that if y_ε is the solution to the ERIS $(Y_\varepsilon, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$, the following implication holds, for the limits $\varepsilon\to 0$,

$$y_\varepsilon(0) \overset{c2}{\to} y(0) = (u(0), z(0), \chi(0)) \quad\Rightarrow\quad \forall t\in(0,T],\quad y_\varepsilon(t)\overset{c2}{\to} y(t) = (u(t), z(t), \chi(t)), \tag{7.43}$$

where $\overset{c2}{\to}$ denotes *"cross" two-scale convergence* that is explained in (8.17) in Section 8.2. Also, $y:[0,T]\to Y$ denotes the solution to an effective continuum ERIS given in terms of the (extended) state space $Y = H^1_0(Q)^d \times L^2(\Omega\times Q)^k \times \left(L^2_{\text{pot}}(\Omega)\otimes L^2(Q)\right)^d$ and effective functionals $\mathcal{E}_0$ and $\mathcal{R}_0$ (see Section 8.2 for definitions). The quadratic term in the effective energy $\mathcal{E}_0$ takes the form

$$\langle \mathbb{A}y, y\rangle_{Y^*,Y} = \left\langle \int_Q A(\omega)\begin{pmatrix}\nabla_s u(x) + \chi_s(\omega,x)\\ z(\omega,x)\end{pmatrix}\cdot\begin{pmatrix}\nabla_s u(x)+\chi_s(\omega,x)\\ z(\omega,x)\end{pmatrix}dx\right\rangle. \tag{7.44}$$

In the limit system, the discrete physical space $Q\cap\varepsilon\mathbb{Z}^d$ is replaced by its continuum counterpart $Q\subset\mathbb{R}^d$ and the coefficients do not include any more rapid oscillations. However, $(Y,\mathcal{E}_0,\mathcal{R}_0)$ is still stochastic and it does not reduce to a classical deterministic ERIS since the internal variable z is not deterministic in general. The proof of our homogenization result relies on the stochastic unfolding procedure, it uses an abstract strategy for asymptotic analysis of ERIS from [MRS08] (cf. Section 3), and it is motivated by the analysis in [MT07], where similarly the periodic unfolding method is employed for periodic homogenization of continuum elasto-plasticity. In the continuum case, similar results have been obtained for deterministic periodic materials in [MT07, Vis08, Han11] (via periodic unfolding) and [Nes07] (viscoplasticity via two-scale convergence), and recently for random materials in [Hei17] (using quenched stochastic two-scale convergence), [HS16, HS18] and [HLL18] (visco-elasticity using stochastic two-scale convergence in the mean). In the stochastic and discrete-to-continuum setting, to the best of our knowledge, this result has not been obtained earlier.

Despite being significantly simpler than the original problem, the effective system presents some computational challenges - the presence of the stochastic gradient χ makes the computation of the solution inaccessible (cf. discussion in Section 7.2). We present an approximation scheme that is motivated by the standard representative volume element (RVE) method (see [Owh03, BP04] and Section 7.2). However, in contrast to problems where the effective coefficients are deterministic quantities and may be replaced by their separately computed approximations, we deal with a genuine two-scale effective formulation. For this reason, the approximation for the effective system is as well given in a specific two-scale form in an extended space. The procedure that we propose is similar to the so-called FE^2-method from numerical homogenization (see, e.g., [Fey99, Mie02, SH13, NW19]). In particular, we consider a parameter $L \gg 1$, which corresponds to the representative volume element size (in practice $\varepsilon \ll \frac{1}{L}$). We introduce a new "averaging" variable $q \in LB \cap \mathbb{Z}^d$ ($B \subset \mathbb{R}^d$) and the expression (7.44) is replaced by

$$\left\langle \fint_{LB^+\cap\mathbb{Z}^d} \int_Q A(\tau_q\omega)\begin{pmatrix}\nabla_{x,s}u(\omega,x)+\nabla_{q,s}\varphi(\omega,q,x)\\ z(\omega,q,x)\end{pmatrix}\cdot\begin{pmatrix}\nabla_{x,s}u(\omega,x)+\nabla_{q,s}\varphi(\omega,q,x)\\ z(\omega,q,x)\end{pmatrix} dx dm(q)\right\rangle.$$

This choice leads to an ERIS $(Y_L, \mathcal{E}_L, \mathcal{R}_L)$ that is defined on an extended space and where the variable χ is replaced by its deterministic and discrete counterpart $\nabla_q\varphi$, where $\varphi(\omega,\cdot,x)$ is defined on a large set $LB \cap \mathbb{Z}^d$ and satisfies homogeneous Dirichlet boundary conditions. In Theorem 8.15, we show that in the limit $L \to \infty$, $(Y_L, \mathcal{E}_L, \mathcal{R}_L)$ converges to $(Y, \mathcal{E}_0, \mathcal{R}_0)$ in a suitable sense, similarly as in (7.43). The proof of this statement is similar to the proof of Theorem 8.12 and relies on the stochastic unfolding procedure.

In Section 8.3, we consider a discrete version of gradient plasticity. Namely, we modify the energy functional $\mathcal{E}_\varepsilon$ by adding a gradient term of the variable z_ε that penalizes large oscillations of the sequence z_ε, i.e., $\mathcal{E}^\gamma_\varepsilon(t,y_\varepsilon) = \mathcal{E}_\varepsilon(t,y_\varepsilon) + \left\langle \int_{Q+\varepsilon\cap\varepsilon\mathbb{Z}^d} G\varepsilon^\gamma\nabla^\varepsilon z_\varepsilon\cdot\varepsilon^\gamma\nabla^\varepsilon z_\varepsilon dm_\varepsilon\right\rangle$ with $\gamma \in (0,1)$. As a result of this, letting $\varepsilon \to 0$, the two-scale limit of the sequence z_ε turns out to be deterministic. Accordingly, we derive a deterministic limit system that takes the form of classical continuum elasto-plasticity. In particular, in Theorem 8.20 we show that as $\varepsilon \to 0$, the modified system $(Y_\varepsilon, \mathcal{E}^\gamma_\varepsilon, \mathcal{R}_\varepsilon)$ converges similarly as in (7.43) to a deterministic limit ERIS $(Y_{\text{hom}}, \mathcal{E}_{\text{hom}}, \mathcal{R}_{\text{hom}})$. Here, $Y_{\text{hom}} = H^1_0(Q)^d \times L^2(Q)^k$, $\mathcal{E}_{\text{hom}}(t,y) = \int_Q A_{\text{hom}}\binom{\nabla_s u}{z}\cdot\binom{\nabla_s u}{z} - l(t)\cdot u dx$, and $\mathcal{R}_{\text{hom}}(y) = \int_Q \rho_{\text{hom}}(z)dx$. In the setting of periodic homogenization, a similar result is obtained in [Han11]. In the stochastic case, as far as the author knows, this result is new.

The effective properties A_{hom} and ρ_{hom} are deterministic, yet still difficult to compute since A_{hom} is given in terms of a stochastic corrector equation (cf. Section 7.2) and $\rho_{\text{hom}}(\cdot)$ is defined as the expectation of $\omega \mapsto \rho(\omega,\cdot)$ that might be unknown in practice. We apply the RVE method and replace the corrector equation with its "cut-off" on a large set $LB\cap\mathbb{Z}^d$ that yields an approximation A_L for A_{hom}. Also, we replace ρ_{hom} by a suitable approximation ρ_L. In this respect, similarly as in Lemma 7.11 based on the stochastic unfolding method, we show $A_L \to A_{\text{hom}}$ and $\rho_L \to \rho$ in a suitable sense (Lemma 8.23). This observation, in combination with standard a priori estimates for ERIS, implies the convergence of the ERIS given in terms of A_L and ρ_L to $(Y_{\text{hom}}, \mathcal{E}_{\text{hom}}, \mathcal{R}_{\text{hom}})$, see Theorem 8.25. The convergence statement $A_L \to A_{\text{hom}}$ is already obtained in the continuum setting in [Owh03, BMW94] (even in a stronger topology), however, we present an alternative proof based on stochastic unfolding.

Section 9: In this section, our analysis focuses on L^2-type gradient flows with λ-convex energy functionals and it covers classical examples of parabolic quasilinear equations such as Allen-Cahn type equations, but also nonlinear equations such as equations driven by the p-Laplace operator with $p \in (1,\infty)$. The evolution is given in terms of the following energy and dissipation functionals,

$$
\begin{aligned}
\mathcal{E}_\varepsilon(y) &= \left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, \nabla y(\omega,x)) + f(\tau_{\frac{x}{\varepsilon}}\omega, x, y(\omega,x))dx \right\rangle, \\
\mathcal{R}_\varepsilon(v) &= \left\langle \int_Q r(\tau_{\frac{x}{\varepsilon}}\omega, x)|v(\omega,x)|^2 dx \right\rangle,
\end{aligned}
$$

defined on the state space $Y = L^2(\Omega \times Q)$ (for unsuitable functions $\mathcal{E}_\varepsilon = \infty$). Here, $Q \subset \mathbb{R}^d$ and $(\Omega, \mathcal{F}, P, \tau)$ is a probability space given as in Section 6.1. We assume p-growth assumptions ($p > 1$) and convexity for the integrand V in its last variable. Also, we assume s-growth assumptions ($s \geq 2$) and λ-convexity ($\lambda \in \mathbb{R}$) for f in its last variable, which results in Λ-convexity of the energy $\mathcal{E}_\varepsilon$ (for suitable $\Lambda \in \mathbb{R}$). Upon assuming some additional standard assumptions, in the limit $\varepsilon \to 0$, we derive a gradient flow given in terms of effective functionals $\mathcal{E}_{\text{hom}}$ and $\mathcal{R}_{\text{hom}}$ defined on the state space $L^2_{\text{inv}}(\Omega) \otimes L^2(Q)$, which in the ergodic case boil down to

$$
\mathcal{E}_{\text{hom}}(y) = \int_Q V_{\text{hom}}(x, \nabla y(x)) + f_{\text{hom}}(x, y(x))dx, \quad \mathcal{R}_{\text{hom}}(v) = \int_Q r_{\text{hom}}(x)|v(x)|^2 dx,
$$

where V_{hom}, f_{hom} and r_{hom} are deterministic effective integrands (see Remark 9.5). In particular, in Theorem 9.3 we show the following well-prepared E-convergence statement:

$$
\begin{aligned}
&\text{If} \quad y_\varepsilon(0) \to y(0) \quad \text{strongly in } Y, \quad \mathcal{E}_\varepsilon(y_\varepsilon(0)) \to \mathcal{E}_{\text{hom}}(y(0)), \\
&\text{then} \quad \forall t \in (0,T], \quad y_\varepsilon(t) \to y(t) \quad \text{strongly in } Y, \quad \mathcal{E}_\varepsilon(y_\varepsilon(t)) \to \mathcal{E}_{\text{hom}}(y(t)).
\end{aligned}
$$

where y_ε and y are the unique EVI solutions to $(Y, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ and $\big(L^2_{\text{inv}}(\Omega) \otimes L^2(Q), \mathcal{E}_{\text{hom}}, \mathcal{R}_{\text{hom}}\big)$, respectively. An important tool in standard approaches to such problems in the deterministic setting is the compact embedding $H^1(Q) \subset\subset L^2(Q)$, which provides a possible access to the limit passage for the nonconvex part of the energy. In contrast, in our stochastic setting, the compact embedding of $L^2(\Omega) \otimes H^1(Q)$ into $L^2(\Omega) \otimes L^2(Q)$ does not hold and therefore we need to work a priori with weak convergence arguments. Fortunately, using the λ-convexity of the energy, the considered gradient flow may be transformed into a formulation given in terms of a new convex time-dependent energy functional as discussed in Section 4. Using this observation and the stochastic unfolding method, we obtain the proof of Theorem 9.3. In this respect, we remark that the stochastic unfolding procedure provides a very simple homogenization tool which mostly bases on l.s.c. and continuity arguments for integral functionals, however, we need to rely on merely weak-type convergence, which works well for convex problems. For nonconvex problems some additional care has to be taken as in the above example. Also, we point out that for problems involving nonconvexity in ∇y, i.e., if $V(\omega, x, \cdot)$ is nonconvex, the stochastic unfolding method alone is not sufficient for homogenization. However, the same difficulty is present in the case of periodic unfolding or periodic two-scale convergence. In the periodic setting homogenization results of this type are obtained for quasilinear parabolic equations, e.g., in [NRJ07, Wou10, FMP12] (via two-scale convergence and unfolding), for reaction-diffusion systems with different diffusion length scales in [MRT14] (via unfolding), for Cahn-Hilliard type equations in [LR18] (via unfolding). In the stochastic case, linear parabolic equations are treated in [ZKO82], quasilinear equations in [DR09, Hei12]. In the nonlinear setting, a related result to ours is obtained in [EP05] where nonlinear parabolic type equations are treated. However, the approach we consider is different, it relies on the gradient flow formulation and we do not rely on differentiability of the integrands V and f and on continuity assumptions on their derivatives.

The effective integrand V_{hom} is given by a homogenization formula in terms of a stochastic gradient and the integrands f_{hom} and r_{hom} are obtained as the expectations of f and r. Similarly as before, these facts make the evaluation of these quantities in general inaccessible. In Section 9.1.1 we discuss an approximation scheme that is based on the RVE strategy for the effective system in a simplified setting of an Allen-Cahn type gradient flow. Namely, in this case the effective gradient system takes the form of the following PDE (equipped with suitable initial and boundary conditions)

$$r_{\text{hom}}\dot{y} - \operatorname{div}(A_{\text{hom}}\nabla y) + f'_{\text{hom}}(y) = 0.$$

We replace the coefficient A_{hom} by its "cut-off" A_L (cf. Lemma 7.11). Also, r_{hom} and $f_{\text{hom}}(\cdot)$ are replaced by suitable averages of their stationary extensions over large sets $LB \subset \mathbb{R}^d$. In particular, we show that the corresponding solutions satisfy $y_L(t) \to y(t)$ in $L^2(\Omega \times Q)$ for all $t \in (0, T]$, if $y_L(0) \to y(0)$ in $L^2(\Omega \times Q)$ in the limit $L \to \infty$, see Theorem 9.8. The proof of this result bases on a standard a priori Gronwall type estimate for the considered equation and on the convergence of the coefficients in a suitable topology (cf. Section 7.2).

References: The two main articles that this part is based on are: [NV18] written by Stefan Neukamm and the author (Section 8), and [HNV19] written by Martin Heida, Stefan Neukamm and the author (Section 9). Nevertheless, the approximation results in Sections 8.2.1, 8.3.1 and 9.1.1 are new and not published yet.

8 Discrete rate-independent systems

In this section, we study the macroscopic rate-independent behavior of periodic networks formed of elasto-plastic springs with random material properties. Below, we shortly present a simple two-dimensional model problem to demonstrate the modeling principles that we follow. As a preparation for the rate-independent evolutionary problem, we first discuss a homogenization procedure for a static convex minimization problem in Section 8.1. Section 8.2 is devoted to homogenization and discrete-to-continuum transition for a network consisting of elasto-plastic springs and we also discuss an approximation scheme for the obtained effective system. In the following Section 8.3, we modify the previous microscopic problem by introducing a gradient regularization term that reduces the homogenized problem to a standard deterministic continuum elasto-plasticity problem. Moreover, we discuss approximations for the effective system. To keep the exposition clutter-free, the proofs in this part are presented at the end of each section.

Model problem

We now present a simple two-dimensional discrete model problem, but in the following Section 8.2 we shall treat more general, multidimensional problems of this type. To explain the model, we first consider a single spring that in its natural state has endpoints $x_0, x_1 \in \mathbb{R}^d$, and thus is aligned with $b := x_1 - x_0$. We describe a deformation of the spring with the help of a displacement function v that maps an endpoint x_i to its new position $x_i + v(x_i)$. As the measure of relative elongation (resp. compression) of the spring, we consider the Cauchy strain $\frac{|(b+|b|\partial_b v)|-|b|}{|b|}$ with $\partial_b v := \frac{v(x_0+b)-v(x_0)}{|b|}$. If the displacement is (infinitesimally) small, i.e., $v = \delta u$ with $0 < \delta \ll 1$ and $u : \{x_0, x_1\} \to \mathbb{R}^d$, we arrive, by rescaling the strain $\frac{1}{\delta}\frac{|(b+|b|\partial_b v)|-|b|}{|b|}$ and passing to the limit $\delta \to 0$, at the *linearized strain* $\frac{b}{|b|} \cdot \partial_b u$. As is usual in linear elasto-plasticity (see, e.g., [HR12, Section 3]), we assume that the linearized strain admits an additive decomposition $\frac{b}{|b|} \cdot \partial_b u = \epsilon + z$, where ϵ and z are its *elastic* and *plastic* parts, respectively. The force (its intensity) exerted by the spring is linear in the elastic strain: $\sigma = a\epsilon$, $a > 0$ being the spring constant. We define a free energy, that describes elasto-plastic materials with linear kinematic hardening, by

$$\mathcal{E}_b(u, z) := \frac{1}{2} a \left(\frac{b}{|b|} \cdot \partial_b u - z \right)^2 + \frac{1}{2} h z^2,$$

where $h > 0$ denotes a hardening parameter. Let $T > 0$ be a finite time horizon. The rate-independent evolution of the elasto-plastic spring under the loading $l : [0, T] \times \{x_0, x_1\} \to \mathbb{R}^d$ is determined by

$$\begin{aligned} &(-1)^i a \left(\frac{b}{|b|} \cdot \partial_b u(t) - z(t) \right) \frac{b}{|b|} + l(t, x_i) = 0, \quad i = 0, 1, \\ &\dot{z}(t) \in \partial I_{[-\sigma_y, \sigma_y]} \left(-\frac{\partial \mathcal{E}_b}{\partial z}(u(t), z(t)) \right). \end{aligned} \tag{8.1}$$

In (8.1), $\sigma_y \geq 0$ is the yield stress of the spring, $\partial I_{[-\sigma_y, \sigma_y]}$ denotes the convex subdifferential of $I_{[-\sigma_y, \sigma_y]}$, which is the indicator function of the set $[-\sigma_y, \sigma_y]$ (see Section A.1 for definitions). Note that the first two equations are force balance equations (inertial terms are disregarded), reasonable in regimes of small displacements, and the second expression is a flow rule for the variable z.

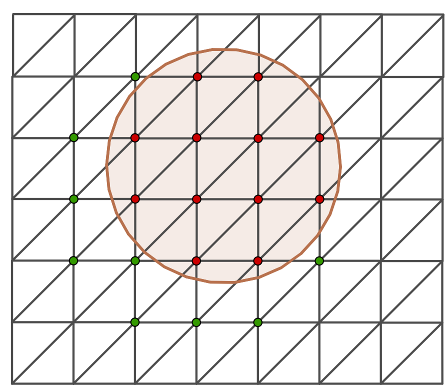


Figure 8.1: *Two-dimensional periodic lattice graph. The shaded region represents $Q \subset \mathbb{R}^2$, $Q^\varepsilon = Q \cap \varepsilon\mathbb{Z}^2$ is the collection of all red dots, $Q^{+\varepsilon}$ is the collection of all red and green dots.*

We consider a network of springs $E = \{e = [x, x + \varepsilon b] : x \in \varepsilon\mathbb{Z}^2, b \in \{e_1, e_2, e_1 + e_2\}\}$, where the nodes $x \in \varepsilon\mathbb{Z}^2$ represent the reference configuration of particles connected by springs. The displacement of the network is described with help of a map $u : \varepsilon\mathbb{Z}^2 \to \mathbb{R}^2$, and the plastic strains of the springs are accounted by an internal variable $z : \varepsilon\mathbb{Z}^2 \to \mathbb{R}^3$, e.g., $z_1(x)$ is the plastic strain of the spring $[x, x + \varepsilon e_1]$. We assume that the particles outside of a set $Q^\varepsilon := Q \cap \varepsilon\mathbb{Z}^d$

are fixed, i.e., $u = 0$ in $\varepsilon\mathbb{Z}^d \setminus Q^\varepsilon$, where $Q \subset \mathbb{R}^d$. Furthermore, we suppose that z is supported in $Q^{+\varepsilon} := \{x \in \varepsilon\mathbb{Z}^2 : (x, x+\varepsilon b) \cap Q \neq \emptyset \text{ for some } b \in \{e_1, e_2, e_1+e_2\}\}$ (see Figure 8.1). A small external loading $\varepsilon l_\varepsilon : [0,T] \times Q^\varepsilon \to \mathbb{R}^2$ acts on the system. According to the evolution law (8.1) for single springs, the evolution of the network is determined by, for $t \in [0,T]$,

$$\begin{aligned} &\sum_{b\in\{e_1,e_2,e_1+e_2\}} -\partial_{-\varepsilon b}\left(|b|a(x,b)\left(\frac{b}{|b|}\cdot\partial_{\varepsilon b}u(t,x) - z_b(t,x)\right)\right)\frac{b}{|b|} + l_\varepsilon(t,x) = 0 \quad \text{in } Q^\varepsilon, \\ &\dot{z}_b(t,x) \in \partial I_{[-\sigma_y(x,b),\sigma_y(x,b)]}\left(-\frac{\partial\mathcal{E}_b}{\partial z_b}(u(t,x),z_b(t,x))\right) \quad \text{in } Q^{+\varepsilon},\ b \in \{e_1,e_2,e_1+e_2\}, \end{aligned} \tag{8.2}$$

which is a superposition of (8.1). We tacitly identify $b \in \{e_1, e_2, e_1+e_2\}$ with indices $i \in \{1,2,3\}$. The coefficients $a(x,b)$, $h(x,b)$, $\sigma_y(x,b)$ describe the properties of the spring $[x, x+\varepsilon b]$. The above equations may be equivalently recast in the global energetic formulation for rate-independent systems with the help of an energy and dissipation functionals, respectively: $\mathcal{E}_\varepsilon : [0,T] \times L^2_0(Q^\varepsilon)^2 \times L^2_0(Q^{+\varepsilon})^3 \to \mathbb{R}$, $\mathcal{R}_\varepsilon : L^2_0(Q^\varepsilon)^2 \times L^2_0(Q^{+\varepsilon})^3 \to [0,\infty)$,

$$\begin{aligned} \mathcal{E}_\varepsilon(t,u,z) &= \int_{Q^{+\varepsilon}} \frac{1}{2}A(x)\left(\nabla^\varepsilon_s u(x) - z(x)\right)\cdot\left(\nabla^\varepsilon_s u(x) - z(x)\right) \\ &\quad + \frac{1}{2}H(x)z(x)\cdot z(x)dm_\varepsilon(x) - \int_{Q^\varepsilon} l_\varepsilon(t,x)\cdot u(x)dm_\varepsilon(x), \\ \mathcal{R}_\varepsilon(u,z) &= \sum_{b\in\{e_1,e_2,e_1+e_2\}} |b|\int_{Q^{+\varepsilon}} \sigma_y(x,b)|z_b(x)|dm_\varepsilon(x). \end{aligned} \tag{8.3}$$

Above, the coefficients are given in the form $A(x) = \operatorname{diag}\left(a(x,e_1), a(x,e_2), \sqrt{2}a(x,e_1+e_2)\right)$, $H(x) = \operatorname{diag}(h(x,e_1), h(x,e_2), \sqrt{2}h(x,e_1+e_2))$, and $\nabla^\varepsilon_s u = \left(\frac{b}{|b|}\cdot\partial_{\varepsilon b}u\right)_{b\in\{e_1,e_2,e_1+e_2\}}$ is the symmetrized gradient introduced in Section 5.1.1. Namely, if we test the first equation in (8.2) with $\dot{u}(t) - \widetilde{u}$, with an arbitrary $\widetilde{u} \in L^2_0(Q^\varepsilon)^2$, we obtain

$$\langle -D_u\mathcal{E}_\varepsilon(t,u(t),z(t)), \dot{u}(t) - \widetilde{u}\rangle_{L^2_0(Q^\varepsilon)^2} = 0. \tag{8.4}$$

Moreover, using the Fenchel equivalence Lemma A.1 (see also Example A.2), the second equation in (8.2) is equivalent to, for $b \in \{e_1, e_2, e_1+e_2\}$,

$$\mathcal{R}_b(x,\dot{z}_b(t,x)) \leq \mathcal{R}_b(x,\widetilde{z}) - \frac{\partial\mathcal{E}_b}{\partial z_b}(u(t,x),z_b(t,x)))\left(\dot{z}_b(t,x) - \widetilde{z}\right), \quad \text{for all } \widetilde{z} \in \mathbb{R}, \tag{8.5}$$

where $\mathcal{R}_b(x,z) = \sigma_y(x,b)|z|$. If we multiply (8.5) by $|b|$, sum it up in b and integrate over $Q^{+\varepsilon}$, it follows that

$$\mathcal{R}_\varepsilon(\dot{u}(t),\dot{z}(t)) \leq \mathcal{R}_\varepsilon(\widetilde{u},\widetilde{z}) - \langle D_z\mathcal{E}_\varepsilon(t,u(t),z(t)), \dot{z}(t) - \widetilde{z}\rangle_{L^2_0(Q^{+\varepsilon})^3} \quad \text{for all } (\widetilde{u},\widetilde{z}) \in L^2_0(Q^\varepsilon)^2 \times L^2_0(Q^{+\varepsilon})^3. \tag{8.6}$$

Summing up (8.4) and (8.6), it follows that (8.2) boils down to

$$0 \in \partial\mathcal{R}_\varepsilon(\dot{u}(t),\dot{z}(t)) + D\mathcal{E}_\varepsilon(t,u(t),z(t)),$$

where $D = (D_u, D_z)$. According to Remark 3.3, this differential inclusion is equivalent to the global energetic formulation (E)-(S) for the system $(Y,\mathcal{E},\mathcal{R}) = \left(L^2_0(Q^\varepsilon)^2 \times L^2_0(Q^{+\varepsilon})^3, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon\right)$. If we consider the constraint $z = 0$, time-independent force $l_\varepsilon(t) = l_\varepsilon$ and 0 initial datum, the above problem reduces to the minimization of the functional $u \mapsto \mathcal{E}_\varepsilon(0, u, 0)$. This corresponds to the static equilibrium of a spring network with only elastic interactions.

In the following, we assume that the coefficients are random fields oscillating on a scale ε. In particular, the deterministic coefficients $A(x)$, $H(x)$ and $\sigma_y(x, b)$ in the above functionals are replaced by realizations of rescaled stationary random fields $A(\tau_{\frac{x}{\varepsilon}}\omega)$, $H(\tau_{\frac{x}{\varepsilon}}\omega)$ and $\sigma_y^b(\tau_{\frac{x}{\varepsilon}}\omega)$. As a consequence of this, the solutions of the corresponding evolutionary equation at each time instance are not deterministic functions but rather random fields on $\Omega \times \varepsilon\mathbb{Z}^2$. Under suitable assumptions (see Theorem 3.5 and 8.2), there exists a unique solution $(u_\varepsilon, z_\varepsilon) \in C^{\mathrm{Lip}}([0,T], (L^2(\Omega) \otimes L^2_0(Q^\varepsilon)^2) \times (L^2(\Omega) \otimes L^2_0(Q^{+\varepsilon})^3))$ to the above described microscopic rate-independent system. Applying the method of stochastic unfolding, we are able to capture the averaged (w.r.t. the probability measure) behavior of the solution $(u_\varepsilon, z_\varepsilon)$ in the limit $\varepsilon \to 0$. Particularly, we show that, upon assuming suitable strong convergence for the initial data and forces, there exists $(u, z, \chi) \in C^{\mathrm{Lip}}([0,T], H^1_0(Q)^2 \times L^2(\Omega \times Q)^3 \times (L^2_{\mathrm{pot}}(\Omega) \otimes L^2(Q))^2)$, which solves an effective rate-independent system on a continuum physical space (see Section 8.2), and for all $t \in [0,T]$

$$(u_\varepsilon(t), z_\varepsilon(t)) \overset{c2}{\to} (u(t), z(t), \chi(t)),$$

where $\overset{c2}{\to}$ denotes "cross" two-scale convergence, which is defined in (8.17) in Section 8.2.

As the below Example 8.1 suggests, even if we start with elasto-plastic springs with linear kinematic hardening, the effective system does not correspond to classical continuum elasto-plasticity with linear-kinematic hardening, but it admits a genuine two-scale formulation (see Section 8.2). We remark that the methods that we use apply as well to systems with different constitutive laws; e.g., one might consider an energy functional with an additional term depending on the gradient of the internal variable z_ε, as is the case in gradient plasticity (see Section 8.3). For this system, in the ergodic case, we even obtain a deterministic continuum elasto-plastic limiting model with linear-kinematic hardening (cf. Figure 8.3).

The function of discrete lattice and network models is twofold: They might be seen as numerical finite difference approximations to continuum models (numerical discretization). On the other hand, such discrete models might be used as a direct modeling approach for problems that involve microstructural discreteness, e.g., truss-like structures (structural discretization). For general discussion of such discrete models we refer to [JB94, OS02, HWK10, BKN13] and the references therein. The convergence results that we establish can be seen as a justification of continuum models for microstructural spring networks that feature uncertainty in the constitutive relations on the microscopic scale. In this context, the method could also be applied to prove the consistency of computational schemes based on the lattice method.

Example 8.1 (One-dimensional chain)**.** *We consider a simplified experiment – a one dimensional chain of n particles connected by elasto-plastic springs of length $\varepsilon = \frac{1}{n}$ with the above constitutive laws. The coefficients $a(\cdot, \frac{x}{\varepsilon})$, $h(\cdot, \frac{x}{\varepsilon})$ and $\sigma(\cdot, \frac{x}{\varepsilon})$ are i.i.d. random variables uniformly distributed on certain intervals. One end of the chain is fixed and the other end is loaded by a periodic loading $2\sin(t)$. The below diagrams in Figure 8.2 present the relationship between the loading (on y-axis; which also equals to the average stress in the chain) and displacement of the end of the chain (on x-axis; that presents also the average strain in the chain). The different curves correspond to 7 different realizations of the coefficients. The diagram suggests that for small ε, the macroscopic* *response of the system might be characterized by a single (effective) hysteresis loop. However, this loop does not correspond to the response of an elasto-plastic spring with linear-kinematic hardening. This means that the effective system (if it exists) exhibits a different type of constitutive law than a single spring in the chain (cf. Figure 8.3).*

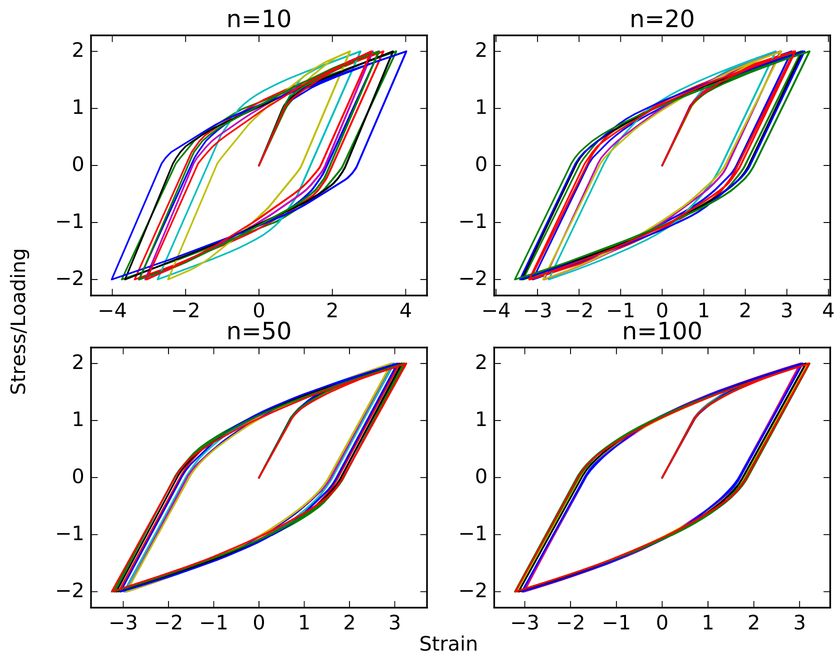


Figure 8.2: *Stress-strain diagrams of a one-dimensional chain of random springs loaded at one end. n is the number of springs considered and the different colors correspond to different realizations of the coefficients.*

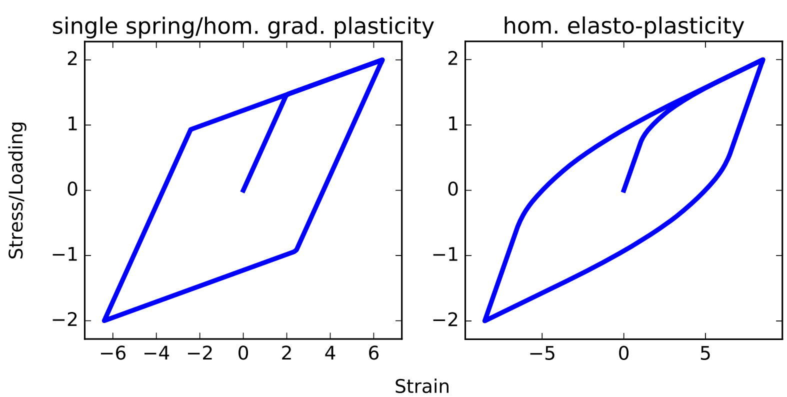


Figure 8.3: *The left diagram is a typical stress-strain curve of a single elasto-plastic spring with linear kinematic hardening. A chain of such springs homogenizes to a model that exhibits hysteresis diagrams of the form as in the right figure. A gradient regularization yields an effective problem with linear kinematic hardening, i.e., the effective system provides stress-strain curves of the form again as in the left figure.*

8.1 Homogenization of static problems

Let $(\Omega, \mathcal{F}, P, \tau)$ be a probability space that satisfies Assumption 5.8, $Q \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary and $(\mathbb{Z}^d, E)$ be a periodic Korn lattice graph with an edge generating set E_0 (see Definition 5.5). The discrete framework that we use below is introduced in Section 5.1.1. We set $Q^{+\varepsilon} := Q \cup \{x \in \mathbb{R}^d : (x, x + \varepsilon b) \cap Q \neq \emptyset \text{ for some } b \in E_0\}$. We consider a set of particles with reference positions at $\varepsilon\mathbb{Z}^d$. It is assumed that the edges εE represent random springs with elastic response (cf. the introduction with internal variable $z = 0$ and loading $l_\varepsilon(t) = l_\varepsilon$). The equilibrium state of the system is determined by a minimization problem which, in a slightly more general setting, reads as

$$\min_{u \in \left(L^p(\Omega) \otimes L^p_0(Q \cap \varepsilon\mathbb{Z}^d)\right)^d} \left\langle \int_{Q^{+\varepsilon} \cap \varepsilon\mathbb{Z}^d} V(\tau_{\frac{x}{\varepsilon}}\omega, \nabla^\varepsilon_s u(\omega, x)) dm_\varepsilon(x) - \int_{Q \cap \varepsilon\mathbb{Z}^d} l_\varepsilon(\omega, x) \cdot u(\omega, x) dm_\varepsilon(x) \right\rangle. \tag{8.7}$$

Let $p \in (1, \infty)$ and $q = \frac{p}{p-1}$. We assume the following:

(A1) $V : \Omega \times \mathbb{R}^k \to \mathbb{R}$ and it holds that $V(\cdot, F)$ is measurable for all $F \in \mathbb{R}^k$ and for P-a.a. ω, $V(\omega, \cdot)$ is convex.

(A2) There exists $c > 0$ such that

$$\frac{1}{c}|F|^p - c \leq V(\omega, F) \leq c(|F|^p + 1) \quad \text{for } P\text{-a.a. } \omega \in \Omega \text{ and all } F \in \mathbb{R}^k.$$

In the case that the loading l_ε converges in a sufficiently strong sense (see Remark 8.4), in order to describe the asymptotic behavior of minimizers in (8.7) in the limit $\varepsilon \to 0$, it is sufficient to consider the energy functional $\mathcal{E}_\varepsilon : \left(L^p(\Omega) \otimes L^p_0(Q \cap \varepsilon\mathbb{Z}^d)\right)^d \to \mathbb{R}$,

$$\mathcal{E}_\varepsilon(u) = \left\langle \int_{Q^{+\varepsilon} \cap \varepsilon\mathbb{Z}^d} V(\tau_{\frac{x}{\varepsilon}}\omega, \nabla^\varepsilon_s u(\omega, x)) dm_\varepsilon(x) \right\rangle.$$

As shown below for $\varepsilon \to 0$, we derive the effective two-scale functional

$$\mathcal{E}_0 : (L^p_{\text{inv}}(\Omega) \otimes W^{1,p}_0(Q))^d \times (L^p_{\text{pot}}(\Omega) \otimes L^p(Q))^d \to \mathbb{R},$$
$$\mathcal{E}_0(u, \chi) = \left\langle \int_Q V(\omega, \nabla_s u(\omega, x) + \chi_s(\omega, x)) dx \right\rangle.$$

Moreover, if we assume that $\langle \cdot \rangle$ is ergodic, the effective energy reduces to a single-scale functional

$$\mathcal{E}_{\text{hom}} : W^{1,p}_0(Q)^d \to \mathbb{R}, \quad \mathcal{E}_{\text{hom}}(u) = \int_Q V_{\text{hom}}(\nabla u(x)) dx,$$

where the homogenized energy density $V_{\text{hom}} : \mathbb{R}^{d \times d} \to \mathbb{R}$ is defined by the corrector problem

$$V_{\text{hom}}(F) = \inf_{\chi \in L^p_{\text{pot}}(\Omega)^d} \langle V(\omega, F_s + \chi_s(\omega)) \rangle. \tag{8.8}$$

The stochastic Korn inequality Lemma 5.12 and the direct method of calculus of variations imply the following lemma.

Lemma 8.2 (Existence of a corrector). *Let $p \in (1,\infty)$ and we assume (A1)-(A2). For any $F \in \mathbb{R}^{d\times d}$, there exists $\chi \in L^p_{\text{pot}}(\Omega)^d$ which attains the infimum in (8.8).*
(See Section 8.1.1 for the proof.)

The main result of this section is the following theorem.

Theorem 8.3 (Two-scale homogenization). *Let $p \in (1,\infty)$ and we assume (A1)-(A2).*

(i) (Compactness) For $u_\varepsilon \in \left(L^p(\Omega) \otimes L^p_0(Q \cap \varepsilon\mathbb{Z}^d)\right)^d$ with $\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) < \infty$, there exist a subsequence (not relabeled), $u \in (L^p_{\text{inv}}(\Omega) \otimes W^{1,p}_0(Q))^d$ and $\chi \in (L^p_{\text{pot}}(\Omega) \otimes L^p(Q))^d$ such that

$$u_\varepsilon \overset{2}{\rightharpoonup} u \quad \text{in } L^p(\Omega \times \mathbb{R}^d)^d, \quad \nabla^\varepsilon u_\varepsilon \overset{2}{\rightharpoonup} \nabla u + \chi \quad \text{in } L^p(\Omega \times \mathbb{R}^d)^{d\times d}. \tag{8.9}$$

(ii) (Lower bound) Assume that the convergence (8.9) holds for the whole sequence u_ε. Then

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}_0(u,\chi).$$

(iii) (Upper bound) For any $u \in (L^p_{\text{inv}}(\Omega) \otimes W^{1,p}_0(Q))^d$ and $\chi \in (L^p_{\text{pot}}(\Omega) \otimes L^p(Q))^d$, there exists a sequence $u_\varepsilon \in \left(L^p(\Omega) \otimes L^p_0(Q \cap \varepsilon\mathbb{Z}^d)\right)^d$ such that

$$\begin{aligned} &u_\varepsilon \overset{2}{\to} u \quad \text{in } L^p(\Omega \times \mathbb{R}^d)^d, \quad \nabla^\varepsilon u_\varepsilon \overset{2}{\to} \nabla u + \chi \quad \text{in } L^p(\Omega \times \mathbb{R}^d)^{d\times d},\\ &\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}_0(u,\chi). \end{aligned}$$

(See Section 8.1.1 for the proof.)

Remark 8.4 (Convergence of minimizers). *Under the assumptions of Theorem 8.3 and if the loadings $l_\varepsilon \in L^q(\Omega \times \varepsilon\mathbb{Z}^d)^d$ satisfy $l_\varepsilon \overset{2}{\to} l$ in $L^q(\Omega \times \mathbb{R}^d)^d$, where $l \in L^q(\Omega \times Q)^d$, the above theorem implies that minimizers u_ε in (8.7) satisfy (up to an extraction of a subsequence)*

$$u_\varepsilon \overset{2}{\rightharpoonup} u \quad \text{in } L^p(\Omega \times \mathbb{R}^d)^d, \qquad \nabla^\varepsilon u_\varepsilon \overset{2}{\rightharpoonup} \nabla u + \chi \quad \text{in } L^p(\Omega \times \mathbb{R}^d)^{d\times d}, \tag{8.10}$$

where $(u,\chi) \in (L^p_{\text{inv}}(\Omega) \otimes W^{1,p}_0(Q))^d \times (L^p_{\text{pot}}(\Omega) \otimes L^p(Q))^d$ is a minimizer of the two-scale functional $\mathcal{I}_0 : (L^p_{\text{inv}}(\Omega) \otimes W^{1,p}_0(Q))^d \times (L^p_{\text{pot}}(\Omega) \otimes L^p(Q))^d \to \mathbb{R}$,

$$\mathcal{I}_0(u,\chi) = \mathcal{E}_0(u,\chi) - \left\langle \int_Q l \cdot u dx \right\rangle.$$

This can be obtained by a standard argument from Γ-convergence, cf. Corollary 7.2. Also, should $\mathcal{I}_0$ have a unique minimizer, e.g., if $V(\omega,\cdot)$ is strictly convex, then (8.10) holds for the whole sequence.

In the ergodic case, the limit is a deterministic functional:

Theorem 8.5 (Ergodic case). *Let $p \in (1,\infty)$. We assume (A1)-(A2) and that $\langle\cdot\rangle$ is ergodic.*

(i) (Compactness and lower bound) Let $u_\varepsilon \in \left(L^p(\Omega) \otimes L^p_0(Q \cap \varepsilon\mathbb{Z}^d)\right)^d$ *satisfy*

$$\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) < \infty.$$

Then, there exists $u \in W^{1,p}_0(Q)^d$ *such that, up to a subsequence,*

$$u_\varepsilon \overset{2}{\rightharpoonup} u \quad \text{in } L^p(\Omega \times \mathbb{R}^d)^d, \quad \mathcal{E}_{\text{hom}}(u) \le \liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon).$$

Moreover, $\langle u_\varepsilon \rangle \to u$ *strongly in* $L^p(\mathbb{R}^d)^d$ *and* $\langle \nabla^\varepsilon u_\varepsilon \rangle \rightharpoonup \nabla u$ *weakly in* $L^p(\mathbb{R}^d)^{d\times d}$.

(ii) (Upper bound) For any $u \in W^{1,p}_0(Q)^d$, *there exists* $u_\varepsilon \in \left(L^p(\Omega) \otimes L^p_0(Q \cap \varepsilon\mathbb{Z}^d)\right)^d$ *such that*

$$u_\varepsilon \overset{2}{\to} u \quad \text{in } L^p(\Omega \times \mathbb{R}^d)^d, \quad \lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}_{\text{hom}}(u).$$

Moreover, $\langle u_\varepsilon \rangle \to u$ *strongly in* $L^p(\mathbb{R}^d)^d$ *and* $\langle \nabla^\varepsilon u_\varepsilon \rangle \rightharpoonup \nabla u$ *weakly in* $L^p(\mathbb{R}^d)^{d\times d}$.

(See Section 8.1.1 for the proof.)

If we, additionally, assume the following assumption, we obtain strong convergence for minimizers.

(A3) For P-a.a. $\omega \in \Omega$, $V(\omega,\cdot)$ is uniformly convex with modulus $(\cdot)^p$, i.e., there exists $c > 0$ that is independent of ω such that for all $F, G \in \mathbb{R}^k$ and $t \in [0,1]$

$$V(\omega, tF + (1-t)G) \le tV(\omega, F) + (1-t)V(\omega, G) - (1-t)tc|F-G|^p.$$

Proposition 8.6 (Strong convergence)**.** *Let the assumptions of Theorem 8.5 and (A3) hold. Let* $l_\varepsilon \in L^q(\Omega \times \varepsilon\mathbb{Z}^d)^d$ *satisfy* $l_\varepsilon \overset{2}{\to} l$ *in* $L^q(\Omega \times \mathbb{R}^d)^d$, *where* $l \in L^q(Q)^d$. *Problem (8.7) admits a unique minimizer* $u_\varepsilon \in \left(L^p(\Omega) \otimes L^p_0(Q \cap \varepsilon\mathbb{Z}^d)\right)^d$, *which satisfies*

$$u_\varepsilon \overset{2}{\to} u \quad \text{in } L^p(\Omega \times \mathbb{R}^d)^d, \quad \nabla^\varepsilon u_\varepsilon \overset{2}{\to} \nabla u + \chi \quad \text{in } L^p(\Omega \times \mathbb{R}^d)^{d\times d}, \tag{8.11}$$

where $u \in W^{1,p}_0(Q)^d$ *is the unique minimizer of*

$$\mathcal{I}_{\text{hom}} : W^{1,p}_0(Q)^d \to \mathbb{R}, \quad \mathcal{I}_{\text{hom}}(u) = \mathcal{E}_{\text{hom}}(u) - \int_Q l \cdot u dx,$$

and $\chi \in \left(L^p_{\text{pot}}(\Omega) \otimes L^p(Q)\right)^d$ *satisfies, for a.a.* x,

$$V_{\text{hom}}(\nabla u(x)) = \langle V(\omega, \nabla_s u(x) + \chi_s(\omega, x)) \rangle.$$

(See Section 8.1.1 for the proof.)

Remark 8.7. *An alternative strategy to the above for obtaining strong convergence for minimizers may be based on the general principle outlined in [Vis84], which bases on the weaker assumption of strict convexity of* $V(\omega,\cdot)$, *cf. Remark 7.8.*

Remark 8.8 (Comparison to previous results)**.** *Stochastic homogenization of discrete integral functionals in the setting of discrete-to-continuum transition has already been studied before. In particular, it is known that in the ergodic case for* P*-a.a.* $\omega \in \Omega$ *the functional given by* $\mathcal{E}_\varepsilon(\omega,\cdot) = \int_{Q+\varepsilon\cap\varepsilon\mathbb{Z}^d} V(\tau_{\frac{x}{\varepsilon}}\omega, \nabla^\varepsilon_s u_\varepsilon(x)) dm_\varepsilon(x)$ *(defined on* $L^p_0(Q \cap \varepsilon\mathbb{Z}^d)^d$*)* Γ*-converges to* $\mathcal{E}_{\text{hom}}$. *This quenched convergence result can be found, e.g., in [ACG11], where even nonconvex integrands are treated. Based on stochastic unfolding, we present a simple alternative proof, but we obtain the weaker "mean" result of Theorem 8.5 as a consequence of Theorem 8.3. While our argument is relatively easy, the analysis of the stronger quenched convergence result is based on the subadditive ergodic theorem [AK81] and it is more involved.*

8.1.1 Proofs

Proof of Lemma 8.2. We consider a minimizing sequence $\chi^n \in L^p_{\mathrm{pot}}(\Omega)^d$, $n \in \mathbb{N}$. Using the growth conditions of V and Lemma 5.12, it follows that $\limsup_{n\to\infty} \|\chi^n\|_{L^p(\Omega)} < \infty$. Therefore, we may extract (a not relabeled subsequence) such that $\chi^n \rightharpoonup \chi$ weakly in $L^p_{\mathrm{pot}}(\Omega)^d$. Furthermore, it follows that $\chi^n_s \rightharpoonup \chi_s$ weakly in $L^p(\Omega)^k$ since $(\cdot)_s$ is a bounded operator. The functional $L^p(\Omega)^k \ni \chi_s \mapsto \langle V(\omega, F_s + \chi_s(\omega))\rangle$ is convex and l.s.c., and therefore it is weakly l.s.c. As a result of this, it follows that $V_{\mathrm{hom}}(F) = \liminf_{n\to\infty} \langle V(\omega, F_s + \chi^n_s(\omega))\rangle \geq \langle V(\omega, F_s + \chi_s(\omega))\rangle$. □

Proof of Theorem 8.3. (i) The growth conditions of V, the Korn property of the lattice (Definition 5.5) and the discrete Poincaré inequality imply that

$$\limsup_{\varepsilon\to 0} \left\langle \int_{\varepsilon\mathbb{Z}^d} |u_\varepsilon(\omega,x)|^p + |\nabla^\varepsilon u_\varepsilon(\omega,x)|^p dm_\varepsilon(x) \right\rangle < \infty.$$

Therefore, the claim follows by Proposition 5.22 (i) and Corollary 5.23.

(ii) The claim directly follows from Lemma 5.18 and Proposition 5.19 (ii).

(iii) The claim follows from Corollary 5.28 (iii), Lemma 5.18 and Proposition 5.19 (i). □

Proof of Theorem 8.5. (i) By Theorem 8.3 there exist $u \in W^{1,p}_0(Q)^d$, $\chi \in (L^p_{\mathrm{pot}}(\Omega) \otimes L^p(Q))^d$, and a weakly two-scale convergent subsequence such that

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) \geq \mathcal{E}_0(u,\chi) \geq \mathcal{E}_{\mathrm{hom}}(u).$$

The convergence for $\langle u_\varepsilon\rangle$ and $\langle \nabla^\varepsilon u_\varepsilon\rangle$ follows directly from Corollary 5.24 and Lemma 5.4.

(ii) It is sufficient to show that for $u \in W^{1,p}_0(Q)^d$, there exists $\chi \in (L^p_{\mathrm{pot}}(\Omega) \otimes L^p(Q))^d$ such that

$$\mathcal{E}_0(u,\chi) = \mathcal{E}_{\mathrm{hom}}(u).$$

Indeed, if this holds, Theorem 8.3 (iii) implies that there exists $u_\varepsilon \in \left(L^p(\Omega) \otimes L^p_0(Q \cap \varepsilon\mathbb{Z}^d)\right)^d$ such that

$$\begin{aligned} &u_\varepsilon \xrightarrow{2} u \quad \text{in } L^p(\Omega \times \mathbb{R}^d)^d, \quad \nabla^\varepsilon u_\varepsilon \xrightarrow{2} \nabla u + \chi \quad \text{in } L^p(\Omega \times \mathbb{R}^d)^{d\times d}, \\ &\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon) = \mathcal{E}_0(u,\chi) = \mathcal{E}_{\mathrm{hom}}(u) \end{aligned}$$

and the convergence for $\langle u_\varepsilon\rangle$ and $\langle \nabla^\varepsilon u_\varepsilon\rangle$ follows from Corollary 5.24 and Lemma 5.4.

To show the above claim, we apply a measurable selection argument, Theorem A.7. We define an integrand $f : Q \times L^p_{\mathrm{pot}}(\Omega)^d \to \mathbb{R}$ given by $f(x,\chi) = \langle V(\omega, \nabla_s u(x) + \chi_s(\omega))\rangle$. f is finite everywhere, for fixed x $f(x,\cdot)$ is convex and l.s.c. and thus continuous. Moreover, the integrand $V : \Omega \times \mathbb{R}^k \to \mathbb{R}$ is a Carathéodory integrand as defined in Remark A.4 (if necessary, we tacitly redefine it by $V(\omega,\cdot) = 0$ for $\omega \in \widetilde{\Omega}$ with $P(\widetilde{\Omega}) = 0$). As a result of this and using the growth conditions, for fixed $\chi \in L^p_{\mathrm{pot}}(\Omega)^d$, the mapping $(\omega,x) \mapsto V(\omega, \nabla_s u(x) + \chi_s(\omega))$ is integrable. Fubini's theorem implies that $x \mapsto \langle V(\omega, \nabla_s u(x) + \chi_s(\omega))\rangle = f(x,\chi)$ is $\mathcal{L}(Q)$-measurable. This means that f is a convex Carathéodory integrand and therefore it is a convex normal integrand (see Definition A.3 and Remark A.4). Moreover, it holds that $\int_Q f(x,0)dx < \infty$, therefore the assumptions of Proposition A.7 are satisfied and we have (see Remark A.8)

$$\mathcal{E}_{\mathrm{hom}}(u) = \int_Q \inf_{\chi \in L^p_{\mathrm{pot}}(\Omega)^d} f(x,\chi)dx = \inf_{\chi \in (L^p_{\mathrm{pot}}(\Omega)\otimes L^p(Q))^d} \int_Q f(x,\chi(x))dx = \inf_{\chi \in (L^p_{\mathrm{pot}}(\Omega)\otimes L^p(Q))^d} \mathcal{E}_0(u,\chi).$$

By the direct method of calculus of variations and using the stochastic Korn inequality 5.12 (cf. proof of Lemma 8.2) the infimum on the right-hand side is attained and we conclude the proof. □

Proof of Proposition 8.6. The existence of unique minimizers for (8.7) and $\mathcal{I}_{\text{hom}}$ follows by the direct method of calculus of variations and using the uniform convexity assumption (A3). Theorem 8.5 implies that (up to a subsequence) $u_\varepsilon \overset{2}{\rightharpoonup} u$ in $L^p(\Omega\times\mathbb{R}^d)^d$, where u is the minimizer of $\mathcal{I}_{\text{hom}}$ (cf. Remark 8.4). Also, for another subsequence we have $\nabla^\varepsilon u_\varepsilon \overset{2}{\rightharpoonup} \nabla u+\chi$ with $\chi\in(L^p_{\text{pot}}(\Omega)\otimes L^p(Q))^d$. Moreover, Theorem 8.3 (ii) and Theorem 8.5 imply that

$$\mathcal{E}_{\text{hom}}(u)=\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(u_\varepsilon)\geq \mathcal{E}_0(u,\chi)\geq \mathcal{E}_{\text{hom}}(u). \tag{8.12}$$

Since for a.a. x, $V_{\text{hom}}(\nabla u(x))\leq \langle V(\omega,\nabla_s u(x)+\chi_s(\omega,x))\rangle$, (8.12) implies that $V_{\text{hom}}(\nabla u(x))=\langle V(\omega,\nabla_s u(x)+\chi_s(\omega,x))\rangle$ for a.a. x. Note that by strict convexity of V, the latter formula uniquely determines the two-scale limit χ. By Theorem 8.3, there exists a strong two-scale recovery sequence $v_\varepsilon\in\left(L^p(\Omega)\otimes L^p_0(Q\cap\varepsilon\mathbb{Z}^d)\right)^d$ for (u,χ). We have

$$\begin{aligned}&\left\langle\int_{\mathbb{R}^d}|\mathcal{T}_\varepsilon u_\varepsilon-u|^p+|\mathcal{T}_\varepsilon\nabla^\varepsilon u_\varepsilon-\nabla u-\chi|^p dx\right\rangle \qquad (8.13)\\ \leq\;& c\left\langle\int_{\mathbb{R}^d}|\mathcal{T}_\varepsilon u_\varepsilon-\mathcal{T}_\varepsilon v_\varepsilon|^p+|\mathcal{T}_\varepsilon\nabla^\varepsilon u_\varepsilon-\mathcal{T}_\varepsilon\nabla^\varepsilon v_\varepsilon|^p+|\mathcal{T}_\varepsilon v_\varepsilon-u|^p+|\mathcal{T}_\varepsilon\nabla^\varepsilon v_\varepsilon-\nabla u-\chi|^p dx\right\rangle.\end{aligned}$$

The last two terms on the right-hand side vanish in the limit $\varepsilon\to 0$ by the properties of v_ε. The first two terms also vanish as $\varepsilon\to 0$ and this follows by a standard argument using uniform convexity: By the isometry property of $\mathcal{T}_\varepsilon$, a discrete Poincaré-Korn inequality following from (5.5), the strong convexity (A3), and since $\nabla^\varepsilon_s u_\varepsilon$ and $\nabla^\varepsilon_s v_\varepsilon$ are supported in $Q^{+\varepsilon}\cap\varepsilon\mathbb{Z}^d$, we have

$$\begin{aligned}\left\langle\int_{\mathbb{R}^d}|\mathcal{T}_\varepsilon u_\varepsilon(\omega,x)-\mathcal{T}_\varepsilon v_\varepsilon(\omega,x)|^p dx\right\rangle &\leq c\left\langle\int_{\varepsilon\mathbb{Z}^d}|\nabla^\varepsilon u_\varepsilon(\omega,x)-\nabla^\varepsilon v_\varepsilon(\omega,x)|^p dm_\varepsilon(x)\right\rangle\\ &\leq c\left\langle\int_{\varepsilon\mathbb{Z}^d}|\nabla^\varepsilon_s u_\varepsilon(\omega,x)-\nabla^\varepsilon_s v_\varepsilon(\omega,x)|^p dm_\varepsilon(x)\right\rangle\\ &\leq c\left(\frac{1}{2}\big(\mathcal{E}_\varepsilon(u_\varepsilon)+\mathcal{E}_\varepsilon(v_\varepsilon)\big)-\mathcal{E}_\varepsilon\left(\frac{1}{2}u_\varepsilon+\frac{1}{2}v_\varepsilon\right)\right).\end{aligned} \tag{8.14}$$

Since u_ε solves (8.7), we have

$$-\mathcal{E}_\varepsilon\left(\frac{1}{2}u_\varepsilon+\frac{1}{2}v_\varepsilon\right)\leq-\mathcal{E}_\varepsilon(u_\varepsilon)+\left\langle\int_{Q\cap\varepsilon\mathbb{Z}^d}l_\varepsilon\cdot u_\varepsilon dm_\varepsilon\right\rangle-\left\langle\int_{Q\cap\varepsilon\mathbb{Z}^d}l_\varepsilon\cdot\left(\frac{1}{2}u_\varepsilon+\frac{1}{2}v_\varepsilon\right)dm_\varepsilon\right\rangle,$$

and thus with (8.14), the first two terms on the right-hand side of (8.13) may be both bounded by the following expression

$$c\left(\frac{1}{2}\big(\mathcal{E}_\varepsilon(v_\varepsilon)-\mathcal{E}_\varepsilon(u_\varepsilon)\big)+\frac{1}{2}\left\langle\int_{Q\cap\varepsilon\mathbb{Z}^d}l_\varepsilon\cdot(u_\varepsilon-v_\varepsilon)\,dm_\varepsilon\right\rangle\right).$$

The second term vanishes as $\varepsilon\to 0$ using strong two-scale convergence of l_ε. Similarly as in the proof of Theorem 8.5, it follows that $\limsup_{\varepsilon\to 0}(\mathcal{E}_\varepsilon(v_\varepsilon)-\mathcal{E}_\varepsilon(u_\varepsilon))\leq 0$ and therefore the claim of the proposition is proved for a subsequence. Convergence for the whole sequence follows by a standard contradiction argument and using the uniqueness of the minimizers (u,χ). □

8.2 Homogenization of elasto-plasticity

Let $(\Omega, \mathcal{F}, P, \tau)$ be a probability space that satisfies Assumption 5.8, $Q \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary and $(\mathbb{Z}^d, E)$ be a periodic Korn lattice graph with an edge generating set E_0 (see Definition 5.5). We set $Q^{+\varepsilon} := Q \cup \{x \in \mathbb{R}^d : (x, x+\varepsilon b) \cap Q \neq \emptyset \text{ for some } b \in E_0\}$. A system of particles connected by springs is represented using $(\varepsilon\mathbb{Z}^d, \varepsilon E)$, where the edges εE represent springs with elasto-plastic response (cf. the introduction). Upon an external loading l, the system evolves according to an ERIS driven by an energy and dissipation functional (we recall the abstract theory in Section 3). The following model is a random and discrete counterpart of the model considered in [MT07], where the periodic continuum case is treated.

- The state space is $Y_\varepsilon = \left(L^2(\Omega) \otimes L^2_0(Q \cap \varepsilon\mathbb{Z}^d)\right)^d \times \left(L^2(\Omega) \otimes L^2_0(Q^{+\varepsilon} \cap \varepsilon\mathbb{Z}^d)\right)^k$, the displacement u_ε and the internal variable z_ε are merged into a joint variable $y_\varepsilon = (u_\varepsilon, z_\varepsilon)$. We equip Y_ε with the scalar product

$$\begin{aligned}\langle y_1, y_2\rangle_{Y_\varepsilon} = &\left\langle \int_{\varepsilon\mathbb{Z}^d} u_1(\omega,x)\cdot u_2(\omega,x) dm_\varepsilon(x)\right\rangle + \left\langle \int_{\varepsilon\mathbb{Z}^d} \nabla^\varepsilon u_1(\omega,x) : \nabla^\varepsilon u_2(\omega,x) dm_\varepsilon(x)\right\rangle \\ &+ \left\langle \int_{\varepsilon\mathbb{Z}^d} z_1(\omega,x)\cdot z_2(\omega,x) dm_\varepsilon(x)\right\rangle.\end{aligned}$$

- The energy functional is $\mathcal{E}_\varepsilon : [0,T]\times Y_\varepsilon \to \mathbb{R}$,

$$\begin{aligned}\mathcal{E}_\varepsilon(t, y_\varepsilon) &= \frac{1}{2}\langle \mathbb{A}_\varepsilon y_\varepsilon, y_\varepsilon\rangle_{Y_\varepsilon^*, Y_\varepsilon} - \left\langle \int_{Q\cap\varepsilon\mathbb{Z}^d} \pi_\varepsilon l(t)(x)\cdot u_\varepsilon(\omega,x) dm_\varepsilon(x)\right\rangle, \\ \langle \mathbb{A}_\varepsilon y_1, y_2\rangle_{Y_\varepsilon^*, Y_\varepsilon} &= \left\langle \int_{Q^{+\varepsilon}\cap\varepsilon\mathbb{Z}^d} A(\tau_{\frac{x}{\varepsilon}}\omega)\begin{pmatrix}\nabla_s^\varepsilon u_1(\omega,x)\\ z_1(\omega,x)\end{pmatrix}\cdot\begin{pmatrix}\nabla_s^\varepsilon u_2(\omega,x)\\ z_2(\omega,x)\end{pmatrix} dm_\varepsilon(x)\right\rangle.\end{aligned}$$

- The dissipation potential is $\mathcal{R}_\varepsilon : Y_\varepsilon \to [0,\infty]$,

$$\mathcal{R}_\varepsilon(\dot{y}_\varepsilon) = \left\langle \int_{Q^{+\varepsilon}\cap\varepsilon\mathbb{Z}^d} \rho(\tau_{\frac{x}{\varepsilon}}\omega, \dot{z}_\varepsilon(\omega,x)) dm_\varepsilon(x)\right\rangle.$$

We assume the following:

(B1) $A \in L^\infty(\Omega, \mathbb{R}^{2k\times 2k}_{\text{sym}})$ and it satisfies: there exists $c>0$ such that $A(\omega)F\cdot F \geq c|F|^2$ for P-a.a. $\omega\in\Omega$ and all $F\in\mathbb{R}^{2k}$.

(B2) $\rho : \Omega\times\mathbb{R}^k \to [0,\infty)$, $\rho(\cdot, F)$ is measurable for all $F\in\mathbb{R}^k$, and for P-a.a. ω, $\rho(\omega,\cdot)$ is convex and positively homogeneous of degree 1, i.e., $\rho(\omega,\alpha F) = \alpha\rho(\omega,F)$ for all $\alpha\geq 0$ and $F\in\mathbb{R}^k$. Also, we assume that there exists $\psi\in L^2(\Omega)$ such that for P-a.a. ω, $\rho(\omega,F)\leq\psi(\omega)(|F|+1)$ for all $F\in\mathbb{R}^k$.

Let $T>0$ be a finite time horizon and we consider the ERIS $(Y_\varepsilon, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$. We denote by $S_\varepsilon(t) := \{y\in Y_\varepsilon : \mathcal{E}_\varepsilon(t,y)\leq \mathcal{E}_\varepsilon(t,\widetilde{y}) + \mathcal{R}_\varepsilon(\widetilde{y}-y) \text{ for all } \widetilde{y}\in Y_\varepsilon\}$ the set of stable states at time $t\in[0,T]$.

Remark 8.9 (Existence and a priori estimates). *If we assume (B1)-(B2), $l\in C^1([0,T], L^2(Q)^d)$ and $y^0_\varepsilon\in S_\varepsilon(0)$, the conditions of Theorem 3.5 are fulfilled and there exists $y_\varepsilon\in C^{\text{Lip}}([0,T], Y_\varepsilon)$, a* *unique energetic solution to the ERIS* $(Y_\varepsilon, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ *with* $y_\varepsilon(0) = y_\varepsilon^0$, *i.e., for all* $t \in [0,T]$ *we have* $y_\varepsilon(t) \in S_\varepsilon(t)$ *and*

$$\mathcal{E}_\varepsilon(t, y_\varepsilon(t)) + \int_0^t \mathcal{R}_\varepsilon(\dot{y}_\varepsilon(s))ds = \mathcal{E}_\varepsilon(0, y_\varepsilon(0)) - \int_0^t \left\langle \int_{Q\cap\varepsilon\mathbb{Z}^d} \pi_\varepsilon \dot{l}(s) \cdot u_\varepsilon(s) dm_\varepsilon \right\rangle ds, \tag{8.15}$$

and, moreover, $\|y_\varepsilon(t) - y_\varepsilon(s)\|_{Y_\varepsilon} \leq c|t-s|$ *for all* $t, s \in [0,T]$, *where* $c > 0$ *is independent of* ε.

The passage to the limit model as $\varepsilon \to 0$ is conducted in the setting of evolutionary Γ-convergence (see Section 3) and involves a discrete-to-continuum transition. The homogenized model is also described by an ERIS:

- The state space is given by $Y = H_0^1(Q)^d \times L^2(\Omega \times Q)^k \times (L^2_{\text{pot}}(\Omega) \otimes L^2(Q))^d$ and we denote the state variable by $y = (u, z, \chi)$.
- The energy functional is

$$\mathcal{E}_0 : [0,T] \times Y \to \mathbb{R}, \quad \mathcal{E}_0(t,y) = \frac{1}{2} \langle \mathbb{A}y, y \rangle_{Y^*,Y} - \int_Q l(t) \cdot u dx,$$
$$\langle \mathbb{A}y, y \rangle_{Y^*,Y} = \int_Q \left\langle A(\omega) \begin{pmatrix} \nabla_s u(x) + \chi_s(\omega,x) \\ z(\omega,x) \end{pmatrix} \cdot \begin{pmatrix} \nabla_s u(x) + \chi_s(\omega,x) \\ z(\omega,x) \end{pmatrix} \right\rangle dx.$$

- The limit dissipation functional is given by

$$\mathcal{R}_0 : Y \to [0,\infty], \quad \mathcal{R}_0(\dot{y}) = \int_Q \langle \rho(\omega, \dot{z}(\omega,x)) \rangle dx.$$

We denote by $S(t) := \{y \in Y : \mathcal{E}_0(t,y) \leq \mathcal{E}_0(t,\widetilde{y}) + \mathcal{R}_0(\widetilde{y} - y) \text{ for all } \widetilde{y} \in Y\}$ the set of stable states at time $t \in [0,T]$.

Remark 8.10 (Positive-definiteness of $\mathbb{A}$)**.** *If we assume (B1), then there exists* $c > 0$ *such that* $\langle \mathbb{A}y, y \rangle_{Y^*,Y} \geq c\|y\|_Y^2$ *for all* $y \in Y$. *Indeed, (B1) implies that*

$$\langle \mathbb{A}y, y \rangle_{Y^*,Y} \geq c \left\langle \int_Q |\nabla_s u(x) + \chi_s(\omega,x)|^2 + |z(\omega,x)|^2 dx \right\rangle.$$

Since u *is deterministic, we have* $\int_Q \langle \nabla_s u(x) \cdot \chi_s(\omega,x) \rangle dx = 0$ *and therefore*

$$\left\langle \int_Q |\nabla_s u(x) + \chi_s(\omega,x)|^2 dx \right\rangle = \int_Q |\nabla_s u(x)|^2 dx + \int_Q \langle |\chi_s(\omega,x)|^2 \rangle dx.$$

This implies the positive-definiteness using the deterministic and stochastic Korn inequalities Remark 5.6 and Lemma 5.12, respectively. Moreover, $\mathbb{A}$ *is symmetric by the symmetry of* A *and* $\mathbb{A}y$ *is bounded since the symmetrization* $(\cdot)_s$ *is a bounded operator.*

Remark 8.11 (Existence and uniqueness)**.** *If we assume (B1)-(B2),* $l \in C^1([0,T], L^2(Q)^d)$ *and* $y^0 \in S(0)$, *then the assumptions of Theorem 3.5 are satisfied (see Remark 8.10) and therefore there* *exists a unique energetic solution $y \in C^{\mathrm{Lip}}([0,T],Y)$ to the ERIS $(Y,\mathcal{E}_0,\mathcal{R}_0)$ with $y(0)=y^0$, i.e., for all $t\in[0,T]$ we have $y(t)\in S(t)$ and*

$$\mathcal{E}_0(t,y(t))+\int_0^t \mathcal{R}_0(\dot y(s))ds=\mathcal{E}_0(0,y(0))-\int_0^t\int_Q \dot l(s)\cdot u(s)dxds. \tag{8.16}$$

We remark that this system in general does not reduce to classical deterministic elasto-plasticity with linear kinematic hardening since the internal variable z is not deterministic. However, we might see it as a deterministic elasto-plasticity model with nonstandard constitutive laws accounting for the microstructure, where we view u as the deterministic solution and the pair (z,χ) represents an internal variable that simultaneously accounts for the plastic behavior and the microstructure. In fact, the evolution of the macroscopic variable u may be described by the equation $-\mathrm{div}\mathcal{P}_x((z(0),\chi(0)),\nabla_s u(\cdot,x))=l(t,x)$, where $\mathcal{P}_x$ is a generalized Prandtl-Ishlinskii type operator given in terms of the evolving corrector quantities (z,χ), cf. [Mie12] for a detailed discussion in the related situation of continuum periodic homogenization of gradient plasticity.

For notational convenience, we introduce the following abbreviation: For $y_\varepsilon\in Y_\varepsilon$ and $y\in Y$,

$$y_\varepsilon \stackrel{c2}{\rightharpoonup} y \quad :\Leftrightarrow \quad u_\varepsilon \stackrel{2}{\rightharpoonup} u,\ \nabla^\varepsilon u_\varepsilon \stackrel{2}{\rightharpoonup} \nabla u+\chi \text{ and } z_\varepsilon \stackrel{2}{\rightharpoonup} z \quad \text{(in the corresp. } L^2\text{-spaces).} \tag{8.17}$$

Also, we use $\stackrel{c2}{\to}$ if the quantities on the right-hand side strongly two-scale converge. The "c" in this shorthand refers to "cross" convergence as is used in the periodic case in [MT07]. The proof of the following homogenization theorem closely follows the strategy developed in [MT07] (see Section 3 for a short outline of the abstract principle and for more references). In that paper, the periodic unfolding method is applied to a similar (continuum) problem with periodic coefficients. The main result of this section is:

Theorem 8.12 (Two-scale homogenization)**.** *Assume that (B1)-(B2) hold, $\langle\cdot\rangle$ is ergodic and $l\in C^1([0,T],L^2(Q)^d)$. Let $y^0_\varepsilon\in S_\varepsilon(0)$ satisfy*

$$y^0_\varepsilon \stackrel{c2}{\to} y^0\in Y.$$

Let $y_\varepsilon\in C^{\mathrm{Lip}}([0,T],Y_\varepsilon)$ be the unique energetic solution to the ERIS $(Y_\varepsilon,\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ with $y_\varepsilon(0)=y^0_\varepsilon$. Then,

$$y^0\in S(0),\ \text{and for all } t\in[0,T]: \quad y_\varepsilon(t)\stackrel{c2}{\to} y(t), \quad \mathcal{E}_\varepsilon(t,y_\varepsilon(t))\to\mathcal{E}_0(t,y(t)),$$

where $y\in C^{\mathrm{Lip}}([0,T],Y)$ is the unique energetic solution to the ERIS $(Y,\mathcal{E}_0,\mathcal{R}_0)$ with $y(0)=y^0$. (See Section 8.2.2 for the proof.)

Remark 8.13 (Nonergodic case)**.** *We remark that the above result holds true in the case that $\langle\cdot\rangle$ is not ergodic (with minor changes in the proof) with a modified state space for the continuum model, specifically $Y=(L^2_{\mathrm{inv}}(\Omega)\otimes H^1_0(Q))^d\times L^2(\Omega\times Q)^k\times(L^2_{\mathrm{pot}}(\Omega)\otimes L^2(Q))^d$.*

8.2.1 Representative volume element approximations

The effective system $(Y,\mathcal{E}_0,\mathcal{R}_0)$ is given in terms of functions from the space of stochastic gradients $L^2_{\mathrm{pot}}(\Omega)$ which is defined on a probability space that is typically infinite-dimensional. This fact makes the computation of the solution inaccessible for standard numerical methods as discussed in Section 7.2. For this reason, we develop an approximation procedure based on the RVE method, cf. Section 7.2. The genuine two-scale nature of this problem requires a careful choice for the approximating system. In particular, for standard (one-scale) problems where the effective coefficients are deterministic quantities, the RVE method consists of replacing the coefficients by their approximations, which may be obtained separately by solving some approximate corrector equations. We refer to Sections 7.2, 8.3.1 and 9.1.1 for such problems. On the other hand, for our two-scale formulation, the approximate correctors are not determined separately, but their computation will be merged into the ERIS. In fact, this is necessary since the corrector χ in our two-scale problem evolves in time, i.e., we deal with a microstructure-evolution problem. In particular, we view $x \in Q$ as a macroscopic variable and we introduce a new variable $q \in \mathbb{Z}^d$ which we see as a microscopic "averaging" variable. To each macroscopic point x, we attach a representative volume domain $LB \cap \mathbb{Z}^d$, $L \gg 1$ is a parameter and $B \subset \mathbb{R}^d$, and the problematic quantity in the energy

$$A\begin{pmatrix} \nabla_s u(x) + \chi_s(x) \\ z(x) \end{pmatrix} \cdot \begin{pmatrix} \nabla_s u(x) + \chi_s(x) \\ z(x) \end{pmatrix}$$

is replaced by the averaged object

$$\fint_{LB\cap\mathbb{Z}^d} A(\tau_q\omega)\begin{pmatrix} \nabla_s u(x) + \nabla_{q,s}\varphi(q,x) \\ z(q,x) \end{pmatrix} \cdot \begin{pmatrix} \nabla_s u(x) + \nabla_{q,s}\varphi(q,x) \\ z(q,x) \end{pmatrix} dm(q),$$

where the symmetrized stochastic gradient is replaced by its deterministic discrete counterpart $\nabla_{q,s}\varphi$. For φ we may consider different choices of boundary conditions, however, to make the exposition simple, we stick to the choice of homogeneous Dirichlet conditions as before (periodic boundary conditions are also applicable). Also, the dissipation potential needs to be correspondingly modified. This procedure is similar to the FE^2-method from numerical homogenization, see, e.g., [Fey99, Mie02, SH13, NW19], where for similar problems a macroscopic discretization of the physical domain Q is considered and to each simplex in it, another (discretized) representative volume domain is assigned, which serves for the computation of the averaged stresses (see also [EGSZ15] for a related stochastic approach). In the following we present the precise setting for the approximation.

Let $L \geq 1$ and $B \subset \mathbb{R}^d$ be open bounded convex with a Lipschitz boundary, the corresponding variable is denoted by $q \in B$ (also $q \in LB \cap \mathbb{Z}^d$) to differentiate it from the macroscopic variable $x \in Q$. We set $B^{+\frac{1}{L}} := B \cup \left\{ q \in \mathbb{R}^d : (q, q + \frac{1}{L}b) \cap B \neq \emptyset \text{ for some } b \in E_0 \right\}$ (we use the shorthand B^+).

- The state space is given by $Y_L = \left(L^2(\Omega) \otimes H^1_0(Q)\right)^d \times \left(L^2(\Omega) \otimes L^2_0(LB^+ \cap \mathbb{Z}^d) \otimes L^2(Q)\right)^k \times \left(L^2(\Omega) \otimes L^2_0(LB \cap \mathbb{Z}^d) \otimes L^2(Q)\right)^d$, and the state variable is denoted by $y_L = (u_L, z_L, \varphi_L)$. We equip $L^2_0(LB^+ \cap \mathbb{Z}^d)^k \times L^2_0(LB \cap \mathbb{Z}^d)^d$ with the scalar product

$$\langle (z_1,\varphi_1),(z_2,\varphi_2)\rangle_L = \fint_{LB^+\cap\mathbb{Z}^d} z_1(q)\cdot z_2(q) + \nabla_q\varphi_1(q) : \nabla_q\varphi_2(q) dm(q),$$

  where ∇_q denotes the discrete gradient w.r.t. the variable q (below with ∇_x we denote the gradient w.r.t. the x variable). The extension of this space in the definition of Y_L is equipped with the corresponding extension of $\langle \cdot, \cdot \rangle_L$.

- The energy is given by $\mathcal{E}_L : [0,T] \times Y_L \to \mathbb{R}$,

$$\mathcal{E}_L(t, y_L) = \frac{1}{2} \langle \mathbb{A}_L y_L, y_L \rangle_{Y_L^*, Y_L} - \left\langle \fint_{LB \cap \mathbb{Z}^d} \int_Q l(t) \cdot u_L dx dm \right\rangle,$$

$$\langle \mathbb{A}_L y_1, y_2 \rangle_{Y_L^*, Y_L} = \left\langle \fint_{LB^+ \cap \mathbb{Z}^d} \int_Q A(\tau_q \omega) \begin{pmatrix} \nabla_{x,s} u_1 + \nabla_{q,s} \varphi_1 \\ z_1 \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x,s} u_2 + \nabla_{q,s} \varphi_2 \\ z_2 \end{pmatrix} dx dm \right\rangle.$$

Above, $\nabla_{x,s}$ and $\nabla_{q,s}$ are the symmetrized gradients in the x and q variables, respectively.

- The dissipation potential is given by $\mathcal{R}_L : Y_L \to [0, \infty]$,

$$\mathcal{R}_L(\dot{y}_L) = \left\langle \fint_{LB^+ \cap \mathbb{Z}^d} \int_Q \rho(\tau_q \omega, \dot{z}_L(\omega, q, x)) dx dm(q) \right\rangle.$$

The set of stable states at time $t \in [0,T]$ is denoted by $S_L(t)$. Under the assumption (B1), we obtain that $\mathbb{A}_L$ is linear bounded and symmetric, and there exists $c > 0$ such that $\langle A_L y_L, y_L \rangle \geq c \|y_L\|_{Y_L}^2$ for all $y_L \in Y_L$. Indeed, this follows analogously as in Remark 8.10 using the continuum and discrete Korn inequalities, and the orthogonality of $\nabla_{x,s} u$ and $\nabla_{q,s} \varphi$.

Remark 8.14 (Existence and a priori estimates)**.** *If we assume (B1)-(B2), $l \in C^1([0,T], L^2(Q)^d)$ and $y_L^0 \in S_L(0)$, then the assumptions of Theorem 3.5 are satisfied and therefore there exists a unique energetic solution $y_L \in C^{\mathrm{Lip}}([0,T], Y_L)$ to the ERIS $(Y_L, \mathcal{E}_L, \mathcal{R}_L)$ with $y_L(0) = y_L^0$. Moreover, $\|y_L(t) - y_L(s)\|_{Y_L} \leq c|t-s|$ for all $t, s \in [0,T]$ and $c > 0$ does not depend on L. We remark that this system is given in the mean formulation, i.e., the functionals are averaged over Ω. However, it also admits an equivalent pointwise P-a.e. formulation, which, for fixed ω, presents a deterministic ERIS suitable for usual finite element computations, see Remark 8.16.*

In the following, $\mathcal{T}_\varepsilon : L^2(\Omega) \otimes L^2(\varepsilon\mathbb{Z}^d) \otimes L^2(Q) \to L^2(\Omega) \otimes L^2(\mathbb{R}^d) \otimes L^2(Q)$ denotes the linear isometric extension of $\mathcal{T}_\varepsilon : L^2(\Omega) \otimes L^2(\varepsilon\mathbb{Z}^d) \to L^2(\Omega) \otimes L^2(\mathbb{R}^d)$ as introduced at the end of Section 5.3 (i.e., "$\mathcal{T}_\varepsilon u(\omega, q, x) = u(\tau_{-\frac{q}{\varepsilon}} \omega, q, x)$"). In the following, we view L as a sequence that tends to ∞. We emphasize that for a function $\varphi \in L^2(\Omega) \otimes L^2(\mathbb{Z}^d) \otimes L^2(Q)$ the second (discrete) variable is denoted by $q \in \mathbb{Z}^d$, i.e., "$\varphi = \varphi(\omega, q, x)$". We reuse the same letter for the second (continuum) variable of $\mathcal{T}_1 \varphi \in L^2(\Omega) \otimes L^2(B) \otimes L^2(Q)$, i.e., "$\mathcal{T}_1 \varphi = \mathcal{T}_1 \varphi(\omega, q, x)$" and $q \in B \subset \mathbb{R}^d$. In this respect, below $\int_{LB} \cdot dq$ denotes integration with respect to the (second) continuum variable $q \in LB$ with respect to the Lebesgue measure. The main result of this section is:

Theorem 8.15 (Convergence of approximation)**.** *Assume that (B1)-(B2) hold, $\langle \cdot \rangle$ is ergodic, $l \in C^1([0,T], L^2(Q)^d)$. Let $y^0 \in Y$ and $y_L^0 \in S_L(0)$ satisfy, as $L \to \infty$,*

$$u_L^0 \to u^0 \quad \text{in } \left(L^2(\Omega) \otimes H_0^1(Q)\right)^d, \quad \left\langle \fint_{LB^+} \int_Q |\mathcal{T}_1 z_L^0 - z^0|^2 dx dq \right\rangle \to 0,$$

$$\frac{1}{L^2} \left\langle \fint_{LB} \int_Q |\mathcal{T}_1 \varphi_L^0|^2 dx dq \right\rangle \to 0, \quad \left\langle \fint_{LB^+} \int_Q |\mathcal{T}_1 \nabla_q \varphi_L^0 - \chi^0|^2 dx dq \right\rangle \to 0.$$

Let $y_L \in C^{\mathrm{Lip}}([0,T], Y_L)$ be the unique energetic solution to the ERIS $(Y_L, \mathcal{E}_L, \mathcal{R}_L)$ with $y_L(0) = y_L^0$. Then, $y^0 \in S(0)$ and for all $t \in [0,T]$,

$$u_L(t) \to u(t) \quad \text{in } \left(L^2(\Omega) \otimes H_0^1(Q)\right)^d, \quad \mathcal{E}_L(t, y_L(t)) \to \mathcal{E}_0(t, y(t)),$$

where y is the energetic solution to the ERIS $(Y, \mathcal{E}_0, \mathcal{R}_0)$ with $y(0) = y^0$. Moreover, for all $t \in [0,T]$, it holds

$$\left\langle \fint_{LB^+} \int_Q |\mathcal{T}_1 z_L(t) - z(t)|^2 dx dq \right\rangle \to 0, \quad \frac{1}{L^2} \left\langle \fint_{LB} \int_Q |\mathcal{T}_1 \varphi_L(t)|^2 dx dq \right\rangle \to 0,$$
$$\left\langle \fint_{LB^+} \int_Q |\mathcal{T}_1 \nabla_q \varphi_L(t) - \chi(t)|^2 dx dq \right\rangle \to 0.$$

(See Section 8.2.2 for the proof.)

Remark 8.16 (Practical computations)**.** *As discussed in Section 7.3.1 (cf. Lemma 7.16), for P-a.a. ω, the solution of the above system $y_L(\omega)$ solves a deterministic ERIS $\left(Y_L^{\det}, \mathcal{E}_L^\omega, \mathcal{R}_L^\omega\right)$ (parametrized by ω) given by*

$$\begin{aligned}
&Y_L^{\det} = H_0^1(Q)^d \times \left(L_0^2(LB^+ \cap \mathbb{Z}^d) \otimes L^2(Q)\right)^k \times \left(L_0^2(LB \cap \mathbb{Z}^d) \otimes L^2(Q)\right)^d, \\
&\mathcal{E}_L^\omega(t,y) = \frac{1}{2} \fint_{LB^+ \cap \mathbb{Z}^d} \int_Q A(\tau_q \omega) \begin{pmatrix} \nabla_{x,s} u + \nabla_{q,s} \varphi \\ z \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x,s} u + \nabla_{q,s} \varphi \\ z \end{pmatrix} dx dm - \int_Q l(t) \cdot u dx, \\
&\mathcal{R}_L^\omega(\dot{y}) = \fint_{LB^+ \cap \mathbb{Z}^d} \int_Q \rho(\tau_q \omega, \dot{z}(q,x)) dx dm(q).
\end{aligned}$$

This observation provides a ground for the practical computation of the solution y_L.

8.2.2 Proofs

Before proving Theorem 8.12, we show, as an auxiliary result, the existence of joint recovery sequences, which implies the stability of two-scale limits of solutions.

Lemma 8.17. *Assume that (B1)-(B2) hold, $\langle \cdot \rangle$ is ergodic, $l \in C^1([0,T], L^2(Q)^d)$. Let $t \in [0,T]$ and $y_\varepsilon \in S_\varepsilon(t)$ such that $y_\varepsilon \overset{c2}{\rightharpoonup} y \in Y$. For any $\widetilde{y} \in Y$ there exists $\widetilde{y}_\varepsilon \in Y_\varepsilon$ such that $\widetilde{y}_\varepsilon \overset{c2}{\rightharpoonup} \widetilde{y}$ and*

$$\lim_{\varepsilon \to 0} \left(\mathcal{E}_\varepsilon(t, \widetilde{y}_\varepsilon) + \mathcal{R}_\varepsilon(\widetilde{y}_\varepsilon - y_\varepsilon) - \mathcal{E}_\varepsilon(t, y_\varepsilon) \right) = \mathcal{E}_0(t, \widetilde{y}) + \mathcal{R}_0(\widetilde{y} - y) - \mathcal{E}_0(t, y).$$

This implies $y \in S(t)$.

Proof. Corollary 5.28 (iii) implies that there exists a sequence $v_\varepsilon \in \left(L^2(\Omega) \otimes L_0^2(Q \cap \varepsilon\mathbb{Z}^d)\right)^d$ with

$$v_\varepsilon \overset{2}{\to} \widetilde{u} - u \quad \text{in } L^2(\Omega \times \mathbb{R}^d)^d, \quad \nabla^\varepsilon v_\varepsilon \overset{2}{\to} \nabla \widetilde{u} - \nabla u + \widetilde{\chi} - \chi \quad \text{in } L^2(\Omega \times \mathbb{R}^d)^{d \times d}.$$

The sequence $g_\varepsilon \in \left(L^2(\Omega) \otimes L_0^2(Q^{+\varepsilon} \cap \varepsilon\mathbb{Z}^d)\right)^k$, given by $g_\varepsilon = \mathbf{1}_{Q^{+\varepsilon}} \mathcal{F}_\varepsilon(\widetilde{z} - z)$, satisfies

$$g_\varepsilon \overset{2}{\to} \widetilde{z} - z \quad \text{in } L^2(\Omega \times \mathbb{R}^d)^k.$$

We define $\widetilde{y}_\varepsilon$ componentwise: $\widetilde{u}_\varepsilon = u_\varepsilon + v_\varepsilon$ and $\widetilde{z}_\varepsilon = z_\varepsilon + g_\varepsilon$. By weak two-scale convergence of y_ε, we have that $\widetilde{y}_\varepsilon \overset{c2}{\rightharpoonup} \widetilde{y}$, and furthermore $\widetilde{y}_\varepsilon - y_\varepsilon \overset{c2}{\to} \widetilde{y} - y$.

The energy functional is quadratic, thus it satisfies

$$\begin{aligned}
&\mathcal{E}_\varepsilon(t,\widetilde{y}_\varepsilon)-\mathcal{E}_\varepsilon(t,y_\varepsilon) \\
=\ & \frac{1}{2}\left\langle \int_{Q^{+\varepsilon}\cap\varepsilon\mathbb{Z}^d} A(\tau_{\frac{x}{\varepsilon}}\omega)\begin{pmatrix}\nabla_s^\varepsilon(\widetilde{u}_\varepsilon-u_\varepsilon)(\omega,x)\\ (\widetilde{z}_\varepsilon-z_\varepsilon)(\omega,x)\end{pmatrix}\cdot\begin{pmatrix}\nabla_s^\varepsilon(\widetilde{u}_\varepsilon+u_\varepsilon)(\omega,x)\\ (\widetilde{z}_\varepsilon+z_\varepsilon)(\omega,x)\end{pmatrix} dm_\varepsilon(x)\right\rangle \\
&-\left\langle \int_{Q\cap\varepsilon\mathbb{Z}^d}\pi_\varepsilon l(t)(x)\cdot(\widetilde{u}_\varepsilon-u_\varepsilon)(\omega,x)dm_\varepsilon(x)\right\rangle.
\end{aligned}\tag{8.18}$$

Using the transformation formula (5.14), we rewrite the first term on the right-hand side as

$$\frac{1}{2}\left\langle \int_{\mathbb{R}^d} A(\omega)\begin{pmatrix}\mathcal{T}_\varepsilon\nabla_s^\varepsilon(\widetilde{u}_\varepsilon-u_\varepsilon)(\omega,x)\\ \mathcal{T}_\varepsilon(\widetilde{z}_\varepsilon-z_\varepsilon)(\omega,x)\end{pmatrix}\cdot\begin{pmatrix}\mathcal{T}_\varepsilon\nabla_s^\varepsilon(\widetilde{u}_\varepsilon+u_\varepsilon)(\omega,x)\\ \mathcal{T}_\varepsilon(\widetilde{z}_\varepsilon+z_\varepsilon)(\omega,x)\end{pmatrix} dx\right\rangle.$$

This expression is a scalar product of strongly and weakly convergent sequences (see Lemma 5.18), and therefore it converges to, as $\varepsilon\to 0$,

$$\begin{aligned}
&\frac{1}{2}\left\langle \int_{\mathbb{R}^d} A\begin{pmatrix}\nabla_s\widetilde{u}_s-\nabla_s u+\widetilde{\chi}_s-\chi_s\\ \widetilde{z}-z\end{pmatrix}\cdot\begin{pmatrix}\nabla_s\widetilde{u}+\nabla_s u+\widetilde{\chi}_s+\chi_s\\ \widetilde{z}+z\end{pmatrix}dx\right\rangle \\
=\ &\frac{1}{2}\Big(\langle\mathbb{A}\widetilde{y},\widetilde{y}\rangle_{Y^*,Y}-\langle\mathbb{A}y,y\rangle_{Y^*,Y}\Big).
\end{aligned}$$

The second term on the right-hand side of (8.18) converges to $-\int_Q l(t)\cdot(\widetilde{u}-u)dx$. Furthermore, by Jensen's inequality and the transformation (5.14), we obtain

$$\begin{aligned}
\mathcal{R}_\varepsilon(\widetilde{y}_\varepsilon-y_\varepsilon) &\le \left\langle \int_{\varepsilon\mathbb{Z}^d}\rho(\omega,\pi_\varepsilon(\widetilde{z}-z)(\omega,x))dm_\varepsilon(x)\right\rangle \\
&\le \left\langle \int_{\varepsilon\mathbb{Z}^d}\fint_{x+\varepsilon\Box}\rho(\omega,\widetilde{z}(\omega,\overline{x})-z(\omega,\overline{x}))d\overline{x}dm_\varepsilon(x)\right\rangle \\
&=\mathcal{R}_0(\widetilde{y}-y).
\end{aligned}$$

On the other hand, using Fatou's lemma and the fact that $\rho(\omega,\cdot)$ is continuous, we obtain

$$\liminf_{\varepsilon\to 0}\mathcal{R}_\varepsilon(\widetilde{y}_\varepsilon-y_\varepsilon)\ge\mathcal{R}_0(\widetilde{y}-y).$$

Above, we also use the usual transformation formula (5.14). This concludes the proof. $\square$

Proof of Theorem 8.12. Step 1. Compactness and stability. First, we set $v_\varepsilon:=(\mathcal{T}_\varepsilon u_\varepsilon,\mathcal{T}_\varepsilon\nabla^\varepsilon u_\varepsilon,\mathcal{T}_\varepsilon z_\varepsilon):[0,T]\to L^2(\Omega\times\mathbb{R}^d)^d\times L^2(\Omega\times\mathbb{R}^d)^{d\times d}\times L^2(\Omega\times\mathbb{R}^d)^k=:H$. Using the a priori estimate from Remark 8.9 and the isometry property of $\mathcal{T}_\varepsilon$, it follows that v_ε is uniformly bounded in $C^{\mathrm{Lip}}([0,T],H)$. Therefore, the Arzelà-Ascoli theorem implies that there exist $v\in C^{\mathrm{Lip}}([0,T],H)$ and a subsequence (not relabeled), such that for all $t\in[0,T]$

$$v_\varepsilon(t)\rightharpoonup v(t)\quad\text{weakly in } H.$$

Moreover, by boundedness of $y_\varepsilon(t)$ and the above, we conclude that for all $t\in[0,T]$, $v(t)=(u(t),\nabla u(t)+\chi(t),z(t))$, for some $y(t)=(u(t),z(t),\chi(t))\in Y$. Here we use the fact that if $z_\varepsilon\in L^2(\Omega)\otimes L^2_0(Q^{+\varepsilon}\cap\varepsilon\mathbb{Z}^d)^k$ converges in the weak two-scale sense, then, similarly as in Corollary

5.23, the limit may be identified with an $L^2(\Omega\times Q)^k$ function. In other words, we have $y_\varepsilon(t) \overset{c2}{\rightharpoonup} y(t)$ and $y(0)=y^0$. Lemma 8.17 implies that $y(t)\in S(t)$ for all $t\in[0,T]$.

Step 2. Energy balance. We fix $t\in(0,T]$. We pass to the limit $\varepsilon\to 0$ in (8.15) and show that y satisfies

$$\mathcal{E}_0(t,y(t)) + \int_0^t \mathcal{R}_0(\dot{y}(s))ds \le \mathcal{E}_0(0,y(0)) - \int_0^t\int_Q \dot{l}(s)\cdot u(s)dxds. \tag{8.19}$$

The (EB) equality of the discrete system reads

$$\begin{aligned}&\frac{1}{2}\left\langle \mathbb{A}_\varepsilon y_\varepsilon(t), y_\varepsilon(t)\right\rangle_{Y_\varepsilon^*,Y_\varepsilon} - \left\langle \int_{Q\cap\varepsilon\mathbb{Z}^d} \pi_\varepsilon l(t)(x)\cdot u_\varepsilon(t)(\omega,x)dm_\varepsilon(x)\right\rangle + \int_0^t \mathcal{R}_\varepsilon(\dot{y}_\varepsilon(s))ds \\ =&\frac{1}{2}\left\langle \mathbb{A}_\varepsilon y_\varepsilon(0), y_\varepsilon(0)\right\rangle_{Y_\varepsilon^*,Y_\varepsilon} - \left\langle \int_{Q\cap\varepsilon\mathbb{Z}^d} \pi_\varepsilon l(0)(x)\cdot u_\varepsilon(0)(\omega,x)dm_\varepsilon(x)\right\rangle \\ &-\int_0^t\left\langle \int_{Q\cap\varepsilon\mathbb{Z}^d} \pi_\varepsilon \dot{l}(s)(x)\cdot u_\varepsilon(s)(\omega,x)dm_\varepsilon(x)\right\rangle ds.\end{aligned} \tag{8.20}$$

The strong two-scale convergence of the initial data implies that the first two terms on the right-hand side converge to $\mathcal{E}(0,y(0))$. The remaining term on the right-hand side converges to $-\int_0^t\int_Q \dot{l}(s)\cdot u(s)dxds$ by the dominated convergence theorem. Moreover, using Proposition 5.19 and the strong convergence of $\pi_\varepsilon l(t)$ we obtain

$$\liminf_{\varepsilon\to 0}\left(\frac{1}{2}\left\langle \mathbb{A}_\varepsilon y_\varepsilon(t), y_\varepsilon(t)\right\rangle_{Y_\varepsilon^*,Y_\varepsilon} - \left\langle \int_{Q\cap\varepsilon\mathbb{Z}^d} \pi_\varepsilon l(t)(x)\cdot u_\varepsilon(t)(\omega,x)dm_\varepsilon(x)\right\rangle\right) \ge \mathcal{E}_0(t,y(t)). \tag{8.21}$$

To treat the last term on the left-hand side of (8.20), we consider a partition $\{t_i\}$ of $[0,t]$. We have

$$\sum_i \mathcal{R}_0(y(t_i)-y(t_{i-1})) \le \liminf_{\varepsilon\to 0}\sum_i \mathcal{R}_\varepsilon(y_\varepsilon(t_i)-y_\varepsilon(t_{i-1})).$$

The above inequality follows by the transformation formula $\mathcal{R}_\varepsilon(y_\varepsilon) = \left\langle\int_{\mathbb{R}^d}\rho(\omega,\mathcal{T}_\varepsilon z_\varepsilon)dx\right\rangle$ and by the fact that the integral functional $\left\langle\int_{\mathbb{R}^d}\rho(\omega,\cdot)\right\rangle$ is weakly l.s.c. Taking the supremum over all partitions $\{t_i\}$ of $[0,t]$ above, exploiting the 1-homogeneity of $\mathcal{R}_0$ and the growth condition from (B2), we obtain

$$\int_0^t \mathcal{R}_0(\dot{y}(s))ds \le \liminf_{\varepsilon\to 0}\int_0^t \mathcal{R}_\varepsilon(\dot{y}_\varepsilon(s))ds. \tag{8.22}$$

This proves (8.19). The other inequality in the (EB) equality of the limit system follows using the stability of y (see Lemma 3.6) and therefore we conclude that y satisfies (8.16). Moreover, using this equality and the fact that the right-hand side of (8.20) converges to $\mathcal{E}_0(0,y(0))-\int_0^t\int_Q \dot{l}(s)\cdot u(s)dxds$, we conclude that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\left(\mathcal{E}_\varepsilon(t,y_\varepsilon(t)) + \int_0^t \mathcal{R}_\varepsilon(\dot{y}_\varepsilon(s))ds\right) \\ =\;& \lim_{\varepsilon\to 0}\left(\mathcal{E}_\varepsilon(0,y_\varepsilon(0)) - \int_0^t\left\langle\int_{Q\cap\varepsilon\mathbb{Z}^d}\pi_\varepsilon\dot{l}(s)\cdot u_\varepsilon(s)dm_\varepsilon\right\rangle\right)ds \\ =\;& \mathcal{E}_0(0,y(0)) - \int_0^t\int_Q \dot{l}(s)\cdot u(s)dxds = \mathcal{E}_0(t,y(t)) + \int_0^t \mathcal{R}_0(\dot{y}(s))ds.\end{aligned} \tag{8.23}$$

This implies that $\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon(t,y_\varepsilon(t)) \leq \mathcal{E}_0(t,y(t)) + \int_0^t \mathcal{R}_0(\dot{y}(s))ds + \limsup_{\varepsilon\to 0} - \int_0^t \mathcal{R}_\varepsilon(\dot{y}_\varepsilon(s))ds$ and therefore using the liminf estimates (8.21) and (8.22), we obtain that

$$\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(t,y_\varepsilon(t)) = \mathcal{E}_0(t,y(t)). \tag{8.24}$$

Step 3. Strong convergence. To obtain strong two-scale convergence, we construct a strong recovery sequence $\widetilde{y}_\varepsilon(t) \in Y_\varepsilon$ for $y(t) \in Y$ for all $t \in [0,T]$, in the sense that

$$\widetilde{y}_\varepsilon(t) \overset{c2}{\to} y(t),$$

cf. the proof of Lemma 8.17. For notational convenience, we drop the "t" from the sequences and we denote $v_\varepsilon := (\mathcal{T}_\varepsilon u_\varepsilon, \mathcal{T}_\varepsilon \nabla^\varepsilon u_\varepsilon, \mathcal{T}_\varepsilon z_\varepsilon)$, $\widetilde{v}_\varepsilon := (\mathcal{T}_\varepsilon \widetilde{u}_\varepsilon, \mathcal{T}_\varepsilon \nabla^\varepsilon \widetilde{u}_\varepsilon, \mathcal{T}_\varepsilon \widetilde{z}_\varepsilon)$ and $v := (u, \nabla u + \chi, z)$. By the triangle inequality, we have

$$\|v_\varepsilon - v\|_H \leq \|v_\varepsilon - \widetilde{v}_\varepsilon\|_H + \|\widetilde{v}_\varepsilon - v\|_H. \tag{8.25}$$

The second term on the right-hand side vanishes in the limit $\varepsilon \to 0$. Also, since the energy is quadratic, we obtain, using the isometry property of $\mathcal{T}_\varepsilon$ and a discrete Poincaré-Korn inequality,

$$\begin{aligned} &\|v_\varepsilon - \widetilde{v}_\varepsilon\|_H^2 \\ \leq\ & c\left(\mathcal{E}_\varepsilon(t,y_\varepsilon) - \mathcal{E}_\varepsilon(t,\widetilde{y}_\varepsilon) + \langle \mathbb{A}_\varepsilon \widetilde{y}_\varepsilon, \widetilde{y}_\varepsilon - y_\varepsilon\rangle_{Y_\varepsilon^*,Y_\varepsilon} + \left\langle \int_{Q\cap\varepsilon\mathbb{Z}^d} \pi_\varepsilon l(t)(x)\cdot(u_\varepsilon - \widetilde{u}_\varepsilon)(\omega,x)dm_\varepsilon(x)\right\rangle\right). \end{aligned}$$

The last two terms on the right-hand side vanish as $\varepsilon \to 0$ (cf. the proof of Lemma 8.17). The sum of the first two terms vanishes as well in the limit $\varepsilon \to 0$ by (8.24) and the fact that $\widetilde{y}_\varepsilon$ is a strong recovery sequence (and therefore $\lim_{\varepsilon\to 0}\mathcal{E}_\varepsilon(t,\widetilde{y}_\varepsilon) = \mathcal{E}_0(t,y)$). This proves the claim of the theorem for a subsequence. To show that the convergence holds for the whole sequence, for a fixed $t \in [0,T]$, we consider $e_\varepsilon(t) := \|v_\varepsilon(t) - v(t)\|_H$. For any subsequence ε' of ε, we can find a further subsequence ε'' such that $e_{\varepsilon''}(t) \to 0$ by the uniqueness of the solution y. From this follows that the whole sequence converges in the sense given in the statement of the theorem. $\square$

In the proof of Theorem 8.15, by a change of variables $q \rightsquigarrow \frac{q}{L}$ we equivalently restate $(Y_L, \mathcal{E}_L, \mathcal{R}_L)$ as a system that is similar to $(Y_\varepsilon, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ on an extended space. As a result of this, the claim follows similarly as Theorem 8.12 and therefore in the proof we omit the details that are completely analogous.

Proof of Theorem 8.15. Step 0. Reduction to a familiar problem. For notational convenience we set $\varepsilon := \frac{1}{L}$, we recall that $\mathcal{T}_\varepsilon$ denotes the extended unfolding operator and we use the usual notation $\overset{2}{\rightharpoonup}$ to denote convergence of unfolded sequences. We define $y_\varepsilon(\cdot,q,\cdot) := \left(u_{\frac{1}{\varepsilon}}(\cdot,\cdot), z_{\frac{1}{\varepsilon}}(\cdot,\frac{1}{\varepsilon}q,\cdot), \varepsilon\varphi_{\frac{1}{\varepsilon}}(\cdot,\frac{1}{\varepsilon}q,\cdot)\right)$ and using a change of variables $\widetilde{q} = \varepsilon q$, a direct computation shows that y_ε is the unique solution to the ERIS $\left(\widetilde{Y}_\varepsilon, \widetilde{\mathcal{E}}_\varepsilon, \widetilde{\mathcal{R}}_\varepsilon\right)$ given by

$$\widetilde{Y}_\varepsilon = \left(L^2(\Omega)\otimes H_0^1(Q)\right)^d \times \left(L^2(\Omega)\otimes L_0^2(B^{+\varepsilon}\cap\varepsilon\mathbb{Z}^d)\otimes L^2(Q)\right)^k \times \left(L^2(\Omega)\otimes L_0^2(B\cap\varepsilon\mathbb{Z}^d)\otimes L^2(Q)\right)^d$$

$$\widetilde{\mathcal{E}}_\varepsilon(t,y) = \frac{1}{2}\left\langle \fint_{B^{+\varepsilon}\cap\varepsilon\mathbb{Z}^d}\int_Q A(\tau_{\frac{q}{\varepsilon}}\omega)\begin{pmatrix}\nabla_{x,s}u + \nabla^\varepsilon_{q,s}\varphi \\ z\end{pmatrix}\cdot\begin{pmatrix}\nabla_{x,s}u + \nabla^\varepsilon_{q,s}\varphi \\ z\end{pmatrix}dxdm_\varepsilon\right\rangle - \left\langle \int_Q l(t)\cdot udx\right\rangle,$$

$$\widetilde{\mathcal{R}}_\varepsilon(\dot{y}) = \left\langle \fint_{B^{+\varepsilon}}\int_Q \rho(\tau_{\frac{q}{\varepsilon}}\omega, \dot{z}(\omega,q,x))dxdm_\varepsilon(q)\right\rangle.$$

In the following steps we show that this system converges to an ERIS, given in terms of $\widetilde{\mathcal{E}}_0 =$"$\fint_B \mathcal{E}_0$" and $\widetilde{\mathcal{R}}_0 =$"$\fint_B \mathcal{R}_0$", that is equivalent to $(Y, \mathcal{E}_0, \mathcal{R}_0)$ for initial data independent of q, cf. Step 5. We define the intermediate system $\left(\widetilde{Y}, \widetilde{\mathcal{E}}_0, \widetilde{\mathcal{R}}_0\right)$ as follows:

$$\widetilde{Y} = H_0^1(Q)^d \times \left(L^2(\Omega) \otimes L^2(B) \otimes L^2(Q)\right)^k \times \left(H_0^1(B) \otimes L^2(Q)\right)^d \times \left(L^2_{\text{pot}}(\Omega) \otimes L^2(B) \otimes L^2(Q)\right)^d,$$

the state variable is denoted by $w = (u, z, \varphi, \chi)$, the energy functional is given by $\widetilde{\mathcal{E}}_0 : [0,T]\times\widetilde{Y} \to \mathbb{R}$,

$$\widetilde{\mathcal{E}}_0(t,w) = \left\langle \fint_B \int_Q A \begin{pmatrix} \nabla_{x,s} u + \nabla_{q,s}\varphi + \chi_s \\ z \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x,s} u + \nabla_{q,s}\varphi + \chi_s \\ z \end{pmatrix} dxdq \right\rangle - \int_Q l(t)\cdot u dx,$$

and the dissipation functional is $\widetilde{\mathcal{R}}_0 : \widetilde{Y} \to [0,\infty]$,

$$\widetilde{\mathcal{R}}_0(\dot{w}) = \left\langle \fint_B \int_Q \rho(\omega, \dot{z}(\omega,q,x)) dxdq \right\rangle.$$

In this proof we tacitly identify the function $(\omega, x) \mapsto u_\varepsilon(\omega, x)$ with the function $(\omega, q, x) \mapsto \mathbf{1}_{B^{+\varepsilon}\cap\varepsilon\mathbb{Z}^d}(q) u_\varepsilon(\omega,x)$ and analogously for any function independent of $q \in \varepsilon\mathbb{Z}^d$ (or $q \in B$).

Step 1. Compactness. We define

$$v_\varepsilon := \left(\mathcal{T}_\varepsilon u_\varepsilon, \mathcal{T}_\varepsilon \nabla_x u_\varepsilon, \mathcal{T}_\varepsilon z_\varepsilon, \mathcal{T}_\varepsilon \varphi_\varepsilon, \mathcal{T}_\varepsilon \nabla_q^\varepsilon \varphi_\varepsilon\right) : [0,T] \to H,$$

where $H := \left(L^2(\Omega)\otimes L^2(\mathbb{R}^d)\otimes L^2(Q)\right)^{d+d\times d+k+d+d\times d}$. Using the a priori estimate from Remark 8.14, the isometry property of $\mathcal{T}_\varepsilon$ and the discrete Poincaré-Korn inequality, it follows that

$$\|v_\varepsilon(t) - v_\varepsilon(s)\|_H \leq c m_\varepsilon(B^{+\varepsilon}\cap\varepsilon\mathbb{Z}^d)|t-s| \quad \text{for all } t,s \in [0,T].$$

We notice that $m_\varepsilon(B^{+\varepsilon}\cap\varepsilon\mathbb{Z}^d) \to |B|$ and therefore the Arzelà-Ascoli theorem yields the existence of a (not relabeled) subsequence and $v \in C^{\text{Lip}}([0,T], H)$ such that for all $t \in [0,T]$,

$$v_\varepsilon(t) \rightharpoonup v(t) \quad \text{weakly in } H.$$

Using the compactness properties of the extended unfolding operator discussed at the end of Section 5.3, in particular (5.18) and (5.19), we obtain that $v(t) = (u(t), \nabla_x u(t), z(t), \varphi(t), \nabla_q\varphi(t) + \chi(t))$ such that $w(t) := (u(t), z(t), \varphi(t), \chi(t)) \in \widetilde{Y}$. The convergence of the initial data implies that $w(0) = \left(u^0, z^0, 0, \chi^0\right)$.

Step 2. Stability. We fix $t \in [0,T]$ that we drop from the notation. We consider an arbitrary $\widetilde{w} = (\widetilde{u}, \widetilde{z}, \widetilde{\varphi}, \widetilde{\chi}) \in \widetilde{Y}$. We apply the extended recovery sequence construction from (5.20) in Section 5.3 to the pair $(\widetilde{\varphi}-\varphi, \widetilde{\chi}-\chi)$ that yields a function $\phi_\varepsilon \in \left(L^2(\Omega)\otimes L^2_0(B\cap\varepsilon\mathbb{Z}^d)\otimes L^2(Q)\right)^d$ such that $\phi_\varepsilon \xrightarrow{2} \widetilde{\varphi}-\varphi$ and $\nabla_q^\varepsilon\phi_\varepsilon \xrightarrow{2} \nabla_q\widetilde{\varphi}-\nabla_q\varphi+\widetilde{\chi}-\chi$. Also, we set $g_\varepsilon(\omega,q,x) := \mathbf{1}_{B^{+\varepsilon}\cap\varepsilon\mathbb{Z}^d}(q)\mathcal{F}_\varepsilon(\widetilde{z}-z)(\omega,q,x)$, which satisfies $g_\varepsilon \xrightarrow{2} \widetilde{z}-z$. We define $\widetilde{y}_\varepsilon \in \widetilde{Y}_\varepsilon$ componentwise $\widetilde{u}_\varepsilon = u_\varepsilon + \widetilde{u} - u$, $\widetilde{z}_\varepsilon = z_\varepsilon + g_\varepsilon$, $\widetilde{\varphi}_\varepsilon = \varphi_\varepsilon + \phi_\varepsilon$. As a result of this, we obtain

$$\begin{aligned}
&\nabla_{x,s}(\widetilde{u}_\varepsilon - u_\varepsilon) \xrightarrow{2} \nabla_{x,s}\left(\widetilde{u}-u\right), \quad \widetilde{z}_\varepsilon - z_\varepsilon \xrightarrow{2} \widetilde{z}-z, \quad \nabla_{q,s}^\varepsilon\left(\widetilde{\varphi}_\varepsilon - \varphi_\varepsilon\right) \xrightarrow{2} \nabla_{q,s}\left(\widetilde{\varphi}-\varphi\right) + \widetilde{\chi}-\chi, \\
&\nabla_{x,s}(\widetilde{u}_\varepsilon + u_\varepsilon) \xrightharpoonup{2} \nabla_{x,s}\left(\widetilde{u}+u\right), \quad \widetilde{z}_\varepsilon + z_\varepsilon \xrightharpoonup{2} \widetilde{z}+z, \quad \nabla_{q,s}^\varepsilon\left(\widetilde{\varphi}_\varepsilon + \varphi_\varepsilon\right) \xrightharpoonup{2} \nabla_{q,s}\left(\widetilde{\varphi}+\varphi\right) + \widetilde{\chi}+\chi.
\end{aligned}$$

As a result of this, as in Lemma 8.17, we use the quadratic structure of the energy and the usual transformation formula using $\mathcal{T}_\varepsilon$ to obtain (shorthand $M_\varepsilon = m_\varepsilon(B^{+\varepsilon} \cap \varepsilon\mathbb{Z}^d)$)

$$\begin{aligned}
& \widetilde{\mathcal{E}}_\varepsilon(t, \widetilde{y}_\varepsilon) - \widetilde{\mathcal{E}}_\varepsilon(t, y_\varepsilon) \\
= \quad & \frac{1}{2M_\varepsilon} \left\langle \int_{\mathbb{R}^d} \int_Q A \mathcal{T}_\varepsilon \begin{pmatrix} \nabla_{x,s} (\widetilde{u}_\varepsilon - u_\varepsilon) + \nabla^\varepsilon_{q,s} (\widetilde{\varphi}_\varepsilon - \varphi_\varepsilon) \\ \widetilde{z}_\varepsilon - z_\varepsilon \end{pmatrix} \cdot \mathcal{T}_\varepsilon \begin{pmatrix} \nabla_{x,s} (\widetilde{u}_\varepsilon + u_\varepsilon) + \nabla^\varepsilon_{q,s} (\widetilde{\varphi}_\varepsilon + \varphi_\varepsilon) \\ \widetilde{z}_\varepsilon + z_\varepsilon \end{pmatrix} \right\rangle \\
& - \left\langle \int_Q l(t) \cdot (\widetilde{u} - u) \right\rangle \\
\to \quad & \widetilde{\mathcal{E}}_0(t, \widetilde{w}) - \widetilde{\mathcal{E}}_0(t, w), \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Moreover, analogously as in Lemma 8.17 we obtain that $\lim_{\varepsilon\to 0} \widetilde{\mathcal{R}}_\varepsilon(\widetilde{y}_\varepsilon - y_\varepsilon) = \widetilde{\mathcal{R}}_0(\widetilde{w} - w)$. These observations imply that $w(t) \in \widetilde{S}(t)$, which is the set of stable states corresponding to the functionals $\widetilde{\mathcal{E}}_0$ and $\widetilde{\mathcal{R}}_0$.

Step 3. Energy balance. Analogously as in Step 2 in the proof of Theorem 8.12 we pass to the limit $\varepsilon \to 0$ in the energy balance equality of the system $\left(\widetilde{Y}_\varepsilon, \widetilde{\mathcal{E}}_\varepsilon, \widetilde{\mathcal{R}}_\varepsilon\right)$ to obtain

$$\widetilde{\mathcal{E}}_0(t, w(t)) + \int_0^t \widetilde{\mathcal{R}}_0(\dot{w}(s)) ds = \widetilde{\mathcal{E}}_0(0, w(0)) - \int_0^t \int_Q \dot{l}(s) \cdot u(s) ds.$$

We remark that the left hand side is treated completely analogously as in Theorem 8.12 using the extended unfolding operator. Also, by assumption the initial energy satisfies $\mathcal{E}_\varepsilon(0, y^0_\varepsilon) \to \widetilde{\mathcal{E}}_0(0, w(0))$. Furthermore, it follows that

$$\widetilde{\mathcal{E}}_\varepsilon(t, y_\varepsilon(t)) \to \widetilde{\mathcal{E}}_0(t, w(t)). \tag{8.26}$$

Steps 2 and 3 show that w is a solution to the ERIS $\left(\widetilde{Y}, \widetilde{\mathcal{E}}_0, \widetilde{\mathcal{R}}_0\right)$ with $w(0) = \left(u^0, z^0, 0, \chi^0\right)$, that turns out to be unique, cf. Step 5. In this respect, the above convergences hold for the entire sequence.

Step 4. Strong convergence. For $w(t) = (u(t), z(t), \varphi(t), \chi(t))$, similarly as in Step 2, we find a strong recovery sequence $\widetilde{y}_\varepsilon = (\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon, \widetilde{\varphi}_\varepsilon) \in Y_\varepsilon$ such that $\widetilde{v}_\varepsilon = \left(\mathcal{T}_\varepsilon \widetilde{u}_\varepsilon, \mathcal{T}_\varepsilon \nabla_x \widetilde{u}_\varepsilon, \mathcal{T}_\varepsilon \widetilde{z}_\varepsilon, \mathcal{T}_\varepsilon \widetilde{\varphi}_\varepsilon, \mathcal{T}_\varepsilon \nabla^\varepsilon_q \widetilde{\varphi}_\varepsilon\right)$ satisfies

$$\widetilde{v}_\varepsilon \to v(t) \quad \text{strongly in } H.$$

Following the argumentation of Step 3 in Theorem 8.12 (using the quadratic structure and the convergence of the energy), we obtain that $v_\varepsilon(t) \to v(t)$ strongly in H.

Step 5. Conclusion: We show that $(u, z, 0, \chi)$ is the unique solution to $\left(\widetilde{Y}, \widetilde{\mathcal{E}}_0, \widetilde{\mathcal{R}}_0\right)$, where $y = (u, z, \chi)$ is the solution to $(Y, \mathcal{E}_0, \mathcal{R}_0)$.

First, we remark that the ERIS $\left(\widetilde{Y}, \widetilde{\mathcal{E}}_0, \widetilde{\mathcal{R}}_0\right)$ satisfies the assumptions of Theorem 3.5, that might be seen using the (pairwise) orthogonality of $\nabla_{x,s} u$, $\nabla_{q,s} \varphi$ and χ_s, cf. Remark 8.10. Therefore, it has a unique solution w that satisfies $w(0) = \left(u^0, z^0, 0, \chi^0\right)$; note that $y^0 \in S(0) \Rightarrow w(0) \in \widetilde{S}(0)$. Second, we consider $y = (u, z, \chi)$, the solution to $(Y, \mathcal{E}_0, \mathcal{R}_0)$ with $y(0) = y^0$. By stability of y, for an arbitrary $\widetilde{w} \in \widetilde{Y}$, we have

$$\begin{aligned}
&\mathcal{E}_0(t,y)\\
\leq\ & \frac{1}{2}\left\langle \int_Q A(\omega)\begin{pmatrix} \nabla_{x,s}\widetilde{u} + \fint_B (\nabla_{q,s}\widetilde{\varphi} + \widetilde{\chi}_s)\, dq \\ \fint_B \widetilde{z} dq \end{pmatrix} \cdot \begin{pmatrix} \nabla_{x,s}\widetilde{u} + \fint_B (\nabla_{q,s}\widetilde{\varphi} + \widetilde{\chi}_s)\, dq \\ \fint_B \widetilde{z} dq \end{pmatrix} dx \right\rangle\\
& - \int_Q l(t)\cdot \widetilde{u} dx + \left\langle \int_Q \rho(\omega, \fint_B \widetilde{z} - z dq) dx \right\rangle\\
\leq\ & \widetilde{\mathcal{E}}_0(t,\widetilde{w}) + \widetilde{\mathcal{R}}_0(\widetilde{w} - (u,z,0,\chi)),
\end{aligned}$$

where the latter is Jensen's inequality. An averaged integration of the above over B, yields $(u(t), z(t), 0, \chi(t)) \in \widetilde{S}(t)$. Similarly, integrating the energy balance equality of $(Y, \mathcal{E}_0, \mathcal{R}_0)$ over B, it follows that $(u(t), z(t), 0, \chi(t))$ satisfies the energy balance equality of $\left(\widetilde{Y}, \widetilde{\mathcal{E}}_0, \widetilde{\mathcal{R}}_0\right)$. As a result of this, we conclude that $(u(t), z(t), 0, \chi(t)) = w(t)$. This fact, (8.26) and the strong convergence from Step 4 conclude the proof. □

8.3 Homogenization of gradient plasticity

The effective rate-independent system from Section 8.2 cannot be equivalently recast as a classical ERIS with deterministic properties as in the case of convex minimization (Theorem 8.5). The reason for this is that the limiting internal variable z is in general not deterministic. The microscopic problem might be regularized by adding a gradient term of the internal variable z_ε and in that way homogenization yields a deterministic limit problem. This strategy was demonstrated in [Han11], where periodic homogenization of gradient plasticity in the continuum setting is discussed. In the following, we show that the same applies in our stochastic, discrete-to-continuum setting.

Let $\gamma \in (0,1)$. The new microscopic system involves the same dissipation potential $\mathcal{R}_\varepsilon$ as before, as well as the same state space Y_ε, yet now equipped with the scalar product

$$\begin{aligned}
&\langle y_1, y_2\rangle_{Y_\varepsilon^\gamma}\\
=\ & \left\langle \int_{\varepsilon\mathbb{Z}^d} u_1(\omega,x)\cdot u_2(\omega,x) dm_\varepsilon(x)\right\rangle + \left\langle \int_{\varepsilon\mathbb{Z}^d} \nabla^\varepsilon u_1(\omega,x) : \nabla^\varepsilon u_2(\omega,x) dm_\varepsilon(x)\right\rangle\\
& + \left\langle \int_{\varepsilon\mathbb{Z}^d} z_1(\omega,x)\cdot z_2(\omega,x) dm_\varepsilon(x)\right\rangle + \left\langle \int_{\varepsilon\mathbb{Z}^d} \varepsilon^\gamma \nabla^\varepsilon z_1(\omega,x) : \varepsilon^\gamma \nabla^\varepsilon z_2(\omega,x) dm_\varepsilon(x)\right\rangle.
\end{aligned}$$

We consider a modified energy functional $\mathcal{E}_\varepsilon^\gamma : [0,T]\times Y_\varepsilon \to \mathbb{R}$

$$\mathcal{E}_\varepsilon^\gamma(t, y_\varepsilon) = \mathcal{E}_\varepsilon(t,y_\varepsilon) + \left\langle \int_{\varepsilon\mathbb{Z}^d} G(\tau_{\frac{x}{\varepsilon}}\omega)\varepsilon^\gamma \nabla^\varepsilon z_\varepsilon(\omega,x) : \varepsilon^\gamma \nabla^\varepsilon z_\varepsilon(\omega,x) dm_\varepsilon(x)\right\rangle,$$

where $G : \Omega \to \mathrm{Lin}\left(\mathbb{R}^{k\times d}, \mathbb{R}^{k\times d}\right)$. We assume the following:

(B3) $G \in L^\infty(\Omega, \mathrm{Lin}(\mathbb{R}^{k\times d}, \mathbb{R}^{k\times d}))$, it is symmetric ($P$-a.e.) and it satisfies the following: There exists $c > 0$ such that $G(\omega)F : F \geq c|F|^2$ for P-a.a. $\omega \in \Omega$ and all $F \in \mathbb{R}^{k\times d}$.

The set of stable states at time $t \in [0,T]$ is denoted by $S_\varepsilon^\gamma(t)$.

Remark 8.18 (Existence and a priori estimates). *If we assume (B1)-(B3), $l \in C^1([0,T], L^2(Q)^d)$ and $y_\varepsilon^0 \in S_\varepsilon^\gamma(0)$, then Theorem 3.5 implies that there exists a unique energetic solution $y_\varepsilon \in C^{\mathrm{Lip}}([0,T], Y_\varepsilon)$ to the ERIS $(Y_\varepsilon, \mathcal{E}_\varepsilon^\gamma, \mathcal{R}_\varepsilon)$ with $y_\varepsilon(0) = y_\varepsilon^0$, i.e., for all $t \in [0,T]$ we have $y_\varepsilon(t) \in S_\varepsilon^\gamma(t)$ and*

$$\mathcal{E}_\varepsilon^\gamma(t, y_\varepsilon(t)) + \int_0^t \mathcal{R}_\varepsilon(\dot{y}_\varepsilon(s))ds = \mathcal{E}_\varepsilon^\gamma(0, y_\varepsilon(0)) - \int_0^t \left\langle \int_{Q\cap\varepsilon\mathbb{Z}^d} \pi_\varepsilon \dot{l}(s) \cdot u_\varepsilon(s) dm_\varepsilon \right\rangle ds, \qquad (8.27)$$

and, moreover, $\|y_\varepsilon(t) - y_\varepsilon(s)\|_{Y_\varepsilon^\gamma} \leq c|t-s|$ for all $t, s \in [0,T]$ and $c > 0$ does not depend on ε.

In the limit $\varepsilon \to 0$, we obtain a deterministic rate-independent system described as follows:

- The state space is $Y = H_0^1(Q)^d \times L^2(Q)^k$ and the state variable is denoted by $y = (u, z)$.
- The energy functional is $\mathcal{E}_{\mathrm{hom}} : [0,T] \times Y \to \mathbb{R}$

  $$\mathcal{E}_{\mathrm{hom}}(t, y) = \int_Q \frac{1}{2} V_{\mathrm{hom}}(\nabla_s u, z)dx - \int_Q l(t) \cdot u dx,$$

  where V_{hom} is given by the corrector problem: For $F_1, F_2 \in \mathbb{R}^k$,

  $$V_{\mathrm{hom}}(F_1, F_2) = \inf_{\chi \in L^2_{\mathrm{pot}}(\Omega)^d} \left\langle A(\omega) \begin{pmatrix} F_1 + \chi_s(\omega) \\ F_2 \end{pmatrix} \cdot \begin{pmatrix} F_1 + \chi_s(\omega) \\ F_2 \end{pmatrix} \right\rangle.$$

  In fact, V_{hom} is quadratic: There exists $A_{\mathrm{hom}} \in \mathbb{R}^{2k\times 2k}_{\mathrm{sym}}$ positive-definite such that $V_{\mathrm{hom}}(F_1, F_2) = A_{\mathrm{hom}} \binom{F_1}{F_2} \cdot \binom{F_1}{F_2}$ for all $F_1, F_2 \in \mathbb{R}^k$. Explicitly, for $i, j \in \{1, ..., 2k\}$, we have

  $$A_{\mathrm{hom}}^{ij} = \left\langle A(\omega) \left(e_i + \begin{pmatrix} \chi_s^i(\omega) \\ 0 \end{pmatrix} \right) \cdot e_j \right\rangle, \qquad (8.28)$$

  where $\chi^i \in L^2_{\mathrm{pot}}(\Omega)^d$ is the unique solution to

  $$\left\langle A(\omega) \left(e_i + \begin{pmatrix} \chi_s^i(\omega) \\ 0 \end{pmatrix} \right) \cdot \begin{pmatrix} \widetilde{\chi}_s(\omega) \\ 0 \end{pmatrix} \right\rangle = 0 \quad \text{for all } \widetilde{\chi} \in L^2_{\mathrm{pot}}(\Omega)^d. \qquad (8.29)$$

- The dissipation potential is given by $\mathcal{R}_{\mathrm{hom}} : Y \to [0, \infty]$

  $$\mathcal{R}_{\mathrm{hom}}(\dot{y}) = \int_Q \rho_{\mathrm{hom}}(\dot{z}(x))dx,$$

  where $\rho_{\mathrm{hom}} : \mathbb{R}^k \to [0, \infty]$, $\rho_{\mathrm{hom}}(F) = \langle \rho(\omega, F) \rangle$, that is convex, l.s.c. and positively 1-homogeneous.

The set of stable states at time $t \in [0,T]$ is denoted by $S_{\mathrm{hom}}(t)$.

Remark 8.19 (Existence and uniqueness). *If we assume (B1)-(B3), $l \in C^1([0,T], L^2(Q)^d)$ and $y^0 \in S_{\mathrm{hom}}(0)$, then Theorem 3.5 implies that there exists $y \in C^{\mathrm{Lip}}([0,T], Y)$, a unique energetic solution to the ERIS $(Y, \mathcal{E}_{\mathrm{hom}}, \mathcal{R}_{\mathrm{hom}})$ with $y(0) = y^0$.*

The main result of this section is:

Theorem 8.20 (One-scale homogenization)**.** *We assume that (B1)-(B3) hold and that $\langle\cdot\rangle$ is ergodic. Let $l \in C^1([0,T], L^2(Q)^d)$, $y_\varepsilon^0 \in S_\varepsilon^\gamma(0)$, $y^0 \in Y$, $\chi^0 \in (L^2_{\text{pot}}(\Omega) \otimes L^2(Q))^d$ satisfy*

$$y_\varepsilon^0 \overset{c2}{\to} (y^0, \chi^0), \quad \varepsilon^\gamma \nabla^\varepsilon z_\varepsilon^0 \overset{2}{\to} 0 \text{ in } L^2(\Omega \times \mathbb{R}^d)^{k\times d}, \quad \mathcal{E}_{\text{hom}}(0, y^0) = \mathcal{E}_0(0, (y^0, \chi^0)). \tag{8.30}$$

Let $y_\varepsilon \in C^{\text{Lip}}([0,T], Y_\varepsilon)$ be the unique energetic solution to the ERIS $(Y_\varepsilon, \mathcal{E}_\varepsilon^\gamma, \mathcal{R}_\varepsilon)$ with $y_\varepsilon(0) = y_\varepsilon^0$. Then $y^0 \in S_{\text{hom}}(0)$ and for all $t \in [0,T]$,

$$u_\varepsilon(t) \overset{2}{\to} u(t) \quad \text{in } L^2(\Omega \times \mathbb{R}^d)^d, \quad z_\varepsilon(t) \overset{2}{\to} z(t) \quad \text{in } L^2(\Omega \times \mathbb{R}^d)^k, \quad \mathcal{E}_\varepsilon^\gamma(t, y_\varepsilon(t)) \to \mathcal{E}_{\text{hom}}(t, y(t)),$$

where $y = (u,z) \in C^{\text{Lip}}([0,T], Y)$ is the unique energetic solution to the ERIS $(Y, \mathcal{E}_{\text{hom}}, \mathcal{R}_{\text{hom}})$ with $y(0) = y^0$.
(See Section 8.3.2 for the proof.)

Remark 8.21 (Existence of admissible initial data)**.** *We remark that for given $y^0 \in Y$, there exists χ^0 that satisfies the third claim in (8.30). Indeed, this can be shown by a measurable selection argument as in the proof of Theorem 8.5. Also, for such (y^0, χ^0) there exists a sequence y_ε^0 satisfying (8.30) that follows by a strong recovery sequence construction (cf. Theorem 8.3 and Step 2 in the proof of Theorem 8.20).*

Remark 8.22 (Convergence of gradients)**.** *The proof of Theorem 8.20 shows that, in addition, we have for all $t \in [0,T]$,*

$$\nabla^\varepsilon u_\varepsilon(t) \overset{2}{\to} \nabla u(t) + \chi(t) \quad \text{in } L^2(\Omega \times \mathbb{R}^d)^{d\times d}, \quad \varepsilon^\gamma \nabla^\varepsilon z_\varepsilon(t) \overset{2}{\to} 0 \quad \text{in } L^2(\Omega \times \mathbb{R}^d)^{k\times d},$$

where $\chi(t) \in (L^2_{\text{pot}}(\Omega) \otimes L^2(Q))^d$ is uniquely determined by the identity, for a.a. $x \in Q$,

$$V_{\text{hom}}(\nabla_s u(t)(x), z(t)(x)) = \left\langle A(\omega) \begin{pmatrix} \nabla_s u(t)(x) + \chi_s(t)(\omega, x) \\ z(t)(x) \end{pmatrix} \cdot \begin{pmatrix} \nabla_s u(t)(x) + \chi_s(t)(\omega, x) \\ z(t)(x) \end{pmatrix} \right\rangle.$$

8.3.1 Representative volume element approximations

The effective system $(Y, \mathcal{E}_{\text{hom}}, \mathcal{R}_{\text{hom}})$ obtained in the previous section is deterministic and it is given in terms of the effective coefficients A_{hom} and ρ_{hom}. As discussed in Section 7.2, the computation of these quantities is difficult in practice, cf. below (i) and (ii). Therefore, we introduce an approximating ERIS that is suitable for numerical analysis and which is based on the RVE method that is discussed in Section 7.2. In contrast to the two-scale problem from Section 8.2, here we deal with a standard (one-scale) formulation and consequently in the approximation it is sufficient to replace the coefficients A_{hom} and ρ_{hom} by their approximations A_L and ρ_L which are random, yet easier for computational purposes; here $L \gg 1$ is a parameter. The argument for the approximation result in the limit $L \to \infty$ might be obtained similarly as in the proof of Theorem 8.15. However, in this section we assume that ρ has a specific form as in the introductory example, cf. (8.3). In this respect, the proof of convergence simplifies. In particular, we first obtain the convergence $(A_L, \rho_L) \to (A_{\text{hom}}, \rho_{\text{hom}})$ in a suitable sense and secondly we conclude convergence for the corresponding solutions of the ERIS using standard a priori estimates which also quantify the speed of convergence of the solutions w.r.t. the rate of $(A_L - A_{\text{hom}}, \rho_L - \rho_{\text{hom}})$.

Specifically, we assume that:

(B4) ρ has the following form $\rho(\omega, F) = \sum_{i=1}^{k} r_i(\omega)|F_i|$, where $r \in L^\infty(\Omega)^k$ and $r_i \geq 0$ for all $i \in \{1, ..., k\}$.

Note that if (B4) holds, (B2) is also fulfilled. In this case, $\rho_{\text{hom}}(F) = \sum_{i=1}^{k} r_{\text{hom},i}|F_i|$, where $r_{\text{hom}} = \langle r \rangle$.

We point out the difficulties by the determination of A_{hom} and r_{hom}, and define their replacements:

(i) A_{hom} is defined by the homogenization formula (8.28) that requires the solution of the corrector equation (8.29). However, this equation is given in terms of stochastic gradients and it is defined on a typically infinite dimensional space Ω that makes the usual finite element approach ineffective, cf. Section 7.2. Let $L \geq 1$ and $B \subset \mathbb{R}^d$ be open bounded convex with Lipschitz boundary. We set $B^+ := B \cup \left\{ x \in \mathbb{R}^d : (x, x + \frac{1}{L}b) \cap B \neq \emptyset \text{ for some } b \in E_0 \right\}$. The replacement for A_{hom} is given by $A_L : \Omega \to \mathbb{R}^{2k\times 2k}_{\text{sym}}$,

$$A_L^{ij}(\omega) = \fint_{LB^+ \cap \mathbb{Z}^d} A(\tau_x \omega) \left(e_i + \begin{pmatrix} \nabla_s \varphi_i(\omega, x) \\ 0 \end{pmatrix} \right) \cdot e_j dm(x),$$

where $\varphi_i(\omega, \cdot) \in L^2_0(LB \cap \mathbb{Z}^d)$ is the Dirichlet corrector, i.e., it is the unique solution to

$$\fint_{LB^+ \cap \mathbb{Z}^d} A(\tau_x \omega) \left(e_i + \begin{pmatrix} \nabla_s \varphi_i(\omega, x) \\ 0 \end{pmatrix} \right) \cdot \begin{pmatrix} \nabla_s \widetilde{\varphi}(x) \\ 0 \end{pmatrix} dm(x) = 0 \quad \text{for all } \widetilde{\varphi} \in L^2_0(LB \cap \mathbb{Z}^d). \tag{8.31}$$

(ii) r_{hom} is simply given by the expectation of r, yet, in practice, we often do not have exact information about the statistical properties of the modeled material, we merely assume that the constitutive laws are described by a stationary random field. In this respect, it is convenient to define an approximation for r_{hom} by a spatial average of a realization of the stationary extension of r, i.e., we define

$$r_L(\omega) = \fint_{LB \cap \mathbb{Z}^d} r(\tau_x \omega) dm(x).$$

Note that, in contrast to A_{hom} and r_{hom}, A_L and r_L are still random, yet more convenient for computational purposes. In the following we see L as a sequence that tends to ∞. We collect the main properties of the approximations:

Lemma 8.23 (Properties of A_L and r_L). *Let $B \subset \mathbb{R}^d$ be open bounded convex with Lipschitz boundary. Let (B1) and (B4) be satisfied and $\langle \cdot \rangle$ be ergodic. Then:*

(i) A_L is a bounded sequence in $L^\infty(\Omega, \mathbb{R}^{2k\times 2k}_{\text{sym}})$ and there exists $c > 0$ independent of L, such that for P-a.a. ω, $A_L(\omega)F \cdot F \geq c|F|^2$ for all $F \in \mathbb{R}^{2k}$, and as $L \to \infty$,

$$A_L \to A_{\text{hom}} \quad \text{strongly in } L^2(\Omega)^{2k\times 2k}.$$

(ii) r_L is a bounded sequence in $L^\infty(\Omega)^k$, $r_{L,i} \geq 0$ for all $i \in \{1, ..., k\}$, and as $L \to \infty$,

$$r_L \to r_{\text{hom}} \quad \text{strongly in } L^2(\Omega)^k.$$

(See Section 8.3.2 for the proof.)

We consider the ERIS given in terms of A_L and r_L:

- The state space is $Y_{\text{ap}} = \left(L^2(\Omega) \otimes H_0^1(Q)\right)^d \times \left(L^2(\Omega) \otimes L^2(Q)\right)^k$ with the corresponding state variable denoted by $y = (u, z)$.
- The energy functional is given by $\mathcal{E}_L : [0, T] \times Y_{\text{ap}} \to \mathbb{R}$,

$$\mathcal{E}_L(t, y) = \frac{1}{2} \langle \mathbb{A}_L y, y \rangle_{Y_{\text{ap}}^*, Y_{\text{ap}}} - \left\langle \int_Q l(t) \cdot u dx \right\rangle,$$
$$\langle \mathbb{A}_L y_1, y_2 \rangle_{Y_{\text{ap}}^*, Y_{\text{ap}}} = \left\langle \int_Q A_L(\omega) \begin{pmatrix} \nabla_s u_1(\omega, x) \\ z_1(\omega, x) \end{pmatrix} \cdot \begin{pmatrix} \nabla_s u_2(\omega, x) \\ z_2(\omega, x) \end{pmatrix} dx \right\rangle.$$

- The dissipation functional is given by $\mathcal{R}_L : Y_{\text{ap}} \to [0, \infty]$,

$$\mathcal{R}_L(\dot{y}) = \left\langle \int_Q \sum_{i=1}^k r_{L,i}(\omega) |\dot{z}_i(\omega, x)| dx \right\rangle.$$

For the system $(Y_{\text{ap}}, \mathcal{E}_L, \mathcal{R}_L)$, the set of stable states at $t \in [0, T]$ is denoted by $S_L(t)$.

Remark 8.24 (Existence and quenched formulation)**.** *If we assume (B1) and (B4), $y_L^0 \in S_L(0)$ and $l \in C^1([0, T], L^2(Q)^d)$, then by Theorem 3.5, there exists a unique energetic solution y_L to the ERIS $(Y_{\text{ap}}, \mathcal{E}_L, \mathcal{R}_L)$ with $y_L(0) = y_L^0$. We remark that this system is given in the mean formulation (i.e., the functionals are averaged in Ω), however, it admits an equivalent pointwise P-a.e. formulation, i.e., for P-a.a. ω, $y_L(\omega)$ solves the deterministic parametrized ERIS given in terms on $A_L(\omega)$ and $\rho_L(\omega)$, cf. Lemma 7.16 in Section 7.3 and Remark 8.16. This fact presents the basis for the computation of the solution y_L.*

The main result of this section is the following theorem that provides convergence for the above described approximation. The proof relies on a standard a priori estimate for ERIS (similar to the one in (3.4), cf. [Mie05, Section 2.2] and [HR12, Section 7.5]) and on Lemma 8.23.

Theorem 8.25 (Convergence of approximation)**.** *Let the assumptions of Lemma 8.23 be satisfied. Let $l \in C^1([0, T], L^2(Q)^d)$, $y_L^0 \in S_L(0)$ and $y \in S_{\text{hom}}(0)$. We consider y_L and y the unique energetic solutions to $(Y_{\text{ap}}, \mathcal{E}_L, \mathcal{R}_L)$ with $y_L(0) = y_L^0$ and to $(Y, \mathcal{E}_{\text{hom}}, \mathcal{R}_{\text{hom}})$ with $y(0) = y^0$, respectively. Then:*

(i) For all $t \in (0, T]$, it holds

$$\begin{aligned} &\|y_L(t) - y(t)\|^2_{Y_{\text{ap}}} \\ \leq \;& c \left(\|A_L\|_{L^\infty(\Omega)^{2k \times 2k}} \left\|y_L^0 - y^0\right\|^2_{Y_{\text{ap}}} + \|r_L - r_{\text{hom}}\|_{L^2(\Omega)^k} + \|A_{\text{hom}} - A_L\|_{L^2(\Omega)^{2k \times 2k}} \right), \end{aligned} \tag{8.32}$$

where $c > 0$ depends only on $\|\dot{l}\|_{L^\infty([0,T], L^2(Q)^d)}$, $\left\|y^0\right\|_Y$, T, $\|r\|_{L^\infty(\Omega)^k}$ and the ellipticity ratio from (B1).

(ii) If $y_L^0 \to y^0$ strongly in Y_{ap}, then for all $t \in (0, T]$,

$$y_L(t) \to y(t) \quad \textit{strongly in } Y_{\text{ap}}.$$

(See Section 8.3.2 for the proof.)

Remark 8.26 (Different choices for A_L)**.** *We remark that the above theorem does not depend on the specific choice of A_L and r_L, but only on the fact that they converge to A_{hom} and r_{hom}. In this view, we may consider other admissible choices for these coefficients, e.g., we might consider different types of boundary conditions in (8.31), periodization in law for the coefficient A or Monte-Carlo type approximations for variance reduction, cf. Section 7.2.*

8.3.2 Proofs

The proof of Theorem 8.20 follows the same strategy and it is very similar to the proof of Theorem 8.12. Therefore, in the proof we leave out the details that are completely analogous.

Proof of Theorem 8.20. Step 1. Compactness. We consider $v_\varepsilon := (\mathcal{T}_\varepsilon u_\varepsilon, \mathcal{T}_\varepsilon \nabla^\varepsilon u_\varepsilon, \mathcal{T}_\varepsilon z_\varepsilon, \mathcal{T}_\varepsilon \varepsilon^\gamma \nabla^\varepsilon z_\varepsilon) : [0,T] \to H := L^2(\Omega \times \mathbb{R}^d)^{d+d\times d+k+k\times d}$. Using the a priori estimate from Remark 8.18 and Corollary 5.23, analogously as in the proof of Theorem 8.12, we obtain that (up to a subsequence) for all $t \in [0,T]$, it holds

$$y_\varepsilon(t) \overset{c2}{\rightharpoonup} (y(t), \chi_1(t)), \quad \varepsilon^\gamma \nabla^\varepsilon z_\varepsilon(t) \overset{2}{\rightharpoonup} \chi_2(t) \quad \text{in } L^2(\Omega \times \mathbb{R}^d)^{k\times d},$$

where $y(t) \in Y$, $\chi_1(t) \in \left(L^2_{\text{pot}}(\Omega) \otimes L^2(Q)\right)^d$ and $\chi_2(t) \in \left(L^2_{\text{pot}}(\Omega) \otimes L^2(Q)\right)^k$ and the mapping $t \mapsto (y(t), \chi_1(t), \chi_2(t))$ is Lipschitz continuous. Also, by the convergence of the initial data it holds $(y(0), \chi_1(0), \chi_2(0)) = \left(y^0, \chi^0, 0\right)$.

Step 2. Stability. We fix $t \in [0,T]$. For an arbitrary $\widetilde{y} \in Y$, a measurable selection argument applies similarly as in the proof of Theorem 8.5 (ii), and we find $\widetilde{\chi} \in \left(L^2_{\text{pot}}(\Omega) \otimes L^2(Q)\right)^d$ such that

$$\mathcal{E}_{\text{hom}}(t, \widetilde{y}) = \mathcal{E}_0(t, (\widetilde{y}, \widetilde{\chi})),$$

$\mathcal{E}_0$ being the energy functional from Section 8.2. Corollary 5.28 (iii) implies that for the couple $(\widetilde{u} - u(t), \widetilde{\chi} - \chi_1(t))$ there exists a sequence $w_\varepsilon \in \left(L^2(\Omega) \otimes L^2_0(Q \cap \varepsilon\mathbb{Z}^d)\right)^d$ such that

$$w_\varepsilon \overset{2}{\to} \widetilde{u} - u(t), \quad \nabla^\varepsilon w_\varepsilon \overset{2}{\to} \nabla \widetilde{u} - \nabla u(t) + \widetilde{\chi} - \chi_1(t).$$

Furthermore, Corollary 5.28 (iv) implies that for $(\widetilde{z} - z(t), -\chi_2(t))$, there exists a sequence $g_\varepsilon \in \left(L^2(\Omega) \otimes L^2_0(Q \cap \varepsilon\mathbb{Z}^d)\right)^k$ such that

$$g_\varepsilon \overset{2}{\to} \widetilde{z} - z(t), \quad \varepsilon^\gamma \nabla^\varepsilon g_\varepsilon \overset{2}{\to} -\chi_2(t).$$

We define $\widetilde{y}_\varepsilon$ componentwise: $\widetilde{u}_\varepsilon = u_\varepsilon + w_\varepsilon$ and $\widetilde{z}_\varepsilon = z_\varepsilon + g_\varepsilon$. Following the steps in the proof of Lemma 8.17, considering the first terms in the energies, we obtain

$$\lim_{\varepsilon\to 0} \left(\mathcal{E}_\varepsilon(t, \widetilde{y}_\varepsilon) - \mathcal{E}_\varepsilon(t, y_\varepsilon(t))\right) = \mathcal{E}_0(t, (\widetilde{y}, \widetilde{\chi})) - \mathcal{E}_0(t, (y(t), \chi_1(t))) = \mathcal{E}_{\text{hom}}(t, \widetilde{y}) - \mathcal{E}_0(t, (y(t), \chi_1(t))).$$

Similarly, for the second terms in the energies, we have

$$\begin{aligned}
&\left\langle \int_{\varepsilon\mathbb{Z}^d} G(\tau_{\frac{x}{\varepsilon}}\omega)\varepsilon^\gamma\nabla^\varepsilon\widetilde{z}_\varepsilon : \varepsilon^\gamma\nabla^\varepsilon\widetilde{z}_\varepsilon dm_\varepsilon \right\rangle - \left\langle \int_{\varepsilon\mathbb{Z}^d} G(\tau_{\frac{x}{\varepsilon}}\omega)\varepsilon^\gamma\nabla^\varepsilon z_\varepsilon(t) : \varepsilon^\gamma\nabla^\varepsilon z_\varepsilon(t) dm_\varepsilon \right\rangle \\
= \; & \left\langle \int_{\varepsilon\mathbb{Z}^d} G(\tau_{\frac{x}{\varepsilon}}\omega)\varepsilon^\gamma\nabla^\varepsilon\left(\widetilde{z}_\varepsilon - z_\varepsilon(t)\right) : \varepsilon^\gamma\nabla^\varepsilon\left(\widetilde{z}_\varepsilon + z_\varepsilon(t)\right) dm_\varepsilon \right\rangle \\
= \; & \left\langle \int_{\mathbb{R}^d} G(\omega)\mathcal{T}_\varepsilon\varepsilon^\gamma\nabla^\varepsilon\left(\widetilde{z}_\varepsilon - z_\varepsilon(t)\right) : \mathcal{T}_\varepsilon\varepsilon^\gamma\nabla^\varepsilon\left(\widetilde{z}_\varepsilon + z_\varepsilon(t)\right) dx \right\rangle \\
\to \; & -\left\langle \int_{\mathbb{R}^d} G\chi_2(t) : \chi_2(t) dx \right\rangle \quad (\text{as } \varepsilon \to 0).
\end{aligned}$$

Furthermore, we have $\mathcal{R}_\varepsilon(\widetilde{y}_\varepsilon - y_\varepsilon(t)) \leq \left\langle \int_{\mathbb{R}^d} \rho(\omega, \mathcal{T}_\varepsilon g_\varepsilon) dx \right\rangle$. (B2) implies that $\rho(\omega, \mathcal{T}_\varepsilon g_\varepsilon(\omega, x)) \leq \psi(\omega)\left(1 + |\mathcal{T}_\varepsilon g_\varepsilon(\omega, x)|\right)$. Since $\psi \in L^2(\Omega)$, the strong convergence of $\mathcal{T}_\varepsilon g_\varepsilon$ in L^2 and the dominated convergence theorem imply that $\limsup_{\varepsilon\to 0} \mathcal{R}_\varepsilon(\widetilde{y}_\varepsilon - y_\varepsilon(t)) \leq \mathcal{R}_{\text{hom}}(\widetilde{y} - y(t))$. Similarly, it follows that $\liminf_{\varepsilon\to 0} \mathcal{R}_\varepsilon(\widetilde{y}_\varepsilon - y_\varepsilon(t)) \geq \mathcal{R}_{\text{hom}}(\widetilde{y} - y(t))$. Collecting the above statements, we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left(\mathcal{E}_\varepsilon^\gamma(t, \widetilde{y}_\varepsilon) + \mathcal{R}_\varepsilon(\widetilde{y}_\varepsilon - y_\varepsilon(t)) - \mathcal{E}_\varepsilon^\gamma(t, y_\varepsilon(t))\right) \\
= \; & \mathcal{E}_{\text{hom}}(t, \widetilde{y}) + \mathcal{R}_{\text{hom}}(\widetilde{y} - y(t)) - \mathcal{E}_0(t, (y(t), \chi_1(t))) - \int_Q \langle G\chi_2(t) : \chi_2(t)\rangle dx \\
\leq \; & \mathcal{E}_{\text{hom}}(t, \widetilde{y}) + \mathcal{R}_{\text{hom}}(\widetilde{y} - y(t)) - \mathcal{E}_{\text{hom}}(t, y(t)).
\end{aligned}$$

As a result of this, we have $y(t) \in S_{\text{hom}}(t)$. Another important fact following from this inequality, in particular using that the left-hand side is nonnegative, is obtained by setting $\widetilde{y} = y(t)$ and using that $\mathcal{R}_{\text{hom}}(0) = 0$:

$$\mathcal{E}_0(t, (y(t), \chi_1(t))) + \int_Q \langle G\chi_2(t) : \chi_2(t)\rangle dx \leq \mathcal{E}_{\text{hom}}(t, y(t)).$$

As a result of this, we conclude that $\chi_1(t)$ is the corrector corresponding to $y(t)$, i.e.,

$$\mathcal{E}_{\text{hom}}(t, y(t)) = \mathcal{E}_0(t, (y(t), \chi_1(t))) \tag{8.33}$$

and, moreover, we obtain that $\chi_2 = 0$.

Step 3. Energy balance. The energy balance equality is obtained in the same manner as in the proof of Theorem 8.12 by using the assumptions on the initial data and using that

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon^\gamma(t, y_\varepsilon(t)) &\geq \mathcal{E}_0(t, (y(t), \chi_1(t))) + \int_Q \langle G(\omega)\chi_2(\omega, x) : \chi_2(\omega, x)\rangle dx \\
&= \mathcal{E}_{\text{hom}}(t, y(t)),
\end{aligned}$$

which is obtained with the help of Proposition 5.19 (i) and by the fact that $\chi_2 = 0$. Also, the inequality

$$\liminf_{\varepsilon\to 0} \int_0^t \mathcal{R}_\varepsilon(\dot{y}_\varepsilon(s)) ds \geq \int_0^t \mathcal{R}_{\text{hom}}(\dot{y}(s)) ds$$

and the convergence of the energy $\mathcal{E}_\varepsilon^\gamma(t, y_\varepsilon(t)) \to \mathcal{E}_{\text{hom}}(t, y(t))$ follow analogously.

Step 4. Strong convergence. We set $v_\varepsilon(t) := (\mathcal{T}_\varepsilon u_\varepsilon(t), \mathcal{T}_\varepsilon \nabla^\varepsilon u_\varepsilon(t), \mathcal{T}_\varepsilon z_\varepsilon(t), \mathcal{T}_\varepsilon \varepsilon^\gamma \nabla^\varepsilon z_\varepsilon(t))$ and $v(t) = (u(t), \nabla u(t) + \chi_1(t), z(t), 0)$. With the help of Corollary 5.28, for $(u(t), \chi_1(t))$ we find a strong recovery sequence $\widetilde{u}_\varepsilon$ such that $\widetilde{u}_\varepsilon \overset{2}{\to} u(t)$ and $\nabla^\varepsilon \widetilde{u}_\varepsilon \overset{2}{\to} \nabla u(t) + \chi_1(t)$ and for $(z(t), 0)$ we find a strong recovery sequence $\widetilde{z}_\varepsilon$ such that $\widetilde{z}_\varepsilon \overset{2}{\to} z(t)$ and $\varepsilon^\gamma \nabla^\varepsilon \widetilde{z}_\varepsilon \overset{2}{\to} 0$. We set $\widetilde{y}_\varepsilon := (\widetilde{u}_\varepsilon, \widetilde{z}_\varepsilon)$ and $\widetilde{v}_\varepsilon = (\mathcal{T}_\varepsilon \widetilde{u}_\varepsilon, \mathcal{T}_\varepsilon \nabla^\varepsilon \widetilde{u}_\varepsilon, \mathcal{T}_\varepsilon \widetilde{z}_\varepsilon, \mathcal{T}_\varepsilon \varepsilon^\gamma \nabla^\varepsilon \widetilde{z}_\varepsilon)$. Note that this choice yields $\lim_{\varepsilon \to 0} \mathcal{E}^\gamma_\varepsilon(t, \widetilde{y}_\varepsilon) = \mathcal{E}_{\mathrm{hom}}(t, y(t))$. The strong convergence $\|v_\varepsilon(t) - \widetilde{v}_\varepsilon\| \to 0$ follows virtually the same lines as in the proof of Theorem 8.12 using the quadratic structure of the energy. As a result of this, we conclude that $v_\varepsilon(t) \to v(t)$. Also, convergence for the whole sequence follows by the uniqueness of the solution for the limit system. The proof is done. □

Proof of Lemma 8.23. (i) For an arbitrary $F \in \mathbb{R}^{2k}$, we have

$$\begin{aligned} A_L(\omega)F \cdot F &= \fint_{LB^+ \cap \mathbb{Z}^d} A(\tau_x \omega) \left(F + \sum_i F_i \begin{pmatrix} \nabla_s \varphi_i(\omega) \\ 0 \end{pmatrix} \right) \cdot \left(F + \sum_i F_i \begin{pmatrix} \nabla_s \varphi_i(\omega) \\ 0 \end{pmatrix} \right) dm \\ &\geq c \fint_{LB^+ \cap \mathbb{Z}^d} \left| F + \sum_i F_i \begin{pmatrix} \nabla_s \varphi_i(\omega) \\ 0 \end{pmatrix} \right|^2 dm \\ &\geq c|F|^2, \end{aligned}$$

where in the first equality we use equation (8.31), the second inequality follows by (B1) and the last inequality is obtained by Jensen's inequality and by the fact that $\nabla_s \varphi_i(\omega)$ averages to 0. Moreover, a standard a priori estimate for (8.31) implies that $\fint_{LB^+ \cap \mathbb{Z}^d} |\nabla_s \varphi_i(\omega)|^2 dm \leq c$, where $c > 0$ depends only on the ellipticity ratio from (B1). As a result of this, we conclude that A_L is a bounded sequence in $L^\infty(\Omega)^{2k \times 2k}$; A_L is indeed $\mathcal{F}$-measurable that follows from the measurability of φ_i, cf. Section 7.3. For notational convenience, in the following we set $\varepsilon := \frac{1}{L}$ and we fix one $i \in \{1, ..., 2k\}$ that we drop from the notation. Note that the rescaled variable $\varphi_\varepsilon(\omega, x) = \varepsilon \varphi(\omega, \frac{x}{\varepsilon})$, φ being the solution of (8.31), satisfies $\varphi_\varepsilon \in L^2(\Omega) \otimes L^2_0(B \cap \varepsilon\mathbb{Z}^d)^d$ and it uniquely solves the following minimization problem

$$\min_{L^2(\Omega) \otimes L^2_0(B \cap \varepsilon \mathbb{Z}^d)^d} \left(\mathcal{E}_\varepsilon(\varphi) = \left\langle \int_{B^{+\varepsilon} \cap \varepsilon \mathbb{Z}^d} V(\tau_{\frac{x}{\varepsilon}} \omega, \nabla^\varepsilon_s \varphi(\omega, x)) dm_\varepsilon(x) \right\rangle \right),$$

where $V(\omega, F) = \frac{1}{2} A(\omega) \left(e_i + \binom{F}{0} \right) \cdot \left(e_i + \binom{F}{0} \right)$ for $\omega \in \Omega$ and $F \in \mathbb{R}^k$. With $p = 2$, the assumptions of Proposition 8.6 are satisfied and therefore we obtain that, as $\varepsilon \to 0$,

$$\nabla^\varepsilon_s \varphi_\varepsilon \overset{2}{\to} \nabla_s u + \chi_s \quad \text{in } L^2(\Omega \times \mathbb{R}^d)^k, \tag{8.34}$$

where $u \in H^1_0(B)$ minimizes the functional $u \mapsto \frac{1}{2} \int_B A_{\mathrm{hom}} \left(e_i + \binom{\nabla_s u}{0} \right) \cdot \left(e_i + \binom{\nabla_s u}{0} \right) dx$ that has a unique minimizer given by $u = 0$. Furthermore, Proposition 8.6 also implies that $\chi \in L^2_{\mathrm{pot}}(\Omega)^d$ solves (8.29). As a result of this, we have, by applying a change of variables $x \rightsquigarrow \varepsilon x$,

$$\begin{aligned} & \left\langle |A_L(\omega) e_i \cdot e_j - A_{\mathrm{hom}} e_i \cdot e_j|^2 \right\rangle \\ = \quad & \left\langle \left| \fint_{B^{+\varepsilon} \cap \varepsilon \mathbb{Z}^d} A(\tau_{\frac{x}{\varepsilon}} \omega) \left(e_i + \begin{pmatrix} \nabla_s \varphi_\varepsilon(\omega, x) \\ 0 \end{pmatrix} \right) \cdot e_j dm_\varepsilon(x) - \left\langle A \left(e_i + \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} \right) \cdot e_j \right\rangle \right|^2 \right\rangle \\ \leq \quad & c \left\langle \fint_{B^{+\varepsilon} \cap \varepsilon \mathbb{Z}^d} \left| A(\tau_{\frac{x}{\varepsilon}} \omega) \begin{pmatrix} \nabla_s \varphi_\varepsilon(\omega, x) - \chi_s(\tau_{\frac{x}{\varepsilon}} \omega) \\ 0 \end{pmatrix} \right|^2 dm_\varepsilon(x) \right\rangle \\ & + c \left\langle \left| \fint_{B^{+\varepsilon} \cap \varepsilon \mathbb{Z}^d} A(\tau_{\frac{x}{\varepsilon}} \omega) \left(e_i + \begin{pmatrix} \chi_s(\tau_{\frac{x}{\varepsilon}} \omega) \\ 0 \end{pmatrix} \right) \cdot e_j dm_\varepsilon(x) - \left\langle A \left(e_i + \begin{pmatrix} \chi_s \\ 0 \end{pmatrix} \right) \cdot e_j \right\rangle \right|^2 \right\rangle, \end{aligned}$$

where the second inequality follows by the triangle inequality. The second term vanishes in the limit $\varepsilon \to 0$ by the von Neumann ergodic theorem Remark 5.9. The first term can be bounded by $c\left\langle \fint_{B+\varepsilon} |\mathcal{T}_\varepsilon \nabla_s \varphi_\varepsilon(\omega,x) - \chi_s(\omega)|^2 dx \right\rangle$ that vanishes in the limit $\varepsilon \to 0$ by (8.34). The claim is proved.

(ii) Measurability, positiveness and boundedness in $L^\infty(\Omega)^k$ are all inherited from the corresponding properties of r. Also, a direct application of von Neumann's ergodic theorem, Remark 5.9, implies the convergence $r_L \to r_{\mathrm{hom}}$ in $L^2(\Omega)^k$. □

Proof of Theorem 8.25. (i) By Remark 3.3, it follows that (for a.a. t, that we drop from the notation)

$$\mathcal{R}_L(\dot{y}_L) \leq \mathcal{R}_L(\widetilde{y}) - \langle D\mathcal{E}_L(y_L), \dot{y}_L - \widetilde{y}\rangle_{Y^*_{\mathrm{ap}},Y_{\mathrm{ap}}} \quad \text{for all } \widetilde{y} \in Y_{\mathrm{ap}}, \tag{8.35}$$

$$\mathcal{R}_{\mathrm{hom}}(\dot{y}) \leq \mathcal{R}_{\mathrm{hom}}(\widetilde{y}) - \langle D\mathcal{E}_{\mathrm{hom}}(y), \dot{y} - \widetilde{y}\rangle_{Y^*,Y} \quad \text{for all } \widetilde{y} \in Y. \tag{8.36}$$

Furthermore, we have (we identify $\mathbb{A}_{\mathrm{hom}}$ with its extension $\mathbb{A}_{\mathrm{hom}} : Y_{\mathrm{ap}} \to Y^*_{\mathrm{ap}}$)

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\left\langle \mathbb{A}_L\left(y_L - y\right), y_L - y\right\rangle_{Y^*_{\mathrm{ap}},Y_{\mathrm{ap}}} \\
= \;& \left\langle \mathbb{A}_L\left(y_L - y\right), \dot{y}_L - \dot{y}\right\rangle_{Y^*_{\mathrm{ap}},Y_{\mathrm{ap}}} \\
= \;& \langle \mathbb{A}_L y_L, \dot{y}_L - \dot{y}\rangle_{Y^*_{\mathrm{ap}},Y_{\mathrm{ap}}} + \langle \mathbb{A}_{\mathrm{hom}} y, \dot{y} - \dot{y}_L\rangle_{Y^*_{\mathrm{ap}},Y_{\mathrm{ap}}} + \langle \left(\mathbb{A}_{\mathrm{hom}} - \mathbb{A}_L\right) y, \dot{y}_L - \dot{y}\rangle_{Y^*_{\mathrm{ap}},Y_{\mathrm{ap}}} \\
\leq \;& \mathcal{R}_L(\dot{y}) - \mathcal{R}_{\mathrm{hom}}(\dot{y}) + \langle \mathcal{R}_{\mathrm{hom}}(\dot{y}_L(\omega))\rangle - \mathcal{R}_L(\dot{y}_L) + \langle \left(\mathbb{A}_{\mathrm{hom}} - \mathbb{A}_L\right) y, \dot{y}_L - \dot{y}\rangle_{Y^*_{\mathrm{ap}},Y_{\mathrm{ap}}}.
\end{aligned} \tag{8.37}$$

The last inequality above is obtained by setting $\widetilde{y} = \dot{y}$ in (8.35), by setting $\widetilde{y} = \dot{y}_L(\omega)$ in (8.36) and integrating it over Ω. Hölder's inequality implies that $\mathcal{R}_L(\dot{y}) - \mathcal{R}_{\mathrm{hom}}(\dot{y}) \leq \|r_L - \langle r\rangle\|_{L^2(\Omega)^k}\|\dot{y}\|_Y$ and $\langle \mathcal{R}_{\mathrm{hom}}(\dot{y}_L(\omega))\rangle - \mathcal{R}_L(\dot{y}_L) \leq \|r_L - \langle r\rangle\|_{L^2(\Omega)^k}\|\dot{y}_L\|_{Y_{\mathrm{ap}}}$. Moreover, we have

$$\begin{aligned}
\langle \left(\mathbb{A}_{\mathrm{hom}} - \mathbb{A}_L\right) y, \dot{y}_L - \dot{y}\rangle_{Y^*_{\mathrm{ap}},Y_{\mathrm{ap}}} &\leq \left\langle \int_Q |A_{\mathrm{hom}} - A_L|^2 |(\nabla_s u, z)|^2 dx \right\rangle^{\frac{1}{2}} \|\dot{y}_L - \dot{y}\|_{Y_{\mathrm{ap}}} \\
&\leq \|A_{\mathrm{hom}} - A_L\|_{L^2(\Omega)^{2k\times 2k}} \|y\|_Y \|\dot{y}_L - \dot{y}\|_{Y_{\mathrm{ap}}},
\end{aligned}$$

where in the last inequality we use that $A_{\mathrm{hom}} - A_L$ does not depend on x and that y is deterministic. The last three inequalities and (8.37) imply that

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\left\langle \mathbb{A}_L\left(y_L - y\right), y_L - y\right\rangle_{Y^*_{\mathrm{ap}},Y_{\mathrm{ap}}} \\
\leq \;& \|r_L - \langle r\rangle\|_{L^2(\Omega)^k}\left(\|\dot{y}\|_Y + \|\dot{y}_L\|_{Y_{\mathrm{ap}}}\right) + \|A_{\mathrm{hom}} - A_L\|_{L^2(\Omega)^{2k\times 2k}}\|y\|_Y \|\dot{y}_L - \dot{y}\|_{Y_{\mathrm{ap}}} \\
\leq \;& c\left(\|r_L - \langle r\rangle\|_{L^2(\Omega)^k} + \|A_{\mathrm{hom}} - A_L\|_{L^2(\Omega)^{2k\times 2k}}\right),
\end{aligned}$$

where the last inequality is obtained using the standard a priori estimate for ERIS and the constant c depends only on $\|\dot{l}\|_{L^\infty([0,T],L^2(Q)^d)}$, $\left\|y^0\right\|_Y$, T and the constants from (B1) and (B4) (all these quantities are independent of t and L). Integration over $(0,t)$ and (B1) imply that for all $t \in (0,T]$, we have (note that c is modified)

$$\|y_L(t) - y(t)\|^2_{Y_{\mathrm{ap}}} \leq c\left(\|A_L\|_{L^\infty(\Omega)^{2k\times 2k}}\left\|y^0_L - y^0\right\|^2_{Y_{\mathrm{ap}}} + \|r_L - \langle r\rangle\|_{L^2(\Omega)^k} + \|A_{\mathrm{hom}} - A_L\|_{L^2(\Omega)^{2k\times 2k}}\right).$$

(ii) The claim follows directly by (i) and Lemma 8.23. □

9 Continuum λ-convex gradient flows

In this section we consider a sequence of gradient flows $(Y, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ where the functionals $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ are, respectively, λ-convex and quadratic integral functionals with random and rapidly oscillating integrands. We refer to Section 4 where we recall the abstract framework for gradient flows. In particular, we apply the modified abstract strategy for asymptotic analysis of gradient flows discussed in Section 4 and the stochastic unfolding procedure in order to obtain a homogenization result. Moreover, in a simplified setting of an Allen-Cahn type equation we consider an approximation scheme for the effective system based on the RVE strategy. To keep the exposition simple, we present the proofs at the end of this section.

9.1 Homogenization of gradient flows

Let $(\Omega, \mathcal{F}, P, \tau)$ be a probability space that satisfies Assumption 6.1 and $Q \subset \mathbb{R}^d$ be open and bounded. Let $p \in (1, \infty)$ and $s \in [2, \infty)$. The system that we consider is defined on a state space $Y = L^2(\Omega \times Q)$. The dissipation functional is given by $\mathcal{R}_\varepsilon : Y \to [0, \infty)$,

$$\mathcal{R}_\varepsilon(v) = \frac{1}{2} \left\langle \int_Q r(\tau_{\frac{x}{\varepsilon}}\omega, x)|v(\omega, x)|^2 dx \right\rangle.$$

The energy functional $\mathcal{E}_\varepsilon : Y \to \mathbb{R} \cup \{\infty\}$ is defined as

$$\mathcal{E}_\varepsilon(y) = \left\langle \int_Q V(\tau_{\frac{x}{\varepsilon}}\omega, x, \nabla y(\omega, x)) + f(\tau_{\frac{x}{\varepsilon}}\omega, x, y(\omega, x)) dx \right\rangle,$$

for $y \in \left(L^p(\Omega) \otimes W_0^{1,p}(Q)\right) \cap L^s(\Omega \times Q) =: \mathrm{dom}(\mathcal{E}_\varepsilon)$ and $\mathcal{E}_\varepsilon = \infty$ otherwise. Above, $r : \Omega \times Q \to \mathbb{R}$, $V : \Omega \times Q \times \mathbb{R}^d \to \mathbb{R}$ and $f : \Omega \times Q \times \mathbb{R} \to \mathbb{R}$ and we consider the following assumptions:

(C1) r is $\mathcal{F} \otimes \mathcal{L}(Q)$-measurable and there exists $c > 0$ such that for a.a. $(\omega, x) \in \Omega \times Q$, we have $\frac{1}{c} \leq r(\omega, x) \leq c$.

(C2) $V(\cdot, \cdot, F)$ is $\mathcal{F} \otimes \mathcal{L}(Q)$-measurable for all $F \in \mathbb{R}^d$, $V(\omega, x, \cdot)$ is convex for a.a. $(\omega, x) \in \Omega \times Q$ and there exists $c > 0$ such that

$$\frac{1}{c}|F|^p - c \leq V(\omega, x, F) \leq c(|F|^p + 1)$$

for a.a. $(\omega, x) \in \Omega \times Q$ and all $F \in \mathbb{R}^d$.

(C3) $f(\cdot,\cdot,\alpha)$ is $\mathcal{F}\otimes\mathcal{L}(Q)$-measurable for all $\alpha\in\mathbb{R}$. There exist $\lambda\in\mathbb{R}$ and $c>0$ such that for a.a. $(\omega,x)\in\Omega\times Q$

$$\begin{aligned}&f(\omega,x,\cdot) \text{ is } \lambda\text{-convex, i.e.,} \quad \alpha\mapsto f(\omega,x,\alpha)-\frac{\lambda}{2}|\alpha|^2 \text{ is convex,}\\ &\frac{1}{c}|\alpha|^s-c\leq f(\omega,x,\alpha)\leq c(|\alpha|^s+1) \quad \text{for all } \alpha\in\mathbb{R}.\end{aligned}$$

We remark that the above assumptions imply that there exists $\Lambda\in\mathbb{R}$ such that $y\mapsto\mathcal{E}_\varepsilon(y)-\Lambda\mathcal{R}_\varepsilon(y)$ is convex, i.e. $\mathcal{E}_\varepsilon$ is Λ-convex w.r.t. $\mathcal{R}_\varepsilon$. In particular, if $\lambda<0$, then we set $\Lambda=\lambda c$, and in the case $\lambda\geq 0$, $\Lambda=\frac{\lambda}{c}$, where c is the constant from (C1). Let $T>0$ be a finite time horizon. Also, the above assumptions imply the existence of a unique solution to the corresponding gradient flow:

Remark 9.1 (Existence and a priori estimates)**.** *If we assume (C1)-(C3) and $y_\varepsilon^0\in\mathrm{dom}(\mathcal{E}_\varepsilon)$, then Theorem 4.3 implies that there exists a unique EVI solution $y_\varepsilon\in H^1(0,T;Y)$ to the gradient flow $(Y,\mathcal{E}_\varepsilon,\mathcal{R}_\varepsilon)$ with $y_\varepsilon(0)=y_\varepsilon^0$, i.e., for a.a. $t\in[0,T]$,*

$$\frac{d}{dt}\mathcal{R}_\varepsilon(y_\varepsilon(t)-\widetilde{y})\leq\mathcal{E}_\varepsilon(\widetilde{y})-\mathcal{E}_\varepsilon(y_\varepsilon(t))-\Lambda\mathcal{R}_\varepsilon(y_\varepsilon(t)-\widetilde{y}) \quad \textit{for all } \widetilde{y}\in Y.$$

Moreover, we have $\mathcal{E}_\varepsilon(y_\varepsilon(t))\leq\mathcal{E}_\varepsilon(y_\varepsilon^0)$, $\int_0^t\mathcal{R}_\varepsilon(\dot{y}_\varepsilon(s))ds\leq\mathcal{E}_\varepsilon(y_\varepsilon^0)-\mathcal{E}_\varepsilon(y_\varepsilon(t))$ for all $t\in[0,T]$. Indeed, $\mathcal{R}_\varepsilon$ and $\mathcal{E}_\varepsilon$ satisfy the assumptions of Theorem 4.3. In particular, using the growth conditions of V and f, the coercivity of the energy follows (we use $s\geq 1$). Moreover, the energy functional is l.s.c. Indeed, this follows using the facts that $\mathcal{R}_\varepsilon(\cdot)$ is continuous and the mappings $y\mapsto\left\langle\int_Q V(\tau_{\frac{x}{\varepsilon}}\omega,x,\nabla y)dx\right\rangle$ and $y\mapsto\left\langle\int_Q f(\tau_{\frac{x}{\varepsilon}}\omega,x,y)dx\right\rangle-\Lambda\mathcal{R}_\varepsilon(y)$ are convex and l.s.c. in $L^p(\Omega)\otimes W^{1,p}(Q)$ and $L^s(\Omega\times Q)$, respectively; here, the growth assumptions and continuity of $V(\omega,x,\cdot)$ and $f(\omega,x,\cdot)$ are helpful.

In the limit $\varepsilon\to 0$, we derive an effective gradient system which is described as follows. The state space is $Y_0=L^2_{\mathrm{inv}}(\Omega)\otimes L^2(Q)$. The effective dissipation potential is given by $\mathcal{R}_{\mathrm{hom}}:Y_0\to[0,\infty)$,

$$\mathcal{R}_{\mathrm{hom}}(v)=\frac{1}{2}\left\langle\int_Q r(\omega,x)|v(\omega,x)|^2dx\right\rangle.$$

The energy functional is $\mathcal{E}_{\mathrm{hom}}:Y_0\to\mathbb{R}\cup\{\infty\}$,

$$\mathcal{E}_{\mathrm{hom}}(y)=\inf_{\chi\in L^p_{\mathrm{pot}}(\Omega)\otimes L^p(Q)}\left\langle\int_Q V\left(\omega,x,\nabla y(\omega,x)+\chi(\omega,x)\right)dx\right\rangle+\left\langle\int_Q f(\omega,x,y(\omega,x))dx\right\rangle \tag{9.1}$$

for $y\in\left(L^p_{\mathrm{inv}}(\Omega)\otimes W^{1,p}_0(Q)\right)\cap\left(L^s_{\mathrm{inv}}(\Omega)\otimes L^s(Q)\right)=:\mathrm{dom}(\mathcal{E}_{\mathrm{hom}})$ and $\mathcal{E}_{\mathrm{hom}}=\infty$ otherwise. We remark that $\mathcal{E}_{\mathrm{hom}}(\cdot)-\Lambda\mathcal{R}_{\mathrm{hom}}(\cdot)$ is convex with the same $\Lambda\in\mathbb{R}$ as for $\mathcal{E}_\varepsilon$.

Remark 9.2 (Existence and uniqueness)**.** *If we assume (C1)-(C3) and $y^0\in\mathrm{dom}(\mathcal{E}_{\mathrm{hom}})$, then Theorem 4.3 yields a unique EVI solution $y\in H^1(0,T;Y_0)$ to $(Y_0,\mathcal{E}_{\mathrm{hom}},\mathcal{R}_{\mathrm{hom}})$ with $y(0)=y^0$. Indeed, $\mathcal{R}_{\mathrm{hom}}$ and $\mathcal{E}_{\mathrm{hom}}$ satisfy the assumptions in Theorem 4.3. In particular, we first notice that for fixed $y\in Y_0$, the minimization problem in (9.1) attains a minimum by the direct method of calculus of variations and using the growth assumptions and convexity of the integrand V. In this respect, $\mathcal{E}_{\mathrm{hom}}$ is proper, coercive (using the growth conditions of f) and $\mathcal{E}_{\mathrm{hom}}(\cdot)$ is Λ-convex w.r.t. $\mathcal{R}_{\mathrm{hom}}$. L.s.c. of $\mathcal{E}_{\mathrm{hom}}$ follows using the growth conditions of f and V, in particular the estimate $\left\langle\int_Q|\nabla y+\chi|^pdx\right\rangle\geq\left\langle\int_Q|P_{\mathrm{inv}}\nabla y+P_{\mathrm{inv}}\chi|^pdx\right\rangle=\left\langle\int_Q|\nabla y|^pdx\right\rangle$ is useful, where the first inequality is Jensen's inequality for the conditional expectation P_{inv}.*

The main result of this section is the following homogenization theorem. In particular, the proof relies on the modified abstract strategy discussed in Section 4 and on the stochastic unfolding procedure.

Theorem 9.3 (Homogenization). *Let $p \in (1, \infty)$, $s \in [2, \infty)$ and $Q \subset \mathbb{R}^d$ be open and bounded. Assume (C1)-(C3), and consider $y^0 \in \mathrm{dom}(\mathcal{E}_{\mathrm{hom}})$, $y^0_\varepsilon \in \mathrm{dom}(\mathcal{E}_\varepsilon)$ such that*

$$y^0_\varepsilon \to y^0 \quad \text{strongly in } Y, \quad \limsup_{\varepsilon \to 0} \mathcal{E}_\varepsilon(y^0_\varepsilon) < \infty.$$

Let y_ε be the unique EVI solution to $(Y, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon)$ with $y_\varepsilon(0) = y^0_\varepsilon$. Then, for all $t \in (0, T]$,

$$y_\varepsilon(t) \to y(t) \quad \text{strongly in } Y,$$

where y is the unique EVI solution to $(Y_0, \mathcal{E}_{\mathrm{hom}}, \mathcal{R}_{\mathrm{hom}})$ with $y(0) = y^0$. Moreover, if we additionally assume that $\mathcal{E}_\varepsilon(y^0_\varepsilon) \to \mathcal{E}_{\mathrm{hom}}(y^0)$, it holds that $\dot{y}_\varepsilon \to \dot{y}$ strongly in $L^2(0, T; Y)$ and $\mathcal{E}_\varepsilon(y_\varepsilon(t)) \to \mathcal{E}_{\mathrm{hom}}(y(t))$ for all $t \in [0, T]$.
(See Section 9.1.2 for the proof.)

Remark 9.4 (Convergence of gradients). *We remark that in the proof we additionally show that $y_\varepsilon(t) \xrightharpoonup{2} y(t)$ in $L^s(\Omega \times Q)$ and in $L^p(\Omega \times Q)$, and $P_{\mathrm{inv}} \nabla y_\varepsilon(t) \rightharpoonup \nabla y(t)$ weakly in $L^p(\Omega \times Q)^d$. Also, if we additionally assume that $V(\omega, x, \cdot)$ is strictly convex, we may obtain that for all $t \in (0, T]$ it holds*

$$\nabla y_\varepsilon(t) \xrightharpoonup{2} \nabla y(t) + \chi(t) \quad \text{in } L^p(\Omega \times Q)^d,$$

where $\chi(t) \in L^p_{\mathrm{pot}}(\Omega) \otimes L^p(Q)$ is the unique solution to the minimization problem

$$\inf_{\chi \in L^p_{\mathrm{pot}}(\Omega) \otimes L^p(Q)} \left\langle \int_Q V(\omega, x, \nabla y(t, \omega, x) + \chi(\omega, x)) dx \right\rangle.$$

Remark 9.5 (Ergodic case). *If we additionally assume that $\langle \cdot \rangle$ is ergodic, the limit system is driven by deterministic functionals. In particular, the state space reduces to $Y_0 = L^2(Q)$. The dissipation potential is given by*

$$\mathcal{R}_{\mathrm{hom}}(v) = \int_Q r_{\mathrm{hom}}(x) |v(x)|^2 dx,$$

where $r_{\mathrm{hom}}(x) = \langle r(\omega, x) \rangle$. The energy functional boils down to (cf. proof of Theorem 7.6)

$$\mathcal{E}_{\mathrm{hom}}(y) = \int_Q V_{\mathrm{hom}}(x, \nabla y(x)) + f_{\mathrm{hom}}(x, y(x)) dx$$

in $W^{1,p}_0(Q) \cap L^s(Q)$ and otherwise ∞. Above, $f_{\mathrm{hom}}(x, \alpha) = \langle f(\omega, x, \alpha) \rangle$ for $x \in Q$ and $\alpha \in \mathbb{R}$, and $V_{\mathrm{hom}}(x, F) = \inf_{\chi \in L^p_{\mathrm{pot}}(\Omega)} \langle V(x, \omega, F + \chi(\omega)) \rangle$ for $x \in Q$, $F \in \mathbb{R}^d$. Moreover, V_{hom} satisfies analogous p-growth conditions as V.

9.1.1 Representative volume element approximations

Even in the ergodic case, cf. Remark 9.5, the effective system from the previous section is not accessible for standard numerical approaches, cf. Section 7.2. In particular, the evaluation of the homogenized integrands r_{hom}, V_{hom} and f_{hom} requires an approximation argument in practice, as discussed for V_{hom} in Section 7.2 and for r_{hom} in a similar situation in Section 8.3.1. In the following, we present an approximation for the effective system in a simplified setting using the RVE method as described in Section 7.2. Namely, we consider an Allen-Cahn type equation in an ergodic setting.

Specifically, we consider the following assumptions:

(D1) r satisfies (C1) and it does not depend on x, i.e., $r(\omega, x) = r(\omega)$ (we abuse the notation by not relabeling the function on the right-hand side).

(D2) $V(\omega, x, F) = \frac{1}{2}A(\omega)F \cdot F$, where $A \in L^\infty(\Omega, \mathbb{R}^{d\times d}_{\text{sym}})$ and there exists $c > 0$ such that

$$A(\omega)F \cdot F \geq c|F|^2 \quad \text{for } P\text{-a.a. } \omega \text{ and all } F \in \mathbb{R}^d.$$

(D3) f satisfies (C3) and it does not depend on x, i.e., $f(\omega, x, \alpha) = f(\omega, \alpha)$ (we abuse the notation by not relabeling the function on the right-hand side). Also, we assume that for P-a.a. ω, $f(\omega, \cdot) \in C^2(\mathbb{R})$ and $\min f(\omega, \cdot) = f(\omega, 0) = 0$.

We remark that (D2) implies (C2) with $p = 2$. In the ergodic case, the effective system, see Remark 9.5, reduces to

$$Y_0 = L^2(Q), \quad \mathcal{R}_{\text{hom}}(v) = \int_Q r_{\text{hom}}|v(x)|^2 dx,$$

$$\mathcal{E}_{\text{hom}}(y) = \int_Q A_{\text{hom}}\nabla y(x) \cdot \nabla y(x) + f_{\text{hom}}(y(x))dx \quad (\text{in its domain } H^1_0(Q) \cap L^s(Q)),$$

where $r_{\text{hom}} = \langle r\rangle$, $f_{\text{hom}}(\alpha) = \langle f(\omega, \alpha)\rangle$ and $A_{\text{hom}} \in \mathbb{R}^{d\times d}_{\text{sym}}$ is given by

$$A^{ij}_{\text{hom}} = \langle A(\omega)\left(e_i + \chi_i(\omega)\right) \cdot e_j\rangle,$$

where $\chi_i \in L^2_{\text{pot}}(\Omega)$ is the solution to the corrector equation

$$\langle A\left(e_i + \chi_i\right) \cdot \widetilde{\chi}\rangle = 0 \quad \text{for all } \widetilde{\chi} \in L^2_{\text{pot}}(\Omega).$$

Similarly as in Section 8.3.1, we replace r_{hom}, A_{hom} and f_{hom} by suitable approximations r_L, A_L and f_L, respectively. In particular, let $L \geq 1$ and $B \subset \mathbb{R}^d$ be open bounded and convex.

(i) The replacement for r_{hom} is given by $r_L : \Omega \to \mathbb{R}$,

$$r_L(\omega) = \fint_{LB} r(\tau_x\omega)dx.$$

(ii) The approximation for A_{hom} is defined by $A_L : \Omega \to \mathbb{R}^{d\times d}$,

$$A^{ij}_L(\omega) = \fint_{LB} A(\tau_x\omega)\left(e_i + \nabla\varphi_i(\omega, x)\right) \cdot e_j dx,$$

where $\varphi_i(\omega, \cdot) \in H^1_0(LB)$ is the unique solution to, for P-a.a. ω,

$$\begin{aligned} -\operatorname{div}\left(A(\tau_x\omega)\left(e_i + \nabla\varphi_i\right)\right) &= 0 \quad \text{in } LB, \\ \varphi_i &= 0 \quad \text{on } \partial LB. \end{aligned}$$

Other possible choices for the approximation A_L are discussed in Section 7.2, e.g., we may consider periodic boundary conditions for the above equation or we may employ periodization in law for the coefficient field A.

(iii) The proxy for f_{hom} is given by the following function $f_L : \Omega \times \mathbb{R} \to \mathbb{R}$,

$$f_L(\omega, \alpha) = \fint_{LB} f(\tau_x \omega, \alpha) dx.$$

In the following we view L as a sequence that tends to ∞. We collect the main properties of the above approximations:

Lemma 9.6 (Properties of r_L, A_L and f_L)**.** *Let $B \subset \mathbb{R}^d$ be open bounded and convex. We assume (D1)-(D3) to hold and $\langle \cdot \rangle$ to be ergodic. Then:*

(i) r_L is $\mathcal{F}$-measurable and there exists $c > 0$ such that $\frac{1}{c} \leq r_L(\omega) \leq c$ for P-a.a. ω and all $L \geq 1$. This c is the same constant as in (D1) (resp. (C1)). Moreover, it holds

$$r_L \to r_{\text{hom}} \quad \textit{strongly in } L^2(\Omega).$$

(ii) (Lemma 7.11, Section 7.2) A_L is a bounded sequence in $L^\infty(\Omega, \mathbb{R}^{d\times d}_{\text{sym}})$ and there exists $c > 0$ such that

$$A_L(\omega) F \cdot F \geq c|F|^2 \quad \textit{for } P\textit{-a.a. } \omega, \textit{ all } F \in \mathbb{R}^d \textit{ and all } L \geq 1.$$

Moreover, it holds

$$A_L \to A_{\text{hom}} \quad \textit{strongly in } L^2(\Omega)^{d\times d}.$$

(iii) $f_L(\cdot, \alpha)$ is $\mathcal{F}$-measurable for all $\alpha \in \mathbb{R}$. For P-a.a. ω, $f_L(\omega, \cdot) \in C^2(\mathbb{R})$, $\min f_L(\omega, \cdot) = f_L(\omega, 0) = 0$ and it is λ-convex with the same $\lambda \in \mathbb{R}$ as in (D3) (resp. (C3)). There exists $c > 0$ such that

$$\frac{1}{c}|\alpha|^s - c \leq f_L(\omega, \alpha) \leq c\left(|\alpha|^s + 1\right) \quad \textit{for } P\textit{-a.a. } \omega, \textit{ all } \alpha \in \mathbb{R}^d \textit{ and all } L \geq 1.$$

(See Section 9.1.2 for the proof.)

We present the gradient flow defined in terms of the above approximate coefficients as follows. The state space is given by $Y = L^2(\Omega \times Q)$, the dissipation functional is $\mathcal{R}_L : Y \to \mathbb{R}$,

$$\mathcal{R}_L(v) = \frac{1}{2}\left\langle \int_Q r_L(\omega)|v(\omega, x)|^2 dx \right\rangle,$$

and the energy functional is defined as $\mathcal{E}_L : Y \to \mathbb{R} \cup \{\infty\}$,

$$\mathcal{E}_L(y) = \left\langle \int_Q \frac{1}{2} A_L(\omega) \nabla y(\omega, x) \cdot \nabla y(\omega, x) + f_L(\omega, y(\omega, x)) dx \right\rangle$$

in $\text{dom}(\mathcal{E}_L) := \left(L^2(\Omega) \otimes H^1_0(Q)\right) \cap L^s(\Omega \times Q)$ and $\mathcal{E}_L = \infty$ otherwise. We remark that $\mathcal{E}_{\text{hom}}(\cdot) - \Lambda \mathcal{R}_{\text{hom}}(\cdot)$ and $\mathcal{E}_L(\cdot) - \Lambda \mathcal{R}_L(\cdot)$ are convex with the same constant $\Lambda \in \mathbb{R}$ as for $\mathcal{E}_\varepsilon$.

Remark 9.7 (Existence and quenched formulation)**.** *Similarly as in Remark 9.1, the properties from Lemma 9.6 imply the existence of a unique EVI solution $y_L \in H^1(0, T; Y)$ to $(Y, \mathcal{E}_L, \mathcal{R}_L)$ with initial datum $y^0_L \in \text{dom}(\mathcal{E}_L)$, cf. Theorem 4.3. This system is well-suited for computational* *purposes, since, as discussed in Section 7.3 (cf. Lemma 7.17), for P-a.a. ω the solution $y_L(\omega)$ solves the deterministic gradient flow given in terms of the state space $L^2(Q)$ and functionals*

$$\mathcal{R}_L^\omega(v) = \frac{1}{2}\int_Q r_L(\omega)|v(x)|^2 dx,$$
$$\mathcal{E}_L^\omega(y) = \int_Q \frac{1}{2}A_L(\omega)\nabla y(x)\cdot\nabla y(x) + f_L(\omega, y(x))dx \quad \textit{(in its domain } H_0^1(Q)\cap L^s(Q)\textit{)}.$$

This parametrized deterministic system may be solved by usual finite element approximations.

The main result of this section is the following theorem that provides convergence for the above described approximation scheme. The proof relies on a standard Gronwall-type a priori estimate for the considered equation and on Lemma 9.6, which follows using stochastic unfolding and von Neumann's ergodic theorem. In particular, in the treatment of the nonlinear term f_L' we utilize the monotonicity of $f_L'(\omega,\cdot)-\lambda(\cdot)$.

Theorem 9.8 (Convergence of approximation)**.** *Let $s\in[2,\infty)$ and $Q\subset\mathbb{R}^d$ be open and bounded. Let the assumptions of Lemma 9.6 be satisfied. Let $y_L^0\in\mathrm{dom}(\mathcal{E}_L)$ and $y^0\in\mathrm{dom}(\mathcal{E}_{\mathrm{hom}})$. We consider y_L and y, the unique EVI solutions to $(Y,\mathcal{E}_L,\mathcal{R}_L)$ with $y_L(0)=y_L^0$ and to $(Y_0,\mathcal{E}_{\mathrm{hom}},\mathcal{R}_{\mathrm{hom}})$ with $y(0)=y^0$, respectively. Then:*

(i) For all $t\in(0,T]$, it holds

$$\begin{aligned}\|y_L(t)-y(t)\|_Y^2 \le & c_1 e^{c_2 t}\left(\left\|y_L^0-y^0\right\|_Y^2 + C_L^{\frac{1}{2}}\|r_{\mathrm{hom}}-r_L\|_{L^2(\Omega)} + C_L^{\frac{1}{2}}\|A_{\mathrm{hom}}-A_L\|_{L^2(\Omega)^{d\times d}}\right)\\ &+ c_1 e^{c_2 t} C_L^{\frac{1}{s}}\left(\int_0^T\left\langle\int_Q |f_{\mathrm{hom}}'(y(t,x))-f_L'(\omega,y(t,x))|^{s^*}dx\right\rangle dt\right)^{\frac{1}{s^*}}, \qquad (9.2)\end{aligned}$$

where $C_L=\mathcal{E}_L(y_L^0)+c_3>0$ and the constants $c_1,c_2,c_3>0$ depend only on $\mathcal{E}_{\mathrm{hom}}(y^0)$, T and the constants from the assumptions (D1)-(D3) (resp. (C1)-(C3)). Above, f_L' denotes the derivative w.r.t. the second variable and $s^*=\frac{s}{s-1}$.

(ii) If $y_L^0\to y^0$ strongly in Y and $\limsup_{L\to\infty}\mathcal{E}_L(y_L^0)<\infty$, then for all $t\in(0,T]$,

$$y_L(t)\to y(t) \quad \textit{strongly in } Y.$$

(See Section 9.1.2 for the proof.)

9.1.2 Proofs

Before presenting the proof of Theorem 9.3, we provide two auxiliary lemmas providing a suitable time-dependent recovery sequence for the proof. Analogously as in Section 5.3 for the discrete case, we extend the (continuum) unfolding operator $\mathcal{T}_\varepsilon : L^p(\Omega\times Q)\to L^p(\Omega\times Q)$ to a (not relabeled) linear isometry $\mathcal{T}_\varepsilon : L^p(0,T;L^p(\Omega\times Q))\to L^p(0,T;L^p(\Omega\times Q))$ that is uniquely characterized by the relation $\mathcal{T}_\varepsilon(\eta\varphi)(t,\cdot)=\eta(t)\mathcal{T}_\varepsilon\varphi(\cdot)$ for all $\eta\in L^p(0,T)$ and $\varphi\in L^p(\Omega\times Q)$.

Lemma 9.9 (Recovery sequence). *Let $p \in (1,\infty)$, $s \in [2,\infty)$ and $Q \subset \mathbb{R}^d$ be open and bounded. For given $w \in L^p(0,T;L^p_{\mathrm{inv}}(\Omega)\otimes W^{1,p}_0(Q)) \cap L^s(0,T;L^s_{\mathrm{inv}}(\Omega)\otimes L^s(Q))$ and $\chi \in L^p(0,T;L^p_{\mathrm{pot}}(\Omega)\otimes L^p(Q))$, there exists a sequence $w_\varepsilon \in L^p(0,T;L^p(\Omega)\otimes W^{1,p}_0(Q)) \cap L^s(0,T;L^s(\Omega\times Q))$ such that*

$$\begin{aligned}&\mathcal{T}_\varepsilon w_\varepsilon \to w \quad \textit{strongly in } L^s(0,T;L^s(\Omega\times Q)),\\ &\mathcal{T}_\varepsilon \nabla w_\varepsilon \to \nabla w + \chi \quad \textit{strongly in } L^p(0,T;L^p(\Omega\times Q)^d).\end{aligned}$$

Proof. Since $\chi \in L^p(0,T;L^p_{\mathrm{pot}}(\Omega)\otimes L^p(Q))$, we can find a sequence $\psi^k = \sum_{i=1}^k \eta^{k,i}\chi^{k,i}$ with $\eta^{k,i} \in C^\infty_c(0,T)$ and $\chi^{k,i} \in L^p_{\mathrm{pot}}(\Omega)\otimes L^p(Q)$, such that

$$\|\psi^k - \chi\|_{L^p(0,T;L^p(\Omega\times Q)^d)} \to 0 \quad \text{as } k \to \infty.$$

According to Lemma 6.16, for each $\chi^{k,i}$ we find $g^{k,i}_{\delta,\varepsilon} \in \left(L^p(\Omega)\otimes W^{1,p}_0(Q)\right) \cap L^s(\Omega\times Q)$ such that

$$\|g^{k,i}_{\delta,\varepsilon}\|_{L^s(\Omega\times Q)} \le \varepsilon c_{k,i}(\delta), \quad \limsup_{\varepsilon\to 0} \|\mathcal{T}_\varepsilon \nabla g^{k,i}_{\delta,\varepsilon} - \chi^{k,i}\|_{L^p(\Omega\times Q)^d} \le \delta.$$

We define $w^k_{\delta,\varepsilon} = w + \sum_{i=1}^k \eta^{k,i} g^{k,i}_{\delta,\varepsilon}$ and we estimate

$$\begin{aligned}&\|\mathcal{T}_\varepsilon w^k_{\delta,\varepsilon} - w\|_{L^s(0,T;L^s(\Omega\times Q))} + \|\mathcal{T}_\varepsilon \nabla w^k_{\delta,\varepsilon} - \nabla w - \chi\|_{L^p(0,T;L^p(\Omega\times Q)^d)}\\ \le\;& \|\sum_{i=1}^k \eta^{k,i} g^{k,i}_{\delta,\varepsilon}\|_{L^s(0,T;L^s(\Omega\times Q))} + \|\sum_{i=1}^k \eta^{k,i}\left(\mathcal{T}_\varepsilon \nabla g^{k,i}_{\delta,\varepsilon} - \chi^{k,i}\right)\|_{L^p(0,T;L^p(\Omega\times Q)^d)}\\ &+ \left\|\psi^k - \chi\right\|_{L^p(0,T;L^p(\Omega\times Q)^d)}\\ \le\;& \varepsilon \sum_{i=1}^k c_{k,i}(\delta) + \sum_{i=1}^k c_{k,i} \left\|\mathcal{T}_\varepsilon \nabla g^{k,i}_{\delta,\varepsilon} - \chi^{k,i}\right\|_{L^p(\Omega\times Q)^d} + \left\|\psi^k - \chi\right\|_{L^p(0,T;L^p(\Omega\times Q)^d)}.\end{aligned}$$

Letting first $\varepsilon \to 0$, second $\delta \to 0$ and finally $k \to \infty$, the right-hand side above vanishes. As a result of this, we extract diagonal sequences $k(\varepsilon)$ and $\delta(\varepsilon)$ such that $w_\varepsilon := w^{k(\varepsilon)}_{\delta(\varepsilon),\varepsilon}$ satisfies the claim of the lemma. □

Lemma 9.10 (Measurable selection). *Let the assumptions of Theorem 9.3 be in effect. Let $\widetilde{\mathcal{E}}_{\mathrm{hom}} : [0,T]\times Y_0 \to \mathbb{R}\cup\{\infty\}$ be given by*

$$\widetilde{\mathcal{E}}_{\mathrm{hom}}(t,w) = e^{2\Lambda t}\mathcal{E}_{\mathrm{hom}}(e^{-\Lambda t}w) - \Lambda \mathcal{R}_{\mathrm{hom}}(w).$$

For given $\xi \in L^2(0,T;Y_0)^*$, there exists $w \in L^p(0,T;L^p_{\mathrm{inv}}(\Omega)\otimes W^{1,p}_0(Q)) \cap L^s(0,T;L^s_{\mathrm{inv}}(\Omega)\otimes L^s(Q))$ such that

$$\int_0^T \widetilde{\mathcal{E}}^*_{\mathrm{hom}}(t,\xi(t))dt = \langle \xi, w\rangle_{L^2(0,T;Y_0)^*,L^2(0,T;Y_0)} - \int_0^T \widetilde{\mathcal{E}}_{\mathrm{hom}}(t,w(t))dt.$$

Above, $\widetilde{\mathcal{E}}^*_{\mathrm{hom}}(t,\xi) = \sup_{w\in Y_0}\left(\langle \xi, w\rangle_{Y_0^*,Y_0} - \widetilde{\mathcal{E}}_{\mathrm{hom}}(t,w)\right)$ is the Legendre-Fenchel transformation of $\widetilde{\mathcal{E}}_{\mathrm{hom}}(t,\cdot)$ (see Appendix A.2). Moreover, there exists $\chi \in L^p(0,T;L^p_{\mathrm{pot}}(\Omega)\otimes L^p(Q))$ such that

$$\begin{aligned}&\int_0^T \inf_{\chi\in L^p_{\mathrm{pot}}(\Omega)\otimes L^p(Q)} \left\langle \int_Q e^{2\Lambda t} V(\omega,x,e^{-\Lambda t}\nabla w(t)+\chi)dx\right\rangle dt\\ =\;& \int_0^T \left\langle \int_Q e^{2\Lambda t} V(\omega,x,e^{-\Lambda t}\nabla w(t)+\chi(t))dx\right\rangle dt.\end{aligned}$$

Proof. First we note that $\widetilde{\mathcal{E}}_{\text{hom}}$ is a convex normal integrand by Lemma 4.5 and $\int_0^T \widetilde{\mathcal{E}}_{\text{hom}}(t,0)dt < \infty$. Therefore, Proposition A.7 implies that

$$\int_0^T \widetilde{\mathcal{E}}^*_{\text{hom}}(t,\xi(t))dt = \sup_{w\in L^2(0,T;Y_0)} \left(\langle \xi, w\rangle_{L^2(0,T;Y_0)^*,L^2(0,T;Y)} - \int_0^T \widetilde{\mathcal{E}}_{\text{hom}}(t,w(t))dt \right). \tag{9.3}$$

Using the direct method of the calculus of variations, with the help of the growth conditions of V and f, we conclude that the supremum on the right-hand side is attained by some $w \in L^2(0,T;Y_0)$. As a result of this, we have $\int_0^T \widetilde{\mathcal{E}}_{\text{hom}}(t,w(t))dt < \infty$, which implies that $w \in L^p(0,T;L^p_{\text{inv}}(\Omega)\otimes W^{1,p}_0(Q)) \cap L^s(0,T;L^s_{\text{inv}}(\Omega)\otimes L^s(Q))$ using the growth assumptions of V and f.

To show the second claim, we define an integrand $\mathcal{I} : [0,T]\times \left(L^p_{\text{pot}}(\Omega)\otimes L^p(Q)\right) \to \mathbb{R}\cup\{\infty\}$ by $\mathcal{I}(t,\chi) = e^{2\Lambda t}\left\langle \int_Q V(\omega,x,e^{-\Lambda t}\nabla w(t)(\omega,x)+\chi(\omega,x)dx\right\rangle$. We remark that $\mathcal{I}$ is finite everywhere (up to considering a suitable representative of ∇w) and for all $t\in[0,T]$, $\mathcal{I}(t,\cdot)$ is convex and l.s.c. (using the growth conditions of V), in fact, $\mathcal{I}(t,\cdot)$ is continuous. Moreover, for each fixed $\chi \in L^p_{\text{pot}}(\Omega)\otimes L^p(Q)$, $\mathcal{I}(\cdot,\chi)$ is $\mathcal{L}(0,T)$-measurable. Indeed, this follows by the observation that $\mathcal{I}(\cdot,\chi)$ is a composition of the mappings $g_1 : [0,T]\to[0,T]\times L^p(\Omega\times Q)^d$, $g_1(t) = \left(t, e^{-\Lambda t}\nabla w(t)+\chi\right)$, and $g_2 : [0,T]\times L^p(\Omega\times Q)^d \to \mathbb{R}$, $g_2(t,\varphi) = e^{2\Lambda t}\left\langle \int_Q V(\omega,x,\varphi(\omega,x))dx\right\rangle$. g_1 is $\left(\mathcal{L}(0,T), \mathcal{L}(0,T)\otimes\mathcal{B}(L^p(\Omega\times Q)^d)\right)$-measurable and g_2 is a Carathéodory integrand and therefore $\left(\mathcal{L}(0,T)\otimes\mathcal{B}(L^p(\Omega\times Q)^d)\right)$-measurable. The above statements imply that $\mathcal{I}$ is a convex Carathéodory integrand, thus a normal convex integrand (see Appendix A.2). As a result of this, Proposition A.7 (in particular Remark A.8) implies that

$$\int_0^T \inf_{\chi\in L^p_{\text{pot}}(\Omega)\otimes L^p(Q)} \mathcal{I}(t,\chi)dt = \inf_{\chi\in L^p(0,T;L^p_{\text{pot}}(\Omega)\otimes L^p(Q))} \int_0^T \mathcal{I}(t,\chi(t))dt.$$

The infimum on the right-hand side is attained at some $\chi \in L^p(0,T;L^p_{\text{pot}}(\Omega)\otimes L^p(Q))$, using the direct method of calculus of variations. This concludes the proof. □

We recommend, as a warm up, the formal discussion in Section 4 for the strategy of the following proof.

Proof of Theorem 9.3. Step 1. Compactness. Note that using the a priori estimates from Remark 9.1 and using the growth conditions of V and f, we obtain for all $t\in[0,T]$,

$$\|y_\varepsilon(t)\|^p_{L^p(\Omega)\otimes W^{1,p}(Q)} + \|y_\varepsilon(t)\|^s_{L^s(\Omega\times Q)} \leq c_1\left(\mathcal{E}_\varepsilon(y^0_\varepsilon)+c_2\right). \tag{9.4}$$

By assumption, the right-hand side is bounded by a constant independent of ε. Also, by the isometry property of $\mathcal{T}_\varepsilon$ and since $s\geq 2$, the above implies that $\|\mathcal{T}_\varepsilon y_\varepsilon(t)\|^s_Y \leq c$. We remark that $\mathcal{T}_\varepsilon y_\varepsilon \in H^1(0,T;Y)$ since $\dot{(\cdot)}$ and $\mathcal{T}_\varepsilon$ commute, i.e., $\frac{d}{dt}(\mathcal{T}_\varepsilon y_\varepsilon) = \mathcal{T}_\varepsilon \dot{y}_\varepsilon$, where on the left-hand side $\mathcal{T}_\varepsilon y_\varepsilon$ is pointwise defined as $\mathcal{T}_\varepsilon y_\varepsilon(t)$ and on the right-hand side $\mathcal{T}_\varepsilon$ is the extension defined on $L^2(0,T;Y)$. As a result of this and using the isometry property of $\mathcal{T}_\varepsilon$, the a priori estimate $\int_0^t \mathcal{R}_\varepsilon(\dot{y}_\varepsilon(s))ds \leq \mathcal{E}_\varepsilon(y^0_\varepsilon) - \mathcal{E}_\varepsilon(y_\varepsilon(t))$ implies that

$$\|\mathcal{T}_\varepsilon y_\varepsilon\|^2_{H^1(0,T;Y)} \leq c, \quad \|\mathcal{T}_\varepsilon y_\varepsilon(t) - \mathcal{T}_\varepsilon y_\varepsilon(s)\|^2_Y \leq c|t-s| \quad \text{for all } s,t\in[0,T].$$

We extract a (not relabeled) subsequence and $y \in H^1(0,T;Y)$ such that $\mathcal{T}_\varepsilon y_\varepsilon \rightharpoonup y$ in $H^1(0,T;Y)$, and this implies that $\mathcal{T}_\varepsilon \dot{y}_\varepsilon \rightharpoonup \dot{y}$ weakly in $L^2(0,T;Y)$. Moreover, we apply the Arzelà-Ascoli theorem to the sequence $\mathcal{T}_\varepsilon y_\varepsilon$ to obtain that (up to another subsequence) for all $t \in [0,T]$,

$$\mathcal{T}_\varepsilon y_\varepsilon(t) \rightharpoonup y(t) \quad \text{weakly in } Y. \tag{9.5}$$

Using the estimates (9.4) and Proposition 6.14 we conclude that $y(t) \in \left(L^p_{\text{inv}}(\Omega) \otimes W^{1,p}_0(Q)\right) \cap (L^s_{\text{inv}}(\Omega) \otimes L^s(Q))$ and $\mathcal{T}_\varepsilon y_\varepsilon(t) \rightharpoonup y(t)$ weakly in $L^s(\Omega\times Q)$ and in $L^p(\Omega\times Q)$ (see also Remark 6.15). This also implies that $y \in H^1(0,T;Y_0)$. Moreover, for each $t \in [0,T]$ we find $\chi(t) \in L^p_{\text{pot}}(\Omega) \otimes L^p(Q)$ and a subsequence $\varepsilon(t)$ such that $\mathcal{T}_{\varepsilon(t)} \nabla y_{\varepsilon(t)}(t) \rightharpoonup \nabla y(t) + \chi(t)$ weakly in $L^p(\Omega \times Q)^d$. This implies that $P_{\text{inv}} \nabla y_\varepsilon(t) \rightharpoonup \nabla y(t)$ weakly in $L^p(\Omega \times Q)^d$ for the whole (sub)sequence ε. Note that the assumption on the initial data implies that $\mathcal{T}_\varepsilon y_\varepsilon(0) \to y^0$ strongly in Y and hence we have $y(0) = y^0$.

In the following we restate the EVI in an equivalent form using Lemma 4.5. For this reason, we define the new variables $u_\varepsilon(t) = e^{\Lambda t} y_\varepsilon(t)$ and $u(t) = e^{\Lambda t} y(t)$. Note that $\dot{u}_\varepsilon(t) = \Lambda e^{\Lambda t} y_\varepsilon(t) + e^{\Lambda t} \dot{y}_\varepsilon(t)$ and analogously for $\dot{u}$. The above convergence statements result in

$$\begin{aligned} &\mathcal{T}_\varepsilon u_\varepsilon \rightharpoonup u \quad \text{weakly in } H^1(0,T;Y), \\ &\mathcal{T}_\varepsilon u_\varepsilon(t) \rightharpoonup u(t) \quad \text{weakly in } L^s(\Omega \times Q) \text{ and } L^p(\Omega \times Q), \quad \text{for all } t \in [0,T]. \end{aligned} \tag{9.6}$$

Step 2. Reduction to a convex problem. We define a new energy $\widetilde{\mathcal{E}}_\varepsilon : [0,T] \times Y \to \mathbb{R} \cup \{\infty\}$ by $\widetilde{\mathcal{E}}_\varepsilon(t,u) = e^{2\Lambda t} \mathcal{E}_\varepsilon(e^{-\Lambda t} u) - \Lambda \mathcal{R}_\varepsilon(u)$ and analogously $\widetilde{\mathcal{E}}_{\text{hom}} : [0,T] \times Y_0 \to \mathbb{R} \cup \{\infty\}$. Lemma 4.5 implies that $\widetilde{\mathcal{E}}_\varepsilon$ and $\widetilde{\mathcal{E}}_{\text{hom}}$ are normal convex integrands. Moreover, it follows that $u_\varepsilon(t)$ satisfies for a.a. t,

$$\frac{d}{dt} \mathcal{R}_\varepsilon(u_\varepsilon(t)) + \widetilde{\mathcal{E}}_\varepsilon(t, u_\varepsilon(t)) + \widetilde{\mathcal{E}}^*_\varepsilon(t, -D\mathcal{R}_\varepsilon(\dot{u}_\varepsilon(t))) = 0, \tag{9.7}$$

where $\widetilde{\mathcal{E}}^*_\varepsilon(t,\xi) = \sup_{w \in Y} \left(\langle \xi, w \rangle_{Y^*,Y} - \widetilde{\mathcal{E}}_\varepsilon(t,w) \right)$ is the Legendre-Fenchel conjugate of $\widetilde{\mathcal{E}}_\varepsilon(t,\cdot)$, which is also a normal convex integrand (see Appendix A.2). Integration of (9.7) over $(0,T)$ yields

$$\mathcal{R}_\varepsilon(u_\varepsilon(T)) + \int_0^T \widetilde{\mathcal{E}}_\varepsilon(t, u_\varepsilon(t)) + \widetilde{\mathcal{E}}^*_\varepsilon(t, -D\mathcal{R}_\varepsilon(\dot{u}_\varepsilon(t))) dt = \mathcal{R}_\varepsilon(u_\varepsilon(0)). \tag{9.8}$$

Step 3. Passage to the limit $\varepsilon \to 0$ in (9.8). Note that $u_\varepsilon(0) = y^0_\varepsilon \overset{2}{\to} y^0 = u(0)$ in Y and therefore using Proposition 6.13 (ii), for the right-hand side of (9.8), we have

$$\lim_{\varepsilon \to 0} \mathcal{R}_\varepsilon(u_\varepsilon(0)) = \mathcal{R}_{\text{hom}}(u(0)). \tag{9.9}$$

The first term on the left-hand side is treated similarly, using Proposition 6.13 (iii) and (9.6), we have

$$\liminf_{\varepsilon \to 0} \mathcal{R}_\varepsilon(u_\varepsilon(T)) \geq \mathcal{R}_{\text{hom}}(u(T)). \tag{9.10}$$

We treat the second term on the left-hand side of (9.8) as follows. By Fatou's lemma we have

$$\begin{aligned} \liminf_{\varepsilon \to 0} \int_0^T \widetilde{\mathcal{E}}_\varepsilon(t, u_\varepsilon(t)) dt \geq &\int_0^T \liminf_{\varepsilon \to 0} \left\langle \int_Q e^{2\Lambda t} V(\tau_{\frac{x}{\varepsilon}} \omega, x, e^{-\Lambda t} \nabla u_\varepsilon(t)) dx \right\rangle dt \\ &+ \int_0^T \liminf_{\varepsilon \to 0} \left\langle \int_Q e^{2\Lambda t} f(\tau_{\frac{x}{\varepsilon}} \omega, x, e^{-\Lambda t} u_\varepsilon(t)) - \frac{\Lambda}{2} r(\tau_{\frac{x}{\varepsilon}} \omega, x) |u_\varepsilon(t)|^2 dx \right\rangle dt. \end{aligned}$$

For fixed fixed t, the $\liminf$ in the first term is a limit for a subsequence $\varepsilon(t)$ and as in Step 1 we find $\chi(t) \in L^p_{\text{pot}}(\Omega) \otimes L^p(Q)$ such that, up to another (not relabeled) subsequence, it holds $\nabla u_{\varepsilon(t)}(t) \overset{2}{\rightharpoonup} \nabla u(t) + e^{\Lambda t}\chi(t)$ in $L^p(\Omega \times Q)^d$. Also, we notice that $e^{2\Lambda t}V(\omega, x, e^{-\Lambda t}\cdot)$ is convex and has p-growth properties and therefore Proposition 6.13 (iii) implies that

$$
\begin{aligned}
\liminf_{\varepsilon \to 0} \left\langle \int_Q e^{2\Lambda t} V(\tau_{\frac{x}{\varepsilon}}\omega, x, e^{-\Lambda t}\nabla u_\varepsilon(t))dx \right\rangle &\geq \left\langle \int_Q e^{2\Lambda t} V(\omega, x, e^{-\Lambda t}\nabla u(t) + \chi(t))dx \right\rangle \\
&\geq \inf_{\chi \in L^p_{\text{pot}}(\Omega)\otimes L^p(Q)} \left\langle \int_Q e^{2\Lambda t} V(\omega, x, e^{-\Lambda t}\nabla u(t) + \chi)dx \right\rangle.
\end{aligned}
$$

On the other hand, we remark that the integrand $e^{2\Lambda t}f(\omega, x, e^{-\Lambda t}\cdot) - \frac{\Lambda}{2}r(\omega, x)|\cdot|^2$ is convex and satisfies s-growth conditions. As a result of this and by (9.6), Proposition 6.13 (iii) yields

$$
\begin{aligned}
&\liminf_{\varepsilon \to 0} \left\langle \int_Q e^{2\Lambda t} f(\tau_{\frac{x}{\varepsilon}}\omega, x, e^{-\Lambda t}u_\varepsilon(t)) - \frac{\Lambda}{2} r(\tau_{\frac{x}{\varepsilon}}\omega, x)|u_\varepsilon(t)|^2 dx \right\rangle \\
\geq \ &\left\langle \int_Q e^{2\Lambda t} f(\omega, x, e^{-\Lambda t}u(t)) - \frac{\Lambda}{2} r(\omega, x)|u(t)|^2 dx \right\rangle.
\end{aligned}
$$

Using the above two statements we conclude that

$$
\liminf_{\varepsilon \to 0} \int_0^T \widetilde{\mathcal{E}}_\varepsilon(t, u_\varepsilon(t))dt \geq \int_0^T \widetilde{\mathcal{E}}_{\text{hom}}(t, u(t))dt. \tag{9.11}
$$

In order to complete the limit passage, it is left to treat the third term on the left-hand side of (9.8). Using Lemma 9.10, we find $w \in L^p(0,T; L^p_{\text{inv}}(\Omega) \otimes W^{1,p}_0(Q)) \cap L^s(0,T; L^s_{\text{inv}}(\Omega) \otimes L^s(Q))$ such that

$$
\int_0^T \widetilde{\mathcal{E}}^*_{\text{hom}}(t, -D\mathcal{R}_{\text{hom}}(\dot{u}(t)))dt = \int_0^T \langle -D\mathcal{R}_{\text{hom}}(\dot{u}(t)), w(t)\rangle_{Y_0^*, Y_0} - \widetilde{\mathcal{E}}_{\text{hom}}(t, w(t))dt.
$$

Moreover, by the second claim of Lemma 9.10, we find $\chi \in L^p(0,T; L^p_{\text{pot}}(\Omega) \otimes L^p(Q))$ such that

$$
\begin{aligned}
&\int_0^T \widetilde{\mathcal{E}}_{\text{hom}}(t, w(t))dt \\
= \ &\int_0^T e^{2\Lambda t} \left\langle \int_Q V(\omega, x, e^{-\Lambda t}\nabla w(t) + \chi(t)) + f(\omega, x, e^{-\Lambda t}w(t)) \right\rangle - \Lambda \mathcal{R}_{\text{hom}}(w(t))dt.
\end{aligned} \tag{9.12}
$$

For the pair $\left(w, e^{\Lambda \cdot}\chi(\cdot)\right)$ ($e^{\Lambda \cdot}$ denotes the function $t \mapsto e^{\Lambda t}$) Lemma 9.9 implies the existence of $w_\varepsilon \in L^p(0,T; L^p(\Omega) \otimes W^{1,p}_0(Q)) \cap L^s(0,T; L^s(\Omega \times Q))$ such that

$$
\begin{aligned}
&\mathcal{T}_\varepsilon w_\varepsilon \to w \quad \text{strongly in } L^s(0,T; L^s(\Omega \times Q)), \\
&\mathcal{T}_\varepsilon \nabla w_\varepsilon \to \nabla w + e^{\Lambda \cdot}\chi \quad \text{strongly in } L^p(0,T; L^p(\Omega \times Q)^d).
\end{aligned} \tag{9.13}
$$

Using the definition of $\widetilde{\mathcal{E}}^*_\varepsilon$, we have

$$
\int_0^T \widetilde{\mathcal{E}}^*_\varepsilon(t, -D\mathcal{R}_\varepsilon(\dot{u}_\varepsilon(t)))dt \geq \int_0^T \langle -D\mathcal{R}_\varepsilon(\dot{u}_\varepsilon(t)), w_\varepsilon(t)\rangle_{Y^*, Y} - \widetilde{\mathcal{E}}_\varepsilon(t, w_\varepsilon(t))dt.
$$

For the first term on the right-hand side we have, using the fact that the extended unfolding operator is unitary,

$$\begin{aligned}
& \int_0^T \langle -D\mathcal{R}_\varepsilon(\dot u_\varepsilon(t)), w_\varepsilon(t)\rangle_{Y^*,Y}\,dt = \int_0^T \left\langle \int_Q r(\omega,x)\mathcal{T}_\varepsilon \dot u_\varepsilon(t)\mathcal{T}_\varepsilon w_\varepsilon(t)dx\right\rangle dt \qquad (9.14)\\
\to\ & \int_0^T \left\langle \int_Q r(\omega,x)\dot u(t)w(t)dx\right\rangle dt = \int_0^T \langle -D\mathcal{R}_{\mathrm{hom}}(\dot u(t)), w(t)\rangle_{Y_0^*,Y_0}\,dt.
\end{aligned}$$

The above convergence follows since (9.14) is a scalar product of a strongly and weakly convergent sequences. Moreover, we remark that an analogous transformation formula to Proposition 6.13 (i) holds for the extended unfolding operator and hence we have

$$\begin{aligned}
& \int_0^T \widetilde{\mathcal{E}}_\varepsilon(t, w_\varepsilon(t))dt\\
=\ & \int_0^T e^{2\Lambda t}\left\langle \int_Q V(\omega,x,e^{-\Lambda t}\mathcal{T}_\varepsilon \nabla w_\varepsilon(t)) + f(\omega,x,e^{-\Lambda t}\mathcal{T}_\varepsilon w_\varepsilon(t)) - \frac{\Lambda r}{2e^{2\Lambda t}}|\mathcal{T}_\varepsilon w_\varepsilon(t)|^2 dx\right\rangle dt\\
\to\ & \int_0^T e^{2\Lambda t}\left\langle \int_Q V(\omega,x,e^{-\Lambda t}\nabla w(t)+\chi(t)) + f(\omega,x,e^{-\Lambda t}w(t)) - \frac{\Lambda r}{2e^{2\Lambda t}}|w(t)|^2 dx\right\rangle dt.
\end{aligned}$$

Above the latter convergence follows completely analogously as in the proof of Proposition 6.13 (ii) using the strong convergences (9.13) and the growth conditions of the integrands (standard argument using Fatou's lemma). By (9.12), the last expression equals $\int_0^T \widetilde{\mathcal{E}}_{\mathrm{hom}}(t,w(t))dt$ and therefore collecting the above statements we conclude that

$$\liminf_{\varepsilon\to 0}\int_0^T \widetilde{\mathcal{E}}^*_\varepsilon(t,-D\mathcal{R}_\varepsilon(\dot u_\varepsilon(t)))dt \geq \int_0^T \widetilde{\mathcal{E}}^*_{\mathrm{hom}}(t,-D\mathcal{R}_{\mathrm{hom}}(\dot u(t)))dt. \qquad (9.15)$$

Collecting (9.9), (9.10), (9.11) and (9.15), we obtain that

$$\begin{aligned}
& \int_0^T \widetilde{\mathcal{E}}_{\mathrm{hom}}(t,u(t)) + \widetilde{\mathcal{E}}^*_{\mathrm{hom}}(t,-D\mathcal{R}_{\mathrm{hom}}(\dot u(t)))dt\\
\leq\ & -\mathcal{R}_{\mathrm{hom}}(u(T)) + \mathcal{R}_{\mathrm{hom}}(u(0)) = \int_0^T \langle -D\mathcal{R}_{\mathrm{hom}}(\dot u(t)), u(t)\rangle_{Y_0^*,Y_0}\,dt.
\end{aligned}$$

On the other hand, it holds $\widetilde{\mathcal{E}}_{\mathrm{hom}}(t,u(t)) + \widetilde{\mathcal{E}}^*_{\mathrm{hom}}(t,-D\mathcal{R}_{\mathrm{hom}}(\dot u(t))) \geq \langle -D\mathcal{R}_{\mathrm{hom}}(\dot u(t)), u(t)\rangle_{Y_0^*,Y_0}$ for a.a. t by the definition of $\widetilde{\mathcal{E}}^*_{\mathrm{hom}}$. As a result of this and of the above inequality, it follows that for a.a. t, it holds

$$\frac{d}{dt}\mathcal{R}_{\mathrm{hom}}(u(t)) + \widetilde{\mathcal{E}}_{\mathrm{hom}}(t,u(t)) + \widetilde{\mathcal{E}}^*_{\mathrm{hom}}(t,-D\mathcal{R}_{\mathrm{hom}}(\dot u(t))) = 0. \qquad (9.16)$$

Since $u(t) = e^{\Lambda t}y(t)$, Lemma 4.5 (ii) implies that y is the unique EVI solution to $(Y_0,\mathcal{E}_{\mathrm{hom}},\mathcal{R}_{\mathrm{hom}})$. Furthermore, using (9.9) and (9.10) we obtain

$$\limsup_{\varepsilon\to 0}\left(-\mathcal{R}_\varepsilon(u_\varepsilon(T)) + \mathcal{R}_\varepsilon(u_\varepsilon(0))\right) \leq -\mathcal{R}_{\mathrm{hom}}(u(T)) + \mathcal{R}_{\mathrm{hom}}(u(0)).$$

Also, exploiting the equality (9.8) and the liminf inequalities (9.11), (9.15), we obtain

$$\begin{aligned}\liminf_{\varepsilon\to 0}\left(-\mathcal{R}_\varepsilon(u_\varepsilon(T))+\mathcal{R}_\varepsilon(u_\varepsilon(0))\right) &\geq \int_0^T \widetilde{\mathcal{E}}_{\mathrm{hom}}(t,u(t))+\widetilde{\mathcal{E}}^*_{\mathrm{hom}}(t,-D\mathcal{R}_{\mathrm{hom}}(\dot u(t)))dt\\ &= -\mathcal{R}_{\mathrm{hom}}(u(T))+\mathcal{R}_{\mathrm{hom}}(u(0)).\end{aligned}$$

This results in

$$\frac{e^{2\Lambda T}}{2}\left\langle\int_Q r(\omega,x)|\mathcal{T}_\varepsilon y_\varepsilon(T)|^2dx\right\rangle=\mathcal{R}_\varepsilon(u_\varepsilon(T)) \quad\to\quad \mathcal{R}_{\mathrm{hom}}(u(T))=\frac{e^{2\Lambda T}}{2}\left\langle\int_Q r(\omega,x)|y(T)|^2dx\right\rangle,$$

where we use that $\mathcal{R}_\varepsilon(u_\varepsilon(0))$ converges. This and (9.5) imply that $\mathcal{T}_\varepsilon y_\varepsilon(T)\to y(T)$ strongly in Y and since $\mathcal{T}_\varepsilon y(T)=y(T)$ by shift-invariance of $y(T)$, we obtain that $y_\varepsilon(T)\to y(T)$ strongly in Y. We may replace T by any $t\in(0,T]$ in the above procedure to obtain $y_\varepsilon(t)\to y(t)$ strongly in Y. Convergence for the whole sequence is obtained by the usual argument using the uniqueness of solutions for the limit problem.

Step 4. Convergence of $\dot y_\varepsilon$ and $\mathcal{E}_\varepsilon(y_\varepsilon(t))$. Using Remark 4.2 and the chain rule as in (4.5), we obtain that for an arbitrary $t\in(0,T]$,

$$\int_0^t \langle D\mathcal{R}_\varepsilon(\dot y_\varepsilon(s)),\dot y_\varepsilon(s)\rangle_{Y^*,Y}\,ds=\mathcal{E}_\varepsilon(y_\varepsilon(0))-\mathcal{E}_\varepsilon(y_\varepsilon(t)).$$

Since $y_\varepsilon(t)\to y(t)$ strongly in Y and (9.6) holds, we obtain that $\liminf_{\varepsilon\to0}\mathcal{E}_\varepsilon(y_\varepsilon(t))\geq\mathcal{E}_{\mathrm{hom}}(y(t))$ using the usual two-scale convergence arguments for the first convex part of the energy and strong convergence of $\mathcal{T}_\varepsilon y_\varepsilon(t)$ for the second Λ-convex part. As a consequence, using the additional assumption $\mathcal{E}_\varepsilon(y_\varepsilon(0))\to\mathcal{E}_{\mathrm{hom}}(y(0))$, we obtain

$$\limsup_{\varepsilon\to0}\int_0^t\langle D\mathcal{R}_\varepsilon(\dot y_\varepsilon(s)),\dot y_\varepsilon(s)\rangle_{Y^*,Y}\,ds\leq\mathcal{E}_{\mathrm{hom}}(y(0))-\mathcal{E}_{\mathrm{hom}}(y(t))=\int_0^t\langle D\mathcal{R}_{\mathrm{hom}}(\dot y(s)),\dot y(s)\rangle_{Y_0^*,Y_0},$$

where in the last equality we use that y is the solution to the limit problem. Note that it holds $\int_0^t\langle D\mathcal{R}_\varepsilon(\dot y_\varepsilon(s)),\dot y_\varepsilon(s)\rangle_{Y^*,Y}\,ds=\int_0^t\left\langle\int_Q r|\mathcal{T}_\varepsilon\dot y_\varepsilon(s)|^2dx\right\rangle ds$ and since $\mathcal{T}_\varepsilon\dot y_\varepsilon\rightharpoonup\dot y$ weakly in $L^2(0,T;Y)$, it follows that

$$\liminf_{\varepsilon\to0}\int_0^t\langle D\mathcal{R}_\varepsilon(\dot y_\varepsilon(s)),\dot y_\varepsilon(s)\rangle_{Y^*,Y}\,ds\geq\int_0^t\langle D\mathcal{R}_{\mathrm{hom}}(\dot y(s)),\dot y(s)\rangle_{Y_0^*,Y_0}\,ds.$$

Combining the last two inequalities and the weak convergence $\mathcal{T}_\varepsilon\dot y_\varepsilon\rightharpoonup\dot y$, we conclude that for all $t\in(0,T]$,

$$\dot y_\varepsilon\to\dot y\quad\text{strongly in } L^2(0,t;Y),\quad \mathcal{E}_\varepsilon(y_\varepsilon(t))\to\mathcal{E}_{\mathrm{hom}}(y(t)).$$

□

Proof of Lemma 9.6. (i) Measurability and boundedness of r_L follow by the corresponding properties of r. The strong convergence $r_L\to r_{\mathrm{hom}}$ follows by a direct application of von Neumann's ergodic theorem, see Remark 6.3.

(ii) This claim is already proved in Lemma 7.11 (the replacement of $\Box$ by B does not affect the proof).

(iii) f_L inherits all of its properties from the corresponding properties of f from (D3) with the same constants λ and c. □

Proof of Theorem 9.8. (i) Based on standard a priori estimates (see Theorem 4.3) and the growth conditions of $\mathcal{E}_L$ and $\mathcal{E}_{\text{hom}}$, we have for all $t \in [0, T]$,

$$\begin{aligned}
&\|y_L(t)\|^2_{L^2(\Omega)\otimes H^1_0(Q)} + \|y_L(t)\|^s_{L^s(\Omega\times Q)} \le c\left(\mathcal{E}_L(y_L(t)) + c_1\right) \le c\left(\mathcal{E}_L(y^0_L) + c_1\right), \\
&\|y(t)\|^2_{H^1_0(Q)} + \|y(t)\|^s_{L^s(Q)} \le c\left(\mathcal{E}_{\text{hom}}(y(t)) + c_1\right) \le c\left(\mathcal{E}_{\text{hom}}(y^0) + c_1\right), \\
&\int_0^T \|\dot{y}(t)\|^2_{Y_0}\, dt \le c\left(\mathcal{E}_{\text{hom}}(y^0) - \mathcal{E}_{\text{hom}}(y(T))\right) \le c\left(\mathcal{E}_{\text{hom}}(y^0) + c_1\right).
\end{aligned} \tag{9.17}$$

According to Remark 4.2, the EVI inequalities of the systems $(Y, \mathcal{E}_L, \mathcal{R}_L)$ and $(Y_0, \mathcal{E}_{\text{hom}}, \mathcal{R}_{\text{hom}})$ are equivalent to

$$-\, r_L \dot{y}_L(t) - \Lambda r_L y_L(t) \in \partial\left(\mathcal{E}_L(y_L(t)) - \Lambda \mathcal{R}_L(y_L(t))\right), \tag{9.18}$$
$$-\, r_{\text{hom}} \dot{y}(t) - \Lambda r_{\text{hom}} y(t) \in \partial\left(\mathcal{E}_{\text{hom}}(y(t)) - \Lambda \mathcal{R}_{\text{hom}}(y(t))\right). \tag{9.19}$$

For simplicity in notation, above we identify r_L and r_{hom} with elements in $\text{Lin}(Y, Y^*)$ and $\text{Lin}(Y_0, Y_0^*)$, respectively. Also, ∂ denotes the convex subdifferential. We remark that $\mathcal{E}_L(\cdot) - \Lambda\mathcal{R}_L(\cdot)$ is a sum of two convex integral functionals:

$$\mathcal{E}_L - \Lambda \mathcal{R}_L = \mathcal{I}_1 + \mathcal{I}_2, \quad \mathcal{I}_1(y) = \frac{1}{2}\left\langle \int_Q A_L \nabla y \cdot \nabla y dx \right\rangle, \quad \mathcal{I}_2(y) = \left\langle \int_Q f_L(\omega, y) - \frac{\Lambda r_L}{2} |y|^2 dx \right\rangle,$$

defined on their corresponding domains. In this regard, its subdifferential may be obtained as follows (cf. [Bré71], [Sho13, Proposition 2.2, Example 2.F])

$$\partial\left(\mathcal{E}_L - \Lambda \mathcal{R}_L\right) = \partial \mathcal{I}_1 + \partial \mathcal{I}_2.$$

Moreover, we may identify the above subdifferentials as follows: (cf. [Sho13, Examples 2.B and 2.E])

- $\text{dom}(\partial \mathcal{I}_1) = \left\{ y \in Y : \ y \in L^2(\Omega) \otimes H^1_0(Q), \ -\text{div} A_L \nabla y \in Y \right\}$, it holds

  $$\xi \in \partial \mathcal{I}_1(u) \quad \tilde{\Leftrightarrow} \quad \xi = -\text{div} A_L \nabla y$$

- $\text{dom}(\partial \mathcal{I}_2) = \{ y \in Y : f'_L(\cdot, y) - \Lambda r_L y \in Y \}$

  $$\xi \in \partial \mathcal{I}_2(y) \quad \tilde{\Leftrightarrow} \quad \xi(\omega, x) = f'_L(\omega, y(\omega, x)) - \Lambda r_L(\omega) y(\omega, x) \quad \text{for a.a. } (\omega, x),$$

  where f' denotes the derivative w.r.t. the second variable.

The analogous statements hold for $\mathcal{E}_{\text{hom}} - \Lambda \mathcal{R}_{\text{hom}}$. As a result of this, we can test equation (9.18) with $y_L(t) - y(t)$ to obtain (in the following we drop the "t" from the notation)

$$\langle r_L \dot{y}_L + \Lambda r_L y_L, y_L - y \rangle_Y + \langle A_L \nabla y_L, (\nabla y_L - \nabla y) \rangle_{Y^d} + \left\langle f'_L(\cdot, y_L) - \Lambda r_L y_L, y_L - y \right\rangle_Y = 0$$

Above, we misuse the notation by identifying Y^* with Y, and seeing A_L as an element in $\text{Lin}(Y^d, Y^d)$. With the help of the monotonicity properties $\langle (f'_L(\cdot, y_L) - \Lambda r_L y_L) - (f'_L(\cdot, y) - \Lambda r_L y), y_L - y \rangle_Y \ge 0$ and $\langle A_L (\nabla y_L - \nabla y), \nabla y_L - \nabla y \rangle_Y \ge 0$, we obtain

$$\langle r_L \dot{y}_L + \Lambda r_L y_L, y_L - y \rangle_Y + \langle A_L \nabla y, (\nabla y_L - \nabla y) \rangle_{Y^d} + \left\langle f'_L(\cdot, y) - \Lambda r_L y, y_L - y \right\rangle_Y \le 0.$$

On the other hand, we test equation (9.19) with $y - y_L(\omega)$ and integrate it over Ω, to obtain (we tacitly identify r_{hom} and A_{hom} with elements of $\text{Lin}(Y,Y)$ and $\text{Lin}(Y^d, Y^d)$)

$$\langle r_{\text{hom}}\dot{y} + \Lambda r_{\text{hom}} y, y - y_L\rangle_Y + \langle A_{\text{hom}}\nabla y, \nabla y - \nabla y_L\rangle_{Y^d} + \left\langle f'_{\text{hom}}(y) - \Lambda r_{\text{hom}} y, y - y_L\right\rangle_Y = 0.$$

Summing up the last two inequalities, and subtracting and adding $\langle r_L \dot{y}, y_L - y\rangle_Y$ to the left-hand side, we compute

$$\begin{aligned}\langle r_L\left(\dot{y}_L - \dot{y}\right), y_L - y\rangle_Y \leq{}& \langle (r_{\text{hom}} - r_L)\,\dot{y}, y_L - y\rangle_Y + \langle (A_{\text{hom}} - A_L)\,\nabla y, \nabla y_L - \nabla y\rangle_{Y^d, Y^d} \\ &+ \left\langle f'_{\text{hom}}(y) - f'_L(\cdot, y), y_L - y\right\rangle_Y - \Lambda\left\langle r_L(y_L - y), y_L - y\right\rangle_Y.\end{aligned}$$

We integrate the above inequality over $(0,t)$ to obtain (also we use the boundedness of r_L)

$$\begin{aligned}&\|y_L(t) - y(t)\|_Y^2 \\ \leq\ & c\left(\|y_L(0) - y(0)\|_Y^2 + \int_0^t \left| \langle (r_{\text{hom}} - r_L)\,\dot{y}, y_L - y\rangle_Y + \langle (A_{\text{hom}} - A_L)\,\nabla y, \nabla y_L - \nabla y\rangle_{Y^d, Y^d}\right| ds\right) \\ &+ c\int_0^t \left|\left\langle f'_{\text{hom}}(y) - f'_L(\cdot, y), y_L - y\right\rangle_Y\right| ds + c\int_0^t \|y_L - y\|_Y^2\, ds. \qquad (9.20)\end{aligned}$$

The second term on the right-hand side can be bounded, using Hölder's inequality, by the following expression

$$\|(r_{\text{hom}} - r_L)\,\dot{y}\|_{L^2(0,T;Y)}\,\|y_L - y\|_{L^2(0,T;Y)} + \|(A_{\text{hom}} - A_L)\,\nabla y\|_{L^2(0,T;Y^d)}\,\|\nabla y_L - \nabla y\|_{L^2(0,T;Y^d)}. \qquad (9.21)$$

Also, since y is deterministic, and $r_{\text{hom}} - r_L$ and $A_{\text{hom}} - A_L$ do not depend on t and x, it follows that $\|(r_{\text{hom}} - r_L)\,\dot{y}\|_{L^2(0,T;Y)} = \|r_{\text{hom}} - r_L\|_{L^2(\Omega)}\,\|\dot{y}\|_{L^2(0,T;Y_0)}$ and $\|(A_{\text{hom}} - A_L)\,\nabla y\|_{L^2(0,T;Y^d)} \leq \|A_{\text{hom}} - A_L\|_{L^2(\Omega)^{d\times d}}\,\|\nabla y\|_{L^2(0,T;Y_0^d)}$. Using these observations and the a priori estimates (9.17), we obtain that (9.21) is bounded by

$$cC_L^{\frac{1}{2}}\left(\|(r_{\text{hom}} - r_L)\|_{L^2(\Omega)} + \|(A_{\text{hom}} - A_L)\|_{L^2(\Omega)^{d\times d}}\right),$$

where $c > 0$ depends only on $\mathcal{E}_{\text{hom}}(y^0)$, T and the constants from the assumptions. Similarly, the third term in (9.20) is bounded by, using Hölder's inequality with (s^*, s),

$$cC_L^{\frac{1}{s}}\left(\int_0^T \left\langle \int_Q |f'_{\text{hom}}(y) - f'_L(\cdot, y)|^{s^*}\right\rangle\right)^{\frac{1}{s^*}}. \qquad (9.22)$$

Using the above described estimates for (9.20) and the Gronwall lemma, the claim follows.

(ii) Using the assumptions and Lemma 9.6 (i) and (ii), the first three terms on the right-hand side of (9.2) vanish in the limit $L \to \infty$. In the following we show that the last term as well vanishes which concludes the proof, in fact we show that (9.22) vanishes.

First, we note that, using the growth assumptions and the λ-convexity of $f(\omega, \cdot)$, it follows that there exists $c > 0$ such that

$$|f'(\omega, \alpha)| \leq c(1 + |\alpha|^{s-1}) \quad \text{for } P\text{-a.a. } \omega \text{ and all } \alpha \in \mathbb{R}. \qquad (9.23)$$

Indeed, since $f(\omega,\cdot)-\frac{\lambda}{2}(\cdot)^2$ is convex and has the same s-growth properties as f (with modified constants), we obtain (see [Dac07, Proposition 2.32])

$$|f(\omega,\alpha)-f(\omega,\beta)-\frac{\lambda}{2}(\alpha^2-\beta^2)|\leq c\left(1+|\alpha|^{s-1}+|\beta|^{s-1}\right)|\alpha-\beta|\quad \text{for all } \alpha,\beta\in\mathbb{R}.$$

If we set $\alpha=\beta+h$ with $h>0$, multiply the above inequality with $\frac{1}{h}$ and consider the limit $h\to 0$, we obtain $|f'(\omega,\beta)-\lambda\beta|\leq c\left(1+|\beta|^{s-1}\right)$, which implies (9.23) (up to modification of c).
Using (9.23), for each (t,x), $\omega\mapsto f'(\omega,y(t,x))$ defines an element in $L^{s^*}(\Omega)$. Therefore, von Neumann's ergodic theorem (see Remark 6.3) implies that for a.a. (t,x), as $L\to\infty$,

$$\left\langle|f'_L(\omega,y(t,x))-f'_{\hom}(y(t,x))|^{s^*}\right\rangle=\left\langle\left|\fint_{LB}f'(\tau_q\omega,y(t,x))dq-\left\langle f'(\cdot,y(t,x))\right\rangle\right|^{s^*}dq\right\rangle\to 0,$$

where we use the fact that $f'_L(\omega,\alpha)=\fint_{LB}f'(\tau_x\omega,\alpha)dx$. Moreover, by (9.23), it follows that $\left\langle|f'_L(\omega,y(t,x))-f'_{\hom}(y(t,x))|^{s^*}\right\rangle\leq c(1+|y(t,x)|^s)$ and since y is the solution to $(Y_0,\mathcal{E}_{\hom},\mathcal{R}_{\hom})$, it holds $y\in L^s(0,T;L^s(Q))$. As a result of this, the dominated convergence theorem implies that y_L satisfies

$$\int_0^T\left\langle\int_Q|f'_{\hom}(y(t,x))-f'_L(\omega,y(t,x))|^{s^*}dx\right\rangle dt\to 0.$$

This concludes the proof. □

Discussion and outlook

Our intention in this thesis is to introduce the stochastic unfolding procedure that we view as a simple and easily accessible method for modeling and homogenization of random heterogeneous materials. We examine two types of evolutionary problems using this method. However, we believe that stochastic unfolding may be also advantageous for applications beyond the scope of this thesis. The framework that we develop mostly fits in the field of applied analysis, in particular we consider variational problems and our analysis relies on input from Γ-convergence. Yet, we believe that this method may be beneficial for a wider audience. In particular, the elementary tools that we use and the swift derivation of effective models using unfolding may be favorable in applied sciences, such as engineering and material science. Also, the operator-theoretic aspect of the unfolding procedure may be exploited in applied operator theory, e.g., in homogenization of abstract operator equations, that is the topic of our recent paper with Stefan Neukamm and Marcus Waurick [NVW19].

We present the unfolding strategy for the setting of problems that involve discrete-to-continuum transition as well as for continuum physical space problems. An interesting topic for future work is the extension of this method to problems that involve domains with another type of singular behavior, e.g., domains with holes or thin structures, which are already studied in the case of periodic unfolding (cf. [CDD+12, Neu10]). An even more exciting, yet challenging, question is the extension of the concept of unfolding to problems which involve random geometry, e.g., random lattices or randomly perforated structures.

In this work we study homogenization for discrete networks of random elasto-plastic springs in the rate-independent setting. Analogous results for the continuum version of linear elasto-plasticity may be obtained using the continuum unfolding procedure. Discrete lattice models and networks are often used as finite difference approximations for continuum models. On the other hand, they might be seen as a direct modeling approach for structures that exhibit a meso-scale discrete nature, e.g., truss-like structures and polymer networks. Modeling failure mechanisms (e.g., fracture, damage and delamination) in such materials is important and we believe that the unfolding procedure may be practical for homogenization of such problems. Also, material anisotropy plays a crucial role in such failure mechanisms and we believe that a discrete modeling viewpoint may be favorable in understanding such effects and even in optimal design and fabrication of materials using modern technologies such as 3D printing. However, in order to treat such problems, some difficulties must be resolved. In particular, models describing failure in materials are driven by nonconvex energies, e.g., the elastic component of the energy density in damage modeling is of the form $A_{\varepsilon}(z)\nabla_s^{\varepsilon}u\cdot\nabla_s^{\varepsilon}u$ where z is an internal damage variable. In the deterministic setting, oftentimes gradient regularizations for z are used to gain strong type compactness (cf. [Han14, HK17]), which we do not have at our disposal for our stochastic problems in the mean formulation. In this respect, we need to rely a priori only on weak compactness arguments. Also, in modeling damage an irreversibility constraint for z has to be implemented (cf. [AE18]), that is the topic of our current work with Goro Akagi and Stefan Neukamm [ANV].

We consider homogenization of an L^2-type gradient flow driven by a λ-convex energy functional. The results that we obtain include classical examples of Allen-Cahn type equations and evolution equations driven by the p-Laplace operator. A significant difference to the deterministic setting is the fact that the compact embedding of $L^2(\Omega)\otimes H^1(Q)$ into $L^2(\Omega)\otimes L^2(Q)$ does not hold. For this reason, we work with weak compactness arguments and in order to handle the nonconvexity of the energy we need to consider a suitable reformulation of the gradient flow in terms of a time-dependent but convex energy, where we strongly rely on λ-convexity. The treatment of genuinely nonconvex problems, e.g., energy densities of the form $W_{\varepsilon}(\nabla u)$ with W_{ε} nonconvex, via unfolding presents difficulties even in the periodic case. We remark that a possible critique for the consideration of the mean formulation for the problem we treat may be the fact that for the pointwise formulation (P-a.e. in ω), we deal with a solution $u(\omega)\in H^1(Q)$, where strong $L^2(Q)$-compactness is available, that allows an easier treatment of the limit passage. However, we point out that the weak convergence method that we present extends to systems where the energy functional features a scaled version of the gradient, e.g., $\varepsilon^{\gamma}\nabla u$. Such models are used to describe problems that involve different diffusion length-scales (cf. [MRT14]). In such problems even in the periodic case strong compactness is a priori not available. An interesting but challenging topic for future work is the understanding of sharp interface limits coupled with homogenization for Allen-Cahn type equations using the unfolding strategy.

For the obtained effective models, we present approximation schemes based on the representative volume element method. In particular, in the case of gradient plasticity and gradient flows, they are based on approximations of the effective coefficients and standard a priori estimates for the equations at hand. On the other hand, in the case of elasto-plasticity the effective system admits a genuine two-scale form and for this reason we develop an approximating system as well in a two-scale form in an extended space, which is motivated by the usual periodization technique. The proofs of the convergence statements also rely on the stochastic unfolding strategy and on von Neumann's ergodic theorem. The examination of convergence rates for such approximations is the topic of many studies in the last decade. In particular, for static linear and monotone problems the theory is well-established and it bases strongly on regularity properties of the corresponding solutions (cf. [GNO15, AKM17, Fis18]). In the case of evolutionary equations the theory is still developing. In particular, it may be an interesting question to examine convergence rates for the two-scale approximations that we propose for effective elasto-plasticity. However, it seems that the unfolding strategy alone is not sufficient for this task and we require the employment of other more involved techniques. Nonetheless, in our opinion, in applied sciences the quick and straightforward derivation of effective models, that the unfolding procedure does, is sometimes more appreciated than the determination of optimal convergence rates using mathematically-involved and lengthy techniques.

A Appendices

A.1 Basics from convex analysis

We briefly recall some notions of convex analysis that are important for our analysis. For detailed studies we refer to the standard literature [Roc15, ET99, CV06].

Let X be a Banach space and its dual space is denoted by X^*. We consider a function $f : X \to \mathbb{R} \cup \{\infty\}$. The effective domain of f, $\mathrm{dom}(f) \subset X$, is defined by $\mathrm{dom}(f) = \{x \in X : \ f(x) < \infty\}$. We say that f is *proper* if $\mathrm{dom}(f) \neq \emptyset$. The epigraph of f, $\mathrm{epi}(f) \subset X \times \mathbb{R}$, is defined by $\mathrm{epi}(f) = \{(x, \alpha) \in X \times \mathbb{R} : \ f(x) \leq \alpha\}$. We say that f is *convex* if

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y) \quad \text{for all } \theta \in [0,1], \text{ and all } x, y \in X.$$

The *convex subdifferential* of a convex function f is a multifunction $\partial f : X \to 2^{X^*}$ given by

$$\partial f(x) = \Big\{ \xi \in X^* : f(x) \leq f(y) + \langle \xi, x - y \rangle_{X^*, X} \quad \text{for all } y \in X \Big\}.$$

The domain of the subdifferential is given by $\mathrm{dom}(\partial f) = \{x \in X : \ \partial f(x) \neq \emptyset\}$.
For a proper function $f : X \to \mathbb{R} \cup \{\infty\}$, we define its *convex conjugate* (it is also known as *Legendre-Fenchel transformation*) by

$$f^* : X^* \to \mathbb{R} \cup \{\infty\}, \quad f^*(\xi) = \sup_{x \in X} \Big(\langle \xi, x \rangle_{X^*, X} - f(x) \Big).$$

We remark that f^* is convex and l.s.c., and its definition directly implies that

$$f(x) + f^*(\xi) \geq \langle \xi, x \rangle_{X^*, X} \quad \text{for all } x \in X, \xi \in X^*. \tag{A.1}$$

The above inequality is commonly referred to as *Fenchel-Young inequality.*

Lemma A.1 (Fenchel equivalence)**.** *Let $f : X \to \mathbb{R} \cup \{\infty\}$ be proper, l.s.c. and convex. Then, the following equivalence holds:*

$$\xi \in \partial f(x) \quad \Leftrightarrow \quad x \in \partial f^*(\xi) \quad \Leftrightarrow \quad f(x) + f^*(\xi) = \langle \xi, x \rangle_{X^*,X} .$$

Example A.2 (1-homogeneous functions)**.** *We say that $f : X \to \mathbb{R}\cup\{\infty\}$ is positively-homogeneous of degree 1 (positive-1-homogeneous) if $f(\alpha x) = \alpha f(x)$ for all $\alpha > 0,\ x \in X$ and $f(0) = 0$. If we assume that $f \geq 0$ and it is positive-1-homogeneous, it follows that the convex conjugate f^* is given by the indicator function of the convex and closed set $\partial f(0)$, i.e.,*

$$f^*(\xi) = I_{\partial f(0)}(\xi) := \begin{cases} 0 & \text{if } \xi \in \partial f(0), \\ \infty & \text{if } \xi \notin \partial f(0). \end{cases}$$

Moreover, if we additionally assume that f is convex and l.s.c., then Lemma A.1 and the form of f^* imply that

$$\xi \in \partial f(x) \quad \Leftrightarrow \quad f(x) = \langle \xi, x \rangle_{X^*,X} \quad \text{and} \quad \xi \in \partial f(0). \tag{A.2}$$

A.2 Normal integrands and integral functionals

In the following we recall some key facts about measurable integrands and conjugates of integral functionals. A detailed and more general theory can be found in [Roc71], see also the references therein.

Let (S, Σ, μ) be a complete measure space with a σ-finite measure μ and let X be a separable reflexive Banach space with dual space X^*. The product-σ-algebra of Σ and $\mathcal{B}(X)$ (Borel σ-algebra on X) is denoted by $\Sigma \otimes \mathcal{B}(X)$. In the following we refer to a function $f : S \times X \to \mathbb{R}\cup\{\infty\}$ as an *integrand*. For $s \in S$, we denote the function $x \mapsto f(s, x)$ by f_s.

Definition A.3 (Normal integrand)**.** *We say that an integrand f is normal if the following two conditions hold:*

(i) f is $\Sigma \otimes \mathcal{B}(X)$-measurable.

(ii) For each $s \in S$, the function f_s is proper and l.s.c.

If additionally, for each $s \in S$, f_s is convex, we say that f is a convex normal integrand.

Note that if f is a normal integrand and $x : S \to X$ is a $(\Sigma, \mathcal{B}(X))$-measurable function, then $s \mapsto f(s, x(s))$ defines a Σ-measurable mapping.

Remark A.4 (Carathéodory integrand)**.** *We call an integrand f Carathéodory if f is finite everywhere, $f(\cdot, x)$ is Σ-measurable for all $x \in X$, and $f(s, \cdot)$ is continuous for all $s \in S$. If an integrand is Carathéodory, then it is normal (for the proof see, e.g., [AB99, Lemma 4.51]).*

The following proposition provides a practical characterization for normality of integrands.

Proposition A.5 ([Roc71, Proposition 1])**.** *An integrand f is normal if and only if the following two conditions hold:*

(i) For each $s \in S$, $\mathrm{epi} f_s$ is closed and nonempty.

(ii) For any closed set $C \subset X \times \mathbb{R}$, it holds $\{s \in S : \operatorname{epi} f_s \cap C \neq \emptyset\} \in \Sigma$.

Let f be a normal integrand. We define $f^* : S \times X^* \to \mathbb{R} \cup \{\infty\}$ to be the convex conjugate of f in its second variable, i.e., $f^*(s,\xi) = f_s^*(\xi)$.

Proposition A.6 ([Roc71, Proposition 2])**.** *Let f be a normal integrand. If for each $s \in S$, f_s^* is proper (this is true if, e.g., $f \geq -c$ for some $c > 0$), then f^* is a convex normal integrand. If f is a convex normal integrand, then $(f^*)^* = f$.*

Let $p \in (1,\infty)$ and $q = \frac{p}{p-1}$ be its dual exponent of integrability. Since μ is σ-finite, we may identify $L^p(S;X)^*$ with $L^q(S;X^*)$ (see [Sho13, Theorem 1.5]). For a given normal integrand f, we define an integral functional $I_f : L^p(S;X) \to \mathbb{R} \cup \{\pm\infty\}$ by

$$I_f(x) = \int_S f(s, x(s)) d\mu(s),$$

if $s \mapsto f(s,x(s))$ is integrable and otherwise we set I_f to be $+\infty$. Analogously, we define $I_{f^*} : L^q(S;X^*) \to \mathbb{R} \cup \{\pm\infty\}$.
The following result is key to our analysis and its proof relies on Proposition A.5 and a measurable selection argument from [Rok49, KRN65, Cas67].

Proposition A.7 ([Roc71, Theorem 2])**.** *Let $p \in (1,\infty)$, $q = \frac{p}{p-1}$. Let f be a normal integrand. If there is an element $x \in L^p(S;X)$ such that $I_f(x) < \infty$, then for all $\xi \in L^q(S;X^*)$, it holds*

$$I_{f^*}(\xi) = \sup_{x \in L^p(S;X)} \Big(\langle \xi, x \rangle_{L^q(S;X^*),L^p(S;X)} - I_f(x) \Big). \tag{A.3}$$

Remark A.8 (Measurable selection)**.** *The above theorem implies a measurable selection principle for parametrized minimization problems. Namely, setting $\xi = 0$ above, we have*

$$\int_S \inf_{x \in X} f(s,x) d\mu(s) = \inf_{x \in L^p(S;X)} \int_S f(s, x(s)) d\mu(s).$$

In particular, if the minimum on the right-hand side is attained, the latter equality implies that there exists a $(\Sigma, \mathcal{B}(X))$-measurable function $x : S \to X$ such that $\inf_{x\in X} f(s,x) = f(s,x(s))$ μ-a.e.

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1. Die folgende Promotionsordnung in ihrer gültigen Fassung erkenne ich an:

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3. Folgende Personen haben die Promotion wissenschaftlich betreut und/oder mich bei der Auswahl und Auswertung des Materials sowie bei der Herstellung des Manuskripts unterstützt:

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