Technische Universität Dresden
Bereich Mathematik und Naturwissenschaften

Homogenization of Rapidly Oscillating Riemannian Manifolds

Dissertation
zur Erlangung des akademischen Grades

Doctor rerum natum
(Dr. rer. nat.)

vorgelegt dem
Bereich Mathematik und Naturwissenschaften
der Technischen Universität Dresden

von
Helmer Hoppe
(geboren am 12. August 1988 in Cottbus)

Gutachter:
Prof. Dr. Stefan Neukamm (Betreuer)
Prof. Dr. Anja Schlömerkemper

eingereicht am 9. Oktober 2020

Die Dissertation wurde von Januar 2015 bis Oktober 2020
im Institut für Wissenschaftliches Rechnen angefertigt.

This thesis is written with LaTeX.
Pictures of manifolds are computed with R *and then plotted with either* gnuplot *or* ParaView.

Abstract

In this thesis we study the asymptotic behavior of bi-Lipschitz diffeomorphic weighted Riemannian manifolds with techniques from the theory of homogenization. To do so we re-interpret the problem as different induced metrics on one reference manifold.

Our analysis is twofold. On the one hand we consider second-order uniformly elliptic operators on weighted Riemannian manifolds. They naturally emerge when studying spectral properties of the Laplace-Beltrami operator on families of manifolds with rapidly oscillating metrics. We appeal to the notion of H-convergence introduced by Murat and Tartar. In our first main result we establish an H-compactness result that applies to elliptic operators with measurable, uniformly elliptic coefficients on weighted Riemannian manifolds. We further discuss the special case of locally periodic coefficients and study the asymptotic spectral behavior of submanifolds of $\mathbb{R}^n$ with rapidly oscillating geometry.

On the other hand we study integral functionals featuring non-convex integrands with non-standard growth on $\mathbb{R}^n$ in a stochastic framework. Our second main result is a Γ-convergence statement under certain assumptions on the statistics of their integrands. Such functionals provide a tool to study the Dirichlet energy on non-uniformly bi-Lipschitz diffeomorphic manifolds. We show Mosco-convergence of the Dirichlet energy and deduce conditions for the spectral behavior of weighted Riemannian manifolds with locally oscillating random structure, especially in the case of submanifolds of $\mathbb{R}^n$.

Acknowledgments

First of all I like to express my deepest gratitude to my supervisor Prof. Stefan Neukamm for his guidance, support, motivation, and abundance of patience. He spent countless discussions with me full of questions, answers and ideas, and it is a great pleasure to work with him.

I am much obliged to Prof. Anja Schlömerkemper for kindly and expeditiously agreeing to review this thesis.

I would like to thank Prof. Jun Masamune for the many interesting and fruitful discussions during his stays in Dresden. I am also very grateful to JProf. Mathias Schäffner for taking his time to discuss and provide useful ideas and comments. Moreover, I want to thank Prof. Toshihiro Uemura for sharing his knowledge and his kind hospitality during my stay at the Kansai University Osaka.

I want to emphasize the support by the Deutsche Forschungsgemeinschaft (DFG) in the context of TU Dresden's Institutional Strategy "The Synergetic University".

Many thanks go to all the people of the Institute of Scientific Computing at the TU Dresden for warmly welcoming me in their midst, even though I am not the one for social activities and team events.

Besides, I am also beholden to the Institute of Analysis, in particular to Prof. Jürgen Voigt and Prof. Ralph Chill, who were a constant inspiration during my diploma studies and beyond, and are responsible for my affection for topology and functional analysis.

And I am deeply grateful to my family: both of my parents, Annette and Fritz, my sister Lydia, and the love of my live Lisa, for supporting me in every possible sense and taking me as I am. Wouldn't be me without you.

Helmer Hoppe, October 2020

Contents

Introduction

In this thesis we study the asymptotic behavior of Riemannian manifolds under different conditions, especially of fast oscillating surfaces, in terms of the Dirichlet energy and the spectrum of the Laplace-Beltrami operator.

The convergence of metric measure spaces in general, and Riemannian manifolds in particular, has attracted an enormous amount of attention and many different notions of convergence has been considered, focusing on different aspects of the geometry and topology of the spaces. Especially convergence of spectral structures has been intensely studied over the last years, and geometric conditions were established, see e.g. [Fuk87; KK94; KK96; KU97; KS03; LMV08; KS08; MV09; Mas11; CCK15; GMS15; Kas17].

Our point of view is different from these geometric examinations. We consider weighted Riemannian manifolds and associate the respective Laplace-Beltrami operators with elliptic differential operators on some reference manifold. This interpretation simplifies the problem of varying manifolds to the setting of elliptic operators with varying coefficients on one manifold, so we can avail ourselves of the techniques of the theory of homogenization. We choose two different approaches: H-convergence of uniformly elliptic coefficient fields, and Mosco-convergence of energy functionals.

On the one hand, we establish a compactness result that shows that any family of uniformly elliptic coefficient fields on a Riemannian manifold admits an H-convergent subsequence. The notion of H-convergence has been introduced in the context of homogenization of elliptic PDEs on $\mathbb{R}^n$ by Murat and Tartar in [MT97]. In our setting it reads, roughly speaking, that the solutions u_ε (and the fluxes $\mathbb{L}_\varepsilon \nabla u_\varepsilon$) of the elliptic second-order PDEs

$$-\operatorname{div}(\mathbb{L}_\varepsilon \nabla u_\varepsilon) = f$$

on a Riemannian manifold converge to the solution of a limiting PDE of the same form. We will see that, applied to the coefficient fields associated with the Laplace-Beltrami operators on uniformly bi-Lipschitz diffeomorphic families of weighted Riemannian manifolds, H-convergence implies convergence of the spectra of the Laplace-Beltrami operators to the spectrum of the Laplace-Beltrami operator on a limiting manifold (as well as Mosco-convergence of the Dirichlet energies).

In this context one should mention that Kuwae and Shioya in [KS03] established spectral convergence for families of manifolds, which are locally bi-Lipschitz diffeomorphic to a reference manifold with Lipschitz constants converging to 1. But for manifolds with rapidly oscillating structures, the diffeomorphisms between the manifolds may be indeed uniformly bi-Lipschitz, but usually not (locally) close to an isometry, so the approach of [KS03] does not apply in this setting. In contrast, our H-compactness result provides spectral convergence at least along subsequences.

In general, the limiting manifold depends on the extracted subsequence. However, under specific conditions on the geometric structure of the manifolds, the limit can be uniquely determined by appealing to suitable homogenization formulas. A natural geometric condition in the flat case is periodicity of the coefficient fields. We show how the notion of periodicity can be translated to coefficient fields on manifolds, and even to families of manifolds itself featuring special structures, by using local coordinate charts.

On the other hand, we present a Γ-convergence result for integral functionals with non-uniformly elliptic random potentials, providing an explicit formula for the limiting potential. While homogenization of uniformly elliptic integral functionals has been studied for long and is well understood (e.g. [Mar78; Gia83; MDM86; Mül87; BD07]), the case of integrands satisfying non-standard growth conditions is still purpose of recent research, see e.g. [MM94; Mar96; AM01; BD07; BF07; JP14], or [KG42; AB00; ACG11; Bis11; GLTV14; NSS17] for the discrete or discrete-to-continuum case. Therefore it is mentionable that our result, even though it is not required for the studies of Riemannian geometries via the Dirichlet energy, covers integral functionals of vector valued functions with non-convex integrands satisfying the growth condition

$$\lambda_{\min}(x)(\tfrac{1}{C}|F|^p - C) \leq W(x,F) \leq \lambda_{\max}(x)C(|F|^p + 1).$$

With Γ-convergence for non-uniformly elliptic integral functionals we gain a method to handle oscillating geometries of manifolds that are not uniformly bi-Lipschitz diffeomorphic, and therefore out of reach for our H-compactness result. Typical situations, where our method applies, are oscillating surfaces with random amplitude (which might yield unbounded volume), or periodic surfaces with random deformation parameters (which might yield unbounded curvature). We discuss examples of Euclidean submanifolds of both types.

Both approaches in particular allow us to treat Riemannian manifolds which oscillate rapidly on a small length scale (periodically or randomly). In several examples we demonstrate how our results can be used to prove Mosco- and spectral convergence and even present algorithms to find the limiting manifold. While our results are of mathematical interest in its own right, they might be also of interest for applications, especially to diffusion models in bio-mechanics. In this context, diffusion and reaction-diffusion processes in biological membranes and through interfaces are studied, see e.g. [AG82; JII87; Sba+06; NRJ07]. One observation made is that “diffusion in biological membranes can appear anisotropic even though it is molecularly isotropic in all observed instances" ([Sba+06]). In accordance with that our examples show that isotropic diffusion on surfaces with rapidly oscillating geometry can yield an-isotropic effective diffusion on large scales.

Outline

Part I. We give a short overview over the most important concepts, which we are going to consider, and their background. In *Chapter 1* we collect some notions of convergence for Riemannian manifolds. In particular, we introduce Hausdorff-, Gromov-Hausdorff, Mosco-, and spectral convergence and discuss how beneficial they are with respect to the asymptotical study of geometries. *Chapter 2* gives a short insight in the theory of both periodic and stochastic homogenization. This puts the later application of homogenization formulas in Section 3.4 into context, as well as provides a reference for the ergodic theorems frequently used in Chapter 5.

Part II. This part is twofold. In *Chapter 3* we deal with uniformly elliptic operators on a Riemannian manifold and present with the H-compactness statement Theorem 3.2.2 our first main result. In the symmetric case (e.g. for the Laplace-Beltrami operator) we therewith deduce Mosco-convergence of the associated energy forms, cf. Proposition 3.2.4, as well as convergence of the spectra of the associated second-order elliptic operators, cf. Propositions 3.2.6 and 3.2.7. In Section 3.4 we address the problem of identifying the limiting coefficient field. In particular, we provide a homogenization formula for manifolds that feature periodicity in local coordinates. We discuss two exemplary structures of periodic coefficient fields in Section 3.5. All proofs of the results in this chapter are presented in Section 3.6. In *Chapter 4* we discuss the application to families of parametrized manifolds that are uniformly bi-Lipschitz diffeomorphic. In particular, for such families, we establish Mosco- and spectral convergence (along subsequences) in Propositions 4.1.4 to 4.1.6 and discuss the special case of families of submanifolds of $\mathbb{R}^n$. In section 4.2 we discuss concrete examples. The proofs are contained in Section 4.3. Part II relies on basically [HMN19] written by Jun Masamune, Stefan Neukamm and the author, but also contains new and unpublished refinements and extensions of the results.

Part III. This part is again twofold. In *Chapter 5* we introduce the setting of rapidly oscillating random integral functionals, whose integrands satisfy a non-uniform elliptic growth condition. Our second main result of the thesis—the Γ-convergence result Theorem 5.2.2—is presented in Section 5.2, and comes together with a compact embedding statement Proposition 5.2.1. The main properties of the limiting potential are processed in Proposition 5.2.4. In Section 5.3 we state some technical lemmas, which are required to show the results. The proofs are collected in Section 5.4. *Chapter 6* is devoted to the application of the Γ-convergence result to families of Riemannian manifolds. Therefor we introduce the notion of manifolds with rapidly oscillating structures and discuss the differences to the uniformly bi-Lipschitz diffeomorphic case. We deduce Mosco-convergence of the associated Dirichlet energies, cf. Proposition 6.1.3, and under some conditions even convergence of the spectra of the Laplace-Beltrami operators, cf. Proposition 6.1.5. We reformulate our results for the special case of submanifolds of $\mathbb{R}^n$, see Corollary 6.1.6, and give some illustrative concrete examples in Section 6.2. The proofs can be found in Section 6.3. The results of Part III are new and yet unpublished.

Notation

- Sequences (x_ε) indexed with ε are usually considered to converge as $\varepsilon \searrow 0$, which should be understood as convergence of every subsequence $(x_{\varepsilon_j})_{j\in\mathbb{N}}$ with $\varepsilon_j \searrow 0$ as $j \to \infty$.

- We write $U' \Subset U$ if U' is an open set, whose closure $\overline{U'}$ is compact and $\overline{U'} \subset U$.

- We frequently use the notation $\fint_U f$ to denote the average $\frac{1}{|U|}\int_U f$, where $|U|$ denotes the Lebesgue measure of U, if not specified otherwise. In this context we often make use of the reference cell $Y := [0,1)^n \subseteq \mathbb{R}^n$.

- We consider weighted Riemannian manifolds (M, g, μ) with metric g and measure μ. We assume that M is n-dimensional (with $n \geq 2$), smooth, connected, without boundary, and that μ has a smooth positive density against the Riemannian volume associated with g. For the background of the analysis on manifolds, we refer to [Gri09; Jos11].

- For a diffeomorphism $h\colon M \to N$ between manifolds M and N we denote its differential by $dh\colon TM \to TN$. In the special case of a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ we denote its derivative by Df.

- $|\xi|_g(x) = \sqrt{g(\xi,\xi)(x)}$ induced norm in T_xM at $x \in M$. If the meaning is clear from the context, we simply write $|\xi|$.

- For a (sufficiently regular) function u and vector field ξ on U, the gradient of u is denoted by $\nabla_g u$ and the divergence of ξ is denoted by $\operatorname{div}_{g,\mu}\xi$, i.e. we have $g(\nabla_g u, \xi) = \xi u = du(\xi)$ and $\int_U g(\operatorname{div}_{g,\mu}\xi, u)\,\mathrm{d}\mu = -\int_U g(\xi, \nabla_g u)\,\mathrm{d}\mu$ provided either u or ξ are compactly supported. In particular, we write $-\triangle_{g,\mu} := -\operatorname{div}_{g,\mu}\nabla_g$ to denote the (weighted) Laplace-Beltrami operator. If the meaning is clear from the context, we shall simply write ∇, div and Δ.

- For $U \subset M$ open we denote by $L^2(U, g, \mu)$ the Hilbert space of square integrable functions and denote by
$$\|u\|^2_{L^2(U,g,\mu)} := \int_U |u|^2 \,\mathrm{d}\mu$$
the associated norm. We denote by $L^2(TU)$ the space of measurable sections ξ of TU such that $|\xi| \in L^2(U, g, \mu)$.

- We denote by $C_c^\infty(U)$ the space of smooth compactly supported functions, and by $W^{1,p}(U, g, \mu)$ the usual Sobolev space on (U, g, μ), i.e. the space of functions $u \in L^2(U, g, \mu)$ with distributional first derivatives in $L^2(U, g, \mu)$, equipped with the norm
$$\|u\|^2_{W^{1,p}(U,g,\mu)} := \int_M |u|^p + |\nabla u|^p \,\mathrm{d}\mu.$$
For $p = 2$, this is a Hilbert space, which we denote by $H^1(U, g, \mu) = W^{1,2}(U, g, \mu)$.

- We denote by $W_0^{1,p}(U, g, \mu)$ (or $H_0^1(U, g, \mu)$) the closure of $C_c^\infty(U)$ in $W^{1,p}(U, g, \mu)$ (or $H^1(U, g, \mu)$, resp.). We denote by $H^{-1}(U, g, \mu)$ the dual space to $H_0^1(U, g, \mu)$ and use the notation $\langle F, u\rangle_{(U,g,\mu)}$ to denote the dual pairing of $F \in H^{-1}(U, g, \mu)$ and $u \in H_0^1(M, g, \mu)$.

We tacitly simply write U (instead of (U, g, μ)), $L^2(U)$, $H^1(U)$, $\|\cdot\|_{L^2(U)}$, $\|\cdot\|_{H^1(U)}$, $\langle\cdot,\cdot\rangle$, if the meaning is clear from the context.

Part I.

Preliminaries

1. Convergence of Riemannian Manifolds

When talking about convergence of Riemannian manifolds, various notions of convergence have been introduced and developed, taking different aspects into account. In the following we give a brief survey of a few concepts. For a more detailed introduction see e.g. [KK94; KK96]

1.1. Hausdorff-Convergence

For two (embedded) n-dimensional submanifolds M and N of $\mathbb{R}^m$ we recall the *Hausdorff-distance*

$$d_H(M,N) := \max\big\{\sup_{x\in M}\inf_{y\in N}|x-y|, \sup_{y\in N}\inf_{x\in M}|x-y|\big\}.$$

A family (M_ε) of n-dimensional submanifolds of $\mathbb{R}^m$ is called to be Hausdorff-convergent to a manifold M_0, if $d_H(M_\varepsilon, M_0) \to 0$ as $\varepsilon \searrow 0$.

Hausdorff-convergence describes the asymptotic behavior of the extrinsic appearance of the manifolds, in the sense that from an external point of view the deviations from the limiting manifold vanish. But it gives no information about the intrinsic geometry of the manifolds, like for instance the length of geodesics or the volume form.

We illustrate this contrast with an elementary, one-dimensional example. For $\varepsilon = \frac{1}{k}$ with $k \in \mathbb{N}$ we consider the 1-dimensional submanifold $M_\varepsilon \subseteq \mathbb{R}^2$,

$$M_\varepsilon := \Big\{\begin{pmatrix} x \\ f_\varepsilon(x)\end{pmatrix}; x \in [0,L]\Big\}, \tag{1.1}$$

where $L \in \mathbb{N}$, $f_\varepsilon(x) := \varepsilon f(\frac{x}{\varepsilon})$ for some smooth, 1-periodic function f, which satisfies $f(0) = f(1) = 0$, but is not identically 0. Then the sequence (M_ε) Hausdorff-converges to the submanifold

$$M_0 := \Big\{\begin{pmatrix} s \\ 0\end{pmatrix}; s \in [0,L]\Big\}.$$

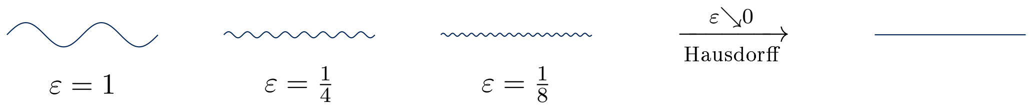


Figure 1.1.: A one-dimensional example. The three pictures on the left show M_ε defined by (1.1) with $f(y) = \frac{1}{2\pi}\sin(2\pi y)$ and $L = 2$ for decreasing values of ε. As $\varepsilon \to 0$ these manifolds Hausdorff-converge to the limiting manifold $M_0 = [0,2] \times \{0\}$, shown on the right. But the spectrum of the Laplace-Beltrami operator on M_ε converges to the spectrum of the Laplace-Beltrami operator on a submanifold $N_0 \subseteq \mathbb{R}^2$, see (1.4). Note that N_0 is (as M_0) a straight line, but its length is $2\rho_0 = \frac{1}{\pi}\int_0^{2\pi}\sqrt{1+\cos^2(y)}\,\mathrm{d}y$—the length of the oscillating curves M_ε which is strictly larger than 2—the length of M_0.

The sequence (M_ε) (for $f(y) = \frac{1}{2\pi}\sin(2\pi y)$ and $L = 2$) and the Hausdorff-limit M_0 are illustrated in Figure 1.1. On the other hand we note that the density of the volume form on M_0 is 1, while the density of the Riemannian volume form ρ_ε associated with M_ε by periodicity weakly-$*$ converges in $L^\infty((0,L))$:

$$\rho_\varepsilon = \sqrt{1+|f_\varepsilon'|^2} = \sqrt{1+|f'(\tfrac{\cdot}{\varepsilon})|^2} \overset{*}{\rightharpoonup} \int_0^1 \sqrt{1+|f'(y)|^2}\,\mathrm{d}y =: \rho_0,$$

and $\rho_0 > 1$, since $f \not\equiv 0$. Moreover, by periodicity (and the conditions on ε and L), the volume of M_ε (which here is just the one-dimensional Hausdorff-measure of M_ε) is independent of ε; more precisely, $\mathrm{vol}_1(M_\varepsilon) = \int_0^L \rho_\varepsilon\,\mathrm{d}y = \int_0^L \rho_0\,\mathrm{d}y = L\rho_0$. But the Hausdorff-limit M_0 has the volume $\mathrm{vol}_1(M_0) = L$. The latter is strictly smaller than the volume of M_ε and the loss of volume is due to the emergence of rapid oscillations in the limit $\varepsilon \searrow 0$.

As the example demonstrates, Hausdorff-convergence of an embedding is not the right choice to study the asymptotic behavior of the intrinsic geometry of manifolds. It is always connected to the embedding and the geometry of the ambient space.

1.2. Gromov-Hausdorff-Convergence

A resort of the problems of Hausdorff-convergence is offered by the Gromov-Hausdorff-distance d_{GH} of two Riemannian manifolds, which is the minimal Hausdorff-distance that can be achieved by any isometric embeddings into any metric space, i.e.

$$d_{GH}(M,N) := \inf\left\{d_H(\phi(M),\psi(N)); X \text{ metric space}, \phi\colon M \overset{\text{iso}}{\to} X,\ \psi\colon N \overset{\text{iso}}{\to} X\right\}.$$

Here "isometric embedding" (denoted by $\overset{\text{iso}}{\to}$) is to be understood in the global sense. To be precise, if d_M denotes the geodesic distance on M, an embedding $\phi\colon M \to X$ into a metric space (X, d) is called isometric, if $d(\phi(x), \phi(y)) = d_M(x, y)$ for every $x, y \in M$. Therefore, if we understand the manifolds M_ε in the example in Section 1.1 equipped with the geodesic distance induced from $\mathbb{R}^2$, they are not isometrically embedded.

The Gromov-Hausdorff distance provides a good measurement on how far two manifold are from being isometric, as it turns out to be a metric on the isometry classes of manifolds. But for a Gromov-Hausdorff-converging sequence of manifold, there is no guarantee that the limit is a manifold, too, and even if it is, it does not need to be of the same dimension. For example a sequence of 2-dimensional rectangles of the same width, but decreasing height can Gromov-Hausdorff-converge to a straight 1-dimensional line. On the other hand, Perelman's Stability Theorem tells that under certain assumptions on the curvature and the volume, almost all manifolds in a Gromov-Hausdorff-convergent sequence are homeomorphic to the limit.

We will not further concern the Gromov-Hausdorff-distance in the following, because there are other concepts being much more convenient to consider for homogenization, as we will see below.

1.3. Spectral Convergence

On a Riemannian manifold (M, g) the intrinsic geometry is strongly related to the heat equation $-\Delta u = \partial_t u$, where $-\Delta$ denotes the Laplace-Beltrami operator on M. The most obvious link is of course Varadhan's formula

$$d(x, y)^2 = \lim_{t\to\infty} -4t \log h(t, x, y),$$

which gives a connection between the heat kernel h on a manifold and the geodesic distance $d(x, y)$ between x and y.

The study of the heat kernel leads to the spectrum of the Laplace-Beltrami operator and the associated eigenfunctions. We call (λ, u) an eigenpair of the Laplace-Beltrami Δ operator on (M, g, μ), consisting of an eigenfunction $u \in H_0^1(M)$ and the corresponding eigenvalue $\lambda \in \mathbb{R}$, if $-\Delta u = \lambda u$ in $H^{-1}(M)$, i.e.

$$\int_M g(\nabla u, \nabla \psi)\, \mathrm{d}\mu = \lambda \int_M u\psi\, \mathrm{d}\mu \qquad \text{for all } \psi \in H_0^1(M).$$

It is well known, that for compact manifolds without boundary the spectrum of the Laplace-Beltrami operator consists only of a real, non-negative point spectrum. We denote by (λ_k) the sequence of increasingly ordered eigenvalues, where eigenvalues are repeated according to their multiplicity, and let (u_k) denote the sequence of associated eigenfunctions, forming a basis of $L^2(M)$. Then the heat kernel on M takes the form

$$h(t,x,y) = \sum_{k=1}^{\infty} e^{-\lambda_k t} u_k(x) u_k(y).$$

The spectrum of the Laplace-Beltrami operator (and the corresponding eigenfunctions) is therefore closely connected to the geodesic distance via the heat kernel.

For different manifolds, the eigenfunctions of the Laplace-Beltrami operator are defined on different spaces. To make them comparable, we introduce in the following the setting of bi-Lipschitz diffeomorphic manifolds and the notion of L^p-convergence for functions defined on the variable spaces.

Definition 1.3.1 (Bi-Lipschitz Diffeomorphic Families of Manifolds)**.** *A family of weighted Riemannian manifolds $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ is called* bi-Lipschitz diffeomorphic*, if there exits a weighted Riemannian manifold (M_0, g_0, μ_0) such that for all $\varepsilon > 0$ there are a diffeomorphism $h_\varepsilon \colon M_0 \to M_\varepsilon$ and a constant $C_\varepsilon > 0$ with*

$$\tfrac{1}{C_\varepsilon} |\xi|_{g_0} \le |dh_\varepsilon(x)\xi|_{g_\varepsilon} \le C_\varepsilon |\xi|_{g_0} \qquad \text{for all } x \in M_0 \text{ and } \xi \in T_x M_0.$$

We call (M_0, g_0, μ_0) reference manifold.

Definition 1.3.2 (Weak and Strong Convergence in L^p on Varying Spaces)**.** *Let $1 \le p < \infty$, and let $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ be a family of weighted Riemannian manifolds being bi-Lipschitz diffeomorphic to a reference manifolds (M_0, g_0, μ_0). For functions $f_\varepsilon \in L^p(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ and $f_0 \in L^p(M_0, g_0, \mu_0)$ we say (f_ε)* weakly converges *to f_0 in L^p, and use the notation*

$$f_\varepsilon \rightharpoonup f_0 \qquad \text{weakly in } L^p((M_\varepsilon, \mu_\varepsilon) \to (M_0, \mu_0)),$$

if

$$\int_{M_\varepsilon} f_\varepsilon (\psi \circ h_\varepsilon^{-1}) \, \mathrm{d}\mu_\varepsilon \to \int_{M_0} f_0 \psi \, \mathrm{d}\mu_0 \qquad \text{for all } \psi \in L^q(M_0, \mu_0), \tag{1.2}$$

where $q = \frac{p}{p-1}$ denotes the dual exponent to p (with $q = \infty$ for $p = 1$). We say (f_ε) strongly converges *to f_0 in L^p, and shortly write*

$$f_\varepsilon \to f_0 \qquad \text{strongly in } L^p((M_\varepsilon, \mu_\varepsilon) \to (M_0, \mu_0)),$$

if

$$\begin{aligned} &\int_{M_\varepsilon} f_\varepsilon (\psi \circ h_\varepsilon^{-1}) \, \mathrm{d}\mu_\varepsilon \to \int_{M_0} f_0 \psi \, \mathrm{d}\mu_0 \qquad \text{for all } \psi \in L^q(M_0, \mu_0), \qquad \text{and} \\ &\int_{M_\varepsilon} |f_\varepsilon|^p \, \mathrm{d}\mu_\varepsilon \to \int_{M_0} |f_0|^p \, \mathrm{d}\mu_0. \end{aligned} \tag{1.3}$$

Note that in the definition of strong L^p-convergence above, it is sufficient to assume the first condition to be satisfied for all $\psi \in C_c^\infty(M_0, g_0, \mu_0)$.

Remark 1.3.3. *If the manifolds M_ε and M_0 are n-dimensional submanifolds of $\mathbb{R}^m$, an alternative way to think about convergence of functions $u_\varepsilon \colon M_\varepsilon \to \mathbb{R}$ would be to extend the functions u_ε to functions on $\mathbb{R}^m$ by $u_\varepsilon = 0$ outside of M_ε, and consider weak convergence of the signed measures $u_\varepsilon \mathrm{d}\mathcal{H}^n$ on $\mathbb{R}^m$, where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff-measure on $\mathbb{R}^m$. However, this only yields a natural notion of convergence for functions defined on (embedded) submanifolds and is, in contrast to Definition 1.3.2, not considerable in the case of abstract manifolds.*

We are now able to formulate the following notion of spectral convergence:

Definition 1.3.4 (Spectral Convergence of Manifolds)**.** *Let $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ for $\varepsilon \geq 0$ be a family of compact, bi-Lipschitz diffeomorphic weighted Riemannian manifolds. Consider the Laplace-Beltrami operators*

$$-\Delta_{g_\varepsilon,\mu_\varepsilon} \colon H_0^1(M_\varepsilon, g_\varepsilon, \mu_\varepsilon) \to H^{-1}(M_\varepsilon, g_\varepsilon, \mu_\varepsilon),$$

and let

$$0 \leq \lambda_{\varepsilon,1} \leq \lambda_{\varepsilon,2} \leq \lambda_{\varepsilon,3} \leq \cdots$$

denote the list of increasingly ordered eigenvalues with eigenvalues being repeated according to their multiplicity. Let $u_{\varepsilon,1}, u_{\varepsilon,2}, u_{\varepsilon,3}, \ldots$ denote the associated eigenfunctions, forming an orthonormal basis of $L^2(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$. We say that the family (M_ε) spectral converges *to M_0 w.r.t. L^2, if for all $k \in \mathbb{N}$,*

$$\lambda_{\varepsilon,k} \to \lambda_{0,k},$$

and if $s \in \mathbb{N}$ is the multiplicity of $\lambda_{0,k}$, i.e.

$$\lambda_{0,k-1} < \lambda_{0,k} = \cdots = \lambda_{0,k+s-1} < \lambda_{0,k+s} \qquad \text{(with the convention } \lambda_{0,0} = -\infty\text{)},$$

there exists a sequence $(\tilde{u}_{\varepsilon,k})_\varepsilon$ of linear combinations of $u_{\varepsilon,k}, \ldots, u_{\varepsilon,k+s-1}$ such that

$$\tilde{u}_{\varepsilon,k} \to u_{0,k} \qquad \text{strongly in } L^2((M_\varepsilon, \mu_\varepsilon) \to (M_0, \mu_0)).$$

Regarding the example in Section 1.1, our results (see Corollary 4.1.7 and Remark 4.1.8) will show that the M_ε spectral converges (w.r.t. L^2) to a weighted Riemannian manifold, whose Riemannian volume form indeed has ρ_0 as the density against the Lebesgue measure. This weighted Riemannian manifold is isometrically isomorphic to a submanifold of $\mathbb{R}^2$, for example to

$$N_0 := \left\{ \begin{pmatrix} x \\ \sqrt{\rho_0^2 - 1}\, x \end{pmatrix} ; x \in [0, L] \right\}, \tag{1.4}$$

which is a straight line with the same volume as M_ε, i.e., $\mathrm{vol}_1(N_0) = \rho_0 L$. Note that N_0 is just one (of many) illustrative isometric embeddings of the limit manifold in $\mathbb{R}^2$.

1.4. Mosco-Convergence

A much weaker approach than spectral convergence is to consider the Dirichlet energy $\mathcal{E}\colon L^2(M) \to \mathbb{R} \cup \{+\infty\}$ on a weighted Riemannian manifold (M, g, μ), given by

$$\mathcal{E}(u) = \begin{cases} \int_M |\nabla u|^2 \,\mathrm{d}\mu, & \text{if } u \in H_0^1(M), \\ +\infty, & \text{otherwise.} \end{cases}$$

The Dirichlet energy is related to the intrinsic geometry of the manifold M in the sense that the minimizers of $\mathcal{E}$ are the solutions in $H_0^1(M)$ of the steady-state heat equation

$$-\Delta u = 0 \qquad \text{in } H^{-1}(M).$$

The asymptotic behavior of minimizers of a series of Dirichlet energies is captured by the notion of Mosco-convergence, which is a common tool to study the convergence properties of the evolution associated to energy forms, see e.g. [KS03; KU97; Kol08; Mas11; Löb15], and has also been generalized to the case of non-symmetric forms (see [Hin98]). We recall that a sequence of functionals $\mathcal{F}_\varepsilon\colon X \to \mathbb{R} \cup \{+\infty\}$ on a topological space X is called Γ-*convergent* to a functional $\mathcal{F}\colon X \to \mathbb{R} \cup \{+\infty\}$, if the following two conditions are satisfied:

(i) (Lower Bound) for every sequence (x_ε) in X with $x_\varepsilon \to x \in X$ we have

$$\liminf_{\varepsilon \searrow 0} \mathcal{F}_\varepsilon(x_\varepsilon) \geq \mathcal{F}(x),$$

(ii) (Recovery Sequence) for every $x \in X$ there is a sequence (x_ε) in X with

$$x_\varepsilon \to x \qquad \text{and} \qquad \mathcal{F}_\varepsilon(x_\varepsilon) \to \mathcal{F}(x) \qquad \text{as } \varepsilon \searrow 0.$$

The sequence $(\mathcal{F}_\varepsilon)$ is called *Mosco-convergent* to $\mathcal{F}$, if it is Γ-convergent w.r.t. the strong and the weak topology on X, i.e.

(i) (Lower Bound) for every sequence (x_ε) in X with $x_\varepsilon \rightharpoonup x \in X$ weakly in X we have

$$\liminf_{\varepsilon \searrow 0} \mathcal{F}_\varepsilon(x_\varepsilon) \geq \mathcal{F}(x),$$

(ii) (Recovery Sequence) for every $x \in X$ there is a sequence (x_ε) in X with

$$x_\varepsilon \to x \quad \text{strongly in } X \qquad \text{and} \qquad \mathcal{F}_\varepsilon(x_\varepsilon) \to \mathcal{F}(x) \qquad \text{as } \varepsilon \searrow 0.$$

A key feature of Γ- and Mosco-convergence, that can easily be seen from the definition, is that cluster points of minimizers are minimizers itself. Remember that $x \in X$ is a minimizer of the functional $\mathcal{F}$, if $\mathcal{F}(x) \leq \mathcal{F}(y)$ for every $y \in X$. An even stronger result holds in the case of *equi-coercive* functionals $\mathcal{F}_\varepsilon$, i.e. for every $t > 0$ the set $\bigcup_{\varepsilon>0}\{x \in X; \mathcal{F}_\varepsilon(x) \leq t\}$ is countably compact in X.

Lemma 1.4.1 (Convergence of Minimizers). *Let $(\mathcal{F}_\varepsilon)$ be a sequence of functionals on a topological space X Γ-converging to $\mathcal{F}$.*

(a) Let $x_\varepsilon \in X$ be a minimizer of $\mathcal{F}_\varepsilon$. If $x_\varepsilon \rightharpoonup x_0$ weakly in X, then x_0 is a minimizer of $\mathcal{F}$, and we have

$$\mathcal{F}_\varepsilon(x_\varepsilon) \to \mathcal{F}_0(x_0).$$

(b) If $(\mathcal{F}_\varepsilon)$ is equi-coercive, then $\mathcal{F}$ is coercive and

$$\lim_{\varepsilon \searrow 0} \inf_{x\in X} \mathcal{F}_\varepsilon(x) = \min_{x \in X} \mathcal{F}(x).$$

If additionally $\mathcal{F}$ has a unique minimizer $x_0 \in X$, then for every minimizer $x_\varepsilon \in X$ of $\mathcal{F}_\varepsilon$ we have $x_\varepsilon \to x_0$.

For a more comprehensive survey over Γ-convergence see e.g. [DM93].

Definition 1.4.2 (Mosco-Convergence of bi-Lipschitz Diffeomorphic Manifolds). *Let $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ be a family of weighted Riemannian manifolds being bi-Lipschitz diffeomorphic to the reference manifold (M_0, g_0, μ_0) via the diffeomorphisms $h_\varepsilon \colon M_0 \to M_\varepsilon$, and let (M, g, μ) be another weighted Riemannian manifold providing a diffeomorphism $h_0 \colon M_0 \to M$. Denote by $\mathcal{E}_\varepsilon \colon L^2(M_\varepsilon, g_\varepsilon, \mu_\varepsilon) \to \mathbb{R} \cup \{+\infty\}$ and $\mathcal{E}_0 \colon L^2(M, g, \mu) \to \mathbb{R} \cup \{+\infty\}$ the Dirichlet energy on M_ε and M, resp., given by*

$$\mathcal{E}_\varepsilon(u) = \begin{cases} \int_{M_\varepsilon} |\nabla_{g_\varepsilon} u|^2_{g_\varepsilon} \,\mathrm{d}\mu_\varepsilon, & \text{if } u \in H^1_0(M_\varepsilon, g_\varepsilon, \mu_\varepsilon), \\ +\infty, & \text{otherwise,} \end{cases}$$

and

$$\mathcal{E}_0(u) = \begin{cases} \int_{M} |\nabla_{g} u|^2_{g} \,\mathrm{d}\mu, & \text{if } u \in H^1_0(M, g, \mu), \\ +\infty, & \text{otherwise.} \end{cases}$$

We say that the family of manifolds (M_ε) Mosco-converges *to M w.r.t. L^2, if the pulled back Dirichlet energies $(\overline{\mathcal{E}}_\varepsilon)$ on M_0, defined by $\overline{\mathcal{E}}_\varepsilon(u) := \mathcal{E}_\varepsilon(u \circ h_\varepsilon)$, Mosco-converge to the pulled back Dirichlet energy $\overline{\mathcal{E}}_0$, given by $\overline{\mathcal{E}}_0(u) := \mathcal{E}_0(u \circ h_0)$ w.r.t. weak and strong $L^2(M_0, g_0, \mu_0)$-convergence.*

In practice, the limiting manifold (M, g, μ) in the Definition above will be the reference manifold M_0 equipped with a different metric and measure. In this setting, the diffeomorphism h_0 becomes the identity on M_0, but of course the integral functionals $\mathcal{E}_0$ and $\overline{\mathcal{E}}_0$ differ, due to the change of geometry.

An alternative way to define Mosco-convergence of manifolds would be to request the Dirichlet energies to Mosco-converge with respect to the weak and strong $L^2((M_\varepsilon, \mu_\varepsilon) \to (M, g))$-convergence from Definition 1.3.2. This is in general not equivalent, if we do not want to make any assumptions about convergence of the measures (μ_ε) to μ.

Mosco-convergence of the manifolds (M_ε) to M implies that if $u_\varepsilon \in H_0^1(M_\varepsilon)$ is a harmonic function on M_ε and $u_\varepsilon \circ h_\varepsilon \rightharpoonup u_0 \circ h_0$ weakly in $L^2(M_0, g_0, \mu_0)$, then $u_0 \in H_0^1(M, g, \mu)$ is a harmonic function on M. (This implication extends also to the Poisson equation with right-hand sides f_ε such that $f_\varepsilon \circ h_\varepsilon$ is strongly L^2-convergent, since Mosco-convergence is stable under continuously convergent perturbations.)

2. Homogenization

As we have seen in the previous chapter, both spectral convergence as well as Mosco-convergence of Riemannian manifolds rely on the study of the Laplace-Beltrami operators. For n-dimensional manifolds $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ being bi-Lipschitz diffeomorphic to a reference manifold (M_0, g_0, μ_0) in the sense of Definition 1.3.1 the Laplace-Beltrami operator on M_ε gives rise to a second-order elliptic operator $-\operatorname{div}(\mathbb{L}_\varepsilon \nabla)$ on M_0 with an elliptic coefficient field $\mathbb{L}_\varepsilon$, i.e.

$$g_0(\xi, \mathbb{L}_\varepsilon \xi) \geq \tfrac{1}{C_\varepsilon^{n+2}} |\xi|_{g_0}^2, \qquad g_0(\xi, \mathbb{L}_\varepsilon^{-1} \xi) \geq C_\varepsilon^{n+2} |\xi|_{g_0}^2 \qquad \text{for every } \xi \in TM_0, \tag{2.1}$$

see Section 4.1 for further details. It is therefore natural to consider homogenization of elliptic operators on the reference manifold with oscillating coefficients and measure.

In the following we collect some basic concepts of the theory of periodic and stochastic homogenization of elliptic PDEs in the flat Euclidean case, i.e. on $\mathbb{R}^n$. See the seminal works [PV79] or [Neu18] for a more detailed presentation of periodic and stochastic homogenization.

2.1. Periodic Homogenization

Consider a measurable coefficient field $a\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ being uniformly elliptic, i.e. there is a constant $C > 0$ such that for a.e. $x \in \mathbb{R}^n$

$$a(x)\xi \cdot \xi \geq \tfrac{1}{C}|\xi|^2 \qquad \text{and} \qquad |a(x)\xi| \leq C|\xi| \qquad \text{for every } \xi \in \mathbb{R}^n.$$

For every bounded open set $U \subseteq \mathbb{R}^n$ and every $f \in H^{-1}(U)$ the equation

$$-\operatorname{div}(a\nabla u) = f \qquad \text{in } H^{-1}(U) \tag{2.2}$$

provides a unique solution $u \in H_0^1(U)$. If the coefficient field a is Y periodic (with the periodicity cell $Y = [0,1)^n$), i.e. $a = a(\cdot + z)$ for every $z \in \mathbb{Z}^n$, it is a standard result (see e.g. [All92, Theorem 2.2]) that there is a constant matrix $a_{\mathrm{hom}} \in \mathbb{R}^{n\times n}$ such that the solutions $u_\varepsilon, u_0 \in H_0^1(U)$ of

$$-\operatorname{div}(a(\tfrac{\cdot}{\varepsilon})\nabla u_\varepsilon) = -\operatorname{div}(a_{\mathrm{hom}}\nabla u_0) = f \qquad \text{in } H^{-1}(U)$$

satisfy

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 & \text{weakly in } H^1(U), \\ a_\varepsilon \nabla u_\varepsilon \rightharpoonup a_{\mathrm{hom}} \nabla u_0 & \text{weakly in } L^2(U). \end{cases}$$

Moreover, the homogenized matrix a_{hom} is characterized by the homogenization formula

$$a_{\mathrm{hom}} e_j = \int_Y a(x)(\nabla \phi_j(x) + e_j)\,\mathrm{d}x, \tag{2.3}$$

where (e_j) is the standard basis in $\mathbb{R}^n$, and the periodic corrector $\phi_j \in H^1_{\mathrm{per}}(Y)$ denotes the unique solution to

$$\int_Y a(x)(\nabla \phi_j(x) + e_j) \cdot \nabla \psi(x)\,\mathrm{d}x = 0 \qquad \text{for all } \psi \in H^1_{\mathrm{per}}(Y), \tag{2.4}$$

where $H^1_{\mathrm{per}}(Y)$ denotes the Hilbert space of Y-periodic functions $\psi \in H^1(Y)$ with zero average, i.e. $\int_Y \psi = 0$.

This behavior is reflected by the definition of H-convergence, which goes back to the seminal work by Murat and Tartar ([MT97]), where the notion is introduced in the flat case $M = \mathbb{R}^n$. It is a generalization of the notion of G-convergence by Spagnolo and De Giorgi.

Definition 2.1.1 (H-Convergence). *We say a sequence (a_ε) of uniformly elliptic coefficient fields* H-converges *to a coefficient field a_{hom}, if for any bounded open set $U \subseteq \mathbb{R}^n$ and any $f \in H^{-1}(U)$ the unique solutions $u_\varepsilon, u_0 \in H^1_0(U)$ to*

$$-\operatorname{div}(a_\varepsilon \nabla u_\varepsilon) = -\operatorname{div}(a_{\mathrm{hom}} \nabla u_0) = f \qquad \textit{in } H^{-1}(U)$$

satisfy

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 & \textit{weakly in } H^1(U), \\ a_\varepsilon \nabla u_\varepsilon \rightharpoonup a_{\mathrm{hom}} \nabla u_0 & \textit{weakly in } L^2(U). \end{cases}$$

In this manner the result above can be summarized as follows:

Lemma 2.1.2 (Periodic Homogenization). *For every uniformly elliptic, Y-periodic coefficient field a the sequence $(a(\frac{\cdot}{\varepsilon}))$ H-converges to the matrix a_{hom} defined by* (2.3).

In [MT97] Murat and Tartar deduce an even stronger H-compactness result, stating that every sequence of (not necessarily periodic) uniformly elliptic coefficient fields provides an H-convergent subsequence. It has been extended to a large class of elliptic equations on $\mathbb{R}^n$ including e.g. linear elasticity [FM86] and monotone operators for vector valued fields ([FMT09]). See also [Wau18] for a variant that applies to non-local operators. Our main goal of part II will be to adapt this result to the case of uniformly elliptic operators on manifolds (Theorem 3.2.2).

The crucial point in the proof of H-compactness (as well as for the periodic result) is to pass to the limit in $\int_U \nabla u_\varepsilon \cdot a(\frac{\cdot}{\varepsilon})^\intercal \nabla\psi$ occurring in the weak formulation of the equation, as it contains the product of two weakly converging sequences. To fix this problem, there are basically two solutions in the literature: On the one hand the notion of two-scale convergence or periodic unfolding, which we will not concern any further, and on the other hand the method of oscillating test functions in combination with the so-called Div-Curl-Lemma ([MT97]):

Lemma 2.1.3 (Div-Curl Lemma). *Let $U \subseteq \mathbb{R}^n$ be open and let (ξ_ε) and (v_ε) be sequences in $L^2(U;\mathbb{R}^n)$ and $H^1(U)$, resp., such that*

$$v_\varepsilon \rightharpoonup v \quad \textit{weakly in } H^1(U) \qquad \textit{and} \qquad \begin{cases} \xi_\varepsilon \rightharpoonup \xi & \textit{weakly in } L^2(U), \\ \operatorname{div}\xi_\varepsilon \to \operatorname{div}\xi & \textit{strongly in } H^{-1}(U). \end{cases}$$

Then

$$\int_U (\xi_\varepsilon \cdot \nabla v_\varepsilon)\psi \to \int_U (\xi \cdot \nabla v)\psi \qquad \textit{for all } \psi \in C_c^\infty(U).$$

2.2. Stochastic Homogenization

The theory of stochastic homogenization deals with random coefficient fields in (2.2) instead of periodic ones. That means, while in periodic homogenization all the information about the coefficients lie in the periodicity cell, the behavior of random coefficients in the limit is only predictable up to a certain probability. In order to make these uncertainty handable, Papanicolaou and Varadhan introduced in [PV79] the following convenient abstract framework:

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$. Assume $\tau\colon \mathbb{R}^n \times \Omega \to \Omega$ to be a group action on Ω, i.e. $\tau_{x+y}\omega = \tau_x\tau_y\omega$ for all $x, y \in \mathbb{R}^n$, and $\omega \in \Omega$, such that the following properties are satisfied:

- (Stationarity) For every random variable $f \in \mathbb{L}^1(\Omega)$ and every $x \in \mathbb{R}^n$ we have

$$\mathbb{E}[f \circ \tau_x] = \mathbb{E}[f].$$

- (Ergodicity) If for $A \in \mathcal{A}$ holds $\tau_x A = A$ for every $x \in \mathbb{R}^n$, then $\mathbb{P}(A) \in \{0, 1\}$.

We call a measurable random coefficient field $a\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ *stationary*, if

$$a(\omega, x+y) = a(\tau_y\omega, x)$$

for every $x, y \in \mathbb{R}^n$ and every $\omega \in \Omega$.

Note that the case of periodic coefficients can be included in this setting by adding a random offset. Precisely, let $a_\#\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be a measurable, Y-periodic coefficient field and set $\Omega := \{\omega\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}; \omega(y) = a_\#(x+y), x \in Y\}$. If we equip Ω with the Borel-σ-algebra, the probability measure generated by uniform distribution of the offsets $x \in Y$, and the group action $\tau_x\omega := \omega(x+\cdot)$ for $x \in \mathbb{R}^n$, $\omega \in \Omega$, we gain a stationary ergodic probability space. Then the coefficient field defined by $a^\omega(x) := \tau_x\omega$ for $x \in \mathbb{R}^n$, $\omega \in \Omega$, is stationary. In particular, for every $\omega \in \Omega$ by construction there is $x \in Y$ such that $a^\omega = a_\#(x+\cdot)$.

For stationary random fields an analog result to the periodic result Lemma 2.1.2 is true (see e.g. [Neu18]).

Lemma 2.2.1 (Stochastic Homogenization)**.** *Let $(\Omega, \mathcal{A}, \mathbb{P}, \tau)$ be a stationary, ergodic probability space, and let a be a stationary random coefficient field. Then there is a deterministic matrix $a_{\text{hom}} \in \mathbb{R}^{n\times n}$ such that for a.e. $\omega \in \Omega$ the sequence $(a^\omega(\frac{\cdot}{\varepsilon}))$ H-converges to a_{hom}. In particular, a_{hom} is characterized by*

$$a_{\text{hom}} e_j = \mathbb{E}\Big[\int_Y a(x)(\nabla\phi_j(x) + e_j)\,\mathrm{d}x\Big],$$

where the stochastic corrector $\phi_j^\omega \in H^1_{\text{loc}}(\mathbb{R}^n)$ is the unique solution of

$$\int_{\mathbb{R}^n} (a^\omega \nabla\phi_j^\omega + e_j)\cdot\nabla\psi = 0 \qquad \text{for all } \psi \in C_c^\infty(\mathbb{R}^n)$$

with sublinear growth, i.e. $\limsup_{R\to\infty} \frac{1}{R^2} \fint_{RY} |\phi^\omega|^2 = 0$, and anchored in the sense that $\int_Y \phi^\omega = 0$.

A key element in the proof is the application of the famous Birkhoff's Ergodic Theorem ([DVJ08, Theorem 12.2.II]), which we recall in the following form for continuous processes:

Lemma 2.2.2 (Birkhoff's Individual Ergodicity Theorem)**.** *Let $(\Omega, \mathcal{A}, \mathbb{P}, \tau)$ be stationary and ergodic. There is a subset $\Omega' \subseteq \Omega$ of full measure such that for every random variable $f \in L^1(\Omega)$ and every open bounded set $A \subseteq \mathbb{R}^n$ we have*

$$\lim_{\varepsilon\searrow 0} \fint_A f(\tau_{\frac{x}{\varepsilon}}\omega)\,\mathrm{d}x = \mathbb{E}[f] \qquad \text{for every } \omega \in \Omega'.$$

In some sense more general is the Subadditive Ergodic Theorem introduced by Akcoglu and Krengel ([AK81]), which makes use of the notion of *stationary subadditive set functionals*. A stationary subadditive set functional is a measurable function $\mathcal{F}\colon \Omega \times \mathfrak{P}(\mathbb{R}^n) \to \mathbb{R}$ such that the following properties are satisfied:

- For every $x \in \mathbb{R}^n$, $A \subseteq \mathbb{R}^n$ and every $\omega \in \Omega$ we have $\mathcal{F}^\omega(x+A) = \mathcal{F}^{\tau_x\omega}(A)$.
- $\sup\left\{\frac{1}{|A|}\mathbb{E}[\mathcal{F}(A)]; A \subseteq \mathbb{R}^n, |A| > 0\right\} < \infty$.
- For every disjoint sets $A, B \subseteq \mathbb{R}^n$ and every $\omega \in \Omega$ we have
$$\mathcal{F}^\omega(A \cup B) \leq \mathcal{F}^\omega(A) + \mathcal{F}^\omega(B).$$

The Subadditive Ergodic Theorem then can be formulated as follows (cf. [AK81, Theorem 2.4, Lemma 3.4 and the remark after the proof of Theorem 2.4]):

Lemma 2.2.3 (Subadditive Ergodic Theorem)**.** *Let $(\Omega, \mathcal{A}, \mathbb{P}, \tau)$ be stationary and ergodic, and let $\mathcal{F}$ be a stationary subadditive set functional. Then there is a subset $\Omega' \subseteq \Omega$ of full measure such that for every cube $Q \subseteq \mathbb{R}^n$ whose vertices are in $\mathbb{Q}^n$ we have*
$$\lim_{\varepsilon \searrow 0} \frac{\mathcal{F}^\omega(\frac{1}{\varepsilon}Q)}{|\frac{1}{\varepsilon}Q|} = \lim_{\varepsilon \searrow 0} \frac{\mathbb{E}[\mathcal{F}(\frac{1}{\varepsilon}Y)]}{|\frac{1}{\varepsilon}Y|},$$
where $Y := [0,1)^d$.

Remark 2.2.4. *If the stationary subadditive set functional in Lemma 2.2.3 has almost surely bounded growth, in the sense that for $\mathbb{P}$-a.e $\omega \in \Omega$ there is a constant $C > 0$ with*
$$\mathcal{F}^\omega(A) \leq C|A|$$
for every compact $A \subseteq \mathbb{R}^n$, then the statement of Lemma 2.2.3 holds for all cubes in $\mathbb{R}^n$ (see for instance the arguments in the proof of Corollary 3.3 in [MM94]).

Part II.

Uniformly bi-Lipschitz Diffeomorphic Manifolds

3. Uniformly Elliptic Operators on a Riemannian Manifold

The intention of this thesis is to study the asymptotic behavior of sequences of bi-Lipschitz diffeomorphic manifolds in terms of the Laplace-Beltrami operator, cf. Chapter 1. In particular, we pull the Laplace-Beltrami operator $-\Delta_{g_\varepsilon,\mu_\varepsilon}$ on $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ back to the reference manifold M_0 by appealing to the diffeomorphism h_ε from Definition 1.3.1. In this way we obtain a family of elliptic operators of the form $-\operatorname{div}(\mathbb{L}_\varepsilon\nabla)$ on M_0 with coefficients $\mathbb{L}_\varepsilon$, see Section 4.1 for further details.

Our first approach is to adapt the method of oscillating test functions on $\mathbb{R}^n$ by Murat and Tartar ([MT97]) to Riemannian manifolds, in order to receive a H-compactness result (Theorem 3.2.2) in the case that the coefficient fields $\mathbb{L}_\varepsilon$ are uniformly elliptic (that is, the constant in (2.1) does not depend on ε). This setting corresponds to uniformal constants in Definition 1.3.1.

This chapter relies basically on [HMN19] by Jun Masamune, Stefan Neukamm and the author, but contains also a way to extend the results to manifolds that couldn't be considered in [HMN19] due to the considered boundary conditions, for example the torus, cf. Proposition 3.2.7 and Lemma 3.6.4.

3.1. Setting

On a weighted Riemannian manifold (M, g, μ) we study families of differential operators of the form

$$-\operatorname{div}(\mathbb{L}\nabla)\colon H_0^1(M) \to H^{-1}(M),$$

where $\mathbb{L}$ denotes a uniformly elliptic coefficient field on M. We make the setting precise with the following definition.

Definition 3.1.1 (Uniformly Elliptic Coefficient Fields). *Let (M, g, μ) be a weighted Riemannian manifold. For $0 < \lambda \leq \Lambda$ we denote by $\mathcal{M}(M, \lambda, \Lambda)$ the set of all measurable coefficient fields $\mathbb{L}\colon M \to \operatorname{Lin}(TM)$ that are* uniformly elliptic *and bounded in the* *sense that for μ-a.e. $x \in M$ and every $\xi \in T_xM$*

$$g(\xi, \mathbb{L}(x)\xi) \geq \lambda |\xi|_g^2, \tag{3.1}$$
$$g(\xi, \mathbb{L}(x)^{-1}\xi) \geq \tfrac{1}{\Lambda} |\xi|_g^2. \tag{3.2}$$

Remark 3.1.2. *The boundedness of a coefficient field $\mathbb{L} \in \mathcal{M}(M, \lambda, \Lambda)$ is a consequence of condition* (3.2)*, since this condition is equivalent to*

$$g(\eta, \mathbb{L}(x)\xi) \leq \Lambda' |\eta||\xi| \qquad \text{for } \mu\text{-a.e. } x \in M \text{ and every } \eta, \xi \in T_xM \tag{3.3}$$

for some constant $\Lambda' > 0$. The reason for the formulation (3.2) *is that the constant Λ is stable under H-convergence (in the sense that $\mathcal{M}(M, \lambda, \Lambda)$ is closed under H-convergence, as we will see), while the constant Λ' is not.*

In order to assure well-posedness of the considered differential equations, throughout this part we will denote for any open set $U \subseteq M$

$$m_0(U) := -\inf \left\{ \frac{\int_U g(\nabla u, \nabla u) \, \mathrm{d}\mu}{\int_U |u|^2 \, \mathrm{d}\mu}; u \in H_0^1(U), \|u\|_{L^2(U)} > 0 \right\} \leq 0. \tag{3.4}$$

This definition of $m_0(U)$ is chosen such that $m > m_0(U)$ if and only if

$$\inf \left\{ \int_U \big(m|u|^2 + g(\nabla u, \nabla u) \big) \, \mathrm{d}\mu; u \in H_0^1(U), \|u\|_{H_0^1(U)} = 1 \right\} > 0,$$

which immediately implies that for $m > \lambda m_0(U)$ the bounded, bilinear form

$$a \colon H_0^1(U) \times H_0^1(U) \to \mathbb{R}, \qquad a(u, v) := m \int_U uv \, \mathrm{d}\mu + \int_U g(\mathbb{L}\nabla u, \nabla v) \, \mathrm{d}\mu$$

is coercive. Therefore, the Lax-Milgram Lemma assures that for every $\mathbb{L} \in \mathcal{M}(U, \lambda, \Lambda)$, $m > \lambda m_0(U)$ and $f \in H^{-1}(U)$ the equation

$$m u_\varepsilon - \operatorname{div}(\mathbb{L}\nabla u) = f \qquad \text{in } H^{-1}(U), \tag{3.5}$$

admits a unique weak solution $u \in H_0^1(U)$ satisfying

$$\|u\|_{H^1(U)} \leq C \|f\|_{H^{-1}(U)} \tag{3.6}$$

for some constant $C > 0$ only depending on U, λ and m.

Remark 3.1.3 (Comments on the Constant $m_0(U)$)**.**
- *The definition of $m_0(U)$ gives a glimpse of the strong connection between* (3.5) *and the spectrum of the Laplace-Beltrami operator on U. In particular, since $\frac{\int_U g(\nabla u, \nabla u) \, \mathrm{d}\mu}{\int_U |u|^2 \, \mathrm{d}\mu}$ is the Rayleigh Quotient, $m_0(U)$ appears to be the negative of the infimum of the spectrum of the Laplace-Beltrami operator on U. If the spectrum is a pure point spectrum, $-m_0(U)$ is actually the lowest eigenvalue.*

- *If $U \Subset M$ is relatively-compact and connected, Poincaré's inequality holds for functions with zero mean, i.e. for all $u \in H^1(U)$*

$$\int_U \left| u - \fint_U u \,\mathrm{d}\mu \right|^2 \mathrm{d}\mu \leq C \int_U |\nabla u|^2 \,\mathrm{d}\mu$$

for some constant $C > 0$ only depending on U. In this case we have $m_0(U) \leq 0$, and in (3.5) any $m > 0$ is admissible. Moreover, if Poincaré's inequality holds for all functions with vanishing boundary conditions, i.e. for all $u \in H_0^1(U)$

$$\int_U |u|^2 \,\mathrm{d}\mu \leq C \int_U |\nabla u|^2 \,\mathrm{d}\mu \tag{3.7}$$

for some constant $C > 0$ only depending on U, then we even have $m_0(U) < 0$ and in (3.5) $m = 0$ is a valid choice.

3.2. Main Results

In the following we state our main results about H-convergence on a manifold. For the sake of readability, we postpone all proofs of this chapter to Section 3.6.

Before formulating our results, we need to translate the definition of H-convergence to the situation of Riemannian manifolds.

Definition 3.2.1 (H-Convergence on a Manifold)**.** *Let (M, g, μ) be a weighted Riemannian manifold and let $0 < \lambda \leq \Lambda$. We say a sequence $(\mathbb{L}_\varepsilon)$ in $\mathcal{M}(M, \lambda, \Lambda)$ H-converges in (M, g, μ) to $\mathbb{L}_0 \in \mathcal{M}(M, \lambda, \Lambda)$, if for any relatively-compact open subset $U \Subset M$ with $m_0(U) < 0$, and any $f \in H^{-1}(U)$, the unique solutions $u_\varepsilon, u_0 \in H_0^1(U)$ to*

$$-\operatorname{div}(\mathbb{L}_\varepsilon \nabla u_\varepsilon) = -\operatorname{div}(\mathbb{L}_0 \nabla u_0) = f \qquad \text{in } H^{-1}(U)$$

satisfy

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 & \text{weakly in } H^1(U), \\ \mathbb{L}_\varepsilon \nabla u_\varepsilon \rightharpoonup \mathbb{L}_0 \nabla u_0 & \text{weakly in } L^2(TU). \end{cases}$$

In that case we write $\mathbb{L}_\varepsilon \xrightarrow{H} \mathbb{L}_0$ in (M, g, μ).

Our main result extends the classical H-compactness result for uniformly elliptic coefficient fields on $\mathbb{R}^n$ in [MT97] to the setting on Riemannian manifolds in the sense of the definition above. In fact, we show a slightly more general version, which immediately implies H-compactness on the manifold.

Theorem 3.2.2. *Let (M, g, μ) be a weighted Riemannian manifold and let $0 < \lambda \leq \Lambda$. Then for every sequence $(\mathbb{L}_\varepsilon)$ in $\mathcal{M}(M, \lambda, \Lambda)$ there exist a subsequence (not relabeled) and $\mathbb{L}_0 \in \mathcal{M}(M, \lambda, \Lambda)$ such that the following holds: For every open subset $U \subseteq M$, $m > \lambda m_0(U)$, and sequences (f_ε) in $L^2(U)$ and (F_ε) in $L^2(TU)$ with*

$$\begin{cases} f_\varepsilon \rightharpoonup f_0 & \text{weakly in } L^2(U), \\ F_\varepsilon \to F_0 & \text{strongly in } L^2(TU), \end{cases}$$

the solutions $u_\varepsilon, u_0 \in H^1_0(U)$ to

$$\begin{aligned} mu_\varepsilon - \operatorname{div}(\mathbb{L}_\varepsilon \nabla u_\varepsilon) &= f_\varepsilon + \operatorname{div} F_\varepsilon \quad \text{in } H^{-1}(U), \\ mu_0 - \operatorname{div}(\mathbb{L}_0 \nabla u_0) &= f_0 + \operatorname{div} F_0 \quad \text{in } H^{-1}(U), \end{aligned} \tag{3.8}$$

satisfy

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 & \text{weakly in } H^1_0(U), \\ \mathbb{L}_\varepsilon \nabla u_\varepsilon \rightharpoonup \mathbb{L}_0 \nabla u_0 & \text{weakly in } L^2(TU). \end{cases}$$

Additionally we have $u_\varepsilon \to u_0$ strongly in $L^2(U)$, if either $H^1_0(U)$ is compactly embedded in $L^2(U)$, or $m \neq 0$ and $f_\varepsilon \to f_0$ strongly in $L^2(U)$.

If in Theorem 3.2.2 we choose $U \Subset M$ relatively-compact and open with $m_0(U) < 0$, we can take $m = 0$ and get H-convergence of $\mathbb{L}_\varepsilon$ to $\mathbb{L}_0$ in the sense of Definition 3.2.1 as a direct consequence (without any further proof):

Corollary 3.2.3. *Let (M, g, μ) be a weighted Riemannian manifold and let $0 < \lambda \leq \Lambda$. Then every sequence in $\mathcal{M}(M, \lambda, \Lambda)$ admits an in M H-convergent subsequence with limit in $\mathcal{M}(M, \lambda, \Lambda)$.*

The statements of Theorem 3.2.2 and Corollary 3.2.3—not the proofs—are actually equivalent in the following sense: On the one hand, we have just seen that H-convergence is a consequence of Theorem 3.2.2. On the other hand, every H-convergent sequence in $\mathcal{M}(M, \lambda, \Lambda)$ admits by Theorem 3.2.2 a subsequence satisfying the assertions of Theorem 3.2.2 with $\mathbb{L}_0$ being the H-limit (by uniqueness of the H-limit, see Proposition 3.3.3 below), thus the assertions are independent of the choice of the subsequence and hold for the entire sequence, cf. also Lemma 3.3.4 below.

Keeping in mind the application to the asymptotics of bi-Lipschitz diffeomorphic manifolds in Chapter 4, we additionally present some relations between H-convergence of coefficient fields and Mosco-convergence of the associated Dirichlet integrals, or convergence of the spectra of the associated operators, resp. We therefor restrict to the case of symmetric coefficient fields $\mathbb{L}_\varepsilon$, i.e.

$$g(\mathbb{L}_\varepsilon \xi, \eta) = g(\xi, \mathbb{L}_\varepsilon \eta) \qquad \text{for all } \xi, \eta \in TM.$$

Proposition 3.2.4 (H-Convergence Implies Mosco-Convergence). *Let (M, g, μ) be a weighted Riemannian manifold and let $0 < \lambda \le \Lambda$. For every sequence $(\mathbb{L}_\varepsilon)$ of symmetric coefficient fields in $\mathcal{M}(M, \lambda, \Lambda)$ the following holds: Suppose $\mathbb{L}_\varepsilon \xrightarrow{H} \mathbb{L}_0$ in M, then the functional $\mathcal{E}_\varepsilon \colon L^2(M) \to \mathbb{R} \cup \{+\infty\}$,*

$$\mathcal{E}_\varepsilon(u) = \begin{cases} \int_M g(\mathbb{L}_\varepsilon \nabla u, \nabla u)\,\mathrm{d}\mu, & \text{if } u \in H_0^1(M), \\ +\infty, & \text{otherwise} \end{cases}$$

Mosco-converges (w.r.t. L^2) to $\mathcal{E}_0 \colon L^2(M) \to \mathbb{R} \cup \{+\infty\}$,

$$\mathcal{E}_0(u) = \begin{cases} \int_M g(\mathbb{L}_0 \nabla u, \nabla u)\,\mathrm{d}\mu, & \text{if } u \in H_0^1(M), \\ +\infty, & \text{otherwise.} \end{cases}$$

Remark 3.2.5. *Mosco-convergence is the natural notion of convergence when handling equations like* (3.8) *with variational methods. For in the case of symmetric coefficients the unique solution of* (3.8) *is characterized as the unique minimizer to the strictly convex and coercive functional $\mathcal{F}_\varepsilon \colon L^2(U) \to \mathbb{R} \cup \{+\infty\}$, $\varepsilon \ge 0$, given by*

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \frac{1}{2}\Big(\mathcal{E}_\varepsilon(u) + m \int_U |u|^2\,\mathrm{d}\mu\Big) - \int_U \big(f_\varepsilon u + g(F_\varepsilon, \nabla u)\big)\,\mathrm{d}\mu, & \text{if } u \in H_0^1(U), \\ +\infty, & \text{otherwise,} \end{cases}$$

with $\mathcal{E}_\varepsilon$ as in Proposition 3.2.4. However, Mosco-convergence of the Dirichlet integrals is a bit weaker than H-convergence of the corresponding coefficient fields, since Mosco-convergence, in combination with Lemma 1.4.1, ensures strong convergence of the solutions u_ε in $L^2(M)$ (or equivalently weak convergence in $H^1(M)$, see e.g. [DM93, Theorem 13.12, cf. Example 13.13]), but gives no information about the fluxes $\mathbb{L}_\varepsilon \nabla u_\varepsilon$. (Though one could use the Div-Curl Lemma, see Lemma 3.3.2 below, to show convergence of the L^2-projection of $\mathbb{L}_\varepsilon \nabla u_\varepsilon$ onto the orthogonal complement of $\{\nabla\phi; \phi \in H_0^1(M)\} \subseteq L^2(TU)$, this would still be weaker than H-convergence.)

Finally, to formulate the consequences for the spectra of operators with H-convergent (symmetric) coefficients, we recall that (λ, u) is called an *eigenpair* of the operator

$$-\operatorname{div}(\mathbb{L}\nabla) \colon H_0^1(U) \to H^{-1}(U),$$

with eigenvalue $\lambda \in \mathbb{R}$ and eigenfunction $u \in H_0^1(U)$, if

$$\int_U g(\mathbb{L}\nabla u, \nabla\psi)\,\mathrm{d}\mu = \lambda \int_U u\psi\,\mathrm{d}\mu \qquad \text{for all } \psi \in H_0^1(U).$$

Proposition 3.2.6 (H-Convergence Implies Spectral Convergence). *Let (M, g, μ) be a weighted Riemannian manifold and $(\mathbb{L}_\varepsilon)$ be a sequence of symmetric coefficient fields in $\mathcal{M}(M, \lambda, \Lambda)$ for some $0 < \lambda \le \Lambda$, H-converging to some coefficient field $\mathbb{L}_0$ in M. Then for every relatively-compact open subset $U \Subset M$ with $m_0(U) < 0$ the following holds:*

(a) *For $\varepsilon \geq 0$ the spectrum of the operator*

$$-\operatorname{div}(\mathbb{L}_\varepsilon \nabla)\colon H_0^1(U) \to H^{-1}(U)$$

only consists of real, strictly positive eigenvalues, denoted in increasing order by

$$0 < \lambda_{\varepsilon,1} \leq \lambda_{\varepsilon,2} \leq \lambda_{\varepsilon,3} \leq \cdots,$$

where eigenvalues are repeated according to their multiplicity, and there is a sequence of associated eigenfunctions $u_{\varepsilon,1}, u_{\varepsilon,2}, u_{\varepsilon,3}, \ldots$ forming an orthonormal basis of $L^2(U)$.

(b) *For all $k \in \mathbb{N}$,*

$$\lambda_{\varepsilon,k} \to \lambda_{0,k} \qquad \text{as } \varepsilon \searrow 0,$$

and if $s \in \mathbb{N}$ denotes the multiplicity of $\lambda_{0,k}$, i.e.

$$\lambda_{0,k-1} < \lambda_{0,k} = \cdots = \lambda_{0,k+s-1} < \lambda_{0,k+s} \qquad (\text{with the convention } \lambda_{0,0} = 0),$$

then there exists a sequence $\tilde{u}_{\varepsilon,k}$ of linear combinations of $u_{\varepsilon,k}, \ldots, u_{\varepsilon,k+s-1}$ such that

$$\tilde{u}_{\varepsilon,k} \to u_{0,k} \qquad \text{strongly in } L^2(U) \text{ as } \varepsilon \searrow 0.$$

The above spectral convergence statement strongly relies on the assumption $m_0(M) < 0$, as this is a necessary condition for zero to be included in the resolvent set of the considered operator. However, in many cases this condition is not satisfied for the entire manifold. To demonstrate how our results might be extended to some of such situations, we consider the n-dimensional torus $\mathbb{T} := \mathbb{R}^n/\mathbb{Z}^n$. Since the spectrum of the operator

$$-\operatorname{div}(\mathbb{L}\nabla)\colon H_0^1(\mathbb{T}) \to H^{-1}(\mathbb{T}),$$

contains zero as an eigenvalue, it is more natural to consider the spectrum of the operator

$$-\operatorname{div}(\mathbb{L}\nabla)\colon H_{\mathrm{per}}^1(\mathbb{T}) \to H^{-1}(\mathbb{T})$$

with

$$H_{\mathrm{per}}^1(\mathbb{T}) := \Big\{ u \in H^1(\mathbb{T});\ \int_{\mathbb{T}} u \,\mathrm{d}\mu = 0 \Big\}.$$

Then an eigenpair (λ, u) of this operator consists of an eigenvalue $\lambda \in \mathbb{R}$ and an eigenfunction $u \in H_{\mathrm{per}}^1(\mathbb{T})$ such that

$$\int_{\mathbb{T}} (\mathbb{L}\nabla u) \cdot \nabla\psi \,\mathrm{d}\mu = \lambda \int_{\mathbb{T}} u\psi \,\mathrm{d}\mu \qquad \text{for all } \psi \in H^1(\mathbb{T}).$$

Proposition 3.2.7 (H-Convergence Implies Spectral Convergence on the Torus)**.** *Let $(\mathbb{T}, g, \mu)$ be the n-dimensional torus equipped with a Riemannian metric and a weighted measure, and let $(\mathbb{L}_\varepsilon)$ be a sequence of symmetric coefficient fields in $\mathcal{M}(\mathbb{T}, \lambda, \Lambda)$ for some $0 < \lambda \leq \Lambda$, H-converging to some coefficient field $\mathbb{L}_0$ in $\mathbb{T}$. Then the following holds:*

(a) *For $\varepsilon \geq 0$ the spectrum of the operator*

$$-\operatorname{div}(\mathbb{L}_\varepsilon \nabla) \colon H^1_{\mathrm{per}}(\mathbb{T}) \to H^{-1}(\mathbb{T})$$

only consists of real, strictly positive eigenvalues, denoted in increasing order by

$$0 < \lambda_{\varepsilon,1} \leq \lambda_{\varepsilon,2} \leq \lambda_{\varepsilon,3} \leq \cdots ,$$

where eigenvalues are repeated according to their multiplicity, and there is a sequence of associated eigenfunctions $u_{\varepsilon,1}, u_{\varepsilon,2}, u_{\varepsilon,3}, \dots$ forming an orthonormal basis of $L^2(\mathbb{T})$.

(b) *For all $k \in \mathbb{N}$,*

$$\lambda_{\varepsilon,k} \to \lambda_{0,k} \qquad \text{as } \varepsilon \searrow 0,$$

and if $s \in \mathbb{N}$ denotes the multiplicity of $\lambda_{0,k}$, i.e.

$$\lambda_{0,k-1} < \lambda_{0,k} = \cdots = \lambda_{0,k+s-1} < \lambda_{0,k+s} \qquad \text{(with the convention } \lambda_{0,0} = 0\text{)},$$

then there exists a sequence $\tilde{u}_{\varepsilon,k}$ of linear combinations of $u_{\varepsilon,k}, \dots, u_{\varepsilon,k+s-1}$ such that

$$\tilde{u}_{\varepsilon,k} \to u_{0,k} \qquad \text{strongly in } L^2(\mathbb{T}) \text{ as } \varepsilon \searrow 0.$$

3.3. Strategy of the Proof and Auxiliary Results

The proof of Theorem 3.2.2 adopts the method of oscillating test-functions by Murat and Tartar, cf. [MT97]. The main difference to the flat case $M = \mathbb{R}^n$ is that the tangent space T_xM changes when x varies in M. This issue can be handled by a localization argument, because in a small neighborhood the tangent space can be spanned by the gradients of finitely many smooth functions. More precisely, if $B \Subset M$ is an open ball with radius smaller than the injectivity radius of M at its center point, then there exist $v_1, \dots, v_n \in C_c^\infty(B)$ such that for all $x \in \frac{1}{2}B$

$$T_x(\tfrac{1}{2}B) = \operatorname{span}\{\nabla v_1(x), \dots, \nabla v_n(x)\}, \tag{3.9}$$

where $\frac{1}{2}B$ denotes the open ball with the same center, but half the radius of B. This allows us to show H-compactness restricted to small balls:

Lemma 3.3.1 (H-Compactness on Small Balls). *Let (M, g, μ) be a weighted Riemannian manifold, $B \Subset M$ be an open ball with radius smaller than the injectivity radius of M at its center, and let $0 < \lambda \leq \Lambda$. For every sequence $(\mathbb{L}_\varepsilon)$ in $\mathcal{M}(M, \lambda, \Lambda)$ there exists a (not relabeled) subsequence of $(\mathbb{L}_\varepsilon)$ that H-converges in $\frac{1}{2}B$ to some $\mathbb{L}_0 \in \mathcal{M}(\frac{1}{2}B, \lambda, \Lambda)$.*

Lemma 3.3.1 is the key moment of the proof of Theorem 3.2.2. It is shown in three steps: We first construct the tensor field $\mathbb{L}_0$ on the small ball $\frac{1}{2}B$, secondly identify it as an H-limit of the sequence $(\mathbb{L}_\varepsilon)$ on $\frac{1}{2}B$, and at last deduce its uniform ellipticity. For the definition of $\mathbb{L}_0$ we introduce the operators

$$\mathcal{L}_\varepsilon^*\colon H_0^1(B) \to H^{-1}(B), \qquad \mathcal{L}_\varepsilon^* u := -\operatorname{div}(\mathbb{L}_\varepsilon^* \nabla), \tag{3.10}$$

where $\mathbb{L}_\varepsilon^*$ denotes the adjoint of the coefficient field $\mathbb{L}_\varepsilon$ defined by

$$g(\mathbb{L}_\varepsilon^* \xi, \eta) = g(\xi, \mathbb{L}_\varepsilon \eta)$$

for all tangent vectors ξ, η. From the uniform ellipticity of the coefficient fields we can deduce by a classic functional analytic result (see Lemma 3.6.1 below) the existence of a linear operator $\mathcal{L}_0^*$, such that its inverse is the limit of $(\mathcal{L}_\varepsilon^*)^{-1}$ in the weak operator topology. Following Murat and Tartar it can be shown that the operator $\mathcal{L}_0^*$ is of the form $-\operatorname{div}(\mathbb{L}_0^*\nabla)$ for some tensor field $\mathbb{L}_0$, utilizing the oscillating test-functions $(\mathcal{L}_\varepsilon^*)^{-1}\mathcal{L}_0^* v_k$ associated with v_k from (3.9). These test-functions will allow to pass to the limit in products of two weakly convergent sequences occurring in the last two steps of the proof of Lemma 3.3.1 by appealing to the following variant of the Div-Curl Lemma for manifolds (cf. Lemma 2.1.3):

Lemma 3.3.2 (Div-Curl Lemma). *Let (M, g, μ) be a weighted Riemannian manifold, and consider sequences (v_ε) in $H^1(M)$, (ξ_ε) in $L^2(TM)$, such that*

$$v_\varepsilon \rightharpoonup v_0 \quad \textit{weakly in } H^1(M) \qquad \textit{and} \qquad \begin{cases} \xi_\varepsilon \rightharpoonup \xi & \textit{weakly in } L^2(TM), \\ \operatorname{div}\xi_\varepsilon \to \operatorname{div}\xi_0 & \textit{in } H^{-1}(M). \end{cases}$$

Then for all $\psi \in C_c^\infty(M)$ we have

$$\int_M g(\xi_\varepsilon, \nabla v_\varepsilon)\psi \,\mathrm{d}\mu \to \int_M g(\xi_0, \nabla v_0)\psi \,\mathrm{d}\mu.$$

Moreover, if $v_\varepsilon, v_0 \in H_0^1(M)$, then

$$\int_M g(\xi_\varepsilon, \nabla v_\varepsilon) \,\mathrm{d}\mu \to \int_M g(\xi_0, \nabla v_0) \,\mathrm{d}\mu.$$

Now in the last two steps, we show that $\mathbb{L}_0$, which is the adjoint of $\mathbb{L}_0^*$, is an H-limit of the sequence $(\mathbb{L}_\varepsilon)$, and that $\mathbb{L}_0 \in \mathcal{M}(\frac{1}{2}B, \lambda, \Lambda)$. This will be done by introducing on a relatively-compact open subset $U \Subset \frac{1}{2}B$ the localized operators

$$\mathcal{L}_\varepsilon\colon H_0^1(U) \to H^{-1}(U), \qquad \mathcal{L}_\varepsilon := -\operatorname{div}(\mathbb{L}_\varepsilon \nabla), \tag{3.11}$$

and showing that $\mathcal{L}_\varepsilon^{-1} \to \mathcal{L}_0^{-1}$ in the weak operator topology on U, which will finish the proof of Lemma 3.3.1.

In order to lift the H-compactness result Lemma 3.3.1 to the entire manifold we cover M by countably many small balls, obtain H-convergence simultaneously on each of these balls by a diagonal argument, and appeal to the following lemma to assure that the H-limits are unique and coincide in the intersections of the balls:

Proposition 3.3.3 (Uniqueness, Locality, Invariance w.r.t. Transposition)**.** *Let (M, g, μ) be a weighted Riemannian manifold, and let $0 < \lambda \leq \Lambda$.*

(a) Consider two sequences $(\mathbb{L}_\varepsilon)$ and $(\widetilde{\mathbb{L}}_\varepsilon)$ in $\mathcal{M}(M, \lambda, \Lambda)$ such that $\mathbb{L}_\varepsilon = \widetilde{\mathbb{L}}_\varepsilon$ for all $\varepsilon > 0$ in some relatively-compact open subset $U \Subset M$. Then from $\mathbb{L}_\varepsilon \overset{H}{\to} \mathbb{L}_0$ and $\widetilde{\mathbb{L}}_\varepsilon \overset{H}{\to} \widetilde{\mathbb{L}}_0$ in U follows $\mathbb{L}_0 = \widetilde{\mathbb{L}}_0$ μ-a.e. in U.

(b) For every sequence $(\mathbb{L}_\varepsilon)$ in $\mathcal{M}(M, g, \mu)$ holds

$$\mathbb{L}_\varepsilon \overset{H}{\to} \mathbb{L}_0 \quad \Leftrightarrow \quad \mathbb{L}_\varepsilon^* \overset{H}{\to} \mathbb{L}_0^* \qquad \text{in } M.$$

The proof of Theorem 3.2.2 will be concluded by the following lemma, allowing us to treat the varying right-hand sides in Theorem 3.2.2, which also automatically occur by stepping from the H-convergence on single balls to H-convergence on the whole manifold.

Lemma 3.3.4. *Let (M, g, μ) be a weighted Riemannian manifold, $U \Subset M$ be a relatively-compact open subset with $m_0(U) < 0$. Let further $(\mathbb{L}_\varepsilon)$ be a sequence in $\mathcal{M}(M, \lambda, \Lambda)$ for some $0 < \lambda \leq \Lambda$, H-converging in M to some $\mathbb{L}_0$. Then for every $f_\varepsilon, f_0 \in L^2(U)$ and $G_\varepsilon, F_\varepsilon, G_0, F_0 \in L^2(TU)$ with*

$$\begin{cases} f_\varepsilon \rightharpoonup f_0 & \text{weakly in } L^2(U), \\ G_\varepsilon \to G_0 & \text{strongly in } L^2(TU), \\ F_\varepsilon \to F_0 & \text{strongly in } L^2(TU), \end{cases}$$

the unique solutions $u_\varepsilon, u_0 \in H_0^1(U)$ to

$$\begin{aligned} -\operatorname{div}(\mathbb{L}_\varepsilon \nabla u_\varepsilon) &= f_\varepsilon - \operatorname{div}(\mathbb{L}_\varepsilon G_\varepsilon) - \operatorname{div} F_\varepsilon & \qquad \text{in } H^{-1}(U), \\ -\operatorname{div}(\mathbb{L}_0 \nabla u_0) &= f_0 - \operatorname{div}(\mathbb{L}_0 G_0) - \operatorname{div} F_\varepsilon & \qquad \text{in } H^{-1}(U) \end{aligned}$$

satisfy

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 & \text{weakly in } H_0^1(U), \\ \mathbb{L}_\varepsilon \nabla u_\varepsilon \rightharpoonup \mathbb{L}_0 \nabla u_0 & \text{weakly in } L^2(TU). \end{cases}$$

3.4. Identification of the Limit via Local Coordinate Charts

Theorem 3.2.2 states H-compactness for general uniformly elliptic coefficient fields, i.e. every family $(\mathbb{L}_\varepsilon)$ of uniformly elliptic coefficient fields contains a H-converging subsequence. As a pure existence statement it contains no information about the H-limit, separate from being uniformly elliptic. Even worse, the limiting coefficient field generally depends on the choice of the subsequence. Hence, if we are looking for H-convergence of the entire sequence $(\mathbb{L}_\varepsilon)$ and maybe even an explicit representation of the limiting coefficient field, we need stronger assumptions on $(\mathbb{L}_\varepsilon)$ than uniform ellipticity.

The theory of homogenization provides several classic results in the flat case $M = \mathbb{R}^n$ assuming the coefficients $\mathbb{L}_\varepsilon$ to have a special oscillating structure, cf. Chapter 2. For example if the coefficient fields are of the form $\mathbb{L}_\varepsilon(x) = \mathbb{L}(\frac{x}{\varepsilon})$ with a periodic coefficient field $\mathbb{L}$, i.e. $\mathbb{L}(\cdot + k) = \mathbb{L}(\cdot)$ a.e. in $\mathbb{R}^n$ for all $k \in \mathbb{Z}^n$, periodic homogenization provides a homogenization formula for the limiting coefficient field (see Lemma 2.1.2), which therefore is independent of the choice of the subsequence and implies H-convergence of the entire sequence. (Besides periodic coefficient fields one could also consider the framework of stochastic homogenization. But since this will be intensively discussed in Part III, we will only focus on the periodic case.)

These homogenization results cannot be directly transferred to manifolds, as it is not clear how to define periodicity of coefficient fields on a general manifold. In order to still benefit from the classic results on $\mathbb{R}^n$, we will express the coefficient fields in local coordinates. We therefor fix a local coordinate chart $(U, \Psi; x^1, x^2, \dots, x^n)$ of M and a relatively-compact open set $A \Subset \Psi(U) \subseteq \mathbb{R}^n$, and set $U' := \Psi^{-1}(A) \subseteq U$. We will suppress the chart Ψ when the meaning is clear from the context. In particular, for the representation of a function u on U in local coordinates we shall simply write u instead of $u \circ \Psi^{-1}$. Then to a coefficient field $\mathbb{L} \in \mathcal{M}(U', \lambda, \Lambda)$ we can associate a coefficient field $a\colon A \to \mathbb{R}^{n\times n}$ with the components

$$a_{ij} := \rho\, g(\mathbb{L}\nabla_g x^i, \nabla_g x^j) \quad \text{for } i,j = 1,\dots,n, \qquad \text{with} \quad \rho = \sigma\sqrt{\det g}, \tag{3.12}$$

where σ denotes the density of μ against the Riemannian volume measure. In this framework we find that the notions of uniform ellipticity and H-convergence simply translate from the manifold setting to the flat setting and vice versa, see Lemma 3.4.1 and Proposition 3.4.2 below.

Lemma 3.4.1. *Let (M, g, μ) be a weighted Riemannian manifold, and let $0 < \lambda \leq \Lambda$. For $\mathbb{L} \in \mathcal{M}(U', \lambda, \Lambda)$ consider $a\colon A \to \mathbb{R}^{n\times n}$ as defined in (3.12). Then there are $0 < \lambda' \leq \Lambda'$ (only depending on Ψ, A, λ and Λ) such that we have*

$$a\xi \cdot \xi \geq \lambda' |\xi|^2 \qquad \text{and} \qquad a^{-1}\xi \cdot \xi \geq \tfrac{1}{\Lambda'}|\xi|^2 \qquad \text{a.e. in } A,$$

for all $\xi \in \mathbb{R}^n$, where "$\cdot$" denotes the standard scalar product in $\mathbb{R}^n$.

Proposition 3.4.2. *Let (M, g, μ) be a weighted Riemannian manifold, and let $0 < \lambda \leq \Lambda$. For $\mathbb{L}_\varepsilon, \mathbb{L}_0 \in \mathcal{M}(U', \lambda, \Lambda)$ consider a_ε, a_0 as defined in* (3.12)*. Then the following assertions are equivalent:*

(i) $(\mathbb{L}_\varepsilon)$ *H-converges to $\mathbb{L}_0$ in (U', g, μ).*

(ii) (a_ε) *H-converges to a_0 in A equipped with the standard Euclidean metric and measure.*

For a_ε we can naturally consider periodic homogenization. We therefor denote by $Y := [0,1)^n$ the periodicity cell and by $H^1_{\text{per}}(Y)$ the space of Y-periodic functions $\phi \in H^1(Y)$ with zero mean, i.e. $\int_Y \phi = 0$. Moreover, we denote by $\mathcal{M}_{\text{per}}(\lambda, \Lambda)$ the class of Y-periodic coefficient fields $a\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ with ellipticity constants $0 < \lambda \leq \Lambda$, i.e.

$$a(\cdot, y) \text{ is continuous for a.e. } y \in \mathbb{R}^n, \tag{3.13}$$

$$a(x, \cdot) \text{ is measurable and } Y\text{-periodic for each } x \in \mathbb{R}^n, \tag{3.14}$$

$$a(x,y)\xi\cdot\xi \geq \lambda|\xi|^2 \text{ and } a(x,y)^{-1}\xi\cdot\xi \geq \tfrac{1}{\Lambda}|\xi|^2 \text{ for each } x \in \mathbb{R}^n, \text{ a.e. } y \in \mathbb{R}^n \text{ and all } \xi \in \mathbb{R}^n. \tag{3.15}$$

From a version of Lemma 2.1.2 (see e.g. [All92, Theorem 2.2]) we know that for $a \in \mathcal{M}_{\text{per}}(\lambda, \Lambda)$ the sequence (a_ε), given by $a_\varepsilon(x) := a(x, \frac{x}{\varepsilon})$, H-converges to the homogenized coefficient field a_{hom} which is characterized by an analog to (2.3). In the examples below (Section 3.5) we will see, that for instance in the natural situation of concentric coefficient fields the following variant of this result will be required, which includes an additional shift in the definition of a_ε. We refer to [HMN19] for a proof.

Lemma 3.4.3. *Let $0 < \lambda \leq \Lambda$, $r \in \mathbb{R}^n$. For $a \in \mathcal{M}_{\text{per}}(\lambda, \Lambda)$, the sequence (a_ε) with $a_\varepsilon(x) := a(x, \frac{x+r}{\varepsilon})$ H-converges on $\mathbb{R}^n$ to a_{hom} defined by*

$$a_{\text{hom}}(x)e_j = \int_Y a(x,y)(\nabla_y \phi_j(x,y) + e_j)\,\mathrm{d}y, \tag{3.16}$$

where (e_j) is the standard basis in $\mathbb{R}^n$, and $\phi_j(x, \cdot) \in H^1_{\text{per}}(Y)$ denotes the unique weak solution to

$$\int_Y a(x,y)(\nabla_y \phi_j(x,y) + e_j)\cdot\nabla_y\psi(y)\,\mathrm{d}y = 0 \qquad \textit{for all } \psi \in H^1_{\text{per}}(Y). \tag{3.17}$$

Now we finally can make the following observation, which is a direct consequence of Proposition 3.4.2 and Lemma 3.4.3, and requires no further proof.

Proposition 3.4.4 (Homogenization Formula). *Let (M, g, μ) be a weighted Riemannian manifold, $0 < \lambda \leq \Lambda$, and let Ψ, U, U' and A be as in the beginning of this section. Let further $\mathbb{L}_\varepsilon, \mathbb{L}_0 \in \mathcal{M}(M, \lambda, \Lambda)$ with $\mathbb{L}_\varepsilon \overset{H}{\to} \mathbb{L}_0$ on M, and suppose local periodicity in the sense that there exists a Y-periodic coefficient field $L\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that for some $r \in \mathbb{R}^n$*

$$g(\mathbb{L}_\varepsilon(x)\nabla_g x^i, \nabla_g x^j) = L_{ij}(\tfrac{r+x}{\varepsilon}) \qquad \text{for a.e. } x \in U.$$

Then $\mathbb{L}_0$ on U' in local coordinates takes the form

$$(a_{\text{hom}})_{ij} = \rho g(\mathbb{L}_0 \nabla_g x^i, \nabla_g x^j) \qquad \text{a.e. in } A,$$

where $a_{\text{hom}}\colon A \to \mathbb{R}^{n\times n}$ is defined by (3.16) with $a(x,y) := \rho(x)L(y)$.

3.5. Examples

As described in the previous section, we want to discuss two structural examples on how (local) periodicity of coefficient fields on manifolds can look like. In particular, we consider laminate-like coefficient fields and identify the H-limit by appealing to homogenization in the flat case via local charts. Note that the coefficient fields in the following examples are intrinsic objects and that the respective H-limit, even though it is studied and expressed in local coordinates, is not bound to charts.

3.5.1. Concentric Laminate-Like Coefficient Fields on Voronoi Tessellated Manifolds

In our first example we assume a Voronoi tessellation on a manifold and consider coefficient fields being rotationally symmetric on each cell w.r.t. the respective center, and depending periodically on the geodesic distance from the center; see Figure 3.1 for some exemplary illustrations of such structures. To make this precise, let (M, g, μ) be an n-dimensional manifold, and let $Z \subseteq M$ be a countable closed subset. For each $z \in Z$ we denote the associated Voronoi cell by M_z, i.e.

$$M_z := \{x \in M; d(x,z) < d(x, Z \setminus \{z\})\},$$

where $d(\cdot,\cdot)$ denotes the geodesic distance on M. Moreover, we assume the Voronoi tessellation to be fine enough such that for μ-a.e. $x_0 \in M$ there exist $z \in Z$ and $\varrho > 0$ such that

$$\text{for all } x \in B_\varrho(x_0) \subseteq M_z \text{ there is exactly one shortest path } \gamma_x \text{ from } x \text{ to } z, \tag{3.18}$$

where $B_\varrho(x_0)$ denotes the open geodesic ball with center x_0 and radius ϱ.

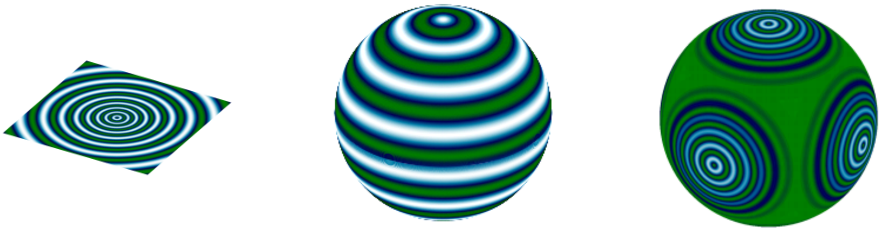


Figure 3.1.: Illustration of coefficient fields with laminate-like structure on Voronoi tesselated manifolds.

Let $0 < \lambda \leq \Lambda$, and consider a sequence $(\mathbb{L}_\varepsilon)$ in $\mathcal{M}(M, \lambda, \Lambda)$ of locally rapidly oscillating coefficient fields, in the sense that $\mathbb{L}_\varepsilon(x) = \mathbb{L}(\frac{d(x,Z)}{\varepsilon})$ for some 1-periodic field $\mathbb{L}$.

By Theorem 3.2.2 $(\mathbb{L}_\varepsilon)$ H-converges (up to a subsequence) to some $\mathbb{L}_0 \in \mathcal{M}(M, \lambda, \Lambda)$. In the following we claim that $\mathbb{L}_0$ coincides μ-a.e. on M with some constant coefficient field, uniquely determined by $\mathbb{L}$, so that the entire sequence $(\mathbb{L}_\varepsilon)$ H-converges to $\mathbb{L}_0$. We will show this by appealing to periodic the homogenization formula in local coordinates. To that end, we construct curvilinear coordinates such that in these coordinates the coefficients are locally close to a laminate. Precisely, we fix $z \in Z$, $x_0 \in M_z$ and construct a coordinate chart $(B_\varrho(x_0), \Psi; x^1, \ldots, x^n)$ such that

$$\begin{cases} \Psi(x_0) = 0, & (3.19)\\ x^1 = d(\cdot, z) - d(x_0, z), & (3.20)\\ g(\nabla_g x^1, \nabla_g x^j) = 0 \quad \text{for } j = 2, \ldots, n, & (3.21)\\ \lim\limits_{x \to x_0} \rho(x) g(\nabla_g x^i, \nabla_g x^j)(x) = \delta_{ij}, & (3.22) \end{cases}$$

where δ denotes the Kronecker symbol. By construction, such chart maps geodesics through z to straight lines parallel to the x^1-axis. In order to construct the chart, we find for the fixed $z \in Z$, $x_0 \in M_z$ a radius $\varrho > 0$ such that (3.18) is satisfied. By (3.20) the first coordinate function x^1 is already determined, namely

$$x^1(x) := d(x, z) - d(x_0, z)$$

for $x \in B_\varrho(x_0)$. Now (3.18) assures that x^1 is differentiable and the level set

$$A_{x_0} := \{x \in B_\rho(x_0); x^1(x) = 0\}$$

is an $n - 1$-dimensional submanifold of M_z including x_0. Moreover, for any point $x \in A_{x_0}$ the tangent space $T_x A_{x_0}$ is orthogonal to the direction of the geodesic $d\gamma_x(0)$, which yields (3.21). We can assume $\varrho > 0$ to be small enough, such that there are local normal coordinates $x^2, \ldots, x^n$ of the submanifold A_{x_0} with $x^j(x_0) = 0$ $(j = 2, \ldots, n)$. Then, by the differentiability of geodesics, these coordinates can be extended to curvilinear coordinates $x^1, \dots, x^n$ on $B_\varrho(x_0)$ (with a probably even smaller ϱ) such that $x^2, \dots, x^n$ are constant on every geodesic γ_x for $x \in B_\varrho(x_0)$, and we find

$$\lim_{x \to x_0} g(\nabla_g x^i, \nabla_g x^j)(x) = \delta_{ij},$$

which implies $\lim_{x \to x_0} \rho(x) = 1$ and therewith yields (3.22).

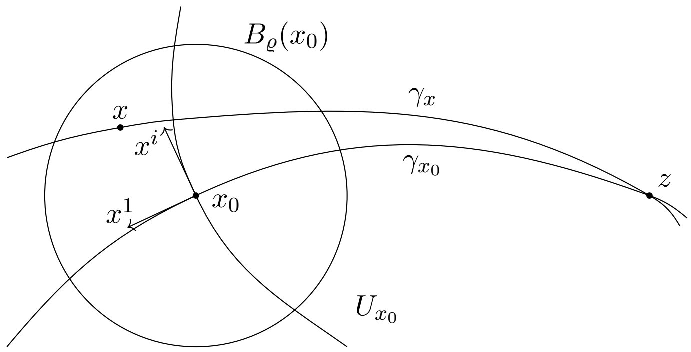


Figure 3.2.: Construction of the local coordinates

Now, using these coordinates the coefficient field on $A := \Psi(B_\varrho(x_0))$ associated with $\mathbb{L}_\varepsilon$ (via (3.12)) takes the form

$$a_\varepsilon(y) = a(y, \tfrac{d(x_0,z)+y_1}{\varepsilon}) \tag{3.23}$$

for some $a\colon A \times \mathbb{R} \to \mathbb{R}^{n\times n}$, which is continuous in the first, and measurable and 1-periodic in the second argument. For, by the concentric structure of $\mathbb{L}_\varepsilon$ and the definition of x^1 we have

$$g(\mathbb{L}_\varepsilon(x)\nabla_g x^i, \nabla_g x^j) = g(\mathbb{L}(\tfrac{d(x,Z)}{\varepsilon})\nabla_g x^i, \nabla_g x^j) = g(\mathbb{L}(\tfrac{d(x_0,Z)+x^1(x)}{\varepsilon})\nabla_g x^i, \nabla_g x^j),$$

and since $x^1(x) = y_1$, we find the desired form (3.23) of a_ε with

$$a_{ij}(y,r) := \rho(y)\, g(\mathbb{L}(r)\nabla_g x^i, \nabla_g x^j)(y), \tag{3.24}$$

where we conveniently write ρ and g in representation of the pushed forward quantities $\rho \circ \Psi^{-1}$ and $g \circ \Psi^{-1}$, respectively.

The homogenized matrix a_{hom} associated with (a_ε) is given by the homogenization formula (3.16) with a as in (3.24). Therefore it can be seen that a_{hom} depends continuously on $y \in A$, and the matrix $a_{\mathrm{hom}}(0)$ is independent of the initial choice of x_0. Moreover, $a_{\mathrm{hom}}(0)$ is explicitly given by weak-$*$ limits in $L^\infty(A)$, (cf. [MT97]):

- $\dfrac{1}{a_{11}(0, \frac{\cdot}{\varepsilon})} \rightharpoonup \dfrac{1}{(a_{\mathrm{hom}})_{11}(0)}$,

- $\dfrac{a_{i1}(0,\frac{\cdot}{\varepsilon})}{a_{11}(0,\frac{\cdot}{\varepsilon})} \rightharpoonup \dfrac{(a_{\mathrm{hom}})_{i1}(0)}{(a_{\mathrm{hom}})_{11}(0)}$ for $i \geq 2$,

- $\dfrac{a_{1j}(0,\frac{\cdot}{\varepsilon})}{a_{11}(0,\frac{\cdot}{\varepsilon})} \rightharpoonup \dfrac{(a_{\mathrm{hom}})_{1j}(0)}{(a_{\mathrm{hom}})_{11}(0)}$ for $j \geq 2$,

- $a_{ij}(0,\frac{\cdot}{\varepsilon}) - \dfrac{a_{i1}(0,\frac{\cdot}{\varepsilon})a_{1j}(0,\frac{\cdot}{\varepsilon})}{a_{11}(0,\frac{\cdot}{\varepsilon})} \rightharpoonup (a_{\mathrm{hom}})_{ij}(0) - \dfrac{(a_{\mathrm{hom}})_{i1}(0)(a_{\mathrm{hom}})_{1j}(0)}{(a_{\mathrm{hom}})_{11}(0)}$ for $i, j \geq 2$.

By Proposition 3.4.4 we have

$$(a_{\mathrm{hom}})_{ij} = \rho\, g(\mathbb{L}_0 \nabla_g x^i, \nabla_g x^j) \qquad \text{a.e. in } A,$$

and we can conclude that $\mathbb{L}_0$ is continuous (μ-a.e.) on $B_\varrho(x_0)$, and thus (3.22) implies

$$g(\mathbb{L}_0(x_0)\nabla_g x^i, \nabla_g x^j)(x_0) = (a_{\mathrm{hom}})_{ij}(0) \qquad \text{for } \mu\text{-a.e. } x_0 \in M.$$

In the special case of $\mathbb{L}$ being diagonal, i.e.

$$\mathbb{L}(r)\nabla_g x^i = \alpha_i(r)\nabla_g x^i$$

for $i = 1, \ldots, n$, the resulting limit $\mathbb{L}_0(x_0)$ is diagonal, too, and simplifies to

$$\begin{aligned} g(\mathbb{L}_0(x_0)\nabla_g x^1, \nabla_g x^1)(x_0) &= \Big(\int_0^1 \tfrac{1}{\alpha_1}\Big)^{-1}, \\ g(\mathbb{L}_0(x_0)\nabla_g x^i, \nabla_g x^i)(x_0) &= \int_0^1 \alpha_i \qquad \text{for } i = 2, \ldots, n. \end{aligned} \tag{3.25}$$

3.5.2. Laminate-Like Coefficient Fields on Spherically Symmetric Manifolds

Another way to define periodic coefficient fields on a manifolds is to assume the manifold to be rotationally symmetric and may therefore be parametrized by the 1-dimensional sphere $\mathbb{S}^1$, and consider coefficient fields being periodic in this parameter, see Figure 3.3. In particular, we fix $0 < R \leq \infty$ and a function $s \in C^\infty([0,R))$ with $s(0) = 0$, $s'(0) = 1$, and $s(r) > 0$ for $r > 0$, and consider the 2-dimensional spherically symmetric manifold

$$M := \{(r,\theta) \in [0,R) \times \mathbb{S}^1\}$$

equipped with the Riemannian metric

$$g = dr^2 + s^2(r)d\theta^2$$

Figure 3.3.: Illustration of the laminate-like structure of the coefficient field on $\mathbb{R}^2$, $\mathbb{S}^2$ and $\mathbb{H}^2$.

in the polar coordinates (r, θ), as described e.g. in [Gri09]. This model covers examples like the plane $\mathbb{R}^2$ (with $R = \infty$ and $s(r) = r$), the 2-dimensional sphere $\mathbb{S}^2$ without pole (with $R = \pi$ and $s(r) = \sin r$), or the hyperbolic plane $\mathbb{H}^2$ (with $R = \infty$ and $s(r) = \sinh r$). For the sake of simplicity we normalize $\mathbb{S}^1$ to have circumference 1.

The coefficient fields $\mathbb{L}_\varepsilon \in \mathcal{M}(M, \lambda, \Lambda)$ we want to consider shall be of the form

$$\mathbb{L}_\varepsilon(r, \theta) = \mathbb{L}_\#\big(r, \theta, \tfrac{\theta}{\varepsilon}\big) \qquad \text{a.e. in } M$$

for some coefficient field $\mathbb{L}_\#$ being continuous in the first two arguments, and measurable and 1-periodic in the third. This means, if $\{\phi(t); t \in \mathbb{R}\}$ denotes the one-parameter group

$$\phi(t)\colon x \mapsto \exp_x(t\nabla_g \theta)$$

for $x \in M \setminus (\{0\} \times \mathbb{S}^1)$, then the coefficient field $\mathbb{L}_\varepsilon$ is rapidly oscillating along ϕ, while it still might macroscopically (continuously) depends on the radius r. Therefore $\mathbb{L}_\varepsilon$ can be called a laminate-like coefficient field on M.

By Theorem 3.2.2 the coefficient fields $(\mathbb{L}_\varepsilon)$ H-converge in M to some coefficient field $\mathbb{L}_0$ along a subsequence. As in the previous example, $\mathbb{L}_0$ is uniquely determined by a homogenization formula, which implies H-convergence to $\mathbb{L}_0$ for the entire sequence. To see this it is sufficient to identify $\mathbb{L}_0$ locally. We therefor fix a bounded open set $U \Subset M$, and, due to the symmetry of M, it is no restriction to assume that its closure $\overline{U}$ does not intersect the curve $\{(r, \theta); \theta = 0\}$. If we denote by Ψ the chart of polar coordinates and set $A := \Psi(U) \subseteq \mathbb{R}^2$, (3.12) provides a coefficient field a_ε on A associated to $\mathbb{L}_\varepsilon$, which has the form $a_\varepsilon(r, \theta) = a_\#(r, \theta, \frac{\theta}{\varepsilon})$, for some function $a_\varepsilon\colon [0, R) \times \mathbb{S}^1 \times \mathbb{R} \to \mathbb{R}^{2\times 2}$, given by

$$a_\#(r, \theta, y) = \begin{pmatrix} s(r) & 0 \\ 0 & s^{-1}(r) \end{pmatrix} \mathbb{L}_\#(r, \theta, y),$$

where we conveniently write $\mathbb{L}_\#(r, \theta, y)$ for the corresponding coefficient matrix in polar coordinates, i.e. $(\mathbb{L}_\#)_{ij} := g(\mathbb{L}_\# \nabla_g x^i, \nabla_g x^j)$ where $(x^1, x^2) = (r, \theta)$. Form the H-convergence of $(\mathbb{L}_\varepsilon)$ to $\mathbb{L}_0$ in U we follow with Proposition 3.4.2 that a_ε H-converges in A to a_0, which is the coefficient field associated with $\mathbb{L}_0$ via (3.12). But due to the special form of a_ε, and since $a_\#$ is continuous in the first two arguments and 1-periodic in the third, we can apply the periodic homogenization formula (3.16) and conclude that a_0 only depends on $\mathbb{L}_\#$ and the metric g, but not on the choice of the subsequence, and hence the same holds for $\mathbb{L}_0$.

As in the previous example we finally consider the special case of diagonal coefficient fields, i.e.

$$\mathbb{L}_\#(r,\theta,y) = \begin{pmatrix} \alpha_\#(y) & 0 \\ 0 & \beta_\#(y) \end{pmatrix}$$

for some $\alpha_\#, \beta_\# \colon \mathbb{R} \to (\lambda, \Lambda)$ being measurable and 1-periodic, which as above is meant to be understood as the representation of $\mathbb{L}_\#$ in polar coordinates. In this case we obtain

$$a_\#(r,\theta,y) = \begin{pmatrix} s(r)\alpha_\#\left(\frac{\theta}{\varepsilon}\right) & 0 \\ 0 & s^{-1}(r)\beta_\#\left(\frac{\theta}{\varepsilon}\right) \end{pmatrix},$$

and application of (3.16) yields

$$a_0(r,\theta) = \begin{pmatrix} s(r)\int_0^1 \alpha_\# & 0 \\ 0 & s^{-1}(r)\left(\int_0^1 \frac{1}{\beta_\#}\right)^{-1} \end{pmatrix}.$$

Thus $\mathbb{L}_0$ is diagonal, too, and explicitly given by

$$\mathbb{L}_0 = \begin{pmatrix} \int_0^1 \alpha_\# & 0 \\ 0 & \left(\int_0^1 \frac{1}{\beta_\#}\right)^{-1} \end{pmatrix}. \tag{3.26}$$

Note that the arithmetic and harmonic mean of $\alpha_\#$ and $\beta_\#$ express the diffusivity orthogonal and aligned to the flow ϕ, respectively.

One should mention that, besides the torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{S}^1 \times \mathbb{S}^1$ does not fit in this model, the same calculations can be done—in both of the spherical parameters.

3.6. Proofs

In the proofs we will pass to various subsequences and it will be necessary to keep track of them. In order to gain a readable notation we will denote by $E \subseteq (0,\infty)$ the index set the original sequence $(\mathbb{L}_\varepsilon) = (\mathbb{L}_\varepsilon)_{\varepsilon\in E}$, and represent subsequences by subsets $E_1, E_2, \ldots \subseteq E$ with a cluster point at zero. We will simply write

$$c_\varepsilon \to c_0 \qquad (\varepsilon \in E_1),$$

and mean that for any sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subseteq E_1$ with $\varepsilon_j \searrow 0$ we have $c_{\varepsilon_j} \to c_0$ as $j \to \infty$.

3.6.1. H-Compactness on Small Balls (Lemma 3.3.1) and the Div-Curl Lemma (Lemma 3.3.2)

At first we prove the manifold version of the Div-Curl Lemma (Lemma 3.3.2), which plays a central role in almost all the proofs of this chapter.

Proof of Lemma 3.3.2. For $\psi \in C_c^\infty(M)$ we can write

$$\int_M g(\xi_\varepsilon, \nabla v_\varepsilon)\psi \,\mathrm{d}\mu = \int_M g(\xi_\varepsilon, \nabla(v_\varepsilon\psi))\,\mathrm{d}\mu - \int_M g(\xi_\varepsilon, v_\varepsilon \nabla\psi)\,\mathrm{d}\mu. \tag{3.27}$$

Regarding the first integral of the right-hand side we use the strong convergence of the divergence of (ξ_ε) and the weak convergence of (v_ε) to see

$$\int_M g(\xi_\varepsilon, \nabla(v_\varepsilon\psi))\,\mathrm{d}\mu \to \int_M g(\xi_0, \nabla(v_0\psi))\,\mathrm{d}\mu = \int_M g(\xi_0, v_0\nabla\psi)\,\mathrm{d}\mu + \int_M (\xi_0, \psi\nabla v_0)\,\mathrm{d}\mu.$$

For the second integral of the right-hand side of (3.27) we note that by Rellich's Theorem on the compact set $U := \operatorname{supp}\psi$ we have strong convergence of v_ε to v_0 in $L^2(U)$, which implies $v_\varepsilon\nabla\psi \to v_0\nabla\psi$ strongly in $L^2(TM)$ and thus

$$\int_M g(\xi_\varepsilon, v_\varepsilon\nabla\psi)\,\mathrm{d}\mu \to \int_M (\xi_0, v_0\nabla\psi)\,\mathrm{d}\mu.$$

In the case $v_\varepsilon, v_0 \in H_0^1(M)$ the statement follows directly with an integration by parts argument. $\square$

Before proving Lemma 3.3.1 we recall the following standard functional analytic result, see e.g. [MT97, Proposition 4]:

Lemma 3.6.1. *Let V be a reflexive separable Banach space and (T_ε) be a sequence of linear operators $T_\varepsilon\colon V \to V'$ that is uniformly bounded and coercive, i.e. there exists $C > 0$ (independent of ε) such that the operator norm of T_ε is bounded by C and*

$$\langle T_\varepsilon v, v\rangle_{V',V} \geq \tfrac{1}{C}\|v\|_V^2 \qquad \textit{for all } v \in V. \tag{3.28}$$

Then there exists a linear bounded operator $T_0\colon V \to V'$ satisfying (3.28) and for a subsequence (not relabeled) we have $T_\varepsilon^{-1} \rightharpoonup T_0^{-1}$ in the weak operator topology, that is for all $f \in V'$ we have

$$T_\varepsilon^{-1} f \rightharpoonup T_0^{-1} f \qquad \textit{weakly in } V.$$

Proof of Lemma 3.3.1.
Step 1: Choice of the subsequence and definition of $\mathbb{L}_0$.
As described in Section 3.3, we consider the operators $\mathcal{L}_\varepsilon^*$ defined in (3.10) and fix

functions $v_1,\dots,v_n \in C_c^\infty(B)$ as in (3.9). We claim the existence of a coefficient field $\mathbb{L}_0$ on $\frac{1}{2}B$ and sequences $(v_{1,\varepsilon}),\dots,(v_{k,\varepsilon})$ of functions in $H_0^1(B)$, such that for a subsequence $E_1 \subseteq E$ we have

$$\begin{cases} v_{k,\varepsilon} \rightharpoonup v_k & \text{weakly in } H_0^1(B) \text{ and strongly in } L^2(B) \quad (\varepsilon \in E_1), \\ (\mathcal{L}_\varepsilon^* v_{k,\varepsilon}) & \text{strongly converges in } H^{-1}(B) \quad (\varepsilon \in E_1), \\ \mathbb{L}_\varepsilon^* \nabla v_{k,\varepsilon} \rightharpoonup \mathbb{L}_0^* \nabla v_k & \text{weakly in } L^2(T(\frac{1}{2}B)) \quad (\varepsilon \in E_1) \end{cases} \tag{3.29}$$

for all $k = 1,\dots,n$. Therefor we note that the uniform ellipticity of $\mathbb{L}_\varepsilon^*$ provides a constant $C > 0$, only depending on B and λ, such that

$$\langle \mathcal{L}_\varepsilon^* u, u\rangle = \int_B g(\mathbb{L}_\varepsilon^* \nabla u, \nabla u)\,\mathrm{d}\mu \geq C\|u\|_{H^1(B)}^2,$$

and thus Lemma 3.6.1 implies the existence of an operator $\mathcal{L}_0^* \colon H_0^1(B) \to H^{-1}(B)$ such that for a subsequence $E_0 \subseteq E$

$$(\mathcal{L}_\varepsilon^*)^{-1} f \rightharpoonup (\mathcal{L}_0^*)^{-1} f \qquad \text{weakly in } H_0^1(B) \quad (\varepsilon \in E_0)$$

for all $f \in H^{-1}(B)$. For $k = 1,\dots,n$ define the oscillating test-functions

$$v_{k,\varepsilon} := (\mathcal{L}_\varepsilon^*)^{-1} \mathcal{L}_0^* v_k \in H_0^1(B),$$

which are bounded uniformly in ε due to the uniform ellipticity of $\mathbb{L}_\varepsilon^*$ and Poincaré's inequality in $H_0^1(B)$ (which holds since $m_0(B) < 0$). Hence we can extract another subsequence $E_1 \subseteq E_0$ such that

$$\begin{cases} v_{k,\varepsilon} \rightharpoonup v_k & \text{weakly in } H_0^1(B) \text{ and strongly in } L^2(B) \quad (\varepsilon \in E_1), \\ \mathbb{L}_\varepsilon^* \nabla v_{k,\varepsilon} \rightharpoonup \xi_k & \text{weakly in } L^2(TB) \quad (\varepsilon \in E_1) \end{cases}$$

for some vector fields $\xi_1,\dots,\xi_n \in L^2(TB)$ and every $k = 1,\dots,n$. We now define the coefficient field $\mathbb{L}_0^*$ via

$$\mathbb{L}_0^* \nabla v_k = \xi_k \qquad \mu\text{-a.e. in } \tfrac{1}{2}B$$

for $k = 1,\dots,n$. The coefficient field $\mathbb{L}_0^*$ is indeed uniquely defined by the identities above, since $\nabla v_1,\dots,\nabla v_n$ span $T(\frac{1}{2}B)$, and we only need to show (3.29). Note that the oscillating test-functions $v_{\varepsilon,k}$ are weakly and strongly convergent along the subsequence E_1 as claimed by (3.29), and constructed such that $\mathcal{L}_\varepsilon^* v_{k,\varepsilon} = \mathcal{L}_0^* v_k$, which trivially implies strong convergence of $(\mathcal{L}_\varepsilon^* v_{k,\varepsilon})$. Finally, the last identity in (3.29) is true by definition of $\mathbb{L}_0^*$.

Step 2: H-convergence of $\mathbb{L}_\varepsilon$ to $\mathbb{L}_0$ in $\frac{1}{2}B$.
We fix a relatively-compact open subset $U \Subset \frac{1}{2}B$ and consider the operator $\mathcal{L}_\varepsilon$ as defined in (3.11). As in step 1, there is an operator $\mathcal{L}_0 \colon H_0^1(U) \to H^{-1}(U)$ such that for a subsequence $E_2 \subseteq E_1$ we have

$$\mathcal{L}_\varepsilon^{-1} \rightharpoonup \mathcal{L}_0^{-1} \qquad \text{in the weak operator topology} \quad (\varepsilon \in E_2). \tag{3.30}$$

We claim

$$\mathcal{L}_0 u_0 = -\operatorname{div}(\mathbb{L}_0 \nabla u_0), \tag{3.31}$$

for all $u_0 \in H_0^1(U)$, which by definition of $\mathcal{L}_\varepsilon$ immediately gives H-convergence. In order to show this claim we set $u_\varepsilon := \mathcal{L}_\varepsilon^{-1}\mathcal{L}_0 u_0 \in H_0^1(U)$, which by (3.30) implies

$$u_\varepsilon \rightharpoonup u_0 \qquad \text{weakly in } H_0^1(U) \text{ and strongly in } L^2(U) \quad (\varepsilon \in E_2). \tag{3.32}$$

Then the uniform ellipticity of $\mathbb{L}_\varepsilon$ assures that the sequence of fluxes $(\mathbb{L}_\varepsilon \nabla u_\varepsilon)$ is bounded in $L^2(TU)$ and hence we can extract another subsequence $E_3 \subseteq E_2$ such that

$$\mathbb{L}_\varepsilon \nabla u_\varepsilon \rightharpoonup J_0 \qquad \text{weakly in } L^2(TU) \quad (\varepsilon \in E_3) \tag{3.33}$$

for some $J_0 \in L^2(TU)$. Since u_ε is constructed such that $-\operatorname{div}\mathbb{L}_\varepsilon \nabla u_\varepsilon = \mathcal{L}_0 u_0$, we find

$$-\operatorname{div} J_0 = \mathcal{L}_0 u_0, \tag{3.34}$$

and thus for any test function $\psi \in C_c^\infty(U)$ (3.29) yields on the one hand

$$\begin{aligned}
\int_U g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \psi \nabla v_{k,\varepsilon})\,\mathrm{d}\mu &= \int_U g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla(\psi v_{k,\varepsilon}))\,\mathrm{d}\mu - \int_U g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, v_{k,\varepsilon}\nabla\psi) \\
&= \langle \mathcal{L}_0 u_0, \psi v_{k,\varepsilon}\rangle - \int_g (\mathbb{L}_\varepsilon \nabla u_\varepsilon, v_{k,\varepsilon}\nabla\psi)\,\mathrm{d}\mu \\
&\to \langle \mathcal{L}_0 u_0, \psi v_k\rangle - \int_U g(J_0, v_k \nabla\psi)\,\mathrm{d}\mu \\
&= \int_U g(J_0, \psi\nabla v_k)\,\mathrm{d}\mu,
\end{aligned}$$

and on the other hand with the Div-Curl Lemma (Lemma 3.3.2)

$$\begin{aligned}
\int_U g(\mathbb{L}_\varepsilon u_\varepsilon, \psi \nabla v_{k,\varepsilon})\,\mathrm{d}\mu &= \int_U g(\psi\nabla u_\varepsilon, \mathbb{L}_\varepsilon^* \nabla v_{k,\varepsilon})\,\mathrm{d}\mu \\
&\to \int_U g(\psi\nabla u_0, \mathbb{L}_0^* \nabla v_k)\,\mathrm{d}\mu \\
&= \int_U g(\mathbb{L}_0 \nabla u_0, \psi\nabla v_k)\,\mathrm{d}\mu.
\end{aligned}$$

Hence, we conclude that

$$\int_U g(\mathbb{L}_0 \nabla u_0, \psi\nabla v_k)\,\mathrm{d}\mu = \int_U g(J_0, \psi\nabla v_k)\,\mathrm{d}\mu,$$

and since this holds for any $\psi \in C_c^\infty(U)$ and $\nabla v_1, \dots, \nabla v_n$ span TU, we get $J_0 = \mathbb{L}_0 \nabla u_0$ μ-a.e. in U, which by (3.34) gives (3.31). Moreover, since J_0 and $\mathcal{L}_0$ are uniquely defined via $\mathbb{L}_0$, the convergences in (3.30), (3.32) and (3.33) are valid for the entire sequence E_1, which in particular does not depend on the choice of U.

Step 3: Uniform ellipticity of $\mathbb{L}_0$.
From (3.32), (3.33) and (3.34) in combination with the Div-Curl Lemma (Lemma 3.3.2) we find for any non-negative $\psi \in C_c^\infty(U)$

$$\int_U g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon)\psi \, \mathrm{d}\mu \to \int_U g(\mathbb{L}_0 \nabla u_0, \nabla u_0)\psi \, \mathrm{d}\mu,$$

which together with (3.1) immediately implies

$$\int_U g(\mathbb{L}_0 \nabla u_0, \nabla u_0)\psi \, \mathrm{d}\mu \geq \lambda \int_U |\nabla u_0|^2 \psi \, \mathrm{d}\mu. \tag{3.35}$$

On the other hand from (3.2) follows

$$\int_U g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon)\psi \, \mathrm{d}\mu = \int_U g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \mathbb{L}_\varepsilon^{-1}\mathbb{L}_\varepsilon \nabla u_\varepsilon)\psi \, \mathrm{d}\mu \geq \Lambda \int_U |\mathbb{L}_\varepsilon \nabla u_0|^2 \psi \, \mathrm{d}\mu,$$

which can be transformed to

$$\int_U g(\mathbb{L}_\varepsilon^{-1} \nabla u_\varepsilon, \nabla u_\varepsilon)\psi \, \mathrm{d}\mu \geq \Lambda \int_U |\nabla u_0|^2 \psi \, \mathrm{d}\mu \tag{3.36}$$

by substituting $\nabla u_0 = \mathbb{L}_0^{-1}\nabla \tilde{u}_0$. Since (3.35) and (3.36) hold for all $u_0 \in H_0^1(U)$ and $\psi \in C_c^\infty(U)$, we conclude that $\mathbb{L}_0 \in \mathcal{M}(U, \lambda, \Lambda)$, and since this is true for all $U \Subset \frac{1}{2}B$ we end up with $\mathbb{L}_0 \in \mathcal{M}(\frac{1}{2}B, \lambda, \Lambda)$. $\square$

3.6.2. Locality, H-Convergence of the Adjoint (Proposition 3.3.3) and Varying Right-Hand Sides (Lemma 3.3.4)

Proof of Proposition 3.3.3.
Step 1: Proof of part (a).
We fix a point $x \in U$, an open ball $B \Subset U$ with center at x and radius smaller than the injectivity radius of M at x (which implies $m_0(B) < 0$), and choose $v_1, \dots, v_n \in C_c^\infty(B)$ as in (3.9). For $k \in \{1, \dots, n\}$ we denote by $u_\varepsilon, \tilde{u}_\varepsilon \in H_0^1(B)$ the unique weak solutions to

$$\begin{aligned} -\operatorname{div}(\mathbb{L}_\varepsilon \nabla u_\varepsilon) &= f \quad \text{in } H^{-1}(B) \quad \text{with} \quad f := -\operatorname{div}(\mathbb{L}_0 \nabla v_k) \in H^{-1}(B), \\ -\operatorname{div}(\widetilde{\mathbb{L}}_\varepsilon \nabla \tilde{u}_\varepsilon) &= \tilde{f} \quad \text{in } H^{-1}(B) \quad \text{with} \quad \tilde{f} := -\operatorname{div}(\widetilde{\mathbb{L}}_0 \nabla v_k) \in H^{-1}(B). \end{aligned}$$

Then the H-convergence of $(\mathbb{L}_\varepsilon)$ and $(\widetilde{\mathbb{L}}_\varepsilon)$ yield

$$\begin{cases} u_\varepsilon \rightharpoonup v_k & \text{and} \quad \tilde{u}_\varepsilon \rightharpoonup v_k & \text{weakly in } H^1(B), \\ \mathbb{L}_\varepsilon \nabla u_\varepsilon \rightharpoonup \mathbb{L}_0 \nabla v_k & \text{and} \quad \widetilde{\mathbb{L}}_\varepsilon \nabla \tilde{u}_\varepsilon \rightharpoonup \widetilde{\mathbb{L}}_0 \nabla v_k & \text{weakly in } L^2(TB), \end{cases}$$

and together with $\mathbb{L}_\varepsilon = \widetilde{\mathbb{L}}_\varepsilon$ on U and $\mathbb{L} \in \mathcal{M}(M, \lambda, \Lambda)$, these convergences imply

$$\mathbb{L}_\varepsilon \nabla u_\varepsilon - \widetilde{\mathbb{L}}_\varepsilon \nabla \tilde{u}_\varepsilon \rightharpoonup (\mathbb{L}_0 - \widetilde{\mathbb{L}}_0)\nabla v_k \qquad \text{weakly in } L^2(B)$$

as well as

$$\mathbb{L}_\varepsilon \nabla u_\varepsilon - \widetilde{\mathbb{L}}_\varepsilon \nabla \tilde{u}_\varepsilon = \mathbb{L}_\varepsilon \nabla (u_\varepsilon - \tilde{u}_\varepsilon) \rightharpoonup 0 \qquad \text{weakly in } L^2(B),$$

which combined give $(\widetilde{\mathbb{L}}_0 - \mathbb{L}_0)\nabla v_k = 0$ μ-a.e. in B. Since this holds true for all $k \in \{1, \dots, n\}$, (3.9) yields $\mathbb{L}_0 = \widetilde{\mathbb{L}}_0$ μ-a.e. in $\frac{1}{2}B$, and since x is arbitrary, the assertion follows.

Step 2: Proof of (b).
We fix a relatively-compact open subset $U \Subset M$ with $m_0(U) < 0$. For $f \in H^{-1}(U)$ we denote by $u_\varepsilon, u_0 \in H_0^1(U)$ the unique weak solutions to

$$\begin{aligned} -\operatorname{div}(\mathbb{L}_\varepsilon^* \nabla u_\varepsilon) &= f \quad \text{in } H^{-1}(U), \\ -\operatorname{div}(\mathbb{L}_0^* \nabla u_0) &= f \quad \text{in } H^{-1}(U). \end{aligned}$$

By a standard energy estimate and the uniform ellipticity of $(\mathbb{L}_\varepsilon^*)$ the solutions (u_ε) and the fluxes $(\mathbb{L}_\varepsilon^* \nabla u_\varepsilon)$ are bounded sequences in $H_0^1(U)$ and $L^2(TU)$, respectively, and we can extract a (not relabeled) subsequence such that

$$\begin{cases} u_\varepsilon \rightharpoonup \tilde{u}_0 & \text{weakly in } H^1(U), \\ \mathbb{L}_\varepsilon^* \nabla u_\varepsilon \rightharpoonup J_0 & \text{weakly in } L^2(TU) \end{cases}$$

for some $\tilde{u}_0 \in H_0^1(U)$ and $J_0 \in L^2(TU)$. It remains to show that $\tilde{u}_0 = u_0$ and $J_0 = \mathbb{L}_0^* \nabla u_0$, because then, since u_0 is independent on the choice of the subsequence, the assertion follows for the entire sequence.

In order to show $\tilde{u}_0 = u_0$ we fix $v_0 \in H_0^1(U)$ and denote by $v_\varepsilon \in H_0^1(U)$ the unique weak solution to

$$-\operatorname{div}(\mathbb{L}_\varepsilon \nabla v_\varepsilon) = \tilde{f} \quad \text{in } H^{-1}(U) \qquad \text{with} \quad \tilde{f} := -\operatorname{div}(\mathbb{L}_0 \nabla v_0) \in H^{-1}(U).$$

Then the H-convergence of $(\mathbb{L}_\varepsilon)$ yields

$$\begin{cases} v_\varepsilon \rightharpoonup v_0 & \text{weakly in } H_0^1(U), \\ \mathbb{L}_\varepsilon \nabla v_\varepsilon \rightharpoonup \mathbb{L}_0 \nabla v_0 & \text{weakly in } L^2(TU), \end{cases}$$

and we find with the definition of u_ε on the one hand

$$\int_U g(\mathbb{L}_\varepsilon^* \nabla u_\varepsilon, \nabla v_\varepsilon)\, \mathrm{d}\mu = \int_U f v_\varepsilon \, \mathrm{d}\mu \to \int_U f v_0 \, \mathrm{d}\mu,$$

and with the Div-Curl Lemma (Lemma 3.3.2) on the other hand

$$\begin{aligned} \int_U g(\mathbb{L}_\varepsilon^* \nabla u_\varepsilon, \nabla v_\varepsilon)\, \mathrm{d}\mu &= \int_U g(\nabla u_\varepsilon, \mathbb{L}_\varepsilon \nabla v_\varepsilon)\, \mathrm{d}\mu \\ &\to \int_U g(\nabla \tilde{u}_0, \mathbb{L}_0 \nabla v_0)\, \mathrm{d}\mu \\ &= \int_U g(\mathbb{L}_0^* \nabla \tilde{u}_0, \nabla v_0)\, \mathrm{d}\mu. \end{aligned}$$

Thus $\tilde{u}_0$ solves the equation $-\operatorname{div}(\mathbb{L}_0^*\nabla\tilde{u}_0) = f$ in $H^{-1}(U)$, and the uniqueness of the weak solution implies $\tilde{u}_0 = u_0$.

We finally show $J_0 = \mathbb{L}_0^*\nabla u_0$. Therefor we fix an open ball $B \Subset U$ with radius less than the injectivity radius of M at its center, and choose $v_1, \dots, v_n \in C_c^\infty(B) \subseteq C_c^\infty(U)$ as in (3.9). As above we denote for $k \in \{1, \dots, n\}$ by $v_\varepsilon \in H_0^1(B)$ the unique weak solution to

$$-\operatorname{div}(\mathbb{L}_\varepsilon\nabla v_\varepsilon) = \tilde{f} \quad \text{in } H^{-1}(U) \qquad \text{with} \quad \tilde{f} := -\operatorname{div}(\mathbb{L}_0\nabla v_k) \in H^{-1}(U),$$

and follow from the H-convergence of $(\mathbb{L}_\varepsilon)$

$$\begin{cases} v_\varepsilon \rightharpoonup v_j & \text{weakly in } H_0^1(B), \\ \mathbb{L}_\varepsilon\nabla v_\varepsilon \rightharpoonup \mathbb{L}_0\nabla v_j & \text{weakly in } L^2(TB). \end{cases}$$

Then for $\psi \in C_c^\infty(B)$ the Div-Curl Lemma (Lemma 3.3.2) yields on the one hand

$$\int_B g(\mathbb{L}_\varepsilon^*\nabla u_\varepsilon, \nabla v_\varepsilon)\psi \,\mathrm{d}\mu \to \int_B g(J_0, \psi\nabla v_j)\,\mathrm{d}\mu,$$

and on the other hand

$$\begin{aligned} \int_B g(\mathbb{L}_\varepsilon^*\nabla u_\varepsilon, \nabla v_\varepsilon)\psi \,\mathrm{d}\mu &= \int_B g(\nabla u_\varepsilon, \mathbb{L}_\varepsilon\nabla v_\varepsilon)\psi\,\mathrm{d}\mu \\ &\to \int_B g(\nabla u_0, \mathbb{L}_0\nabla v_j)\psi\,\mathrm{d}\mu \\ &= \int_B g(\mathbb{L}_0^*\nabla u_0, \psi\nabla v_j)\,\mathrm{d}\mu. \end{aligned}$$

Since these hold true for all $\psi \in C_c^\infty(B)$, $k \in \{1, \dots, n\}$, we can follow because of (3.9) that $J_0 = \mathbb{L}_0^*\nabla u_0$ in $\frac{1}{2}B$, and since U is open and the center of B was arbitrary, we can conclude equality in U. □

Proof of Lemma 3.3.4. By a standard energy estimate, we can find a (not relabeled) subsequence such that

$$\begin{cases} u_\varepsilon \rightharpoonup \tilde{u}_0 & \text{weakly in } H^1(U), \\ \mathbb{L}_\varepsilon\nabla u_\varepsilon \rightharpoonup J_0 & \text{weakly in } L^2(TU) \end{cases}$$

for some $\tilde{u}_0 \in H_0^1(U)$ and $J_0 \in L^2(TU)$. We will show that $\tilde{u}_0 = u_0$ and $J_0 = \mathbb{L}_0\nabla u_0$, because then, since u_0 is uniquely determined and independent of the choice of the subsequence, we can conclude the assertion for the entire sequence.

To show $\tilde{u}_0 = u_0$ we fix $v_0 \in H_0^1(U)$ and denote by $v_\varepsilon \in H_0^1(U)$ the unique weak solution to

$$-\operatorname{div}(\mathbb{L}_\varepsilon^*\nabla v_\varepsilon) = \tilde{f} \quad \text{in } H^{-1}(U) \qquad \text{with} \quad \tilde{f} := -\operatorname{div}(\mathbb{L}_0^*\nabla v_0) \in H^{-1}(U).$$

Since $\mathbb{L}_\varepsilon^* \xrightarrow{H} \mathbb{L}_0^*$ by Proposition 3.3.3, we find (with the compact embedding of $H_0^1(U)$ in $L^2(U)$)

$$\begin{cases} v_\varepsilon \rightharpoonup v_0 & \text{weakly in } H_0^1(U) \text{ and strongly in } L^2(U), \\ \mathbb{L}_\varepsilon^* \nabla v_\varepsilon \rightharpoonup \mathbb{L}_0^* \nabla v_0 & \text{weakly in } L^2(TU). \end{cases}$$

Now the Div-Curl Lemma (Lemma 3.3.2) yields on the one hand

$$\begin{aligned} \int_U g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla v_\varepsilon)\,\mathrm{d}\mu &= \int_U \big(f_\varepsilon v_\varepsilon + g(G_\varepsilon, \mathbb{L}_\varepsilon^* \nabla v_\varepsilon) + g(F_\varepsilon, \nabla v_\varepsilon)\big)\,\mathrm{d}\mu \\ &\to \int_U \big(f_0 v_0 + g(G_0, \mathbb{L}_0^* \nabla v_0) + g(F_0, \nabla v_0)\big)\,\mathrm{d}\mu \\ &= \int_U \big(f_0 v_0 + g(\mathbb{L}_0 G_0 + F_0, \nabla v_0)\big)\,\mathrm{d}\mu, \end{aligned}$$

and, with $u_\varepsilon \rightharpoonup \tilde{u}_0$ weakly in $H^1(U)$, on the other hand

$$\begin{aligned} \int_U g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla v_\varepsilon)\,\mathrm{d}\mu &= \int_U g(\nabla u_\varepsilon, \mathbb{L}_\varepsilon^* \nabla v_\varepsilon)\,\mathrm{d}\mu \\ &\to \int_U g(\nabla \tilde{u}_0, \mathbb{L}_0^* \nabla v_0)\,\mathrm{d}\mu \\ &= \int_U g(\mathbb{L}_0 \nabla \tilde{u}_0, \nabla v_0)\,\mathrm{d}\mu. \end{aligned}$$

That means, $\tilde{u}_0$ solves the limiting equation, and the uniqueness of the weak solution implies $\tilde{u}_0 = u_0$.

Moreover, with the same arguments as in the last paragraph of the proof of Proposition 3.3.3 (b), we also deduce that $J_0 = \mathbb{L}_0 \nabla u_0$, which completes the argument. □

3.6.3. Proof of Theorem 3.2.2

As described in Section 3.3, to prove Theorem 3.2.2 we lift the H-convergence result on small balls (Lemma 3.3.1) to H-convergence on the whole manifold. We first cover M with countably many small balls, stick the H-limits on the individual balls together to one coefficient field $\mathbb{L}_0$ on M by appealing to the uniqueness of the H-limit (Proposition 3.3.3 (a)), and choose a subsequence H-converging to $\mathbb{L}_0$ on every ball. Then the convergence of the solutions and the fluxes will follow from Lemma 3.3.4 using a partition of unity motivated by the covering of M.

Step 1: Choice of the subsequence and definition of $\mathbb{L}_0$.
Let $(B_j)_{j\in\mathbb{N}}$ denote a countable covering of M by open balls such that

- $4B_j \Subset M$, where $4B_j$ denotes the open ball with the same center as B_j and four times the radius of B_j,

- the radius of B_j is smaller than a quarter of the injectivity radius of M at the center of B_j.

The existence of such covering is assured by Vitali's Covering Lemma, see e.g. [Ste93]. For every $j \in \mathbb{N}$ Lemma 3.3.1, applied to $4B_j$, provides a subsequence of $(\mathbb{L}_\varepsilon)$ H-converging to some $\mathbb{L}_{j,0} \in \mathcal{M}(2B_j, \lambda, \Lambda)$ in $2B_j$. Thus we can fix a diagonal subsequence $E_1 \subseteq E$ (cf. the notation of subsequences in the beginning of Section 3.6) such that $(\mathbb{L}_\varepsilon)$ H-converges to $\mathbb{L}_{j,0}$ in $2B_j$ for every $j \in \mathbb{N}$. By the uniqueness of the H-limit (Proposition 3.3.3 (a)) we have $\mathbb{L}_{j,0} = \mathbb{L}_{k,0}$ μ-a.e. in $B_j \cap B_k$, so we can choose a coefficient field $\mathbb{L}_0 \in \mathcal{M}(M, \lambda, \Lambda)$ with $\mathbb{L}_0(x) = \mathbb{L}_{j,0}(x)$ for μ-a.e. $x \in B_j$, $j \in \mathbb{N}$.

Step 2: Convergence of the solutions and the fluxes.
We fix $U \subseteq M$ open with $m > \lambda m_0(U)$, and sequences (f_ε) in $L^2(U)$ and (F_ε) in $L^2(TU)$ with

$$\begin{cases} f_\varepsilon \rightharpoonup f_0 & \text{weakly in } L^2(U), \\ F_\varepsilon \to F_0 & \text{strongly in } L^2(TU). \end{cases}$$

Let $u_\varepsilon \in H^1_0(U)$ be the unique weak solution to

$$m u_\varepsilon - \operatorname{div}(\mathbb{L}_\varepsilon \nabla u_\varepsilon) = f_\varepsilon - \operatorname{div} F_\varepsilon \qquad \text{in } H^{-1}(U). \tag{3.37}$$

The boundedness of (f_ε) and (F_ε) in $L^2(U)$ resp. $L^2(TU)$ ensures by (3.6) boundedness of (u_ε) in $H^1(U)$ and thus, with the uniform ellipticity (3.2), boundedness of $(\mathbb{L}_\varepsilon \nabla u_\varepsilon)$ in $L^2(TU)$. We therefore can extract a subsequence $E'' \subseteq E'$ such that

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 & \text{weakly in } H^1_0(U), \\ \mathbb{L}_\varepsilon \nabla u_\varepsilon \rightharpoonup J_0 & \text{weakly in } L^2(TU) \end{cases} \tag{3.38}$$

for some $u_0 \in H^1(U)$ and $J_0 \in L^2(TU)$. It remains to claim that u_0 is the unique weak solution in $H^1_0(U)$ to

$$m u_0 - \operatorname{div}(\mathbb{L}_0 \nabla u_0) = f_0 - \operatorname{div} F_0 \qquad \text{in } H^{-1}(U) \tag{3.39}$$

and that $J_0 = \mathbb{L}_0 \nabla u_0$, with $\mathbb{L}_0$ defined in step 1. Let $\varphi_j \in C^\infty_c(M)$ denote a partition of unity subordinate to (B_j), i.e. $\operatorname{supp} \varphi_j \Subset B_j$ and $\sum_{j=1}^\infty \varphi_j = 1$. Then for every

$\psi \in H_0^1(U)$ and $j \in \mathbb{N}$ we calculate using (3.37)

$$
\begin{aligned}
&\int_U g(\mathbb{L}_\varepsilon \nabla(\varphi_j u_\varepsilon), \nabla\psi)\,\mathrm{d}\mu \\
&\quad= \int_U g(u_\varepsilon \mathbb{L}_\varepsilon \nabla\varphi_j, \nabla\psi)\,\mathrm{d}\mu + \int_U g(\varphi_j \mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla\psi)\,\mathrm{d}\mu \\
&\quad= \int_U g(u_\varepsilon \mathbb{L}_\varepsilon \nabla\varphi_j, \nabla\psi)\,\mathrm{d}\mu + \int_U g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla(\varphi_j\psi))\,\mathrm{d}\mu - \int_U g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \psi\nabla\varphi_j)\,\mathrm{d}\mu \\
&\quad= \int_U g(u_\varepsilon \mathbb{L}_\varepsilon \nabla\varphi_j, \nabla\psi)\,\mathrm{d}\mu + \int_U \big((f_\varepsilon - m u_\varepsilon)\varphi_j\psi + g(F_\varepsilon, \nabla(\varphi_j\psi))\big)\,\mathrm{d}\mu \\
&\qquad - \int_U g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \psi\nabla\varphi_j)\,\mathrm{d}\mu \\
&\quad= \int_U g(\mathbb{L}_\varepsilon (u_\varepsilon \nabla\varphi_j), \nabla\psi)\,\mathrm{d}\mu + \int_U \Big((f_\varepsilon - m u_\varepsilon)\varphi_j + g((F_\varepsilon - \mathbb{L}_\varepsilon \nabla u_\varepsilon), \nabla\varphi_j)\Big)\psi\,\mathrm{d}\mu \\
&\qquad + \int_U g(\varphi_j F_\varepsilon, \nabla\psi)\,\mathrm{d}\mu \\
&\quad= \int_U g(\mathbb{L}_\varepsilon G_{j,\varepsilon}, \nabla\psi)\,\mathrm{d}\mu + \int_U f_{j,\varepsilon}\psi\,\mathrm{d}\mu + \int_U g(F_{j,\varepsilon}, \nabla\psi)\,\mathrm{d}\mu,
\end{aligned}
\tag{3.40}
$$

where

$$
F_{j,\varepsilon} := \varphi_j F_\varepsilon, \qquad G_{j,\varepsilon} := u_\varepsilon \nabla\varphi_j, \qquad f_{j,\varepsilon} := (f_\varepsilon - m u_\varepsilon)\varphi_j + g((F_\varepsilon - \mathbb{L}_\varepsilon \nabla u_\varepsilon), \nabla\varphi_j).
$$

Moreover, we set $v_{j,\varepsilon} := \varphi_j u_\varepsilon$. Then $v_{j,\varepsilon} \in H_0^1(B_j)$ is by (3.40) the unique solution in $H_0^1(B_j)$ to

$$
-\operatorname{div}(\mathbb{L}_\varepsilon \nabla v_{j,\varepsilon}) = f_{j,\varepsilon} - \operatorname{div}(\mathbb{L}_\varepsilon G_{j,\varepsilon}) - \operatorname{div} F_{j,\varepsilon} \qquad \text{in } H^{-1}(B_j).
$$

From (3.38), the compact embedding $H_0^1(B_j) \hookrightarrow L^2(B_j)$, and the convergence of (f_ε) and (F_ε), we deduce that

$$
\begin{cases}
v_{j,\varepsilon} \rightharpoonup v_{j,0} := \varphi_j u_0 & \text{weakly in } H^1(B_j), \\
f_{j,\varepsilon} \rightharpoonup f_{j,0} := (f_0 - m u_0)\varphi_j + g((F_0 - J_0), \nabla\varphi_j) & \text{weakly in } L^2(B_j), \\
G_{j,\varepsilon} \to G_{j,0} := u_0 \nabla\varphi_j & \text{strongly in } L^2(TB_j), \\
F_{j,\varepsilon} \to F_{j,0} := \varphi_j F_0 & \text{strongly in } L^2(TB_j).
\end{cases}
\tag{3.41}
$$

Since by step 1 we have $\mathbb{L}_\varepsilon \xrightarrow{H} \mathbb{L}_0$ on $2B_j$, Lemma 3.3.4 implies that $v_{j,0} \in H_0^1(B_j)$ is the unique weak solution to

$$
-\operatorname{div}(\mathbb{L}_0 \nabla v_{j,0}) = f_{j,0} - \operatorname{div}(\mathbb{L}_0 G_{j,0}) - \operatorname{div} F_{j,0} \qquad \text{in } H^{-1}(B_j),
$$

and

$$
\mathbb{L}_\varepsilon \nabla v_{j,\varepsilon} \rightharpoonup \mathbb{L}_0 \nabla v_{j,0} \qquad \text{weakly in } L^2(TB_j). \tag{3.42}
$$

Since $\sum_{j=1}^{\infty} \varphi_j = 1$, and therewith $\sum_{j=1}^{\infty} \nabla\varphi_j = 0$, we find that

$$\sum_{j=1}^{\infty} v_{j,0} = u_0, \qquad \sum_{j=1}^{\infty} F_{j,0} = F_0, \qquad \sum_{j=1}^{\infty} G_{j,0} = 0, \qquad \sum_{j=1}^{\infty} f_{j,0} = (f_0 - mu_0).$$

Now summation of (3.42) yields $\mathbb{L}_\varepsilon \nabla u_\varepsilon \rightharpoonup \mathbb{L}_0 \nabla u_0 = J_0$ weakly in $L^2(TU)$. Moreover, for any test-function $\psi \in C_c^\infty(U)$ we have on the one hand

$$\begin{aligned}
\int_U g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla\psi)\,\mathrm{d}\mu &= \sum_{j=1}^{\infty} \int_U g(\mathbb{L}_\varepsilon \nabla v_{j,\varepsilon}, \nabla\psi)\,\mathrm{d}\mu \\
&\to \sum_{j=1}^{\infty} \int_U g(\mathbb{L}_0 \nabla v_{j,0}, \nabla\psi)\,\mathrm{d}\mu \\
&= \int_U g(\mathbb{L}_0 \nabla u_0, \nabla\psi)\,\mathrm{d}\mu
\end{aligned}$$

and on the other hand, by summation of (3.40), and by (3.41),

$$\begin{aligned}
\int_U g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla\psi)\,\mathrm{d}\mu &= \sum_{j=1}^{\infty} \int_{B_j} g(\mathbb{L}_\varepsilon \nabla v_{j,\varepsilon}, \nabla\psi)\,\mathrm{d}\mu \\
&= \sum_{j=1}^{\infty} \int_{B_j} g\big((\mathbb{L}_\varepsilon G_{j,\varepsilon}, \nabla\psi) + g(F_{j,\varepsilon}, \nabla\psi) + f_{j,\varepsilon}\psi\big)\,\mathrm{d}\mu \\
&\to \sum_{j=1}^{\infty} \int_{B_j} \big(g(\mathbb{L}_0 G_{j,0} + F_{j,0}, \nabla\psi) + f_{0,j}\psi\big)\,\mathrm{d}\mu \\
&= \int_U \big(g(F_0, \nabla\psi) + (f_0 - mu_0)\psi\big)\,\mathrm{d}\mu,
\end{aligned}$$

which together yield (3.39). From the uniqueness of the solution we deduce that the convergence holds for the entire subsequence E_1.

Finally we note that if either $H_0^1(U)$ is compactly embedded in $L^2(U)$ or $m \neq 0$ and $f_\varepsilon \to f_0$ strongly in $L^2(U)$, then we even have $u_\varepsilon \to u_0$ strongly in $L^2(U)$. The first implication is a direct consequence of $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(u)$. For the second implication we note that by the Div-Curl-Lemma (Lemma 3.3.2) from $\mathbb{L}_\varepsilon \nabla u_\varepsilon \rightharpoonup \mathbb{L}_0 \nabla u_0$ follows

$$\int_U (\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon) \to \int_U (\mathbb{L}_0 \nabla u_0, \nabla u_0),$$

and thus with (3.37) and (3.39)

$$\begin{aligned}
m\int_U |u_\varepsilon|^2\,\mathrm{d}\mu &= m\int_U |u_\varepsilon|^2\,\mathrm{d}\mu + \int_U g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon)\,\mathrm{d}\mu - \int_U g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon)\,\mathrm{d}\mu \\
&= \int_U f_\varepsilon u_\varepsilon\,\mathrm{d}\mu + \int_U g(F_\varepsilon, \nabla u_\varepsilon)\,\mathrm{d}\mu - \int_U g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla u_\varepsilon)\,\mathrm{d}\mu \\
&\to \int_U f_0 u_0\,\mathrm{d}\mu + \int_U g(F_0, \nabla u_0)\,\mathrm{d}\mu - \int_U g(\mathbb{L}_0 \nabla u_0, \nabla u_0)\,\mathrm{d}\mu = m\int_U |u_0|^2.
\end{aligned}$$

Since $m \neq 0$, this implies $\|u_\varepsilon\|_{L^2(U)} \to \|u_0\|_{L^2(U)}$, which together with the weak convergence in $L^2(U)$ yields strong convergence in $L^2(U)$. □

3.6.4. Mosco- and Spectral Convergence (Propositions 3.2.4, 3.2.6 and 3.2.7)

We first recall that Mosco-convergence is equivalent to resolvent convergence of the associated operator, which we want to formulate as follows (cf. e.g. [DM93, Chapter 13]). Therefor for an operator $\mathcal{L}\colon H_0^1(M) \to H^{-1}(M)$ we denote for $\lambda > 0$ the associated resolvent by $\mathcal{R}^\lambda := (\lambda + \mathcal{L})^{-1}\colon L^2(M) \to L^2(M)$. Then the following holds (see [Mos94, Theorem 2.4.1]):

Lemma 3.6.2. *For operators $\mathcal{L}_\varepsilon, \mathcal{L}_0\colon H_0^1(M) \to H^{-1}(M)$ following two conditions are equivalent:*

(i) The functionals $\mathcal{E}_\varepsilon\colon L^2(M) \to \mathbb{R} \cup \{+\infty\}$, given by

$$\mathcal{E}_\varepsilon(u) = \begin{cases} \langle \mathcal{L}_\varepsilon u, u\rangle, & \text{if } u \in H_0^1(M), \\ +\infty, & \text{otherwise}, \end{cases}$$

Mosco-converge (w.r.t. L^2) to the functional $\mathcal{E}_0\colon L^2(M) \to \mathbb{R} \cup \{+\infty\}$, given by

$$\mathcal{E}_\varepsilon(u) = \begin{cases} \langle \mathcal{L}_0 u, u\rangle, & \text{if } u \in H_0^1(M), \\ +\infty, & \text{otherwise}. \end{cases}$$

(ii) For any $\lambda > 0$, the associated resolvents $\mathcal{R}_\varepsilon^\lambda$ converge to $\mathcal{R}_0^\lambda$ in the strong operator topology of $L^2(M)$.

Proof of Proposition 3.2.4. We apply Lemma 3.6.2 to the operators $\mathcal{L}_\varepsilon u := -\operatorname{div}(\mathbb{L}_\varepsilon \nabla u)$. In order to prove convergence of the resolvents in the strong operator topology on $L^2(M)$, for $\lambda > 0$ and $f_\varepsilon \to f_0$ strongly in $L^2(M)$ we set $u_\varepsilon := \mathcal{R}_\varepsilon^\lambda f_\varepsilon \in H_0^1(M)$. That means, u_ε is the unique weak solution to

$$\lambda u_\varepsilon - \operatorname{div}(\mathbb{L}_\varepsilon \nabla u_\varepsilon) = f_\varepsilon \qquad \text{in } H^{-1}(M),$$

and from the H-convergence of $(\mathbb{L}_\varepsilon)$ (and Theorem 3.2.2) we deduce $u_\varepsilon \to u_0$ strongly in $L^2(M)$, where $u_0 \in H_0^1(M)$ is the unique weak solution to

$$\lambda u_0 - \operatorname{div}(\mathbb{L}_0 \nabla u_0) = f_0 \qquad \text{in } H^{-1}(M).$$

In other words, $\mathcal{R}_\varepsilon^\lambda f_\varepsilon = u_\varepsilon \to u_0 = \mathcal{R}_0^\lambda f_0$ strongly in $L^2(M)$. □

For the proofs of the spectral convergence statements Propositions 3.2.6 and 3.2.7 we access to the resolvents of the operators, too, on which we will apply the following statement (see [JKO12, Lemma 11.3 and Theorem 11.5]):

Lemma 3.6.3. *Let (M, g, μ) be a weighted Riemannian manifold and consider positive, compact, self-adjoint operators $\mathcal{R}_\varepsilon, \mathcal{R}_0 \colon L^2(M) \to L^2(M)$ such that their operator norm $\|\mathcal{R}_\varepsilon\|_{\mathcal{L}(L^2(M))}$ is uniformly bounded for all $\varepsilon > 0$. Denote (for $\varepsilon \geq 0$) by $(\lambda_{\varepsilon,k})_{k\in\mathbb{N}}$ the decreasingly ordered sequence of eigenvalues of $\mathcal{R}_\varepsilon$, where eigenvalues are repeated according to their multiplicity, and let $(u_{\varepsilon,k})_{k\in\mathbb{N}}$ be a sequence of associated eigenfunctions, forming an orthonormal basis of $L^2(M)$. If*

$$f_\varepsilon \rightharpoonup f_0 \quad \text{weakly in } L^2(M) \qquad \Rightarrow \qquad \mathcal{R}_\varepsilon f_\varepsilon \to \mathcal{R}_0 f_0 \quad \text{strongly in } L^2(M),$$

then for all $k \in \mathbb{N}$

$$\lambda_{\varepsilon,k} \to \lambda_{0,k} \qquad \text{as } \varepsilon \searrow 0,$$

and if $s \in \mathbb{N}$ denotes the multiplicity of $\lambda_{0,k}$, i.e.

$$\lambda_{0,k-1} > \lambda_{0,k} = \cdots = \lambda_{0,k+s-1} > \lambda_{0,k+s} \qquad \text{(with the convention } \lambda_{0,0} = \infty\text{)},$$

there exists a sequence $\tilde{u}_{\varepsilon,k}$ of linear combinations of $u_{\varepsilon,k}, \dots, u_{\varepsilon,k+s-1}$ such that

$$\tilde{u}_{\varepsilon,k} \to u_{0,k} \qquad \text{strongly in } L^2(M) \text{ as } \varepsilon \searrow 0.$$

Proof of Proposition 3.2.6.
Step 1: Proof of part (a).
We fix a relatively-compact open subset $U \Subset M$ with $m_0(U) < 0$. The latter ensures that for the operator $\mathcal{L}_\varepsilon := -\operatorname{div}(\mathbb{L}_\varepsilon \nabla) \colon H^1_0(U) \to H^{-1}(U)$ we can consider the associated resolvent $\mathcal{R}_\varepsilon := \mathcal{L}_\varepsilon^{-1} \colon L^2(U) \to L^2(U)$. This resolvent is a compact, self-adjoint operator on $L^2(U)$ and, due to the uniform ellipticity of the coefficient fields, it is positive and its operator norm is bounded by a constant independent on ε. Thus the Spectral Theorem implies that the spectrum of $\mathcal{R}_\varepsilon$ consist only of a real, strictly positive point spectrum, which is bounded from above by a constant independent on ε, and there is an orthonormal basis of $L^2(U)$ consisting of eigenfunctions of $\mathcal{R}_\varepsilon$. Now it is sufficient to note that (λ, u) is an eigenpair of $\mathcal{R}_\varepsilon$ if and only if $(\frac{1}{\lambda}, u)$ is an eigenpair of $\mathcal{L}_\varepsilon$.

Step 2: Proof of part (b).
To apply Lemma 3.6.3 to the resolvents $\mathcal{R}_\varepsilon$ defined in step 1, it only remains to show that $f_\varepsilon \rightharpoonup f_0$ weakly in $L^2(U)$ implies $\mathcal{R}_\varepsilon f_\varepsilon \to \mathcal{R}_0 f_0$ strongly in $L^2(U)$. Indeed, if we set $u_\varepsilon := \mathcal{R}_\varepsilon f_\varepsilon \in H^1_0(U)$ and $u_0 := \mathcal{R}_0 f_0 \in H^1_0(U)$, we find that u_ε and u_0 are the unique weak solutions to

$$\begin{aligned} -\operatorname{div}(\mathbb{L}_\varepsilon \nabla u_\varepsilon) &= f_\varepsilon \quad \text{in } H^{-1}(U), \\ -\operatorname{div}(\mathbb{L}_0 \nabla u_0) &= f_0 \quad \text{in } H^{-1}(U), \end{aligned}$$

and the H-convergence of the coefficient fields (together with Lemma 3.3.4) implies $\mathcal{R}_\varepsilon f_\varepsilon = u_\varepsilon \rightharpoonup u_0 = \mathcal{R}_0 f_0$ weakly in $H^1(U)$. Now the strong convergence in $L^2(U)$ is a consequence of the compact embedding $H_0^1(U) \hookrightarrow L^2(U)$. □

Since the proof of Proposition 3.2.6 relies strongly on the fact that $m_0(M) < 0$, to prove Proposition 3.2.7 we need the following variant of the H-compactness statement Theorem 3.2.2:

Lemma 3.6.4. *Let (M, g, μ) be a weighted Riemannian manifold, and let $(\mathbb{L}_\varepsilon)$ be a family of coefficient fields on M, H-converging in M to some coefficient field $\mathbb{L}_0$. Let further H_ε and H_0 be closed subspaces of $H^1(M, g, \mu)$ such that for every sequence (u_ε) with $u_\varepsilon \in H_\varepsilon$ from $u_\varepsilon \to u_0$ strongly in $L^2(M, g, \mu)$ follows $u_0 \in H_0$. Suppose that for every $f_\varepsilon, f_0 \in H^{-1}(M)$ the equations*

$$\begin{aligned} -\operatorname{div}(\nabla u_\varepsilon) &= f_\varepsilon \quad \text{in } H^{-1}(M), \\ -\operatorname{div}(\nabla u_0) &= f_0 \quad \text{in } H^{-1}(M), \end{aligned}$$

admit unique weak solutions u_ε in H_ε and u_0 in H_0. Then $f_\varepsilon \rightharpoonup f_0$ weakly in $L^2(M, g, \mu)$ implies $u_\varepsilon \to u_0$ strongly in $L^2(M, g, \mu)$.

Proof of Lemma 3.6.4. We first note that due to the weak convergence of (f_ε) in $L^2(M)$ the solutions u_ε are uniformly bounded in $H^1(M)$, so we can find a (not relabeled) subsequence, such that $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(M)$ and $\mathbb{L}_\varepsilon \nabla u_\varepsilon \rightharpoonup J_0$ weakly in $L^2(TM)$ for some $u_0 \in H^1(M)$ and some $J_0 \in L^2(TM)$. We now fix a countable covering (U_i) of M consisting of relatively-compact open subsets $U_i \Subset M$ with $m_0(U_i) < 0$, and consider a partition of unity (φ_i) in $C_c^\infty(M)$ subordinate to this covering, i.e. $\operatorname{supp} \varphi_i \Subset U_i$ and $\sum_{i=1}^\infty \varphi_i = 1$. Then for every $\psi \in H^1(M)$ we find

$$\begin{aligned} &\int_M g(\mathbb{L}_\varepsilon \nabla(\varphi_i u_\varepsilon), \nabla\psi)\, \mathrm{d}\mu \\ &\quad = \int_M g(u_\varepsilon \mathbb{L}_\varepsilon \nabla\varphi_i, \nabla\psi)\, \mathrm{d}\mu + \int_M g(\varphi_i \mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla\psi)\, \mathrm{d}\mu \\ &\quad = \int_M g(u_\varepsilon \mathbb{L}_\varepsilon \nabla\varphi_i, \nabla\psi)\, \mathrm{d}\mu + \int_M g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla(\varphi_i\psi))\, \mathrm{d}\mu - \int_M g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \psi\nabla\varphi_i)\, \mathrm{d}\mu \\ &\quad = \int_M g(u_\varepsilon \mathbb{L}_\varepsilon \nabla\varphi_i, \nabla\psi)\, \mathrm{d}\mu + \int_M f_\varepsilon \varphi_i \psi\, \mathrm{d}\mu - \int_M g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \psi\nabla\varphi_i)\, \mathrm{d}\mu \\ &\quad = \int_M g(\mathbb{L}_\varepsilon G_{i,\varepsilon}, \nabla\psi)\, \mathrm{d}\mu + \int_M f_{i,\varepsilon}\psi\, \mathrm{d}\mu, \end{aligned}$$

with

$$G_{i,\varepsilon} := u_\varepsilon \nabla\varphi_i \qquad \text{and} \qquad f_{i,\varepsilon} := f_\varepsilon\varphi_i - g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla\varphi_i),$$

that means $\varphi_i u_\varepsilon \in H_0^1(U_i)$ is the unique weak solution to

$$-\operatorname{div}(\mathbb{L}_\varepsilon \nabla(\varphi_i u_\varepsilon)) = f_{i,\varepsilon} - \operatorname{div}(\mathbb{L}_\varepsilon G_{i,\varepsilon}) \qquad \text{in } H^{-1}(U_i).$$

Since from the compact embedding $H_0^1(U_i) \hookrightarrow L^2(U_i)$ we have

$$\begin{cases} \varphi_i u_\varepsilon \rightharpoonup \varphi_i u_0 & \text{weakly in } H^1(U_i), \\ f_{i,\varepsilon} \rightharpoonup f_{i,0} := f_0\varphi_i - g(J_0, \nabla\varphi_i) & \text{weakly in } L^2(U_i), \\ G_{i,\varepsilon} \to G_{i,0} := u_0 \nabla\varphi_i & \text{strongly in } L^2(TU_i), \end{cases}$$

the H-convergence of $(\mathbb{L}_\varepsilon)$ and Lemma 3.3.4 yield that $\varphi_i u_0 \in H_0^1(U_i)$ is the unique weak solution to

$$-\operatorname{div}(\mathbb{L}_0 \nabla(\varphi_i u_0)) = f_{i,0} - \operatorname{div}(\mathbb{L}_0 G_{i,0}) \qquad \text{in } H^{-1}(U_i),$$

and we have $\mathbb{L}_\varepsilon \nabla(\varphi_i u_\varepsilon) \rightharpoonup \mathbb{L}_0 \nabla(\varphi_i u_0)$ weakly in $L^2(TU_i)$. Moreover, the compact embedding $H_0^1(U_i) \hookrightarrow L^2(U_i)$ guarantees even strong convergence $\varphi_i u_\varepsilon \to \varphi_i u_0$ in $L^2(U_i)$. Now, since $\sum_{i=1}^\infty \varphi_i = 1$ and $\sum_{i=1}^\infty \nabla\varphi_i = 0$, a summation argument yields $u_\varepsilon \to u_0$ strongly in $L^2(M)$ (for the entire sequence), and $\mathbb{L}_\varepsilon \nabla u_\varepsilon \rightharpoonup \mathbb{L}_0 \nabla u_0$ weakly in $L^2(TM)$. It remains to show that $u_0 = \mathcal{R}_0 f_0$, i.e. that $u_0 \in H_0$ is the unique weak solution to

$$-\operatorname{div}(\mathbb{L}_0 \nabla u_0) = f_0 \qquad \text{in } H^{-1}(M).$$

But this can be seen by

$$\int_M g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla\psi)\,\mathrm{d}\mu \to \int_M g(\mathbb{L}_0 \nabla u_0, \nabla\psi)\,\mathrm{d}\mu$$

and

$$\begin{aligned} \int_M g(\mathbb{L}_\varepsilon \nabla u_\varepsilon, \nabla\psi)\,\mathrm{d}\mu &= \sum_{i=1}^\infty \int_{U_i} g(\mathbb{L}_\varepsilon \nabla(\varphi_i u_\varepsilon), \nabla\psi)\,\mathrm{d}\mu \\ &= \sum_{i=1}^\infty \int_{U_i} \big(f_{i,\varepsilon}\psi + g(\mathbb{L}_\varepsilon G_{i,\varepsilon}, \nabla\psi)\big)\,\mathrm{d}\mu \\ &\to \sum_{i=1}^\infty \int_{U_i} \big(f_{i,0}\psi + g(\mathbb{L}_0 G_{i,0}, \nabla\psi)\big)\,\mathrm{d}\mu \\ &= \int_M f_0\psi\,\mathrm{d}\mu \end{aligned}$$

for all $\psi \in H^1(M)$. □

Proof of Proposition 3.2.7.
Step 1: Proof of part (a).
This step follows the same argumentation as step 1 in the proof of Proposition 3.2.6 above. Since for every $f \in H^{-1}(\mathbb{T})$ the equation

$$-\operatorname{div}(\mathbb{L}_\varepsilon \nabla u) = f \qquad \text{in } H^{-1}(\mathbb{T})$$

admits a unique weak solution $u \in H^1_{\text{per}}(\mathbb{T})$, the operator $\mathcal{L}_\varepsilon := -\operatorname{div}(\mathbb{L}_\varepsilon \nabla) \colon H^1_{\text{per}}(\mathbb{T}) \to H^{-1}(\mathbb{T})$ is invertible and we can consider the associated resolvent $\mathcal{R}_\varepsilon := \mathcal{L}_\varepsilon^{-1} \colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$, which is a positive, compact, self-adjoint operator on $L^2(\mathbb{T})$ and its operator norm is bounded by a constant independent of ε. The Spectral Theorem implies that the spectrum of the resolvent consists only of real, strictly positive eigenvalues, that are bounded from above by a constant independent of ε, and there is an orthonormal basis of $L^2(\mathbb{T})$ consisting of eigenfunctions of $\mathcal{R}_\varepsilon$. The assertion follows since (λ, u) is an eigenpair of $\mathcal{R}_\varepsilon$ if and only if $(\frac{1}{\lambda}, u)$ is an eigenpair of $\mathcal{L}_\varepsilon$.

Step 2: Proof of part (b).
As in step 2 in the proof of Proposition 3.2.6 we want to apply Lemma 3.6.3 to the resolvent operators $\mathcal{R}_\varepsilon$ defined in step 1. We therefor note that $u_\varepsilon := \mathcal{R}_\varepsilon f_\varepsilon \in H^1_{\text{per}}(\mathbb{T})$ and $u_0 := \mathcal{R}_0 f_0 \in H^1_{\text{per}}(\mathbb{T})$ are the unique weak solutions to

$$\begin{aligned} -\operatorname{div}(\mathbb{L}_\varepsilon \nabla u_\varepsilon) &= f_\varepsilon \quad \text{in } H^{-1}(\mathbb{T}), \\ -\operatorname{div}(\mathbb{L}_0 \nabla u_0) &= f_0 \quad \text{in } H^{-1}(\mathbb{T}), \end{aligned}$$

and thus the H-convergence of $(\mathbb{L}_\varepsilon)$ in connection with Lemma 3.6.4 yields that $f_\varepsilon \rightharpoonup f_0$ weakly in $L^2(\mathbb{T})$ implies $\mathcal{R}_\varepsilon f_\varepsilon = u_\varepsilon \to u_0 = \mathcal{R}_0 f_0$ strongly in $L^2(\mathbb{T})$. Now Lemma 3.6.3, applied to the space $H_\varepsilon = H_0 = H^1_{\text{per}}(M)$, concludes the proof. □

3.6.5. Local Coordinates and Homogenization Formula (Lemma 3.4.1 and Proposition 3.4.2

Proof of Lemma 3.4.1. We fix $x \in U'$ and denote by $\overline{\xi} = (\overline{\xi}^1, \dots, \overline{\xi}^n), \overline{\eta} = (\overline{\eta}^1, \dots, \overline{\eta}^n) \in \mathbb{R}^n$ the vectors associated to $\xi, \eta \in T_x M$ via

$$\overline{\xi}^i = g(\xi, \nabla_g x^i) \qquad \text{and} \qquad \overline{\eta}^i = g(\eta, \nabla_g x^i)$$

for $i = 1, \dots, n$. There is a constant $C > 0$ such that

$$\tfrac{1}{C}|\overline{\xi}|^2 \le \sum_{i,j=1}^n g^{ij}(x)\overline{\xi}^i\overline{\xi}^j = g(\xi,\xi)(x) \le C|\overline{\xi}|^2 \qquad \text{and} \qquad \tfrac{1}{C} \le \rho(x) \le C,$$

where (g^{ij}) denotes the inverse of the matrix representation (g_{ij}) of g in local coordinates, i.e. $g_{ij} = g(\nabla_g x^i, \nabla_g x^j)$. Note that the constant C does not depend on x, since the metric $g(\cdot,\cdot)(x)$ is continuous in x, Ψ is a diffeomorphism, and $A \Subset \Psi(U)$ is relatively-compact. Then the uniform ellipticity of $\mathbb{L}$ directly implies

$$a\overline{\xi} \cdot \overline{\xi} = \rho g(\mathbb{L}\xi, \xi) \ge \lambda \rho g(\xi,\xi) \ge \tfrac{1}{C'}|\overline{\xi}|^2$$

and

$$a\overline{\xi} \cdot \overline{\eta} = \rho g(\mathbb{L}\xi, \eta) \le \Lambda \rho |\xi|_g |\eta|_g \le C'|\overline{\xi}||\overline{\eta}|$$

for some $C' > 0$, which gives the assertion. □

Proof of Proposition 3.4.2. We will only show the implication $(ii) \Rightarrow (i)$, since the opposite direction can be proved in the same way.

Suppose that $m_0(U') < 0$ For $f \in L^2(U')$ we denote by $u_\varepsilon \in H_0^1(U')$ the unique weak solution to

$$-\operatorname{div}_{g,\mu}(\mathbb{L}_\varepsilon \nabla_g u_\varepsilon) = f \qquad \text{in } H^{-1}(U').$$

Then for every $\psi \in C_c^\infty(A)$ we have

$$\int_A a(x)\nabla u_\varepsilon(x)\cdot\nabla\psi(x)\,\mathrm{d}x = \int_{U'} g(\mathbb{L}_\varepsilon \nabla_g u_\varepsilon, \nabla_g \psi)\,\mathrm{d}\mu = \int_{U'} f\psi\,\mathrm{d}\mu = \int_A f(x)\psi(x)\rho(x)\,\mathrm{d}x, \tag{3.43}$$

that means, $u_\varepsilon \in H_0^1(A)$ is the unique weak solution to

$$-\operatorname{div}(a\nabla u_\varepsilon) = \rho f \qquad \text{in } H^{-1}(A).$$

From $a_\varepsilon \xrightarrow{H} a_0$ on A we conclude

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 & \text{weakly in } H^1(A), \\ a_\varepsilon \nabla u_\varepsilon \rightharpoonup a_0 \nabla u_0 & \text{weakly in } L^2(A), \end{cases} \tag{3.44}$$

where $u_0 \in H_0^1(A)$ is the unique weak solution to

$$-\operatorname{div}(a_0\nabla u_0) = \rho f \qquad \text{in } H^{-1}(A),$$

which by arguments similar to (3.43) implies that $u_0 \in H_0^1(U')$ is the unique weak solution to

$$-\operatorname{div}_{g,\mu}(\mathbb{L}_0 \nabla_g u_0) = f \qquad \text{in } H^{-1}(U').$$

We first note that (3.44) immediately gives

$$u_\varepsilon \rightharpoonup u_0 \qquad \text{weakly in } H^1(U'),$$

so it only remains to show the convergence of the fluxes. Therefor we fix $\eta \in L^2(TU')$ and set $\overline{\eta} = (\overline{\eta}^1, \ldots, \overline{\eta}^n) \in L^2(A)$ with $\overline{\eta}^i := g(\eta, \nabla_g x^i)$ for $i = 1, \ldots, n$. Then we find

$$\begin{aligned} \int_{U'} g(\mathbb{L}_\varepsilon \nabla_g u_\varepsilon, \eta)\,\mathrm{d}\mu &= \int_A a_\varepsilon(x)\nabla u_\varepsilon \cdot \overline{\eta}\,\mathrm{d}x \to \int_A a_0(x)\nabla u_0 \cdot \overline{\eta}\,\mathrm{d}x \\ &= \int_{U'} g(\mathbb{L}_0 \nabla_g u_0, \eta)\,\mathrm{d}\mu, \end{aligned}$$

which means

$$\mathbb{L}_\varepsilon \nabla_g u_\varepsilon \rightharpoonup \mathbb{L}_0 \nabla_g u_0 \qquad \text{weakly in } L^2(TU'). \qquad \square$$

4. Application to Uniformly bi-Lipschitz Diffeomorphic Manifolds

In this chapter we apply the H-compactness result Theorem 3.2.2 to the coefficient fields of the pulled back Laplace-Beltrami operators. Precisely we consider the case of uniformly bi-Lipschitz diffeomorphic manifolds (i.e. with ε-uniformal constant in Definition 1.3.1, see Definition 4.1.2 below), which results in uniformly elliptic coefficient fields (in the sense of (3.1) and (3.2)) on the reference manifold. The H-compactness result (Theorem 3.2.2) in combination with Propositions 3.2.4 and 3.2.6 will yield Mosco- and spectral compactness in the sense of Definitions 1.3.4 and 1.4.2.

For the sake of readability we postpone every proof to Section 4.3.

This chapter is based on the article [HMN19] of Jun Masamune, Stefan Neukamm and the author, also contains a way to extend the results to a wider class of manifolds, demonstrated on the torus.

4.1. Setting and Results

To study the applications of our H-compactness result Theorem 3.2.2 to Mosco- and spectral convergence of families of bi-Lipschitz diffeomorphic manifolds we first make the relation between the Laplace-Beltrami operator on a manifold and the corresponding coefficient field on the reference manifold concrete by formulating a transformation lemma. In order to do so we introduce the following notation, which we will keep for the rest of this chapter: For two weighted Riemannian manifolds (M, g, μ) and (M_0, g_0, μ_0) with a diffeomorphism $h\colon M_0 \to M$, we denote by $\overline{f} := f \circ h$ the pullback of a function f on M along h. Moreover, we denote by $(dh^{-1})^*\colon TM_0 \to TM$ the adjoint of the differential $dh^{-1}\colon TM \to TM_0$ of h^{-1} given by

$$g((dh^{-1})^*\xi, \eta)(h(x)) = g_0(\xi, dh^{-1}\eta)(x) \qquad \text{for all } \xi \in T_xM_0,\ \eta \in T_{h(x)}M.$$

Lemma 4.1.1 (Transformation Lemma). *Let (M, g, μ) and (M_0, g_0, μ_0) be weighted Riemannian manifolds, and denote by σ and σ_0 the densities of μ and μ_0 w.r.t. the Riemannian volume measures associated with g and g_0, respectively. Let further $h\colon M_0 \to$*

M be a diffeomorphism. We define a density function ρ and a coefficient field $\mathbb{L}$ on M_0 by

$$\rho := \tfrac{\overline{\sigma}}{\sigma_0}\sqrt{\tfrac{\det \overline{g}}{\det g_0}} \qquad \textit{and} \qquad g_0(\mathbb{L}\xi,\eta) = \rho\,\overline{g}((dh^{-1})^*\xi,(dh^{-1})^*\eta),$$

as well as a metric $\hat{g}_0$ and a measure $\hat{\mu}_0$ on M_0 by

$$d\hat{\mu}_0 := \rho d\mu_0 \qquad \textit{and} \qquad \hat{g}_0(\mathbb{L}\xi,\eta) := \rho\, g_0(\xi,\eta),$$

Then the following are equivalent:

(a) $u \in H^1(M)$ is a solution to

$$(m-\Delta_{g,\mu})u = f \qquad \textit{in } H^{-1}(M,g,\mu).$$

(b) $\overline{u} \in H^1(M_0)$ is a solution to

$$(m\rho - \operatorname{div}_{g_0,\mu_0}(\mathbb{L}\nabla_{g_0}))\overline{u} = \rho\overline{f} \qquad \textit{in } H^{-1}(M_0,g_0,\mu_0).$$

(c) $\overline{u} \in H^1(M_0)$ is a solution to

$$(m-\Delta_{\hat{g}_0,\hat{\mu}_0})\overline{u} = \overline{f} \qquad \textit{in } H^{-1}(M_0,\hat{g}_0,\hat{\mu}_0).$$

As one can see, the coefficient field $\mathbb{L}$ in Lemma 4.1.1 is strongly related to the differential dh^{-1} of the inverse of the diffeomorphism between the manifolds. Obviously, for an arbitrary bi-Lipschitz diffeomorphic family of Riemannian manifolds the corresponding coefficient fields on the reference manifold are not necessarily uniformly elliptic. So, in order to apply Theorem 3.2.2 to the family of coefficient fields, it is natural to introduce uniform restrictions to the diffeomorphism in the following sense:

Definition 4.1.2 (Uniformly bi-Lipschitz Diffeomorphic Families of Manifolds). *A family of weighted Riemannian manifolds $(M_\varepsilon,g_\varepsilon,\mu_\varepsilon)$ is called* uniformly bi-Lipschitz diffeomorphic*, if there are a weighted Riemannian manifold (M_0,g_0,μ_0) and a constant $C>0$ such that for every ε there are diffeomorphisms $h_\varepsilon\colon M_0 \to M_\varepsilon$ with*

$$\tfrac{1}{C}|\xi|_{g_0} \le |dh_\varepsilon(x)\xi|_{g_\varepsilon} \le C|\xi|_{g_0} \qquad \textit{for all } x\in M_0 \textit{ and } \xi \in T_xM_0. \tag{4.1}$$

We call (M_0,g_0,μ_0) reference manifold.

From the construction in Lemma 4.1.1 it is easy to see that in this setting the Laplace-Beltrami operators on the manifolds M_ε correspond to elliptic operators on the reference manifold M_0 of the form $-\operatorname{div}(\mathbb{L}_\varepsilon\nabla)$ with uniformly elliptic coefficient fields $\mathbb{L}_\varepsilon$, precisely

$$g_0(\xi,\mathbb{L}_\varepsilon\xi) \ge \tfrac{1}{C^{n+2}}|\xi|^2_{g_0}, \qquad g_0(\xi,\mathbb{L}_\varepsilon^{-1}\xi) \ge C^{n+2}|\xi|^2_{g_0} \qquad \text{for every } \xi \in TM_0,$$

with the constant C from Definition 4.1.2. We are now in position to formulate the corresponding H-compactness result to Theorem 3.2.2 for uniformly bi-Lipschitz diffeomorphic manifolds.

Lemma 4.1.3 (H-Compactness of Uniformly bi-Lipschitz Diffeomorphic Manifolds). *Let (M_0, g_0, μ_0) be a weighted Riemannian manifold such that $H_0^1(M_0, g_0, \mu_0)$ is compactly embedded in $L^2(M_0, g_0, \mu_0)$. Assume the family $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ of weighted Riemannian manifolds to be uniformly bi-Lipschitz diffeomorphic to the reference manifold (M_0, g_0, μ_0) via the diffeomorphisms $h_\varepsilon \colon M_0 \to M_\varepsilon$, and denote by σ_ε and σ_0 the densities of μ_ε and μ_0 w.r.t. the Riemannian volume measures associated with g_ε and g_0, respectively. Define the density ρ_ε and the coefficient field $\mathbb{L}_\varepsilon$ on M_0 by*

$$\rho_\varepsilon := \tfrac{\overline{\sigma}_\varepsilon}{\sigma_0}\sqrt{\tfrac{\det \overline{g}_\varepsilon}{\det g_0}} \qquad \text{and} \qquad g_0(\mathbb{L}_\varepsilon \xi, \eta) = \rho_\varepsilon\, \overline{g}_\varepsilon((dh_\varepsilon^{-1})^*\xi, (dh_\varepsilon^{-1})^*\eta). \tag{4.2}$$

Then there exists a (not relabeled) subsequence such that the following holds:

(a) There is a density ρ_0 and a uniformly elliptic coefficient field $\mathbb{L}_0$ on M_0 such that $\rho_\varepsilon \overset{*}{\rightharpoonup} \rho_0$ weakly-$*$ in $L^\infty(M_0, g_0, \mu_0)$, and $\mathbb{L}_\varepsilon \overset{H}{\to} \mathbb{L}_0$ in (M_0, g_0, μ_0).

(b) Define a measure $\hat{\mu}_0$ and a metric $\hat{g}_0$ on M_0 via the identities

$$\mathrm{d}\hat{\mu}_0 := \rho_0 \mathrm{d}\mu_0 \qquad \text{and} \qquad \hat{g}_0(\mathbb{L}_0\xi, \eta) = \rho_0\, g_0(\xi, \eta).$$

Let $m > m_0(M_0, g_0, \mu_0)$ (with m_0 as in (3.4)) and let $u_\varepsilon \in H_0^1(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ and $u_0 \in H_0^1(M_0, \hat{g}_0, \hat{\mu}_0)$ denote the unique solutions to

$$(m - \Delta_{g_\varepsilon,\mu_\varepsilon})u_\varepsilon = f_\varepsilon \qquad \text{in } H^{-1}(M_\varepsilon, g_\varepsilon, \mu_\varepsilon), \tag{4.3}$$

$$(m - \Delta_{\hat{g}_0,\hat{\mu}_0})u_0 = f_0 \qquad \text{in } H^{-1}(M_0, \hat{g}_0, \hat{\mu}_0). \tag{4.4}$$

Then

$$f_\varepsilon \rightharpoonup f_0 \qquad \text{weakly in } L^2((M_\varepsilon, \mu_\varepsilon) \to (M_0, \hat{\mu}_0))$$

implies

$$u_\varepsilon \to u_0 \qquad \text{strongly in } L^2((M_\varepsilon, \mu_\varepsilon) \to (M_0, \hat{\mu}_0)).$$

As pointed out in Section 3.2 (cf. Propositions 3.2.4, 3.2.6 and 3.2.7), H-compactness provides a tool to gain Mosco- and spectral convergence in the sense of Definitions 1.3.4 and 1.4.2. Indeed, since Mosco-convergence of the manifolds M_ε is equivalent to Mosco-convergence of the pulled back Dirichlet energies on the reference manifold, the following result is a direct consequence of Lemma 4.1.3 in combination with Proposition 3.2.4, and there is no further proof required:

Proposition 4.1.4 (Mosco-Convergence). *In the setting of Lemma 4.1.3 the family $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ Mosco-converges to $(M_0, \hat{g}_0, \hat{\mu}_0)$ w.r.t. L^2.*

The spectral convergence result Proposition 3.2.6 cannot be directly translated like the Mosco-convergence result, because the underlying eigenvalue equation

$$-\operatorname{div}_{g_\varepsilon,\mu_\varepsilon}(\nabla_{g_\varepsilon} u) = \lambda u \qquad \text{in } H^{-1}(M_\varepsilon)$$

reads on the reference manifold by Lemma 4.1.1

$$-\operatorname{div}_{g_0,\mu_0}(\mathbb{L}_\varepsilon \nabla_{g_0} \overline{u}) = \lambda \rho_\varepsilon \overline{u} \qquad \text{in } H^{-1}(M_0),$$

which is obviously not the eigenvalue equation associated with the considered operator $-\operatorname{div}_{g_0,\mu_0}(\mathbb{L}_\varepsilon \nabla_{g_0})$ treated in Proposition 3.2.6. However, the proof of Proposition 3.2.6 can be extended to the case of varying manifolds.

Proposition 4.1.5 (Spectral Convergence)**.** *If in the setting of Lemma 4.1.3 M_0 is compact with $m_0(M_0, g_0, \mu_0) < 0$, then the family $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ spectral converges to $(M_0, \hat{g}_0, \hat{\mu}_0)$ w.r.t. L^2.*

As in Section 3.2 we consider the n-dimensional torus to show how to extend our results to manifolds that do not satisfy the condition $m_0(M_0, g_0, \mu_0) < 0$ by turning away from the Dirichlet-Laplace-Beltrami operator; cf. the discussion before Proposition 3.2.7 for more details.

Proposition 4.1.6 (Spectral Convergence for the Torus)**.** *Assume in the setting of Lemma 4.1.3 the reference manifold to be the n-dimensional torus $M_0 = \mathbb{T}$, consider the operators*

$$\begin{cases} -\Delta_{g_\varepsilon,\mu_\varepsilon} \colon H^1_{\text{per}}(M_\varepsilon, g_\varepsilon, \mu_\varepsilon) \to H^{-1}(M_\varepsilon, g_\varepsilon, \mu_\varepsilon) & \text{for } \varepsilon > 0, \\ -\Delta_{\hat{g}_0,\hat{\mu}_0} \colon H^1_{\text{per}}(\mathbb{T}, \hat{g}_0, \hat{\mu}_0) \to H^{-1}(\mathbb{T}, \hat{g}_0, \hat{\mu}_0) & \text{for } \varepsilon = 0, \end{cases}$$

and let

$$0 < \lambda_{\varepsilon,1} \le \lambda_{\varepsilon,2} \le \lambda_{\varepsilon,3} \le \cdots,$$

denote the list of increasingly ordered eigenvalues with eigenvalues being repeated according to their multiplicity. Let $u_{\varepsilon,1}, u_{\varepsilon,2}, u_{\varepsilon,3}, \ldots$ denote the associated eigenfunctions. Then for all $k \in \mathbb{N}$,

$$\lambda_{\varepsilon,k} \to \lambda_{0,k},$$

and if $s \in \mathbb{N}$ is the multiplicity of $\lambda_{0,k}$, i.e.

$$\lambda_{0,k-1} < \lambda_{0,k} = \cdots = \lambda_{0,k+s-1} < \lambda_{0,k+s} \qquad \text{(with the convention } \lambda_{0,0} = 0\text{)},$$

there exists a sequence $(\tilde{u}_{\varepsilon,k})_\varepsilon$ of linear combinations of $u_{\varepsilon,k}, \ldots, u_{\varepsilon,k+s-1}$ such that

$$\tilde{u}_{\varepsilon,k} \to u_{0,k} \qquad \text{strongly in } L^2((M_\varepsilon, \hat{\mu}_\varepsilon) \to (\mathbb{T}, \hat{\mu}_0)).$$

We finally summarize some useful results and explicit formulas concerning the special case of submanifolds of $\mathbb{R}^m$. We do not give a proof of the following corollary, as it is just a re-formulation of Lemmas 4.1.1 and 4.1.3 and Propositions 4.1.4 to 4.1.6 in this situation.

Corollary 4.1.7. *Consider the setting of Lemma 4.1.3, and assume that*

- M_ε *are* n*-dimensional submanifolds of the Euclidean space* $\mathbb{R}^m$ *with* g_ε *and* μ_ε *induced by the standard metric and measure of* $\mathbb{R}^m$*;*
- *the reference manifold* M_0 *is a subset of the Euclidean space* $\mathbb{R}^n$*, i.e.,* $M_0 \subseteq \mathbb{R}^n$*,* $g_0(\xi, \eta) := \xi \cdot \eta$*, and* $\mathrm{d}\mu_0 = \mathrm{d}x$*.*

Then:

(a) The formulas (4.2) *become*

$$\rho_\varepsilon = \sqrt{\det(dh_\varepsilon^\intercal dh_\varepsilon)} \qquad \text{and} \qquad \mathbb{L}_\varepsilon = \rho_\varepsilon (dh_\varepsilon^\intercal dh_\varepsilon)^{-1},$$

where dh_ε *denotes the Jacobian of* h_ε*.*

(b) There are a density ρ_0 *on* M_0 *and a coefficient field* $\mathbb{L}_0 \in \mathcal{M}(M_0, \frac{1}{C_0}, C_0)$ *(with* $C_0 > 0$ *only depending on the dimension* n *and the constant* C *in* (4.1)*) such that*

$$\begin{aligned} \rho_\varepsilon &= \sqrt{\det(dh_\varepsilon^\intercal dh_\varepsilon)} \overset{*}{\rightharpoonup} \rho_0 && \text{weakly-}* \text{ in } L^\infty(M_0), \\ \mathbb{L}_\varepsilon &= \rho_\varepsilon (dh_\varepsilon^\intercal dh_\varepsilon)^{-1} \overset{H}{\to} \mathbb{L}_0 && \text{on } M_0, \end{aligned}$$

for a (not relabeled) subsequence.

(c) For the subsequence in (b), the manifolds (M_ε) *Mosco-converge w.r.t.* L^2 *and the limiting Riemannian manifold* $(M_0, \hat{g}_0, \hat{\mu}_0)$ *is given by*

$$\mathrm{d}\hat{\mu}_0 = \rho_0 \mathrm{d}x \qquad \text{and} \qquad \hat{g}_0(\xi, \eta) = \rho_0 \mathbb{L}_0^{-1} \xi \cdot \eta.$$

(d) If additionally $M_0 \subseteq \mathbb{R}^n$ *is a bounded open set with a non-empty Lipschitz boundary, then the manifolds* (M_ε) *spectral converge w.r.t.* L^2 *to the limiting manifold in (c) (along the subsequence from (b)).*

(e) The conclusion of (d) about spectral convergence also holds in the case $M_0 = \mathbb{T}$*.*

Remark 4.1.8 (Realizability of $(M_0, \hat{g}_0, \hat{\mu}_0)$). *In general, the measure $\hat{\mu}_0$ has a non-trivial density against the Riemannian volume measure associated to $\hat{g}_0$. But if we have*

$$\det \mathbb{L}_0 = \rho_0^{n-2},$$

which implies

$$\sqrt{\det \hat{g}_0} = \sqrt{\det(\rho_0 \mathbb{L}_0^{-1})} = \sqrt{\rho_0^n \rho_0^{2-n}} = \rho_0,$$

then $\hat{\mu}_0$ is the Riemannian volume measure associated with $\hat{g}_0$, and therefore, if the limiting metric $\hat{g}_0$ is smooth, Nash's Embedding Theorem guarantees that $(M_0, \hat{g}_0, \hat{\mu}_0)$ is realizable in $\mathbb{R}^m$ with m large enough, i.e., there is an isometry $h_0 \colon (M_0, \hat{g}_0, \hat{\mu}_0) \to \mathbb{R}^m$ such that $N_0 := h_0(M_0)$ is an n-dimensional submanifold of $\mathbb{R}^m$ (with induced metric and measure from $\mathbb{R}^m$). Such an embedding is characterized by the identity

$$dh_0^\intercal dh_0 = \rho_0 \mathbb{L}_0^{-1}. \tag{4.5}$$

Note that if one introduces a different reference manifold $\widetilde{M_0}$ with a diffeomorphism $\psi \colon \widetilde{M_0} \to M_0$, one can consider $\tilde{h}_\varepsilon := h_\varepsilon \circ \psi \colon \widetilde{M_0} \to M_\varepsilon$ instead of h_ε, but these (bi-Lipschitz) diffeomorphisms do not necessarily satisfy the uniform ellipticity conditions (4.1)*. However, going through the formulas in Corollary 4.1.7 one ends up with the isometric embedding $\tilde{h}_0 = h_0 \circ \psi \colon \widetilde{M_0} \to \mathbb{R}^m$, which represents the same limiting manifold. Thus, in practice, the calculations to identify the limiting manifold can be done with diffeomorphisms which are not uniformly elliptic in the sense of* (4.1)*, as long as there exist uniformly elliptic diffeomorphisms (see for instance the examples of perturbed spheres in Section 4.2 below).*

4.2. Examples

We want to adopt the abstract results for layered structures discussed in Section 3.5 to produce some concrete examples of spectral (and Mosco-) convergent 2-dimensional submanifolds of $\mathbb{R}^3$.

4.2.1. Concentric Laminate-Like Perturbations of Voronoi Tesselated Manifolds

A graphical surface with concentric corrugations

Following Definition 4.1.2 we start with the reference manifold M_0, which we want to be the flat rectangle

$$M_0 = \{(r, \theta); r \in (\delta, R), \theta \in [0, 1)\}$$

for some $R > \delta > 0$. Now we define a family $M_\varepsilon = h_\varepsilon(M_0)$ of 2-dimensional submanifolds of $\mathbb{R}^3$ (with standard metric and measure induced from $\mathbb{R}^3$) via $h_\varepsilon \colon M_0 \to \mathbb{R}^3$,

$$h_\varepsilon(r,\theta) = \begin{pmatrix} r\sin 2\pi\theta \\ r\cos 2\pi\theta \\ \varepsilon f(\frac{r}{\varepsilon}) \end{pmatrix}, \tag{4.6}$$

for $\varepsilon \in \{\frac{1}{k}, k \in \mathbb{N}\}$, with a smooth, 1-periodic function $f \colon \mathbb{R} \to \mathbb{R}$. Note that the excluded circle around the origin with radius δ ensures that the defined manifolds are indeed uniformly bi-Lipschitz diffeomorphic. In Figure 4.1 we present M_ε for some values of ε with the periodic function $f(y) := \psi(y - \lfloor y \rfloor)$, where $\lfloor y \rfloor$ denotes the integer part of y, i.e. $\lfloor y \rfloor \in \mathbb{Z}$ with $\lfloor y \rfloor \leq y < \lfloor y \rfloor + 1$, and $\psi \colon \mathbb{R} \to \mathbb{R}_{\geq 0}$ denotes the mollifier

$$\psi(t) := \begin{cases} \exp\left(1 - \frac{1}{1-(2t-1)^2}\right), & \text{if } 0 < t < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.7}$$

$\varepsilon = \frac{1}{4}$ $\varepsilon = \frac{1}{8}$ $\varepsilon = \frac{1}{16}$

Figure 4.1.: A family of graphical surfaces with concentric periodic corrugations. The three pictures on the left show M_ε defined via (4.6) with $f(y) = \psi(y - \lfloor y \rfloor)$ for ψ as in (4.7) and decreasing values of ε. The picture on the right shows the limiting surface N_0 defined via (4.8). As $\varepsilon \searrow 0$ the spectrum of the Laplace-Beltrami operator on M_ε converges to the spectrum of the Laplace-Beltrami operator on N_0.

We follow Corollary 4.1.7 and calculate

$$dh_\varepsilon^\intercal dh_\varepsilon = \begin{pmatrix} f'(\frac{r}{\varepsilon})^2 + 1 & 0 \\ 0 & 4\pi^2 r^2 \end{pmatrix},$$

which gives the density

$$\rho_\varepsilon = \sqrt{\det(dh_\varepsilon^\intercal dh_\varepsilon)} = 2\pi r \sqrt{f'(\tfrac{r}{\varepsilon})^2 + 1},$$

and the coefficient field

$$\mathbb{L}_\varepsilon = \rho_\varepsilon (dh_\varepsilon^\intercal dh_\varepsilon)^{-1} = \begin{pmatrix} \left(\frac{1}{2\pi r}\sqrt{f'(\frac{r}{\varepsilon})^2 + 1}\right)^{-1} & 0 \\ 0 & \frac{1}{2\pi r}\sqrt{f'(\frac{r}{\varepsilon})^2 + 1} \end{pmatrix}.$$

One can see that $\rho_\varepsilon \overset{*}{\rightharpoonup} \rho_0$ weakly-$*$ in $L^\infty(M_0)$ with

$$\rho_0(r) = 2\pi r \int_0^1 \sqrt{f'(y)^2 + 1}\,\mathrm{d}y,$$

and using (3.26) we find that $\mathbb{L}_\varepsilon \xrightarrow{H} \mathbb{L}_0$ with

$$\mathbb{L}_0 = \begin{pmatrix} \left(\frac{1}{2\pi r}\int_0^1 \sqrt{f'(y)^2+1}\,\mathrm{d}y\right)^{-1} & 0 \\ 0 & \frac{1}{2\pi r}\int_0^1 \sqrt{f'(y)^2+1}\,\mathrm{d}y \end{pmatrix} = \begin{pmatrix} \frac{4\pi^2 r^2}{\rho_0(r)} & 0 \\ 0 & \frac{\rho_0(r)}{4\pi^2 r^2} \end{pmatrix}.$$

Thus the limiting metric on M_0 is given by

$$\hat{g}_0(\xi,\eta) = \rho_0 \mathbb{L}_0^{-1}\xi\cdot\eta = \begin{pmatrix} \frac{\rho_0(r)^2}{4\pi^2 r^2} & 0 \\ 0 & 4\pi^2 r^2 \end{pmatrix} \xi\cdot\eta.$$

As described in Remark 4.1.8 we finally can find an isometric embedding $h_0\colon M_0 \to \mathbb{R}^3$ via $dh_0^\intercal dh_0 = \rho_0 \mathbb{L}_0^{-1}$, namely

$$h_0(r,\theta) = \begin{pmatrix} r\sin 2\pi\theta \\ r\cos 2\pi\theta \\ \int_0^r \sqrt{\frac{\rho_0(t)^2}{4\pi^2 t^2} - 1}\,\mathrm{d}t \end{pmatrix}. \tag{4.8}$$

That means, the submanifold $N_0 := h_0(M_0)$ of $\mathbb{R}^3$ (with the standard measure and metric induced from $\mathbb{R}^3$), which is illustrated in Figure 4.1, is the spectral (and Mosco-) limit of the family (M_ε). Note that the form of h_0 does not depend on the initial choice of δ, so we can pass to the reference manifold $M_0 = (0,R)\times[0,1)$, and the excluded origin of the manifolds M_ε coincides with the apex of the cone-shaped manifold N_0.

A sphere with radial perturbations oscillating with the latitude

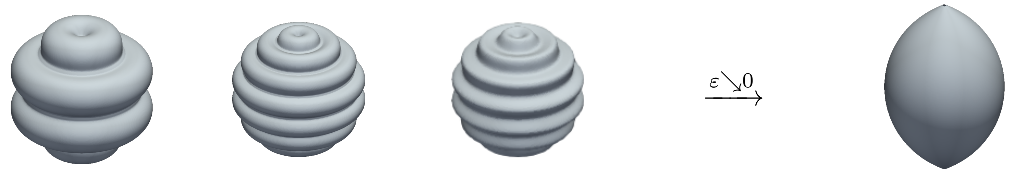
Instead of a graph as in the example above we now consider a 2-dimensional (pointed) sphere with radial perturbations in the same manner. If we took the (unperturbed) sphere as the reference manifold, one could immediately see that for continuously differentiable perturbations with uniformly bounded derivatives, the generated manifolds are uniformly bi-Lipschitz diffeomorphic. Thus, in order to simplify the calculations, Remark 4.1.8 allows us to choose another reference manifold instead of the sphere. We set

$$M_0 = \{(\varphi,\theta); \varphi\in(\delta,1-\delta), \theta\in[0,1)\}$$

and define the family of submanifolds $M_\varepsilon := h_\varepsilon(M_0) \subseteq \mathbb{R}^3$ (with the induced metric and measure) via $h_\varepsilon\colon M_0 \to \mathbb{R}^3$, with

$$h_\varepsilon(\varphi,\theta) = \left(1+\varepsilon f(\tfrac{\varphi}{\varepsilon})\right) \begin{pmatrix} \sin\pi\varphi\sin 2\pi\theta \\ \sin\pi\varphi\cos 2\pi\theta \\ \cos\pi\varphi \end{pmatrix} \tag{4.9}$$

for $\varepsilon \in \{\frac{1}{k}; k \in \mathbb{N}\}$, with a smooth, 1-periodic function $f\colon \mathbb{R} \to [0, \infty)$. As in the example above we excluded a neighborhood around the poles to create a uniformly bi-Lipschitz setting (which can be verified easily by considering the sphere as reference manifold, see the discussion in the beginning of this example). In Figure 4.2 we again choose $f(y) = \psi(y - \lfloor y \rfloor)$ with ψ as in (4.7) to illustrate M_ε for some values of ε.



$\varepsilon = \frac{1}{4}$ $\varepsilon = \frac{1}{8}$ $\varepsilon = \frac{1}{16}$

Figure 4.2.: A family of spheres with periodic radial perturbations oscillating with the latitude. The three pictures on the left show M_ε defined by (4.9) with $f(y) = \psi(y - \lfloor y \rfloor)$ with ψ as in (4.7) and decreasing values of ε. The picture on the right shows the limiting surface N_0 defined via (4.10).

As in the previous example we can calculate the limiting measure with density

$$\rho_0(\varphi) = 2\pi \sin \pi\varphi \int_0^1 \sqrt{f'(y)^2 + \pi^2}\, \mathrm{d}y$$

and the metric

$$\hat{g}_0 = \rho_0 \mathbb{L}_0^{-1} = \begin{pmatrix} \frac{\rho_0^2}{4\pi^2 \sin^2 \pi\varphi} & 0 \\ 0 & 4\pi^2 \sin^2 \pi\varphi \end{pmatrix}.$$

We can find an isometric embedding $h_0\colon M_0 \to \mathbb{R}^3$ via $dh_0^\intercal dh_0 = \rho_0 \mathbb{L}_0^{-1}$, namely

$$h_0(\varphi, \theta) = \begin{pmatrix} \sin \pi\varphi \sin 2\pi\theta \\ \sin \pi\varphi \cos 2\pi\theta \\ \int_0^\varphi \sqrt{\frac{\rho_0(t)^2}{4\pi^2 \sin^2 \pi t} - 4\pi^2 \cos^2 \pi t}\, \mathrm{d}t \end{pmatrix}. \tag{4.10}$$

Thus the submanifold $N_0 := h_0(M_0)$ of $\mathbb{R}^3$, pictured in Figure 4.2, is the spectral (and Mosco-) limit of the sequence (M_ε). As in the example above, we can extend the embedding h_0 to the reference manifold $M_0 = (0, 1) \times [0, 1)$ to get sphere-like manifolds, whose (excluded) poles coincide with the (excluded) poles of the manifold N_0.

A locally corrugated graphical surface

We want to demonstrate the local character of the limiting process by an example with oscillations in several Voronoi cells. We fix an open, bounded set $Y \subseteq \mathbb{R}^2$, and consider a finite set $Z \Subset Y$ of isolated points. For every point $z \in Z$ we utilize a smooth function

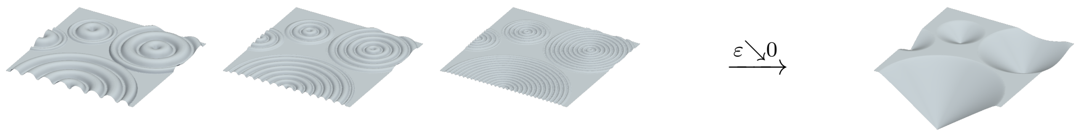
$\psi_z \colon [0,\infty) \to [0,1]$ to define a rotationally symmetric cut-off function $\psi_z(|\cdot - z|)$ such that

$$\begin{cases} \psi_z(0) = 1, \\ \operatorname{supp} \psi_z(|\cdot - z|) \cap \operatorname{supp} \psi_{z'}(|\cdot - z'|) = \emptyset \text{ for all } z' \in Z \setminus \{z\}. \end{cases}$$

Now we take $M_0 := Y \setminus Z$ as the reference manifold and define the submanifold $M_\varepsilon := h_\varepsilon(M_0)$ of $\mathbb{R}^3$ via

$$h_\varepsilon(x) := \sum_{z \in Z} \varepsilon f\big(\tfrac{|x-z|}{\varepsilon}\big) \psi_z(|x-z|), \tag{4.11}$$

with some smooth, 1-periodic function $f \colon \mathbb{R} \to \mathbb{R}$. In Figure 4.3 we choose $f(y) = \psi(y - \lfloor y \rfloor)$ with ψ as in (4.7) to illustrate M_ε for some values of ε.



$\varepsilon = \frac{1}{2}$ $\quad$ $\varepsilon = \frac{1}{4}$ $\quad$ $\varepsilon = \frac{1}{8}$

Figure 4.3.: A family of locally corrugated graphical surfaces. The three pictures on the left show M_ε defined via (4.11) with $f(y) = \psi(y - \lfloor y \rfloor)$ with ψ as in (4.7) and decreasing values of ε. The picture on the right shows the limiting surface N_0 defined via (4.12).

In each Voronoi cell we can do the same calculations as in the previous examples, and get a function $h_0 \colon M_0 \to \mathbb{R}$,

$$h_0(x) := x \mapsto \sum_{z \in Z} \int_0^{|x-z|} \sqrt{\tfrac{\rho_{0,z}(t)^2}{t^2} - 1}\, \mathrm{d}t, \tag{4.12}$$

where $\rho_{0,z}(r) = r \int_0^1 \sqrt{f'(y)^2 \psi_z(r)^2 + 1}\, \mathrm{d}y$. The graph $h_0(M_0)$, which is shown in Figure 4.3, is the spectral (and Mosco-) limit of the family (M_ε).

4.2.2. Laminate-Like Perturbations of Spherically Symmetric Manifolds

A graphical surface with star-shaped corrugations

Analogously to the first of the previous two examples, we consider the reference manifold

$$M_0 := \{(r, \theta); r \in (\delta, R), \theta \in [0, 1)\}$$

for some $R > \delta > 0$, and define the family of submanifolds $M_\varepsilon = h_\varepsilon(M_0)$ of $\mathbb{R}^3$ via $h_\varepsilon \colon M_0 \to \mathbb{R}^3$,

$$h_\varepsilon(r,\theta) = \begin{pmatrix} r\sin 2\pi\theta \\ r\cos 2\pi\theta \\ \varepsilon f(\frac{\theta}{\varepsilon}) \end{pmatrix}, \tag{4.13}$$

for $\varepsilon \in \{\frac{1}{k}; k \in \mathbb{N}\}$, with some smooth, 1-periodic function $f \colon \mathbb{R} \to \mathbb{R}$. Due to the exclusion of the neighborhood of the origin these manifolds are uniformly bi-Lipschitz diffeomorphic. In Figure 4.4 we choose $f(y) = \psi(y - \lfloor y \rfloor)$ with ψ as in (4.7) to illustrate M_ε for some values of ε.

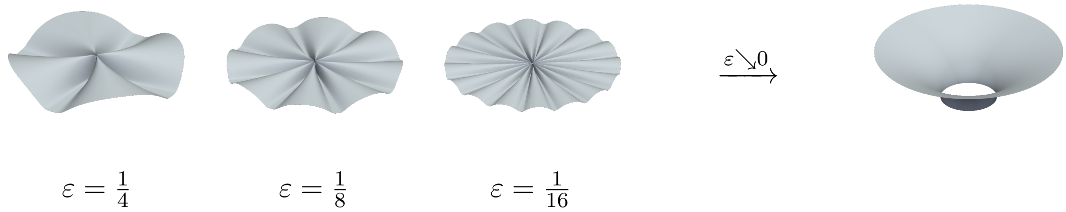


Figure 4.4.: A family of graphical surfaces with star-shaped periodic corrugations. The three pictures on the left show M_ε defined by (4.13) with $f(y) = \psi(y - \lfloor y \rfloor)$ as in (4.7) and decreasing values of ε. The picture on the right shows the limiting surface N_0 defined via (4.14).

Following the path described in Corollary 4.1.7 we calculate

$$\rho_\varepsilon = \sqrt{\det(dh_\varepsilon^\intercal dh_\varepsilon)} = \sqrt{f'(\tfrac{\theta}{\varepsilon})^2 + 4\pi^2 r^2}$$

and the coefficient field

$$\mathbb{L}_\varepsilon = \rho_\varepsilon (dh_\varepsilon^\intercal dh_\varepsilon)^{-1} = 1/\rho_\varepsilon \begin{pmatrix} f'(\frac{\theta}{\varepsilon})^2 + 4\pi^2 r^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

We find $\rho_\varepsilon \overset{*}{\rightharpoonup} \rho_0$ weakly-$*$ in $L^\infty(M_0)$ with

$$\rho_0(r) = \int_0^1 \sqrt{f'(y)^2 + 4\pi^2 r^2}\,\mathrm{d}y,$$

and using (3.25) we see $\mathbb{L}_\varepsilon \overset{H}{\to} \mathbb{L}_0$ with

$$\mathbb{L}_0 = \begin{pmatrix} \int_0^1 \sqrt{f'(y)^2 + 4\pi^2 r^2}\,\mathrm{d}y & 0 \\ 0 & \left(\int_0^1 \sqrt{f'(y)^2 + 4\pi^2 r^2}\,\mathrm{d}y\right)^{-1} \end{pmatrix} = \begin{pmatrix} \rho_0(r) & 0 \\ 0 & \frac{1}{\rho_0(r)} \end{pmatrix},$$

and get the limiting metric

$$\hat{g}_0(\xi,\eta) = \rho_0 \mathbb{L}_0^{-1} \xi \cdot \eta = \begin{pmatrix} 1 & 0 \\ 0 & \rho_0^2 \end{pmatrix} \xi \cdot \eta.$$

We finally find, according to Remark 4.1.8, an isometric embedding $h_0\colon M_0 \to \mathbb{R}^3$ of the limiting manifold such that $dh_0^\intercal dh_0 = \rho_0 \mathbb{L}_0^{-1}$, namely

$$h_0(r,\theta) = \begin{pmatrix} \frac{\rho_0(r)}{2\pi}\sin 2\pi\theta \\ \frac{\rho_0(r)}{2\pi}\cos 2\pi\theta \\ \int_0^r \sqrt{1-\frac{\rho_0'(t)^2}{4\pi^2}}\,\mathrm{d}t \end{pmatrix}. \tag{4.14}$$

The submanifold $N_0 := h_0(M_0)$ of $\mathbb{R}^3$, which is shown in Figure 4.4, is the spectral (and Mosco-) limit of (M_ε). As in the previous two examples, we can pass to the reference manifold $M_0 = (0,R)\times[0,1)$ and find that the still excluded origin of the manifolds M_ε coincides with a circle of radius $\lim_{r\searrow 0}\rho_0(r)$ in the boundary of N_0.

A sphere with radial perturbations oscillating with the longitude

Similar to what we did above, we want to consider the case of a radially perturbed sphere. We start with the reference manifold

$$M_0 = \{(\varphi,\theta);\varphi\in(\delta,1-\delta),\theta\in[0,1)\}$$

for some $\delta > 0$, and define the family of submanifolds $M_\varepsilon := h_\varepsilon(M_0)$ of $\mathbb{R}^3$ via $h_\varepsilon\colon M_0 \to \mathbb{R}^3$,

$$h_\varepsilon(\varphi,\theta) = \big(1+\varepsilon f(\tfrac{\theta}{\varepsilon})\big)\begin{pmatrix}\sin\pi\varphi\sin 2\pi\theta \\ \sin\pi\varphi\cos 2\pi\theta \\ \cos\pi\varphi\end{pmatrix} \tag{4.15}$$

for $\varepsilon \in \{\frac{1}{k}; k\in\mathbb{N}\}$, with some smooth, 1-periodic function $f\colon \mathbb{R}\to[0,\infty)$. Again, the exclusion of the neighborhoods of the two poles assure a uniformly bi-Lipschitz setting. In Figure 4.5 we take $f(y) = \psi(y-\lfloor y\rfloor)$ with ψ as in (4.7) to illustrate M_ε for some values of ε.

$\varepsilon = \frac{1}{4}$ $\varepsilon = \frac{1}{8}$ $\varepsilon = \frac{1}{16}$

Figure 4.5.: A family of spheres with periodic radial perturbations oscillating with the longitude. The three pictures on the left show M_ε defined by (4.15) with $f(y) = (y-\lfloor y\rfloor)$ with ψ as in (4.7) and decreasing values of ε. The picture on the right shows the limiting surface N_0 defined via (4.16).

The same computations as in the previous example provide the limiting density

$$\rho_0(\varphi) = \pi \int_0^1 \sqrt{f'(y)^2 + 4\pi^2 \sin^2 \pi\varphi}\, \mathrm{d}y$$

and the metric

$$\hat{g}_0 = \rho_0 \mathbb{L}_0^{-1} = \begin{pmatrix} \pi^2 & 0 \\ 0 & \frac{\rho_0^2}{\pi^2} \end{pmatrix},$$

and again we can find an isometric embedding $h_0 \colon M_0 \to \mathbb{R}^3$ via $dh_0^\intercal dh_0 = \rho_0 \mathbb{L}_0^{-1}$, namely

$$h_0(\varphi, \theta) = \begin{pmatrix} \frac{\rho_0(\varphi)}{2\pi^2} \sin 2\pi\theta \\ \frac{\rho_0(\varphi)}{2\pi^2} \cos 2\pi\theta \\ \int_0^\varphi \sqrt{\pi^2 - \frac{\rho_0'(t)^2}{4\pi^4}}\, \mathrm{d}t \end{pmatrix}. \tag{4.16}$$

The submanifold $N_0 := h_0(M_0)$ of $\mathbb{R}^3$, pictured in Figure 4.2, is the spectral (and Mosco-) limit of the sequence (M_ε). As in the examples above we can pass to the reference manifold $M_0 = (0,1) \times [0,1)$ and find that the excluded poles of the manifolds M_ε coincide with two circles forming the boundary of N_0.

4.2.3. Laminate-Like Normal Perturbations of the Torus

Perturbations oscillating with the latitude

We finally consider the embedded 2-dimensional torus and add some periodic perturbation in outer normal direction. To be explicit, we start with the periodicity cell of the torus (we ignore the periodic boundary conditions for now) as the reference manifold

$$M_0 := \{(\varphi, \theta); \varphi \in [0,1), \theta \in [0,1)\},$$

and define for submanifolds $M_\varepsilon := h_\varepsilon(M_0)$ of $\mathbb{R}^3$ (with the induced metric and measure) via

$$h_\varepsilon(\varphi, \theta) = \begin{pmatrix} \big(R + (r + \varepsilon f(\frac{\varphi}{\varepsilon})) \cos 2\pi\varphi\big) \cos 2\pi\theta \\ \big(R + (r + \varepsilon f(\frac{\varphi}{\varepsilon})) \cos 2\pi\varphi\big) \sin 2\pi\theta \\ (r + \varepsilon f(\frac{\varphi}{\varepsilon})) \sin 2\pi\varphi \end{pmatrix} \tag{4.17}$$

for some $R > r > 0$, $\varepsilon \in \{\frac{1}{k}; k \in \mathbb{N}\}$, and with some smooth, 1-periodic function $f \colon \mathbb{R} \to [0, R-r)$. Obviously the manifolds M_ε are uniformly bi-Lipschitz diffeomorphic to the torus $\mathbb{T}$. In Figure 4.6 we choose $f(y) = (R-r)\psi(y - \lfloor y \rfloor)$ with ψ as in (4.7) to illustrate the manifolds M_ε for some values of ε.

The same calculations as in the previous examples yield the limiting density

$$\rho_0(\varphi) = 2\pi(R + r\cos 2\pi\varphi) \int_0^1 \sqrt{f'(y)^2 + 4\pi^2 r^2}\, \mathrm{d}y$$

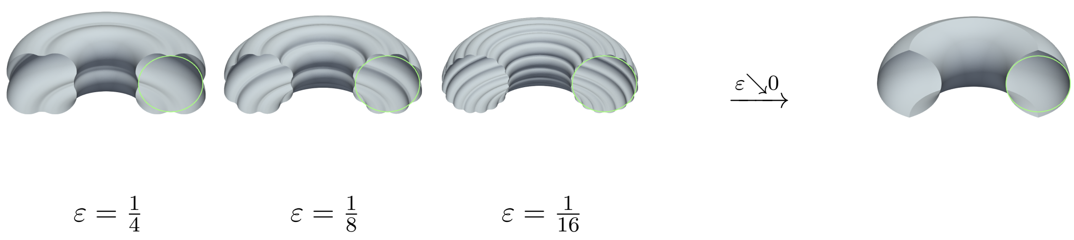


Figure 4.6.: A family of tori with periodic normal perturbations oscillating with the latitude. The three pictures on the left show a section through M_ε defined by (4.17) with $f(y) = (R-r)\psi(y - \lfloor y \rfloor)$ with ψ as in (4.7) for decreasing values of ε. The picture on the right shows a section through the limiting surface N_0 defined via (4.18), while the green circle indicates the shape of the original torus.

and the metric

$$\hat{g}_0 = \rho_0 \mathbb{L}_0^{-1} = \begin{pmatrix} \frac{\rho_0^2}{4\pi^2(R+\cos 2\pi\varphi)^2} & 0 \\ 0 & 4\pi^2(R+\cos 2\pi\varphi)^2 \end{pmatrix}.$$

We find an isometric embedding $h_0 \colon M_0 \to \mathbb{R}^3$ via $dh_0^\intercal dh_0 = \rho_0 \mathbb{L}_0^{-1}$, namely

$$h_0(\varphi, \theta) = \begin{pmatrix} (R + r\cos 2\pi\varphi)\cos 2\pi\theta \\ (R + r\cos 2\pi\varphi)\sin 2\pi\theta \\ \int_0^\varphi \sqrt{\frac{\rho_0(t)^2}{4\pi^2(R+\cos 2\pi t)^2} - 4\pi^2 r^2 \sin^2 2\pi t}\,\mathrm{d}t \end{pmatrix},$$

but this embedding ignores the periodic boundary conditions. By manipulating the sign of the square root in the integral of the third component of h_0, we can achieve a torus shaped embedding, but we loose the differentiability (on a zero set), as can be seen in Figure 4.6. To be precise, there is a unique $\hat{\varphi} \in (0, \frac{1}{2})$ such that

$$\int_0^{\hat{\varphi}} \sqrt{\frac{\rho_0(t)^2}{4\pi^2(R+\cos 2\pi t)^2} - 4\pi^2 \sin^2 2\pi t}\,\mathrm{d}t = \int_{\hat{\varphi}}^{\frac{1}{2}} \sqrt{\frac{\rho_0(t)^2}{4\pi^2(R+\cos 2\pi t)^2} - 4\pi^2 \sin^2 2\pi t}\,\mathrm{d}y,$$

and if we define $s(t) := 1 - 2\mathbb{1}_{[\hat{\varphi}, 1-\hat{\varphi})}(y)$, a periodic embedding is given by

$$h_0(\varphi, \theta) = \begin{pmatrix} (R + r\cos 2\pi\varphi)\cos 2\pi\theta \\ (R + r\cos 2\pi\varphi)\sin 2\pi\theta \\ \int_0^\varphi s(t)\sqrt{\frac{\rho_0(t)^2}{4\pi^2(R+\cos 2\pi t)^2} - 4\pi^2 \sin^2 2\pi t}\,\mathrm{d}t \end{pmatrix}. \tag{4.18}$$

We have to emphasize that the spectral (and Mosco-) limiting Riemannian manifold is actually the torus $\mathbb{T}$ equipped with the metric $\hat{g}_0$ and the measure $\hat{\mu}_0$. The behalf of an embedding as the one above is only to illustrate the geometric structure of the limit.

Perturbations oscillating with the longitude

Instead of perturbations oscillating with φ as in the previous example, we can also consider perturbations oscillating with θ, but macroscopically still depending on φ. Again we start with the periodicity cell of the torus as the reference manifold

$$M_0 := \{(\varphi, \theta); \varphi \in [0,1), \theta \in [0,1)\},$$

and define the submanifolds $M_\varepsilon := h_\varepsilon(M_0)$ of $\mathbb{R}^3$ via

$$h_\varepsilon(\varphi, \theta) = \begin{pmatrix} \big(R + (r + \varepsilon f(\varphi, \frac{\theta}{\varepsilon})) \cos 2\pi\varphi\big) \cos 2\pi\theta \\ \big(R + (r + \varepsilon f(\varphi, \frac{\theta}{\varepsilon})) \cos 2\pi\varphi\big) \sin 2\pi\theta \\ (r + \varepsilon f(\varphi, \frac{\theta}{\varepsilon})) \sin 2\pi\varphi \end{pmatrix} \tag{4.19}$$

for $R > r > 0$, $\varepsilon \in \{\frac{1}{k}; k \in \mathbb{N}\}$ and some smooth function $f \colon \mathbb{R} \times \mathbb{R} \to [0, R-r)$, being 1-periodic in both arguments. In Figure 4.7 we used as function f the periodic continuation of

$$(\varphi, y) \mapsto (R-r)\psi\Big(2(y - \tfrac{1}{2})\frac{R + r\cos 2\pi\varphi}{R-r} + \tfrac{1}{2}\Big). \tag{4.20}$$

for $\varphi \in [0,1]$, $y \in [0,1)$ and ψ as in (4.7). This function is chosen such that the width of the perturbation is the same around the torus, see Figure 4.7.

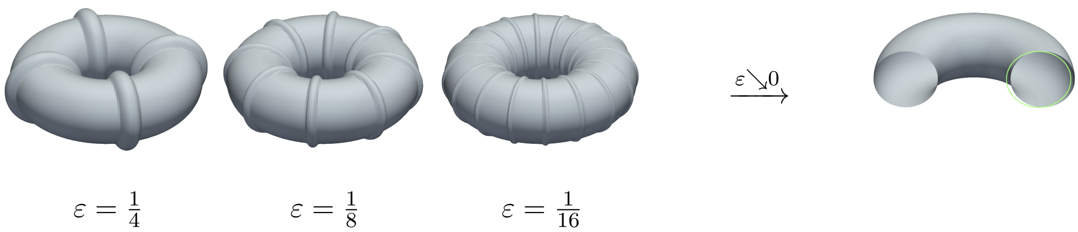


Figure 4.7.: A family of tori with periodic normal perturbations oscillating with the longitude. The three pictures on the left show M_ε defined by (4.19) with f being the periodic continuation of (4.20) for decreasing values of ε. The picture on the right shows a section through the limiting surface N_0 defined via (4.21), while the green circle indicates the shape of the original torus.

Doing the same calculations as above we end up with the density

$$\rho_0(\varphi) = 2\pi r \int_0^1 \sqrt{\partial_2 f(\varphi, y)^2 + 4\pi^2(R + r\cos 2\pi\varphi)^2}\,\mathrm{d}y$$

and the limiting metric

$$\hat{g}_0 = \rho_0 \mathbb{L}_0^{-1} = \begin{pmatrix} 4\pi^2 r^2 & 0 \\ 0 & \frac{\rho_0^2}{4\pi^2 r^2} \end{pmatrix}$$

An isometric embedding $h_0\colon M_0 \to \mathbb{R}^3$ of the limiting manifold can be found via $dh_0^{\intercal} dh_0 = \rho_0 \mathbb{L}^{-1}$, namely

$$h_0(\varphi, \varphi) = \begin{pmatrix} \frac{\rho_0(\varphi)}{4\pi^2 r} \cos 2\pi\theta \\ \frac{\rho_0(\varphi)}{4\pi^2 r} \sin 2\pi\theta \\ \int_0^{\varphi} \sqrt{4\pi^2 r^2 - \frac{\rho_0'(t)^2}{16\pi^4 r^2}}\, \mathrm{d}t \end{pmatrix}, \tag{4.21}$$

but taking the periodic boundary conditions into account, we have to do the same manipulations to the integral as in the example above, resulting in the manifold pictured in Figure 4.7.

4.3. Proofs

The foundation of the proofs in this chapter is the transformation result Lemma 4.1.1, which we need to prove first. It is an application of the integral representation formula and the definition of $(dh^{-1})^*$, cf. the beginning of Section 4.1 for the notation.

Proof of Lemma 4.1.1.
Step 1: Proof of (a)⇔(b).
Since $h\colon M_0 \to M$ is a diffeomorphism, the integral transformation formula yields for any function $f \in L^1(M, g, \mu)$

$$\int_M f \,\mathrm{d}\mu = \int_{M_0} (f \circ h) \rho \,\mathrm{d}\mu_0.$$

To show the equivalence of the statement (a) and (b) it only remains to show

$$\overline{g}(\nabla_{\overline{g}} u, \nabla_{\overline{g}} \psi)\, \rho = g_0(\mathbb{L} \nabla_{g_0} \overline{u}, \nabla_{g_0} \overline{\psi})$$

for any test function $\psi \in C_c^{\infty}(M)$. To that end we first claim $\nabla_{\overline{g}} u = (dh^{-1})^* \nabla_{g_0} \overline{u}$ (and that the same holds for ψ). Indeed, using the definition of the gradient and the adjoint, we have

$$\overline{g}(\nabla_{\overline{g}} u, \xi) = du(\xi) = d(u \circ h)(dh^{-1}\xi) = g_0(\nabla_{g_0} \overline{u}, dh^{-1}\xi) = \overline{g}((dh^{-1})^* \nabla_{g_0} \overline{u}, \xi).$$

Together with the definition of $\mathbb{L}$ we conclude

$$\overline{g}(\nabla_{\overline{g}} u, \nabla_{\overline{g}} \psi)\, \rho = \overline{g}((dh^{-1})^* \nabla_{g_0} \overline{u}, (dh^{-1})^* \nabla_{g_0} \overline{\psi})\, \rho = g_0(\mathbb{L} \nabla_{g_0} \overline{u}, \nabla_{g_0} \overline{\psi}).$$

Step 2: Proof of (b)⇔(c).
By the definition of $\hat{\mu}_0$ it suffices to show

$$g_0(\mathbb{L} \nabla_{g_0} \overline{u}, \nabla_{g_0} \overline{\psi}) = \hat{g}_0(\nabla_{\hat{g}_0} \overline{u}, \nabla_{\hat{g}_0} \overline{\psi})\, \rho.$$

We first observe $\mathbb{L}\nabla_{g_0}\overline{u} = \rho\nabla_{\hat{g}_0}\overline{u}$, which can be seen by the following direct computation, using the definition of $\hat{g}_0$ and of the gradient:

$$\hat{g}_0(\mathbb{L}\nabla_{g_0}\overline{u}, \xi) = \rho\, g_0(\nabla_{g_0}\overline{u}, \xi) = \rho\, d\overline{u}(\xi) = \rho\, \hat{g}_0(\nabla_{\hat{g}_0}\overline{u}, \xi).$$

Again with the definition of the gradient we finally get

$$g_0(\mathbb{L}\nabla_{g_0}\overline{u}, \nabla_{g_0}\overline{\psi}) = \rho\, g_0(\nabla_{\hat{g}_0}\overline{u}, \nabla_{g_0}\overline{\psi}) = \rho\, d\overline{\psi}(\nabla_{\hat{g}_0}\overline{u}) = \rho\, \hat{g}_0(\nabla_{\hat{g}_0}\overline{u}, \nabla_{\hat{g}_0}\overline{\psi}). \qquad \square$$

Now with the transformation lemma (Lemma 4.1.1) the proof of the H-compactness result Lemma 4.1.3 becomes an application Theorem 3.2.2 on the reference manifold, where we only have to care about the well-posedness and the convergence of the right-hand sides of the corresponding problem, which is done using the compact embedding of $H_0^1(M_0, g_0, \mu_0)$ in $L^2(M_0, g_0, \mu_0)$.

Proof of Lemma 4.1.3.
Step 1: Proof of (a).
By the definition (4.2), there is a constant $C_0 > 0$ (only depending on the constant C in Definition 4.1.2 and the dimension n) such that $\frac{1}{C_0} \leq \rho_\varepsilon \leq C_0$ a.e. in M_0 and $\mathbb{L}_\varepsilon \in \mathcal{M}(M_0, \frac{1}{C_0}, C_0)$. Therefore we can extract a subsequence, such that $\rho_\varepsilon \overset{*}{\rightharpoonup} \rho_0$ for some density $\rho_0 \in L^\infty(M_0)$ with $\frac{1}{C_0} \leq \rho_0 \leq C_0$, and, by our H-compactness result Theorem 3.2.2, $\mathbb{L}_\varepsilon \overset{H}{\to} \mathbb{L}_0$ in (M_0, g_0, μ_0) for some coefficient field $\mathbb{L}_0 \in \mathcal{M}(M_0, \frac{1}{C_0}, C_0)$.

Step 2: Proof of (b).
We use Lemma 4.1.1 (a)$\Leftrightarrow$(b) to re-formulate (4.3) as

$$\big(\overline{m} - \operatorname{div}_{g_0,\mu_0}(\mathbb{L}_\varepsilon\nabla_{g_0})\big)\overline{u}_\varepsilon = \rho_\varepsilon\overline{f}_\varepsilon - (\rho_\varepsilon m - \overline{m})\overline{u}_\varepsilon \qquad \text{in } H^{-1}(M_0, g_0, \mu_0) \tag{4.22}$$

for any constant $\overline{m}$, which we can choose large enough to guarantee well-posedness. We find by a standard energy estimate that $(\overline{u}_\varepsilon)$ is bounded in $H^1(M_0, g_0, \mu_0)$, and we can extract a subsequence such that $\overline{u}_\varepsilon \rightharpoonup \overline{u}_0$ weakly in $H^1(M_0, g_0, \mu_0)$ for some $\overline{u}_0 \in H_0^1(M_0, g_0, \mu_0)$. Due to the compact embedding of $H_0^1(M_0, g_0, \mu_0)$ in $L^2(M_0, g_0, \mu_0)$ this implies also $\overline{u}_\varepsilon \to \overline{u}_0$ strongly in $L^2(M_0, g_0, \mu_0)$. Moreover, since $f_\varepsilon \rightharpoonup f_0$ weakly in $L^2((M_\varepsilon, \mu_\varepsilon) \to (M_0, \hat{\mu}_0))$, we have $\rho_\varepsilon\overline{f}_\varepsilon \rightharpoonup \rho_0 f_0$ weakly in $L^2(M_0, g_0, \mu_0)$ (cf. Definition 1.3.2), and thus we get for the right-hand side in (4.22)

$$\rho_\varepsilon\overline{f}_\varepsilon - (\rho_\varepsilon m - \overline{m})\overline{u}_\varepsilon \rightharpoonup \rho_0 f_0 - (\rho_0 m - \overline{m})\overline{u}_0 \qquad \text{weakly in } L^2(M_0, g_0, \mu_0).$$

Now, for $\overline{m}$ large enough, we can deduce from $\mathbb{L}_\varepsilon \overset{H}{\to} \mathbb{L}_0$ with Theorem 3.2.2 that $\overline{u}_0$ is the unique weak solution to

$$\big(\overline{m} - \operatorname{div}_{g_0,\mu_0}(\mathbb{L}_0\nabla_{g_0})\big)\overline{u}_0 = \rho_0 f_0 - (\rho_0 m - \overline{m})\overline{u}_0 \qquad \text{in } H^{-1}(M_0, g_0, \mu_0), \tag{4.23}$$

so we conclude $\overline{u}_\varepsilon \to \overline{u}_0$ strongly in $L^2(M_0, g_0, \mu_0)$ for the entire sequence. Moreover, Lemma 4.1.1 (b)$\Leftrightarrow$(c) tells that (4.23) is equivalent to (4.4), and thus $\overline{u}_0 = u_0$. It finally remains to show that $u_\varepsilon \to u_0$ in $L^2((M_\varepsilon,\mu_\varepsilon) \to (M_0,\hat\mu_0))$. This follows from $\overline{u}_\varepsilon\rho_\varepsilon \rightharpoonup u_0\rho_0$ weakly in $L^2(M_0,g_0,\mu_0)$, because

$$\int_{M_\varepsilon} u_\varepsilon(\psi\circ h_\varepsilon^{-1})\,\mathrm{d}\mu_\varepsilon = \int_{M_0} \overline{u}_\varepsilon\psi\rho_\varepsilon\,\mathrm{d}\mu_0 \to \int_{M_0} u_0\psi\rho_0\,\mathrm{d}\mu_0 = \int_{M_0} u_0\psi\,\mathrm{d}\hat\mu_0$$

for all $\psi \in C_c^\infty(M_0)$, and

$$\int_{M_\varepsilon} |u_\varepsilon|^2\,\mathrm{d}\mu_\varepsilon = \int_{M_0} \overline{u}_\varepsilon\,\overline{u}_\varepsilon\rho_\varepsilon\,\mathrm{d}\mu_0 \to \int_{M_0} u_0\,u_0\rho_0\,\mathrm{d}\mu_0 = \int_{M_0} |u_0|^2\,\mathrm{d}\hat\mu_0. \qquad \square$$

The proofs of the spectral convergence results Propositions 4.1.5 and 4.1.6 are similar to the ones of Propositions 3.2.6 and 3.2.7. But since we have to deal with operators defined on different manifolds, we require a slightly more general version of Lemma 3.6.3 (which is also a consequence of [JKO12, Lemma 11.3 and Theorem 11.5]):

Lemma 4.3.1. *Let $(M_\varepsilon,g_\varepsilon,\mu_\varepsilon)$ be a family of weighted Riemannian manifolds, being bi-Lipschitz diffeomorphic to the reference manifold $(M_0,\hat g_0,\hat\mu_0)$ via the diffeomorphisms $h_\varepsilon\colon M_\varepsilon \to M_0$, and let $\mathcal{R}_\varepsilon,\mathcal{R}_0\colon L^2(M_\varepsilon) \to L^2(M_\varepsilon)$ be positive, compact, self-adjoint operators such that their operator norms $\|\mathcal{R}_\varepsilon\|_{\mathcal{L}(L^2(M_\varepsilon))}$ are uniformly bounded for all $\varepsilon > 0$. Denote by $(\lambda_{\varepsilon,k})_{k\in\mathbb{N}}$ the decreasingly ordered sequence of eigenvalues of $\mathcal{R}_\varepsilon$, where eigenvalues are repeated according to their multiplicity, and let $(u_{\varepsilon,k})_{k\in\mathbb{N}}$ be a sequence of associated eigenfunctions, forming an orthonormal basis of $L^2(M_\varepsilon)$. If $\mu_\varepsilon \overset{*}{\rightharpoonup} \hat\mu_0$, and if*

$$f_\varepsilon \rightharpoonup f_0 \qquad \textit{weakly in } L^2((M_\varepsilon,\mu_\varepsilon) \to (M_0,\hat\mu_0))$$

implies

$$\mathcal{R}_\varepsilon f_\varepsilon \to \mathcal{R}_0 f_0 \qquad \textit{strongly in } L^2((M_\varepsilon,\mu_\varepsilon) \to (M_0,\hat\mu_0)),$$

then for all $k \in \mathbb{N}$

$$\lambda_{\varepsilon,k} \to \lambda_{0,k} \qquad \textit{as } \varepsilon \searrow 0,$$

and if $s \in \mathbb{N}$ denotes the multiplicity of $\lambda_{0,k}$, i.e.

$$\lambda_{0,k-1} > \lambda_{0,k} = \cdots = \lambda_{0,k+s-1} > \lambda_{0,k+s} \qquad \textit{(with the convention } \lambda_{0,0} = \infty\textit{)},$$

there exists a sequence $\tilde u_{\varepsilon,k}$ of linear combinations of $u_{\varepsilon,k},\dots,u_{\varepsilon,k+s-1}$ such that

$$\tilde u_{\varepsilon,k} \to u_{0,k} \qquad \textit{strongly in } L^2((M_\varepsilon,\mu_\varepsilon) \to (M_0,\hat\mu_0)) \textit{ as } \varepsilon \searrow 0.$$

Proof of Proposition 4.1.5. Since the manifolds $(M_\varepsilon,g_\varepsilon,\mu_\varepsilon)$ are uniformly bi-Lipschitz diffeomorphic to (M_0,g_0,μ_0), we can deduce that with M_0 also M_ε is compact and satisfies $m_0(M_\varepsilon,g_\varepsilon,\mu_\varepsilon) < 0$, so for every $f_\varepsilon \in H^{-1}(M_\varepsilon,g_\varepsilon,\mu_\varepsilon)$ there is a unique weak solution $u_\varepsilon \in H_0^1(M_\varepsilon,g_\varepsilon,\mu_\varepsilon)$ to

$$-\operatorname{div}_{g_\varepsilon,\mu_\varepsilon}(\nabla_{g_\varepsilon}u_\varepsilon) = f_\varepsilon \qquad \text{in } H^{-1}(M_\varepsilon,g_\varepsilon,\mu_\varepsilon)$$

and we can consider the resolvent operator $\mathcal{R}_\varepsilon\colon L^2(M_\varepsilon, g_\varepsilon, \mu_\varepsilon) \to L^2(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ with $\mathcal{R}_\varepsilon f_\varepsilon := u_\varepsilon$. With the same argument we can consider the resolvent operator $\mathcal{R}_0$ associated with the Laplace-Beltrami operator on $(M_0, \hat{g}_0, \hat{\mu}_0)$.

The resolvent operators defined above are positive, compact, self-adjoint, and a standard energy estimate shows that the operator norms $\|\mathcal{R}_\varepsilon\|_{\mathcal{L}(L^2(M_\varepsilon))}$ are uniformly bounded. Moreover, Lemma 4.1.3 tells that $\mu_\varepsilon \overset{*}{\rightharpoonup} \hat{\mu}_0$, and that $f_\varepsilon \rightharpoonup f_0$ weakly in $L^2((M_\varepsilon, \mu_\varepsilon) \to (M_0, \hat{\mu}_0))$ implies $\mathcal{R}_\varepsilon f_\varepsilon = u_\varepsilon \to u_0 = \mathcal{R}_0 f_0$ strongly in $L^2((M_\varepsilon, \mu_\varepsilon) \to (M_0, \hat{\mu}_0))$. Thus the assertion follows from Lemma 4.3.1 together with the observation that (λ, u) is an eigenpair of $\mathcal{R}_\varepsilon$ (resp. $\mathcal{R}_0$) if and only if $(\frac{1}{\lambda}, u)$ is an eigenpair of the Laplace-Beltrami operator on $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ (resp. on $(M_0, \hat{g}_0, \hat{\mu}_0)$). □

The proof of Proposition 4.1.6 is more crucial, as we cannot directly apply Lemma 4.1.3, cf. also the proof of Proposition 3.2.7. Instead we need to modify the above proof of Lemma 4.1.3 by replacing the application of Theorem 3.2.2 with Lemma 3.6.4.

Proof of Proposition 4.1.6. For every $f_\varepsilon \in H^{-1}(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$, $f_0 \in H^{-1}(\mathbb{T}, \hat{g}_0, \hat{\mu}_0)$ the equations

$$\begin{aligned} -\operatorname{div}_{g_\varepsilon, \mu_\varepsilon}(\nabla_{g_\varepsilon} u_\varepsilon) &= f_\varepsilon \quad \text{in } H^{-1}(M_\varepsilon, g_\varepsilon, \mu_\varepsilon), \\ -\operatorname{div}_{\hat{g}_0, \hat{\mu}_0}(\nabla_{\hat{g}_0} u_0) &= f_0 \quad \text{in } H^{-1}(\mathbb{T}, \hat{g}_0, \hat{\mu}_0) \end{aligned}$$

admit unique weak solutions $u_\varepsilon \in H^1_{\text{per}}(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ and $u_0 \in H^1_{\text{per}}(\mathbb{T}, \hat{g}_0, \hat{\mu}_0)$. Thus we can consider the associated resolvent operators $\mathcal{R}_\varepsilon\colon L^2(M_\varepsilon, g_\varepsilon, \mu_\varepsilon) \to L^2(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$, $\mathcal{R}_\varepsilon f_\varepsilon := u_\varepsilon$ and $\mathcal{R}_0\colon L^2(\mathbb{T}, \hat{g}_0, \hat{\mu}_0) \to L^2(\mathbb{T}, \hat{g}_0, \hat{\mu}_0)$, $\mathcal{R}_0 f_0 := u_0$, which are positive, compact, self-adjoint operators with uniformly bounded operator norms. Since $\mu_\varepsilon \overset{*}{\rightharpoonup} \hat{\mu}_0$, by Lemma 4.3.1 it would be sufficient to show that $f_\varepsilon \rightharpoonup f_0$ weakly in $L^2((M_\varepsilon, \mu_\varepsilon) \to (\mathbb{T}, \hat{\mu}_0))$ implies $\mathcal{R}_\varepsilon f_\varepsilon \to \mathcal{R}_0 f_0$ strongly in $L^2((M_\varepsilon, \mu_\varepsilon) \to (\mathbb{T}, \hat{\mu}_0))$, because then Lemma 4.3.1 applies and the assertion follows, since (λ, u) is an eigenpair of $\mathcal{R}_\varepsilon$ (resp. $\mathcal{R}_0$) if and only if $(\frac{1}{\lambda}, u)$ is an eigenpair of the Laplace-Beltrami operator on $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ (resp. on $(\mathbb{T}, \hat{g}_0, \hat{\mu}_0)$).

The rest of the proof is a variant of the proof of Lemma 4.1.3 (b). Assume $f_\varepsilon \rightharpoonup f_0$ weakly in $L^2((M_\varepsilon, \mu_\varepsilon) \to (\mathbb{T}, \hat{\mu}_0))$. By Lemma 4.1.1 (a)⇔(b), the equations

$$\begin{aligned} -\operatorname{div}_{g_0, \mu_0}(\mathbb{L}_\varepsilon \nabla_{g_0} \overline{u}_\varepsilon) &= \rho_\varepsilon \overline{f}_\varepsilon \quad \text{in } H^{-1}(\mathbb{T}, g_0, \mu_0), \\ -\operatorname{div}_{g_0, \mu_0}(\mathbb{L}_0 \nabla_{g_0} u_0) &= \rho_0 f_0 \quad \text{in } H^{-1}(\mathbb{T}, g_0, \mu_0) \end{aligned} \tag{4.24}$$

admit the unique weak solutions $\overline{u}_\varepsilon \in H_\varepsilon$ and $u_0 \in H_0$, where

$$\begin{aligned} H_\varepsilon &:= \Big\{ u \in H^1(\mathbb{T}, g_0, \mu_0); \fint_{\mathbb{T}} u \rho_\varepsilon \, \mathrm{d}\mu_0 = 0 \Big\}, \\ H_0 &:= \Big\{ u \in H^1(\mathbb{T}, g_0, \mu_0); \fint_{\mathbb{T}} u \rho_0 \, \mathrm{d}\mu_0 = 0 \Big\}. \end{aligned}$$

Since $\rho_\varepsilon \overset{*}{\rightharpoonup} \rho_0$ weakly-$*$ in $L^\infty(\mathbb{T}, g_0, \mu_0)$, the spaces H_ε and H_0 satisfy the assumptions of Lemma 3.6.4. Moreover, we find for the right-hand sides in (4.24) $\rho_\varepsilon \overline{f}_\varepsilon \rightharpoonup \rho_0 f_0$ weakly in $L^2(\mathbb{T}, g_0, \mu_0)$, so we can conclude $\overline{u}_\varepsilon \to u_0$ strongly in $L^2(\mathbb{T}, g_0, \mu_0)$. But by definition this coincides with $\mathcal{R}_\varepsilon f_\varepsilon = u_\varepsilon \to u_0 = \mathcal{R}_0 f_0$ strongly in $L^2((M_\varepsilon, \mu_\varepsilon) \to (\mathbb{T}, \hat{\mu}_0))$, and the proof is complete. $\square$

Part III.

Rapidly Oscillating Random Manifolds

5. Integral Functionals with Non-Uniformal Growth

While the arguments for the H-compactness method in Part II relied on the uniform ellipticity of the coefficient fields associated with the Laplace-Beltrami operator, we choose another approach in this part. Instead of uniformity of the ellipticity constants, we consider a stochastic framework and assume the constants to have bounded moments. This allows the application to manifolds that possibly, but rather unlikely degenerate, see Chapter 6 for details.

In [NSS17] Neukamm, Schäffner and Schlömerkemper present a Γ-convergence statement for energies with degenerate potentials on discrete lattices, where they use techniques from [Mül87] and [FM92]. This chapter is devoted to adapt their approach to $\mathbb{R}^n$ and make a few generalizations, which are possible due to the continuity of the underlying space, and necessary with respect to the application to Dirichlet energies on manifolds.

The results of this chapter are all new and unpublished.

5.1. Setting

Let $(\Omega, \mathcal{A}, \mathbb{P}, \tau)$ be a stationary, ergodic probability space. For every $\varepsilon > 0$, $\omega \in \Omega$ and bounded Lipschitz domain $A \subseteq \mathbb{R}^n$ we consider the energy functional $\mathcal{E}^\omega(\cdot, A)\colon L^p(A) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{E}_\varepsilon^\omega(u, A) := \begin{cases} \int_A W^\omega(x, \frac{x}{\varepsilon}, Du(x))\,\mathrm{d}x & u \in W^{1,p}(A), \\ +\infty & \text{otherwise}, \end{cases} \tag{5.1}$$

with the measurable stationary potential $W\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{m\times n} \to \mathbb{R}$, i.e.

$$W^{\tau_y \omega} = W^\omega(\cdot, \cdot + y, \cdot)$$

for every $\omega \in \Omega$, $y \in \mathbb{R}^n$. (See Remark 5.1.6 below for a short comment on the measurability of W.) We will conveniently drop the index ω when it is clear from the context, especially to make the proof more readable. On the potential W we make the following assumptions:

Assumption 5.1.1. *There are $1 < p < \infty$, a constant $C > 0$, exponents $\alpha \geq 1$ and $\beta \geq \frac{1}{p-1}$, and measurable stationary functions $\lambda_{\min}, \lambda_{\max} \colon \Omega \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{>0}$, i.e.*

$$\lambda_{\min}^{\tau_y \omega} = \lambda_{\min}^{\omega}(\cdot, \cdot + y) \qquad \text{and} \qquad \lambda_{\max}^{\tau_y \omega} = \lambda_{\max}^{\omega}(\cdot, \cdot + y),$$

for e.a. $\omega \in \Omega$ and every $y \in \mathbb{R}^n$, such that the following conditions are satisfied:

- *(Non-Uniformly Elliptic p-Growth)*

$$\lambda_{\min}^{\omega}(x, y)\,(\tfrac{1}{C}|F|^p - C) \leq W^{\omega}(x, y, F) \leq \lambda_{\max}^{\omega}(x, y)\, C(|F|^p + 1) \tag{5.2}$$

  for a.e. $\omega \in \Omega$, and every $x, y \in \mathbb{R}^n$ and $F \in \mathbb{R}^{m \times n}$,

- *(Mild $\lambda_{\max}$-Convexity)*

$$W^{\omega}(x, y, tF + (1-t)G) \leq C\big(W^{\omega}(x, y, F) + W^{\omega}(x, y, G) + \lambda_{\max}^{\omega}(x, y)\big) \tag{5.3}$$

  for a.e. $\omega \in \Omega$, and every $x, y \in \mathbb{R}^n$, $F, G \in \mathbb{R}^{m \times n}$ and all $0 \leq t \leq 1$,

- *(Weak $\lambda_{\max}$-Δ_2-Property)*

$$W^{\omega}(x, y, 2F) \leq C\big(W^{\omega}(x, y, F) + \lambda_{\max}^{\omega}(x, y)\big) \tag{5.4}$$

  for a.e. $\omega \in \Omega$, and every $x, y \in \mathbb{R}^n$ and $F \in \mathbb{R}^{n \times m}$,

- *(Moment Bounds on $\lambda_{\max}$ and $\lambda_{\min}^{-1}$)*

$$\operatorname*{ess\,sup}_{x, y \in \mathbb{R}^n} \mathbb{E}[\lambda_{\max}(x, y)^{\alpha} + \lambda_{\min}(x, y)^{-\beta}] < \infty. \tag{5.5}$$

For technical reasons we need to distinguish between the cases of the dimension $m = 1$ or $m > 1$, that is whether the function u in (5.1) is scalar or vector valued. In the latter one we need to further restrict the exponents α and β, or better to say the proof can be improved for scalar functions so we can drop the additional assumption on α and β in that case.

Assumption 5.1.2. *The exponents α and β in (5.5) satisfy*

$$\frac{1}{\alpha} + \frac{1}{\beta} \geq \begin{cases} \frac{p}{n}, & \text{if } m > 1, \\ 0, & \text{if } m = 1. \end{cases} \tag{5.6}$$

Remark 5.1.3 (Comments on Assumption 5.1.1)**.**

- *The Non-Uniformly Elliptic p-Growth Condition (5.2) is much weaker than the degenerate elliptic p-growth condition assumed in [NSS17], where* $\lambda_{\min} = \lambda_{\max}$*. It is a necessary generalization with respect to the desired application, because for instance in Chapter 6 we will see that for any 2-dimensional Euclidean submanifold the Dirichlet potential is a strictly convex quadratic form, which corresponds to a* 2×2*-matrix with determinant equal to* 1*. Thus the two eigenvalues are inverse to each other, so either the eigenvalues are uniformly bounded and the potential is uniformly elliptic, or one eigenvalue tends to zero while the other one tends to infinity and the potential is non-uniformly elliptic. The degenerate case with* $\lambda_{\min} = \lambda_{\max}$ *cannot occur.*

- *Conditions of the* Δ_2*-type were introduced by Orlicz in [Orl32] and are frequently used in the context of Orlicz-spaces. However, in our setting the Weak* $\lambda_{\max}$*-*Δ_2*-Property (5.4), as well es the Mild* $\lambda_{\max}$*-Convexity (5.3), are included for technical reasons and only arise at one point, namely in the proof of a technical lemma, the Gluing Construction (Lemma 5.3.5 or 5.3.6, resp.). It is required to fix a gap in the strategy of [NSS17] coming from relaxing the degenerate ellipticity to the non-uniform ellipticity. But it is no restriction against [NSS17], since in the degenerate elliptic case with* $\lambda_{\min} = \lambda_{\max} =: \lambda$ *the Mild* $\lambda_{\max}$*-Convexity and the Weak* $\lambda_{\max}$*-*Δ_2*-Property are already included by the p-Growth Condition (5.2). For the latter one implies for* $0 < t < 1$

$$\begin{aligned} W(x, y, tF + (1-t)G) &\lesssim \lambda(x, y)(t|F|^p + (1-t)|G|^p + 1) \\ &\lesssim W(x, y, F) + W(x, y, G) + \lambda(x, y), \end{aligned}$$

where $\lesssim$ *means* $\leq$ *up to a constant depending only on* C *and* p*, as well as*

$$W(x, y, 2F) \lesssim \lambda(x, y)(2^p|F|^p + 1) \lesssim W(x, y, F) + \lambda(x, y).$$

- *Even though the potentials we are going to consider in the application are strictly convex, Assumption 5.1.1 covers also cases of non-convex potentials. One example is the model double-well potential* $W(x, y, F) = H(x, y)|F|^4 - |F|^2$ *for* $H(x, y) \geq H_0 > 1$*. One can easily check that for* $C \geq \frac{H_0}{H_0 - 1}$*,* $\lambda_{\min} = \lambda_{\max} = \frac{H}{C}$ *the Growth Condition (5.2) is satisfied (which by the arguments in the previous point implies the Mild* $\lambda_{\max}$*-Convexity (5.3) and the Weak* $\lambda_{\max}$*-*Δ_2*-Property (5.4)). Thus, it only depends on the moments of* H *whether Assumption 5.1.1 is fulfilled.*

- *The Moment Bounds on* $\lambda_{\max}$ *and* $\lambda_{\min}^{-1}$ *are the only condition where the probability measure on* Ω *enters. From the p-Growth Condition (5.2) with* $|F|^p \to \infty$ *follows* $\lambda_{\min} \leq C^2 \lambda_{\max}$*. Thus the Moment Bounds on* $\lambda_{\max}$ *and* $\lambda_{\min}^{-1}$ *(5.5) imply* *moment bounds for $\lambda_{\min}$ and $\lambda_{\max}^{-1}$, too. For convenience we assume w.l.o.g.*

$$\left.\begin{array}{l}\operatorname*{ess\,sup}_{x,y\in\mathbb{R}^n}\mathbb{E}[\lambda_{\min}(x,y)^{\alpha}]\\ \operatorname*{ess\,sup}_{x,y\in\mathbb{R}^n}\mathbb{E}[\lambda_{\min}(x,y)^{-\beta}]\end{array}\right\}\leq C \quad \text{and} \quad \left.\begin{array}{l}\operatorname*{ess\,sup}_{x,y\in\mathbb{R}^n}\mathbb{E}[\lambda_{\max}(x,y)^{\alpha}]\\ \operatorname*{ess\,sup}_{x,y\in\mathbb{R}^n}\mathbb{E}[\lambda_{\max}(x,y)^{-\beta}]\end{array}\right\}\leq C. \tag{5.7}$$

with the same constant as in Assumption 5.1.1.

Finally, to handle the spatial inhomogeneity, we demand the following form of continuity.

Assumption 5.1.4 (Spatial Continuity)**.** *There is a function $\rho\colon [0,\infty)\to(0,1)$ with $\lim_{\delta\to 0}\rho(\delta)=0$, such that for a.e. $\omega\in\Omega$, and every $x_1,x_2,y\in\mathbb{R}^n$ and $F\in\mathbb{R}^{n\times m}$ we have*

$$|W^{\omega}(x_1,y,F)-W^{\omega}(x_2,y,F)|\leq\rho(|x_1-x_2|)\left(1+W^{\omega}(x_1,y,F)+W^{\omega}(x_2,y,F)\right) \tag{5.8}$$

Remark 5.1.5. *While Condition (5.8) is a natural formulation, it is often more convenient to use the following version of the Spatial Continuity Condition (5.8): There is a function $\rho'\colon[0,\infty)\to(0,\infty)$ with $\lim_{\delta\to 0}\rho'(\delta)=0$, such that for a.e. $\omega\in\Omega$, and every $x_1,x_2,y\in\mathbb{R}^n$ and $F\in\mathbb{R}^{n\times m}$ we have*

$$|W^{\omega}(x_1,y,F)-W^{\omega}(x_2,y,F)|\leq\rho'(|x_1-x_2|)\left(1+2\min\{W^{\omega}(x_1,y,F),W^{\omega}(x_2,y,F)\}\right). \tag{5.9}$$

To see the equivalence we assume w.l.o.g. $W^{\omega}(x_1,y,F)\geq W^{\omega}(x_2,y,F)$ and conveniently write $\delta:=|x_1-x_2|$. Then we can solve Condition (5.8) for $W^{\omega}(x_1,y,F)$ and get

$$W^{\omega}(x_1,y,F)\leq\tfrac{\rho(\delta)}{1-\rho(\delta)}+\left(\tfrac{2\rho(\delta)}{1-\rho(\delta)}+1\right)W^{\omega}(x_2,y,F),$$

which is exactly Condition (5.9) with $\rho'=\frac{\rho}{(1-\rho)}$.

Remark 5.1.6 (Measurability of the Potential W)**.** *Since W is stationary, it is of the form $W^{\omega}(x,y,F)=a(\tau_y\omega,x,F)$ for some function $a\colon\Omega\times\mathbb{R}^n\times\mathbb{R}^{m\times n}\to\mathbb{R}$. We assume measurability of W in the sense of $\mathcal{A}\otimes\mathcal{B}(\mathbb{R}^n)\otimes\mathcal{B}(\mathbb{R}^{m\times n})$-measurability of a, where $\mathcal{B}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R}^{m\times n})$ denote the Borel-σ-algebra on $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$, resp. This assures the existence of all integrals occurring in this chapter as functionals with values in $\mathbb{R}\cup\{-\infty,+\infty\}$. A sufficient condition, which is always satisfied in the applications to the convergence of manifolds, is that a is a Carathéodory function, by which we mean:*

- *For every $x\in\mathbb{R}^n$, $F\in\mathbb{R}^{m\times n}$ is the function $\omega\mapsto a(\omega,x,F)$ $\mathcal{A}$-measurable.*

- *For every $\omega \in \Omega$, $F \in \mathbb{R}^{m\times n}$ is the function $x \mapsto a(\omega, x, F)$ continuous, (which is already covered by the Spatial Continuity Assumption 5.1.4).*

- *For every $\omega \in \Omega$, $x \in \mathbb{R}^n$ is the function $F \mapsto a(\omega, x, F)$ continuous.*

See e.g. [Roc71, §2] for more details about measurability of Carathéodory integrands.

5.2. Main Results

The first result is the following compactness statement.

Proposition 5.2.1 (Compactness). *Let $A \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. In the situation of Assumptions 5.1.1, 5.1.2 and 5.1.4 from*

$$u^\varepsilon \rightharpoonup u \quad \text{in } L^1(A) \qquad \text{and} \qquad \limsup_{\varepsilon\searrow 0} \mathcal{E}^\omega_\varepsilon(u^\varepsilon, A) < \infty \quad \text{for a.e. } \omega \in \Omega$$

follows

$$u \in W^{1,p}(A) \qquad \text{and} \qquad u^\varepsilon \to u \quad \text{in } L^q_{\mathrm{loc}}(A)$$

for every $q \geq 1$ with

$$\frac{1}{q} \geq \Big(1 + \frac{1}{\beta}\Big)\frac{1}{p} - \frac{1}{n}.$$

In particular, every sequence in $W^{1,p}_0(A)$ with uniformly bounded energy contains a strongly $L^q(A)$-convergent subsequence with limit in $W^{1,p}_0(A)$, which implies that $(\mathcal{E}_\varepsilon(\cdot, A))$ is equi-coercive on $W^{1,p}_0(A)$ w.r.t. the $L^q(A)$-topology.

Our main result is the statement of Γ-convergence to a deterministic integral functional in the above setting.

Theorem 5.2.2 (Γ-Convergence). *In the given situation of Assumptions 5.1.1, 5.1.2 and 5.1.4 the energy functionals $\mathcal{E}_\varepsilon$ a.s. Γ-converge to some deterministic energy functional $\mathcal{E}_{\mathrm{hom}}$ of the form*

$$\mathcal{E}_{\mathrm{hom}}(u, A) = \begin{cases} \int_A W_{\mathrm{hom}}(x, Du(x))\,\mathrm{d}x, & u \in W^{1,p}(A), \\ +\infty, & \text{otherwise} \end{cases}$$

with the homogenized potential

$$W_{\mathrm{hom}}(x, F) = \lim_{k\to\infty} \mathbb{E}\Big[\inf_{\phi\in W^{1,p}_0(kY)} \fint_{kY} W(x, y, F + D\phi(y))\,\mathrm{d}y\Big], \tag{5.10}$$

where $Y := [0, 1)^n$. Specifically there is a set $\Omega_0 \subseteq \Omega$ of full measure such that for every bounded Lipschitz domain $A \subseteq \mathbb{R}^n$ and every $\omega \in \Omega_0$ we have

(i) *for every* $u \in W^{1,p}(A)$ *there is a sequence* (u^ε) *in* $W^{1,p}(A)$ *with* $u^\varepsilon - u \in W_0^{1,p}(A)$ *such that*

$$u^\varepsilon \to u \text{ in } L^{\frac{\beta}{\beta+1}p}(A) \qquad \text{and} \qquad \mathcal{E}_\varepsilon^\omega(u^\varepsilon, A) \to \mathcal{E}_{\hom}(u, A),$$

(ii) *for every* (u^ε) *in* $W^{1,p}(A)$ *with* $u^\varepsilon \rightharpoonup u$ *in* $L^1(A)$ *we have*

$$\liminf_{\varepsilon \searrow 0} \mathcal{E}_\varepsilon^\omega(u^\varepsilon, A) \geq \mathcal{E}_{\hom}(u, A).$$

Remark 5.2.3. *The statement of Theorem 5.2.2 is actually a mixture between strong* $L^{\frac{\beta}{\beta+1}p}(A)$*-*$\Gamma$*-convergence and weak* $L^1(A)$*-*Γ*-convergence, and can therefore be interpreted as* $L^{\frac{\beta}{\beta+1}p}(A)$*-Mosco-convergence. However, if* A *is compact, Theorem 5.2.2 in connection with the compact embedding Proposition 5.2.1 implies even* $L^q(A)$*-Mosco-convergence for* $\frac{1}{q} \geq (1 + \frac{1}{\beta})\frac{1}{p} - \frac{1}{n}$.

We also state the major properties like continuity and ellipticity of the homogenized potential $W_{\hom}$.

Proposition 5.2.4 (Properties of the Homogenized Potential)**.** *The homogenized potential* $W_{\hom}$ *in Theorem 5.2.2 has the following properties:*

(a) $W_{\hom}$ *satisfies a uniform p-growth condition, i.e. there is a constant* $C' > 0$ *such that for every* $x \in \mathbb{R}^n$ *and* $F \in \mathbb{R}^{n \times m}$ *we have*

$$\tfrac{1}{C'}|F|^p - C' \leq W_{\hom}(x, F) \leq C'(|F|^p + 1).$$

In particular the constant can be given explicitly as $C' := C^{\frac{\gamma+1}{\gamma}}$ *with* $\gamma := \min\{\alpha, \beta\}$.

(b) $W_{\hom}$ *satisfies the same spatial continuity as* W*, i.e. for every* $x_1, x_2 \in \mathbb{R}^n$ *and* $F \in \mathbb{R}^{n \times m}$ *we have*

$$\big|W_{\hom}(x_1, F) - W_{\hom}(x_2, F)\big| \leq \rho(|x_1 - x_2|)\big(1 + W_{\hom}(x_1, F) + W_{\hom}(x_2, F)\big).$$

(c) $W_{\hom}$ *is continuous with respect to* F*, i.e. for every* $x \in \mathbb{R}^n$ *and* $F \in \mathbb{R}^{n \times m}$ *we have*

$$F_k \to F \qquad \Rightarrow \qquad W_{\hom}(x, F_k) \to W_{\hom}(x, F).$$

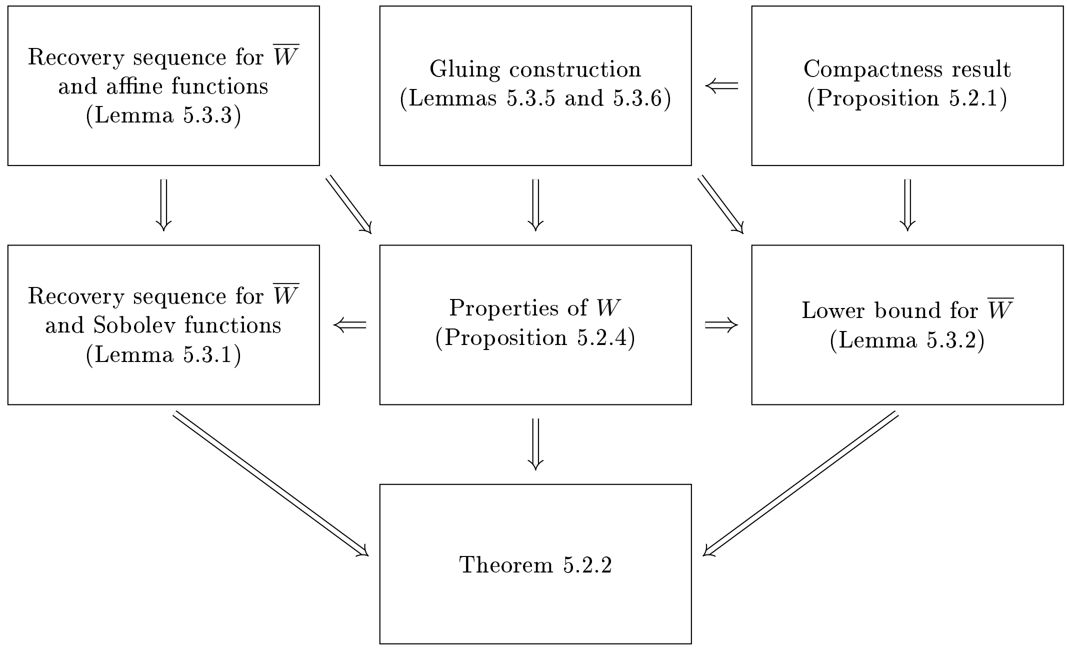


Figure 5.1.: Interplay of the lemmas in the proof of Theorem 5.2.2.

5.3. Strategy of the Proof and Auxiliary Results

An overview of the general strategy of the proof of Theorem 5.2.2 and how the numerous lemmas will interplay is illustrated in Figure 5.1.

We will first show the Compactness Result Proposition 5.2.1, since it allows us in the proof of the Lower Bound in Theorem 5.2.2 to lift the weak L^1-convergence to strong L^1-convergence.

To prove the Γ-convergence result Theorem 5.2.2 we would like to use the approach of [Mül87] combined with a blow-up technique for the lower bound ([FM92], see also [DG16; BMS08]) as done in [NSS17]. The crucial point is that these techniques only apply for spatially homogeneous potentials, so we need to reduce the problem to that case.

The strategy is to discretize the potential W in the spatial argument in the following sense: We part the considered set A into finitely many small cubes with small diameter $\delta > 0$. Then for each such cube Q we can take any point $x_0 \in Q$ and work with the spatially homogeneous potential $W(x_0, \cdot, \cdot)$ instead of $W(\cdot, \cdot, \cdot)$, making a discretization error which by the Spatial Continuity of W and W_{hom} (Condition (5.8) and Proposition 5.2.4 (b)) runs out to vanish as $\delta \searrow 0$.

To be explicit we first prove Γ-convergence for spatially homogeneous energy functionals

$$\overline{\mathcal{E}}_\varepsilon(u, A) = \int_A \overline{W}(\tfrac{x}{\varepsilon}, Du(x))\,\mathrm{d}x$$

with potentials $\overline{W}\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \to \mathbb{R}$ satisfying the same conditions (5.2) and (5.7) we assumed for W, except for the Spatial Continuity Condition (5.8). The over-lined notation is used to make clear whether we deal with the reduced spatially homogeneous potential or with the original spatially inhomogeneous setting. Note that for the spatially homogeneous potential $\overline{W}$ the corresponding functions $\lambda_{\min}$ and $\lambda_{\max}$ do not depend on x as well, which we emphasize by writing $\overline{\lambda}_{\min}$ and $\overline{\lambda}_{\max}$ resp. In this situation we can prove Γ-convergence to the homogenized energy functional

$$\overline{\mathcal{E}}_{\mathrm{hom}}(u, A) = \int_A \overline{W}_{\mathrm{hom}}(Du(x))\,\mathrm{d}x$$

with the homogenized potential

$$\overline{W}_{\mathrm{hom}}(F) = \lim_k \mathbb{E}\Big[\inf_{\phi\in W_0^{1,p}(kY)} \fint_{kY} \overline{W}(y, F + D\phi(y))\,\mathrm{d}y\Big]$$

as stated in Lemmas 5.3.1 and 5.3.2.

Lemma 5.3.1 (Recovery Sequence for $\overline{W}$)**.** *There is a set $\overline{\Omega}_0 \subseteq \Omega$ of full measure such that the following holds: Let $A \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Then for every $u \in W^{1,p}(A)$ there is a sequence (u^ε) in $W^{1,p}(A)$ with $u^\varepsilon - u \in W_0^{1,p}(A)$ such that*

$$u^\varepsilon \to u \text{ in } L^{\frac{\beta}{\beta+1}p}(A) \qquad \text{and} \qquad \overline{\mathcal{E}}^\omega_\varepsilon(u^\varepsilon, A) \to \overline{\mathcal{E}}_{\mathrm{hom}}(u, A) \quad \text{for all } \omega \in \overline{\Omega}_0.$$

Lemma 5.3.2 (Lower Bound for $\overline{W}$)**.** *There is a set $\overline{\Omega}_0 \subseteq \Omega$ of full measure such that the following holds: Let $A \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Then for every sequence (u^ε) in $W^{1,p}(A)$ with $u^\varepsilon \rightharpoonup u$ in $L^1(A)$ we have*

$$\liminf_{\varepsilon\searrow 0} \overline{\mathcal{E}}^\omega_\varepsilon(u^\varepsilon, A) \geq \overline{\mathcal{E}}_{\mathrm{hom}}(u, A) \qquad \text{for all } \omega \in \overline{\Omega}_0.$$

The proof of Lemma 5.3.2 is done by blow-up as in [NSS17] and makes use of a technical statement (Gluing Construction Lemma 5.3.5 for vector valued functions or Lemma 5.3.6 for scalar valued functions resp.) stated below.

The proof of the recovery sequence statement Lemma 5.3.1 needs to be done in two steps (also similar to [NSS17]): We first show the existence of a recovery sequence only for affine functions (Lemma 5.3.3), which in general follows the approach of [Mül87], and then extend this result to Sobolev functions.

Lemma 5.3.3 (Recovery Sequence for $\overline{W}$ and Affine Functions). *For every $F \in \mathbb{R}^{n\times m}$ there is a set $\overline{\Omega}_F \subseteq \Omega$ of full measure such that the following holds: Let $A \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain. Then there is a sequence (u^ε) in $W^{1,p}(A)$ with $u^\varepsilon - Fx \in W_0^{1,p}(A)$ such that*

$$u^\varepsilon \to Fx \text{ in } L^{\frac{\beta}{\beta+1}p}(A) \qquad \text{and} \qquad \overline{\mathcal{E}}_\varepsilon^\omega(u^\varepsilon, A) \to |A|\overline{W}_{\hom}(F) \quad \text{for all } \omega \in \overline{\Omega}_F.$$

We lift this result to Sobolev functions by approximation with piecewise affine functions. Therefor we need the Continuity of $\overline{W}_{\hom}$ (Proposition 5.2.4 (c)), so we will prove the properties of the homogenized potential (Proposition 5.2.4) first, which we will do in the original setting of a spatially inhomogeneous potential.

The proof of the Continuity of the homogenized potential requires the following two technical lemmas: the Equi-Integrability of $\overline{\lambda}$ (Lemma 5.3.4) and the Gluing Construction (Lemma 5.3.5 for vector valued functions or Lemma 5.3.6 for scalar valued functions resp.), which will also be used in the proof of the Lower Bound (Lemma 5.3.2).

Lemma 5.3.4 (Equi-Integrability of $\overline{\lambda}$). *There is a set $\overline{\Omega}_0 \subseteq \Omega$ of full measure such that the following holds: Let $A \subseteq \mathbb{R}^m$ be a bounded Lipschitz domain and let $\overline{\lambda}$ be either $\overline{\lambda}_{\min}$ or $\overline{\lambda}_{\max}$. Then we have*

$$\lim_{N\to\infty} \lim_{\varepsilon\searrow 0} \tfrac{1}{|A|} \int_{A_N^\varepsilon} \overline{\lambda}^\omega(\tfrac{x}{\varepsilon})\,\mathrm{d}x = \lim_{N\to\infty} \mathbb{E}\Big[\int_{Y_N} \overline{\lambda}(y)\,\mathrm{d}y\Big] = 0 \qquad \text{for all } \omega \in \overline{\Omega}_0,$$

where $A_N^\varepsilon := \{x \in A; \overline{\lambda}(\frac{x}{\varepsilon}) > N\}$ and $Y_N := \{y \in Y; \overline{\lambda}(y) > N\}$.

The proofs of the following Gluing Constructions for vector valued or scalar valued functions resp. are quite technical and they are the reason for Assumption 5.1.2. Besides they are also the only point where the Mild $\lambda_{\max}$-Convexity (5.3) and the Weak $\lambda_{\max}$-Δ_2-Property (5.4) show up.

Lemma 5.3.5 (Gluing Construction, Vector Valued Case). *There are a set $\overline{\Omega}_0 \subseteq \Omega$ of full measure and a sequence $(\mathcal{O}_N)$ in $(0,\infty)$ with $\mathcal{O}_N \to 0$ as $N \to \infty$ such that the following holds: Let Q be a cube of side length l and let (u^ε) be a sequence in $W^{1,p}(Q;\mathbb{R}^m)$ with $u^\varepsilon \rightharpoonup u$ weakly in $L^1(Q)$ and let $\overline{u} \in W^{1,\infty}(Q;\mathbb{R}^m)$. Then for every $\varepsilon > 0$ and $N \in \mathbb{N}$ there is a function $\phi_N^\varepsilon \in W_0^{1,p}(Q;\mathbb{R}^m)$ such that for all $\omega \in \overline{\Omega}_0$*

$$\begin{aligned}\limsup_{\varepsilon\searrow 0} &\Big(\tfrac{1}{|Q|}\overline{\mathcal{E}}_\varepsilon^\omega(\overline{u}+\phi_N^\varepsilon, Q) - \tfrac{1}{|Q|}\overline{\mathcal{E}}_\varepsilon^\omega(u^\varepsilon, Q)\Big)\\ &\lesssim \tfrac{1}{N}\limsup_{\varepsilon\searrow 0} \tfrac{1}{|Q|}\overline{\mathcal{E}}_\varepsilon^\omega(u^\varepsilon, Q) + \tfrac{N^2}{l|Q|}\|u-\overline{u}\|_{L^p(Q)}^p + \big(1 + \|D\overline{u}\|_{L^\infty(Q)}^p\big)\mathcal{O}_N,\end{aligned}$$

where $\lesssim$ means $\leq$ up to a constant only depending on the constant C from Assumption 5.1.1.

Lemma 5.3.6 (Gluing Construction, Scalar Valued Case). *There are a set $\overline{\Omega}_0 \subseteq \Omega$ of full measure and a sequence $(\mathcal{O}_N)$ in $(0,\infty)$ with $\mathcal{O}_N \to 0$ as $N \to \infty$ such that the following holds: Let Q be a cube of side length l and let (u^ε) be a sequence in $W^{1,p}(Q;\mathbb{R})$ with $u^\varepsilon \to u$ in $L^1(Q)$ and let $\overline{u} \in W^{1,\infty}(Q;\mathbb{R})$. Then for every $\varepsilon > 0$ and $N \in \mathbb{N}$ there is a function $\phi_N^\varepsilon \in W_0^{1,p}(Q;\mathbb{R})$ such that for all $\omega \in \overline{\Omega}_0$*

$$\begin{aligned}\limsup_{\varepsilon\searrow 0} & \left(\tfrac{1}{|Q|}\overline{\mathcal{E}}_\varepsilon^\omega(\overline{u}+\phi_N^\varepsilon, Q) - \tfrac{1}{|Q|}\overline{\mathcal{E}}_\varepsilon^\omega(u^\varepsilon, Q)\right) \\ & \lesssim \tfrac{1}{N}\limsup_{\varepsilon\searrow 0}\tfrac{1}{|Q|}\overline{\mathcal{E}}_\varepsilon^\omega(u^\varepsilon, Q) + \left(1+\|\nabla\overline{u}\|_{L^\infty(Q)}^p\right)\left(\tfrac{N^3}{l|Q|}\|u-\overline{u}\|_{L^1(Q)} + \mathcal{O}_N\right),\end{aligned}$$

where $\lesssim$ means $\leq$ up to a constant only depending on the constant C from Assumption 5.1.1.

Finally, to avoid having to construct the respective zero sets in the proofs, we want to discuss the choice of the occurring sets Ω_0, $\overline{\Omega}_0$ and $\overline{\Omega}$ here once at this central point.

Remark 5.3.7 (Construction of the Sets Ω_0, $\overline{\Omega}_0$ and $\overline{\Omega}_F$). *The Moment bounds on $\lambda_{\min}$ and $\lambda_{\max}$ (Condition (5.7)) in connection with Birkhoff's Pointwise Ergodic Theorem (Lemma 2.2.2) ensures for every $x \in \mathbb{R}^n$ and every bounded Lipschitz domain $A \subseteq \mathbb{R}^n$*

$$\lim_{\varepsilon\searrow 0} \fint_A \lambda_{\min}(x, \tfrac{y}{\varepsilon})^\alpha \,\mathrm{d}y = \mathbb{E}[\lambda_{\min}^\alpha(x)] \leq C \qquad a.s.$$

and

$$\lim_{\varepsilon\searrow 0} \fint_A \lambda_{\min}(x, \tfrac{y}{\varepsilon})^{-\beta} \,\mathrm{d}y = \mathbb{E}[\lambda_{\min}^{-\beta}(x)] \leq C \qquad a.s.,$$

and the same holds for $\lambda_{\max}$. Here the zero sets only depend on x, so we can choose for every $x \in \mathbb{R}^n$ a set $\Omega(x) \subseteq \Omega$ of full measure such that for all $\omega \in \Omega(x)$

$$\begin{aligned} &\lim_{\varepsilon\searrow 0} \fint_A \lambda_{\min}^\omega(x, \tfrac{y}{\varepsilon})^\alpha \,\mathrm{d}y \leq C, \qquad \lim_{\varepsilon\searrow 0} \fint_A \lambda_{\max}^\omega(x, \tfrac{y}{\varepsilon})^\alpha \,\mathrm{d}y \leq C, \\ &\lim_{\varepsilon\searrow 0} \fint_A \lambda_{\min}^\omega(x, \tfrac{y}{\varepsilon})^{-\beta} \,\mathrm{d}y \leq C, \quad \lim_{\varepsilon\searrow 0} \fint_A \lambda_{\max}^\omega(x, \tfrac{y}{\varepsilon})^{-\beta} \,\mathrm{d}y \leq C. \end{aligned} \tag{5.11}$$

The proof of Lemma 5.3.4 will give rise to shrink the set $\Omega(x)$ even further using the same argument to ensure additionally equi-integrability of $\lambda_{\min}^\omega$ and $\lambda_{\max}^\omega$ for $\omega \in \Omega(x)$.

While in the application of Birkhoff's Pointwise Ergodic Theorem above the zero sets only depend on x, the application of the Subadditive Ergodic Theorem (Lemma 2.2.3) will yield zero sets depending on W, x and F as well: For the subadditive set function $\mathcal{F}$ defined by

$$\mathcal{F}^\omega(U) := \inf_{\phi\in W_0^{1,p}(U)} \int_U W^\omega(x, y, F + D\phi(y)) \,\mathrm{d}y$$

the Subadditive Ergodic Theorem gives for every cube $Q \subseteq \mathbb{R}^m$

$$W_{\text{hom}}(x, F) = \lim_{\varepsilon \searrow 0} \inf_{\phi \in W_0^{1,p}(\frac{1}{\varepsilon}Q)} \fint_{\frac{1}{\varepsilon}Q} W^\omega(x, y, F + D\phi(y))\,\mathrm{d}y \tag{5.12}$$

on a subset of Ω with full measure only depending on W, x and F. We denote by $\Omega_F(x)$ the intersection of this set with the set $\Omega(x)$ above, so we have (5.12) as well as (5.11) and the equi-integrability of $\lambda_{\min}$ and $\lambda_{\max}$. In the case of a spatially homogeneous potential $\overline{W}$ we accordingly write $\overline{\Omega}_F$.

Finally we set $\Omega_0 := \bigcap_{x \in \mathbb{Q}^n} \bigcap_{F \in \mathbb{Q}^{n \times m}} \Omega_F(x)$ (and $\overline{\Omega}_0 := \bigcap_{F \in \mathbb{Q}^{n \times m}} \overline{\Omega}_F$ resp.), so we have (5.11), the equi-integrability of $\lambda_{\min}$ and $\lambda_{\max}$ and (5.12) for every $x \in \mathbb{Q}^n$, $F \in \mathbb{Q}^{n \times m}$, which will be enough due to the continuity of W_{hom} (Proposition 5.2.4 (b) and (c)).

5.4. Proofs

As being said in the beginning, we will from now on conveniently drop the index ω in the proofs where it is clear from the situation.

5.4.1. The Compactness Result (Proposition 5.2.1)

We split the proof of Proposition 5.2.1 into the following three lemmas, which will yield the result immediately:

Lemma 5.4.1. *Let $A \subseteq \mathbb{R}^m$ be a bounded Lipschitz domain. Then from*

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } L^1(A) \qquad \text{and} \qquad \limsup_{\varepsilon \searrow 0} \mathcal{E}_\varepsilon^\omega(u^\varepsilon, A) < \infty \quad \text{for a.e. } \omega \in \Omega$$

follows $u \in W^{1,p}(A)$.

Lemma 5.4.2. *Let $A \subseteq \mathbb{R}^m$ be a bounded Lipschitz domain. Then from*

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } L^1(A) \qquad \text{and} \qquad \limsup_{\varepsilon \searrow 0} \mathcal{E}_\varepsilon^\omega(u^\varepsilon, A) < \infty \quad \text{for a.e. } \omega \in \Omega$$

follows $u^\varepsilon \to u$ in $L^q_{\text{loc}}(A)$ for every $q \geq 1$ with $\frac{1}{q} \geq \left(1 + \frac{1}{\beta}\right)\frac{1}{p} - \frac{1}{n}$.

Lemma 5.4.3. *Let $A \subseteq \mathbb{R}^m$ be a bounded Lipschitz domain, (u^ε) a sequence in $W^{1,p}(A)$. Then from*

$$\limsup_{\varepsilon \searrow 0} \mathcal{E}_\varepsilon^\omega(u^\varepsilon, A) < \infty \qquad \textit{for a.e. } \omega \in \Omega$$

follows $\limsup_{\varepsilon \searrow 0} \|u^\varepsilon\|_{L^1(A)} < \infty$.

The proofs follow closely proofs of [NSS17, Lemmas 3.14 (last step), 3.3 and 3.21] with an added continuity argument to treat the spatial inhomogeneity.

Proof of Lemma 5.4.1. We will show $u \in W^{1,p}(A)$ with a duality argument. To be explicit we will find a constant $\widetilde{C} > 0$ with $\int_A \psi \partial_j u \leq \widetilde{C} \|\psi\|_{L^{\frac{p}{p-1}}(A)}$ for $j = 1, \dots, n$, $\psi \in C_c^\infty(A)$ by smuggling in $\lambda_{\min}$ with Hölder's inequality and then taking advantage of the assumed boundedness of the energy, for $\int_A \lambda_{\min} |Du^\varepsilon|^p$ can be estimated by the energy with the Growth Condition (5.2).

Step 1: Duality argument.
Let $\psi \in C_c^\infty(A)$ and $j \in \{1, \dots, n\}$. As mentioned in Remark 5.3.7 the Moment Bounds Condition (5.7) provides control over moments of $\lambda(x_0, \frac{\cdot}{\varepsilon})$ for every point $x_0 \in \mathbb{R}^n$ and $\omega \in \Omega(x_0)$. Thus we fix $x_0 \in A$, $\omega \in \Omega(x_0)$ and use the lower semi-continuity of the L^1-norm and Hölder's inequality to smuggle in $\lambda_{\min}(x_0, \frac{\cdot}{\varepsilon})$, i.e.

$$\begin{aligned} &\|\psi \partial_j u\|_{L^1(A)} \\ &\quad \leq \liminf_{\varepsilon \searrow 0} \|\psi \partial_j u^\varepsilon\|_{L^1(A)} \\ &\quad \leq \liminf_{\varepsilon \searrow 0} \Big(\int_A \lambda_{\min}(x_0, \tfrac{x}{\varepsilon}) |Du^\varepsilon(x)|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} \Big(\int_A \lambda_{\min}(x_0, \tfrac{x}{\varepsilon})^{-\frac{1}{p-1}} |\psi(x)|^{\frac{p}{p-1}} \,\mathrm{d}x \Big)^{\frac{p-1}{p}}. \end{aligned} \tag{5.13}$$

We will show in Step 2 below, that from the boundedness of the energy follows

$$E := \Big(\limsup_{\varepsilon \searrow 0} \int_A \lambda_{\min}(x_0, \tfrac{x}{\varepsilon}) |Du^\varepsilon(x)|^p \,\mathrm{d}x \Big)^{\frac{1}{p}} < \infty. \tag{5.14}$$

To treat the second integral in (5.13) we use a two-scale-argument. Therefor we set for $\delta > 0$

$$Z_\delta := \{z \in \delta\mathbb{Z}^n; Q_\delta(z) \cap A \neq \emptyset\}, \qquad \text{with } Q_\delta(z) := z + \delta Y,$$

and therewith write

$$\int_A \lambda_{\min}(x_0, \tfrac{x}{\varepsilon})^{-\frac{1}{p-1}} |\psi(x)|^{\frac{p}{p-1}} \,\mathrm{d}x \leq \sum_{z \in Z_\delta} \sup_{x \in Q_\delta(z)} |\psi(x)|^{\frac{p}{p-1}} \int_{Q_\delta(z)} \lambda_{\min}(x_0, \tfrac{x}{\varepsilon})^{-\frac{1}{p-1}} \,\mathrm{d}x.$$

Now by Hölder's inequality and the arguments from Remark 5.3.7 we get

$$\begin{aligned}\limsup_{\varepsilon\searrow 0}\int_{Q_\delta(z)}\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})^{-\frac{1}{p-1}}\,\mathrm{d}x &\leq \delta^n\Big(\limsup_{\varepsilon\searrow 0}\fint_{Q_\delta(z)}\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})^{-\beta}\,\mathrm{d}x\Big)^{\frac{1}{\beta(p-1)}}\\ &\leq C^{\frac{1}{\beta(p-1)}}\delta^n,\end{aligned}$$

and since ψ is continuous, we conclude with $\delta\searrow 0$

$$\begin{aligned}&\limsup_{\varepsilon\searrow 0}\Big(\int_A\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})^{-\frac{1}{p-1}}|\psi(x)|^{\frac{p}{p-1}}\,\mathrm{d}x\Big)^{\frac{p-1}{p}}\\ &\quad\leq C^{\frac{1}{\beta p}}\Big(\lim_{\delta\searrow 0}\sum_{z\in Z_\delta}\sup_{x\in Q_\delta(z)}|\psi(x)|^{\frac{p}{p-1}}\delta^n\Big)^{\frac{p-1}{p}}\\ &\quad= C^{\frac{1}{\beta p}}\|\psi\|_{L^{\frac{p}{p-1}}(A)}.\end{aligned}\tag{5.15}$$

Hence from (5.13) to (5.15) we get

$$\int_A\psi(x)\partial_j u(x)\,\mathrm{d}x\leq\limsup_{\varepsilon\searrow 0}\|\psi\partial_j u^\varepsilon\|_{L^1(A)}\leq EC^{\frac{1}{\beta p}}\|\psi\|_{L^{\frac{p}{p-1}}(A)}$$

that is $\partial_j u\in L^p(A)$ by duality, and by Poincaré's inequality also $u\in L^p(A)$.

Step 2: Proof of (5.14).
As mentioned in the beginning of the proof, we want to estimate E by $\mathcal{E}_\varepsilon(u^\varepsilon,A)$. To that end we start with applying the Growth Condition (5.2), which gives

$$\int_A\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})|Du^\varepsilon(x)|^p\,\mathrm{d}x\leq C\int_A W(x_0,\tfrac{x}{\varepsilon},Du^\varepsilon(x))\,\mathrm{d}x+C^2\int_A\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})\,\mathrm{d}x.\tag{5.16}$$

Hölder's inequality and the arguments in Remark 5.3.7 yield

$$\limsup_{\varepsilon\searrow 0}\int_A\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})\,\mathrm{d}x\leq|A|\Big(\limsup_{\varepsilon\searrow 0}\fint_A\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})^\alpha\,\mathrm{d}x\Big)^{\frac{1}{\alpha}}\leq C^{\frac{1}{\alpha}}|A|<\infty.\tag{5.17}$$

Now to estimate $\int_A W(x_0,\frac{\cdot}{\varepsilon},Du^\varepsilon)$ by the $\mathcal{E}_\varepsilon(u^\varepsilon,A)$ we have to replace x_0 by x, and to do so we use the Spatial Continuity of W (Condition (5.9)), which gives

$$\begin{aligned}&\int_A W(x_0,\tfrac{x}{\varepsilon},Du^\varepsilon(x))\,\mathrm{d}x\\ &\quad\leq\int_A W(x,\tfrac{x}{\varepsilon},Du^\varepsilon(x))\,\mathrm{d}x+\int_A\rho'(|x_0-x|)\big(1+2W(x,\tfrac{x}{\varepsilon},Du^\varepsilon(x))\big)\,\mathrm{d}x\\ &\quad\leq\big(1+2\rho'(\operatorname{diam}A)\big)\mathcal{E}_\varepsilon(u^\varepsilon,A)+\rho'(\operatorname{diam}A)|A|.\end{aligned}\tag{5.18}$$

Taking (5.16) to (5.18) together and recalling that A is bounded we conclude

$$\begin{aligned}\limsup_{\varepsilon\searrow 0}&\int_A \lambda_{\min}(x_0,\tfrac{x}{\varepsilon})|Du^\varepsilon(x)|^p\,\mathrm{d}x\\&\le \big(1+2\rho'(\operatorname{diam}A)\big)\limsup_{\varepsilon\searrow 0}\mathcal{E}_\varepsilon(u^\varepsilon,A)+\big(C^{2+\frac{1}{\alpha}}+\rho'(\operatorname{diam}A)\big)|A|\\&<\infty.\end{aligned}$$ □

Proof of Lemma 5.4.2.
Since $u\in W^{1,p}(A)$ by Lemma 5.4.1, the Gagliardo-Nirenberg-Sobolev inequality yields $u\in L^q(A)$. Thus for every $\delta>0$ there is a function $v\in C^1(A)$ with $\|u-v\|_{L^q(A)}<\frac{\delta}{2}$. We claim that every compact subset $A'\Subset A$

$$\limsup_{\varepsilon\searrow 0}\|u^\varepsilon-v\|_{L^q(A')}\le\|u-v\|_{L^q(A)}. \tag{5.19}$$

Note that this implies the assertion, since

$$\limsup_{\varepsilon\searrow 0}\|u^\varepsilon-u\|_{L^q(A')}<\limsup_{\varepsilon\searrow 0}\|u^\varepsilon-v\|_{L^q(A')}+\tfrac{\delta}{2}\le\|u-v\|_{L^q(A)}+\tfrac{\delta}{2}<\delta.$$

To prove (5.19) we conveniently write $\bar u^\varepsilon:=u^\varepsilon-v$ and $\bar u:=u-v$. We will use the Poincaré-Sobolev inequality combined with a two-scale-argument, so for $\eta>0$ we set

$$Z_\eta:=\{z\in\eta\mathbb{Z}^n;Q_\eta(z)\cap A'\neq\emptyset\},\qquad\text{with }Q_\eta(z):=z+\eta Y.$$

Then for η small enough we have $A'\subseteq\bigcup_{z\in Z_\eta}Q_\eta(z)\subseteq A$ and thus

$$\|\bar u^\varepsilon\|_{L^q(A')}\le\Big(\sum_{z\in Z_\eta}\int_{Q_\eta(z)}\big|\bar u^\varepsilon(x)-⨍_{Q_\eta(z)}\bar u^\varepsilon\big|^q\,\mathrm{d}x\Big)^{\frac1q}+\Big(\sum_{z\in Z_\eta}\big|⨍_{Q_\eta(z)}\bar u^\varepsilon\big|^q\eta^n\Big)^{\frac1q}. \tag{5.20}$$

From $u^\varepsilon\rightharpoonup u$ weakly in $L^1(A)$ we also immediately have $\bar u^\varepsilon\rightharpoonup\bar u$ weakly in $L^1(A)$ and $⨍_{Q_\eta(z)}\bar u^\varepsilon\to⨍_{Q_\eta(z)}\bar u$. Thus Hölder's inequality yields

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\Big(\sum_{z\in Z_\eta}\big|⨍_{Q_\eta(z)}\bar u^\varepsilon\big|^q\eta^n\Big)^{\frac1q}&=\Big(\sum_{z\in Z_\eta}\big|⨍_{Q_\eta(z)}\bar u\big|^q\eta^n\Big)^{\frac1q}\\&\le\Big(\sum_{z\in Z_\eta}\int_{Q_\eta(z)}|\bar u(x)|^q\,\mathrm{d}x\Big)^{\frac1q}\\&\le\|\bar u\|_{L^q(A)}.\end{aligned}\tag{5.21}$$

We now claim

$$\lim_{\eta\searrow 0}\limsup_{\varepsilon\searrow 0}\Big(\sum_{z\in Z_\eta}\int_{Q_\eta(z)}\big|\bar u^\varepsilon(x)-⨍_{Q_\eta(z)}\bar u^\varepsilon\big|^q\,\mathrm{d}x\Big)^{\frac1q}=0, \tag{5.22}$$

which together with (5.20) and (5.21) yields

$$\limsup_{\varepsilon\searrow 0}\|\bar u^\varepsilon\|_{L^q(A')}=\lim_{\eta\searrow 0}\limsup_{\varepsilon\searrow 0}\|\bar u^\varepsilon\|_{L^q(A')}\le\|\bar u\|_{L^q(A)},$$

that is (5.19). We start with applying the Sobolev-Poincaré inequality and than, similar to the proof of Lemma 5.4.1, use Hölder's inequality to smuggle in $\lambda_{\min}(x_0,\frac{\cdot}{\varepsilon})$ for some fixed $x_0\in A$, $\omega\in\Omega(x_0)$, i.e.

$$\begin{aligned}
\Big(\fint_{Q_\eta(z)}\big|\bar u^\varepsilon(x)-\fint_{Q_\eta(z)}\bar u^\varepsilon\big|^q\,\mathrm{d}x\Big)^{\frac1q}
&\lesssim\eta\Big(\fint_{Q_\eta(z)}|Du^\varepsilon(x)|^{\frac{\beta}{\beta+1}p}\,\mathrm{d}x\Big)^{\frac{\beta+1}{\beta}\frac1p}\\
&\le\eta\Big(\fint_{Q_\eta(z)}\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})^{-\beta}\,\mathrm{d}x\Big)^{\frac{1}{\beta p}}\Big(\fint_{Q_\eta(z)}\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})|Du^\varepsilon(x)|^p\,\mathrm{d}x\Big)^{\frac1p},
\end{aligned}$$

where $\lesssim$ means $\le$ up to a constant only depending on n and p. With the arguments in Remark 5.3.7 we note

$$\lim_{\varepsilon\searrow 0}\fint_{Q_\eta(z)}\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})^{-\beta}\,\mathrm{d}x\le C,$$

thus

$$\begin{aligned}
&\limsup_{\varepsilon\searrow 0}\Big(\sum_{z\in Z_\eta}\int_{Q_\eta(z)}|\bar u^\varepsilon(x)-\fint_{Q_\eta(z)}\bar u^\varepsilon|^q\,\mathrm{d}x\Big)^{\frac1q}\\
&\qquad\lesssim\eta^{n(\frac1q-\frac1p-\frac1n)}\limsup_{\varepsilon\searrow 0}\Big(\sum_{z\in Z_\eta}\Big(\int_{Q_\eta(z)}\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})|Du^\varepsilon(x)|^p\,\mathrm{d}x\Big)^{\frac qp}\Big)^{\frac1q},
\end{aligned}$$

where $\lesssim$ means $\le$ up to a constant depending only on n, p, β and C. Now we distinguish the two cases $q\ge p$ and $q<p$. On the one hand if $q\ge p$ we use Hölder's inequality and the fact that $\#Z_\eta\eta^n\le|A|$ to get

$$\begin{aligned}
\Big(\sum_{z\in Z_\eta}\Big(\int_{Q_\eta(z)}\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})|Du^\varepsilon(x)|^p\,\mathrm{d}x\Big)^{\frac qp}\Big)^{\frac1q}
&\le(\#Z_\eta)^{\frac1q-\frac1p}\Big(\sum_{z\in Z_\eta}\int_{Q_\eta(z)}\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})|Du^\varepsilon(x)|^p\,\mathrm{d}x\Big)^{\frac1p}\\
&\le\eta^{-n(\frac1q-\frac1p)}|A|\Big(\int_A\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})|Du^\varepsilon(x)|^p\,\mathrm{d}x\Big)^{\frac1p}
\end{aligned}$$

and thus

$$\begin{aligned}
&\limsup_{\varepsilon\searrow 0}\Big(\sum_{z\in Z_\eta}\int_{Q_\eta(z)}|\bar u^\varepsilon(x)-\fint_{Q_\eta(z)}\bar u^\varepsilon|^q\,\mathrm{d}x\Big)^{\frac1q}\\
&\qquad\lesssim\eta|A|\limsup_{\varepsilon\searrow 0}\Big(\int_A\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})|Du^\varepsilon(x)|^p\,\mathrm{d}x\Big)^{\frac1p}.
\end{aligned}$$

On the other hand if $q < p$ we use the continuous embedding $\ell^{\frac{p}{q}}(Z_\eta) \subseteq \ell^1(Z_\eta)$ and get

$$\Big(\sum_{z\in Z_\eta}\Big(\int_{Q_\eta(z)}\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})|Du^\varepsilon(x)|^p\,\mathrm{d}x\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} \leq \Big(\sum_{z\in Z_\eta}\int_{Q_\eta(z)}\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})|Du^\varepsilon(x)|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}.$$

and thus

$$\begin{aligned}&\limsup_{\varepsilon\searrow 0}\Big(\sum_{z\in Z_\eta}\int_{Q_\eta(z)}|\bar u^\varepsilon(x) - {\textstyle\fint_{Q_\eta(z)}}\bar u^\varepsilon|^q\,\mathrm{d}x\Big)^{\frac{1}{q}}\\ &\qquad\lesssim \eta^{n(\frac{1}{q}-\frac{1}{p}+\frac{1}{n})}\limsup_{\varepsilon\searrow 0}\Big(\int_A\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})|Du^\varepsilon(x)|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}.\end{aligned}$$

Since $\frac{1}{q}-\frac{1}{p}+\frac{1}{n} \geq \frac{1}{\beta p} > 0$ and

$$\limsup_{\varepsilon\searrow 0}\int_A\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})|Du^\varepsilon(x)|^p\,\mathrm{d}x < \infty,$$

as can be seen in the proof of Lemma 5.4.1 Step 2, in both cases we have (5.22). □

Proof of Lemma 5.4.3. The assertion is a consequence of the Gagliardo-Nirenberg-Sobolev inequality, Hölder's inequality, the Moment Bound Condition (5.7) and (5.14), which is applicable in this case. Precisely we have, since $\frac{\beta}{\beta+1}p \geq 1$ and $1 \geq \frac{\beta+1}{\beta}\frac{1}{p} - \frac{1}{n}$ we have

$$\|u^\varepsilon\|_{L^1(A)} \lesssim \|Du^\varepsilon\|_{L^{\frac{\beta}{\beta+1}p}(A)},$$

where $\lesssim$ means $\leq$ up to a constant only depending on A, p and n. We can further estimate using Hölder's inequality and the Growth Condition (5.2) by

$$\begin{aligned}\|Du^\varepsilon\|_{L^{\frac{\beta}{\beta+1}p}(A)} &= \Big(\int_A\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})^{-\frac{\beta}{\beta+1}}\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})^{\frac{\beta}{\beta+1}}|Du^\varepsilon(x)|^{\frac{\beta}{\beta+1}p}\,\mathrm{d}x\Big)^{\frac{\beta+1}{\beta}\frac{1}{p}}\\ &\leq \Big(\int_A\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})^{-\beta}\,\mathrm{d}x\Big)^{\frac{1}{\beta}\frac{1}{p}}\Big(\int_A\lambda_{\min}(x_0,\tfrac{x}{\varepsilon})|Du^\varepsilon(x)|^p\,\mathrm{d}x\Big)^{\frac{1}{p}}\end{aligned}$$

for every $x_0 \in A$. Now the assertion follows from (5.14) and the Moment Bound Condition (5.7). □

5.4.2. Recovery Sequence for Spatially Homogeneous Potentials and Affine Functions (Lemma 5.3.3)

We adapt the technique from the proof of [Mül87, Lemma 2.1 (a)] to the stochastic case using the Subadditive Ergodic Theorem as in Remark 5.3.7 to replace periodicity in the argumentation.

Proof of Lemma 5.3.3.
Step 1: Construction of (u^ε) with boundary values.
To define the recovery sequence (u^ε) we part A into small cubes and then on each cube take a minimizing sequence for $\inf_{\phi\in W_0^{1,p}} \fint \overline{W}(y, F + d\phi(y))\,\mathrm{d}y$, which occurs in the definition of $\overline{W}_{\mathrm{hom}}$. In order to do so we set for $\delta > 0$

$$Z_\delta := \{z \in \tfrac{\delta}{\sqrt{n}}\mathbb{Z}^n; Q_\delta(z) \subseteq A\}, \qquad \text{with } Q_\delta(z) := z + \tfrac{\delta}{\sqrt{n}}Y.$$

Then

$$A_\delta := \bigcup_{z\in Z_\delta} Q_\delta(z) \subseteq A$$

consists of cubes with diameter δ and we have $|A \setminus A_\delta| \to 0$ as $\delta \to 0$. For each $z \in Z_\delta$ and $\varepsilon > 0$ we find $\phi^\varepsilon_{\delta,z} \in W_0^{1,p}(\frac{1}{\varepsilon}Q_\delta(z))$ such that

$$\lim_{\varepsilon\searrow 0} \int_{\frac{1}{\varepsilon}Q_\delta(z)} \overline{W}(y, F + D\phi^\varepsilon_{\delta,z}(y))\,\mathrm{d}y = \lim_{\varepsilon\searrow 0} \inf_{\phi\in W_0^{1,p}(\frac{1}{\varepsilon}Q_\delta(z))} \int_{\frac{1}{\varepsilon}Q_\delta(z)} \overline{W}(y, F + D\phi(y))\,\mathrm{d}y. \tag{5.23}$$

Now we can define $u^\varepsilon_\delta \in W^{1,p}(A)$ by

$$u^\varepsilon_\delta(x) := Fx + \sum_{z\in Z_\delta} \varepsilon\phi^\varepsilon_{\delta,z}(\tfrac{x}{\varepsilon}),$$

where we use the convention that $\phi^\varepsilon_{\delta,z} = 0$ outside of $Q_\delta(z)$. Obviously $u^\varepsilon_\delta(x) = Fx$ for $x \in A \setminus A_\delta$ and thus $u^\varepsilon_\delta - Fx \in W_0^{1,p}(A)$. In Steps 2 and 3 below we will show that

$$\lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0} \overline{\mathcal{E}}_\varepsilon(u^\varepsilon_\delta, A) = \overline{\mathcal{E}}_{\mathrm{hom}}(F, A) \qquad \text{and} \qquad \lim_{\delta\searrow 0}\limsup_{\varepsilon\searrow 0} \|u^\varepsilon_\delta - Fx\|_{L^{\frac{\beta}{\beta+1}p}(A)} = 0,$$

hence we can find a diagonal sequence $\delta(\varepsilon)$ such that $u^\varepsilon := u^\varepsilon_{\delta(\varepsilon)}$ gives the desired recovery sequence.

Step 2: Convergence of the energy.
For every $z \in Z_\delta$ the arguments in Remark 5.3.7 together with (5.23) yields

$$\begin{aligned} \overline{W}_{\mathrm{hom}}(F) &= \lim_{\varepsilon\searrow 0} \fint_{\frac{1}{\varepsilon}Q_\delta(z)} \overline{W}(y, F + D\phi^\varepsilon_{\delta,z}(y))\,\mathrm{d}y \\ &= \lim_{\varepsilon\searrow 0} \fint_{Q_\delta(z)} \overline{W}(\tfrac{x}{\varepsilon}, F + D\phi^\varepsilon_{\delta,z}(\tfrac{x}{\varepsilon}))\,\mathrm{d}x \end{aligned} \tag{5.24}$$

and thus

$$\begin{aligned} \lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0} \int_{A_\delta} \overline{W}(\tfrac{x}{\varepsilon}, Du^\varepsilon_\delta(x))\,\mathrm{d}x &= \lim_{\delta\searrow 0} \sum_{z\in Z_\delta} \lim_{\varepsilon\searrow 0} \int_{Q_\delta(z)} \overline{W}(\tfrac{x}{\varepsilon}, F + D\phi^\varepsilon_{\delta,z}(\tfrac{x}{\varepsilon}))\,\mathrm{d}x \\ &= \lim_{\delta\searrow 0} \sum_{z\in Z_\delta} |Q_\delta(z)|\overline{W}_{\mathrm{hom}}(F) \\ &= |A|\overline{W}_{\mathrm{hom}}(F). \end{aligned}$$

Now it only remains to note that by the Growth Condition (5.2) and the arguments in Remark 5.3.7 we have

$$\begin{aligned}\lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\int_{A\setminus A_\delta}\overline{W}(\tfrac{x}{\varepsilon},Du_\delta^\varepsilon(x))\,\mathrm{d}x &\le C\lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\int_{A\setminus A_\delta}\overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})(|F|^p+1)\,\mathrm{d}x\\ &= C^2(|F|^p+1)\lim_{\delta\searrow 0}|A\setminus A_\delta|\\ &=0.\end{aligned}$$

Step 3: $L^{\frac{\beta}{\beta+1}p}$-convergence of (u_δ^ε).
To control the $L^{\frac{\beta}{\beta+1}p}$-norm of $u_\delta^\varepsilon - Fx$ we use Poincaré's inequality on the level of the cubes $Q_\delta(z)$, so we can benefit from the small scale of the diameter δ, i.e.

$$\begin{aligned}\int_A |u_\delta^\varepsilon(x)-Fx|^{\frac{\beta}{\beta+1}p}\,\mathrm{d}x &= \sum_{z\in Z_\delta}\int_{Q_\delta(z)}|\varepsilon\phi_{\delta,z}^\varepsilon(\tfrac{x}{\varepsilon})|^{\frac{\beta}{\beta+1}p}\,\mathrm{d}x\\ &\le \sum_{z\in Z_\delta}\delta^{\frac{\beta}{\beta+1}p}\int_{Q_\delta(z)}|D\phi_{\delta,z}^\varepsilon(\tfrac{x}{\varepsilon})|^{\frac{\beta}{\beta+1}p}\,\mathrm{d}x\\ &\le \delta^{\frac{\beta}{\beta+1}p}\sum_{z\in Z_\delta}\Big(|Q_\delta(z)||F|^{\frac{\beta}{\beta+1}p}+\int_{Q_\delta(z)}|F+D\phi_{\delta,z}^\varepsilon(\tfrac{x}{\varepsilon})|^{\frac{\beta}{\beta+1}p}\,\mathrm{d}x\Big)\\ &\le \delta^{\frac{\beta}{\beta+1}p}\Big(|A||F|^{\frac{\beta}{\beta+1}p}+\sum_{z\in Z_\delta}\int_{Q_\delta(z)}|Du_\delta^\varepsilon(x)|^{\frac{\beta}{\beta+1}p}\,\mathrm{d}x\Big).\end{aligned}$$

We use Hölder's inequality to smuggle in $\overline{\lambda}_{\min}$ so we can apply the Growth Condition (5.2), which gives

$$\begin{aligned}&\int_{Q_\delta(z)}|Du_\delta^\varepsilon(x)|^{\frac{\beta}{\beta+1}p}\,\mathrm{d}x\\ &\quad\le \Big(\int_{Q_\delta(z)}\overline{\lambda}_{\min}(\tfrac{x}{\varepsilon})^{-\beta}\,\mathrm{d}x\Big)^{\frac{1}{\beta+1}}\Big(\int_{Q_\delta(z)}\overline{\lambda}_{\min}(\tfrac{x}{\varepsilon})|Du_\delta^\varepsilon(x)|^p\,\mathrm{d}x\Big)^{\frac{\beta+1}{\beta}}\\ &\quad\le |Q_\delta(z)|\Big(\fint_{Q_\delta(z)}\overline{\lambda}_{\min}(\tfrac{x}{\varepsilon})^{-\beta}\,\mathrm{d}x\Big)^{\frac{1}{\beta+1}}\\ &\qquad\cdot\Big(C\fint_{Q_\delta(z)}\overline{W}(\tfrac{x}{\varepsilon},Du_\delta^\varepsilon(x))\,\mathrm{d}x+C^2\fint_{Q_\delta(z)}\overline{\lambda}_{\min}(\tfrac{x}{\varepsilon})\,\mathrm{d}x\Big)^{\frac{\beta}{\beta+1}}.\end{aligned}$$

With the arguments in Remark 5.3.7 and Höder's inequality we see

$$\limsup_{\varepsilon\searrow 0}\fint_{Q_\delta(z)}\overline{\lambda}_{\min}(\tfrac{x}{\varepsilon})^{-\beta}\,\mathrm{d}x\le C$$

and

$$\limsup_{\varepsilon\searrow 0}\fint_{Q_\delta(z)}\overline{\lambda}_{\min}(\tfrac{x}{\varepsilon})\,\mathrm{d}x\le\limsup_{\varepsilon\searrow 0}\Big(\fint_{Q_\delta(z)}\overline{\lambda}_{\min}(\tfrac{x}{\varepsilon})^{\alpha}\,\mathrm{d}x\Big)^{\frac{1}{\alpha}}\le C^{\frac{1}{\alpha}},$$

and from (5.24)

$$\lim_{\varepsilon \searrow 0} \fint_{Q_\delta(z)} \overline{W}(\tfrac{x}{\varepsilon}, Du_\delta^\varepsilon(x))\,\mathrm{d}x = \overline{W}_{\mathrm{hom}}(F).$$

Together this means

$$\begin{aligned}
&\limsup_{\varepsilon \searrow 0} \int_A |u_\delta^\varepsilon(x) - Fx|^{\frac{\beta}{\beta+1}p}\,\mathrm{d}x \\
&\quad \le \delta^{\frac{\beta}{\beta+1}p}\Big(|A||F|^{\frac{\beta}{\beta+1}p} + C \sum_{z\in Z_\delta} |Q_\delta(z)|\big(\overline{W}_{\mathrm{hom}}(F) + C^{\frac{\alpha+1}{\alpha}}\big)^{\frac{\beta}{\beta+1}}\Big) \\
&\quad \le \delta^{\frac{\beta}{\beta+1}p}\Big(|A||F|^{\frac{\beta}{\beta+1}p} + C|A|\big(\overline{W}_{\mathrm{hom}}(F) + C^{\frac{\alpha+1}{\alpha}}\big)^{\frac{\beta}{\beta+1}}\Big)
\end{aligned}$$

and we finally conclude as desired

$$\lim_{\delta \searrow 0} \limsup_{\varepsilon \searrow 0} \|u_\delta^\varepsilon - Fx\|_{L^{\frac{\beta}{\beta+1}p}(A)} = 0. \qquad \square$$

5.4.3. Technical Lemmas: Equi-Integrability of $\overline{\lambda}$ (Lemma 5.3.4) and Gluing Constructions (Lemmas 5.3.5 and 5.3.6)

This is the most technical part of the proof of Theorem 5.2.2. We basically follow the proofs of [NSS17, Lemmas 3.22 and 3.23]. The main differences are due to the non-uniformity of the growth condition, as one cannot (informally speaking) estimate

$$\mathcal{E}(F+G) \le \int \lambda(|F|^p + |G|^p) \le \mathcal{E}(F) + \mathcal{E}(G)$$

as in the degenerate case where $\lambda_{\max} = \lambda_{\min}$. A work around is given by the interplay of the Growth Condition (5.2) with the Mild $\lambda_{\max}$-Convexity (5.3) and the Weak $\lambda_{\max}$-Δ_2-Property (5.4).

Proof of Lemma 5.3.5. Despite Lemma 5.3.5 states only the existence of a sequence $(\mathcal{O}_N)$, the following proof will show that one such sequence can be given explicitly by

$$\mathcal{O}_N := 1 - (\tfrac{N-1}{N})^n.$$

Step 1: Construction of ϕ_N^ε.
We want to define $\phi_N^\varepsilon \in W_0^{1,p}(Q)$ such that $\overline{u} + \phi_N^\varepsilon = u^\varepsilon$ on most part of Q, as there the left hand side of the assertion would vanish. Such a construction can be done by cutting off $u^\varepsilon - \overline{u}$ near the boundary of Q. But since we a priori do not know where to cut $u^\varepsilon - \overline{u}$ exactly, we minimize the energy over several cut-off functions. To do so, we set for $j = 1, \dots, N$

$$Q_0 := Q \qquad \text{and} \qquad Q_j := \{x \in Q; \mathrm{dist}(x, \partial Q) > j\tfrac{l}{2N^2}\}.$$

Then $Q_N \subseteq Q_j \subseteq Q_{j-1} \subseteq Q_0 = Q$ and

$$|Q \setminus Q_N| = \big(1 - (\tfrac{N-1}{N})^m\big)|Q| \leq \mathcal{O}_N |Q|.$$

Now we choose cut-off functions $\psi_j \in C_c^\infty(Q_{j-1})$ with

$$\psi_j = 1 \text{ on } Q_j, \qquad \psi_j = 0 \text{ on } Q \setminus Q_{j-1}, \qquad \|\nabla \psi_j\|_{L^\infty(Q)} \leq \tfrac{3N^2}{l}$$

and set

$$w^\varepsilon := u^\varepsilon - \overline{u} \qquad \text{and} \qquad \phi^{\varepsilon,j} := \psi_j w^\varepsilon \in W_0^{1,p}(Q).$$

Since for any given $N \in \mathbb{N}$ and $\varepsilon > 0$ we have only finitely many functions $\phi^{\varepsilon,j}$, we can find $j(N,\varepsilon) \in \{1, \ldots, N\}$ such that the energy difference $\overline{\mathcal{E}}_\varepsilon(\overline{u}+\phi^{\varepsilon,j(N,\varepsilon)}, Q) - \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q)$ is minimal, and we set $\phi_N^\varepsilon := \phi^{\varepsilon,j(N,\varepsilon)}$. Then of course the energy difference can be estimated against the arithmetic mean of all energy differences, i.e.

$$\overline{\mathcal{E}}_\varepsilon(\overline{u} + \phi_N^\varepsilon, Q) - \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q) \leq \tfrac{1}{N} \sum_{j=1}^N \Big(\overline{\mathcal{E}}_\varepsilon(\overline{u} + \phi^{\varepsilon,j}, Q) - \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q)\Big).$$

With the idea to split the energies like

$$\overline{\mathcal{E}}_\varepsilon(\cdot, Q) = \overline{\mathcal{E}}_\varepsilon(\cdot, Q_j) + \overline{\mathcal{E}}_\varepsilon(\cdot, Q_{j-1} \setminus Q_j) + \overline{\mathcal{E}}_\varepsilon(\cdot, Q \setminus Q_{j-1}),$$

we claim that

$$\tfrac{1}{N} \limsup_{\varepsilon \searrow 0} \sum_{j=1}^N \Big(\overline{\mathcal{E}}_\varepsilon(\overline{u} + \phi^{\varepsilon,j}, Q_j) - \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q_j)\Big) = 0, \tag{5.25}$$

$$\begin{aligned} &\tfrac{1}{N} \limsup_{\varepsilon \searrow 0} \sum_{j=1}^N \Big(\overline{\mathcal{E}}_\varepsilon(\overline{u} + \phi^{\varepsilon,j}, Q_{j-1} \setminus Q_j) - \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q_{j-1} \setminus Q_j)\Big) \\ &\quad \lesssim \tfrac{1}{N} \limsup_{\varepsilon \searrow 0} \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q \setminus Q_N) + \tfrac{N^2}{l} \|u - \overline{u}\|_{L^p(Q)}^p + \big(\|D\overline{u}\|_{L^\infty(Q)}^p + 1\big)\mathcal{O}_N |Q| \end{aligned} \tag{5.26}$$

and

$$\tfrac{1}{N} \limsup_{\varepsilon \searrow 0} \sum_{j=1}^N \Big(\overline{\mathcal{E}}_\varepsilon(\overline{u} + \phi^{\varepsilon,j}, Q \setminus Q_{j-1}) - \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q \setminus Q_{j-1})\Big) \lesssim \big(\|D\overline{u}\|_{L^\infty(Q)}^p + 1\big)\mathcal{O}_N |Q|, \tag{5.27}$$

because then we will immediately get

$$\begin{aligned} &\limsup_{\varepsilon \searrow 0} \Big(\tfrac{1}{|Q|}\overline{\mathcal{E}}_\varepsilon(\overline{u} + \phi_N^\varepsilon, Q) - \tfrac{1}{|Q|}\overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q)\Big) \\ &\quad \leq \tfrac{1}{N} \limsup_{\varepsilon \searrow 0} \sum_{j=1}^N \tfrac{1}{|Q|}\Big(\overline{\mathcal{E}}_\varepsilon(\overline{u} + \phi^{\varepsilon,j}, Q) - \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q)\Big) \\ &\quad \lesssim \tfrac{1}{N} \limsup_{\varepsilon \searrow 0} \tfrac{1}{|Q|}\overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q \setminus Q_N) + \tfrac{N^2}{l|Q|} \|u - \overline{u}\|_{L^p(Q)}^p + \big(\|D\overline{u}\|_{L^\infty(Q)}^p + 1\big)\mathcal{O}_N, \end{aligned}$$

which in fact is the assertion, taking into account that

$$\overline{\mathcal{E}}_\varepsilon(\cdot, Q \setminus Q_N) = \overline{\mathcal{E}}_\varepsilon(\cdot, Q) - \overline{\mathcal{E}}_\varepsilon(\cdot, Q_N)$$

and that by the Growth Condition (5.2) and the arguments in Remark 5.3.7

$$\begin{aligned}\liminf_{\varepsilon\searrow 0} \tfrac{1}{|Q|}\overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q_N) &\geq \liminf_{\varepsilon\searrow 0} \tfrac{1}{|Q|}\int_{Q_N} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\big(\tfrac{1}{C}|Du^\varepsilon(x)|^p - C\big)\,\mathrm{d}x \\ &\gtrsim -\limsup_{\varepsilon\searrow 0} \fint_Q \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\,\mathrm{d}x \\ &\gtrsim -1.\end{aligned}$$

Step 2.1: Proof of (5.25).
Here the construction of $\phi^{\varepsilon,j}$ pays off, since we immediately see $\overline{u} + \phi^{\varepsilon,j} = u^\varepsilon$ on Q_j and thus

$$\overline{\mathcal{E}}_\varepsilon(\overline{u} + \phi^{\varepsilon,j}, Q_j) - \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q_j) = 0.$$

Step 2.2: Proof of (5.27).
Again we benefit from the construction of $\phi^{\varepsilon,j}$ and find $\overline{u} + \phi^{\varepsilon,j} = \overline{u}$ on $Q \setminus Q_{j-1}$. So we can use the Growth Condition (5.2), to get on the one hand

$$\overline{\mathcal{E}}_\varepsilon(\overline{u} + \phi^{\varepsilon,j}, Q \setminus Q_{j-1}) \lesssim \int_{Q\setminus Q_N} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\big(|D\overline{u}(x)|^p + 1\big)\,\mathrm{d}x,$$

and on the other hand, with $\overline{\lambda}_{\min} \lesssim \overline{\lambda}_{\max}$,

$$-\overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q \setminus Q_{j-1}) \leq -\int_{Q\setminus Q_{j-1}} \overline{\lambda}_{\min}(\tfrac{x}{\varepsilon})\big(\tfrac{1}{C}|Du^\varepsilon(x)|^p - C\big)\,\mathrm{d}x \lesssim \int_{Q\setminus Q_N} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\,\mathrm{d}x.$$

Now the arguments in Remark 5.3.7 and the definition of $\mathcal{O}_N$ yield

$$\begin{aligned}&\tfrac{1}{N}\limsup_{\varepsilon\searrow 0}\sum_{j=1}^N \Big(\overline{\mathcal{E}}_\varepsilon(\overline{u} + \phi^{\varepsilon,j}, Q \setminus Q_{j-1}) - \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q \setminus Q_{j-1})\Big) \\ &\quad\lesssim \int_{Q\setminus Q_N} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\big(|D\overline{u}(x)|^p + 1\big)\,\mathrm{d}x \\ &\quad\lesssim \big(\|D\overline{u}\|^p_{L^\infty(Q)} + 1\big)\mathcal{O}_N|Q|.\end{aligned}$$

Step 2.3: Proof of (5.26).
Similar to the previous step we find

$$-\,\overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q_{j_1} \setminus Q_1) \lesssim \int_{Q\setminus Q_N} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon}), \tag{5.28}$$

which will be covered so we only need to concentrate on $\overline{\mathcal{E}}_\varepsilon(\overline{u}+\phi^{\varepsilon,j}, Q_{j-1}\setminus Q_j)$. By the definition of $\phi^{\varepsilon,j}$ on $Q_{j-1}\setminus Q_j$ we find

$$D(\overline{u}+\phi^{\varepsilon,j}) = (1-\psi_j)D\overline{u} + \psi_j Du^\varepsilon + w^\varepsilon \nabla\psi_j^\intercal.$$

Thus the Mild $\lambda_{\max}$-Convexity (5.3) (with $F = 2(1-\psi_j)D\overline{u} + 2\psi_j Du^\varepsilon$, $G = 2w^\varepsilon\nabla\psi_j^\intercal$, $t=\frac{1}{2}$) and the Weak $\lambda_{\max}$-Δ_2-Property (5.4) yield

$$\begin{aligned}
&\overline{\mathcal{E}}_\varepsilon(\overline{u}+\phi^{\varepsilon,j}, Q_{j-1}\setminus Q_j)\\
&\quad\lesssim \int_{Q_{j-1}\setminus Q_j} \big(\overline{W}(\tfrac{x}{\varepsilon},(1-\psi_j)D\overline{u}+\psi_j Du^\varepsilon) + \overline{W}(\tfrac{x}{\varepsilon}, w^\varepsilon\nabla\psi_j^\intercal) + \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\big)\,\mathrm{d}x\\
&\quad\lesssim \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q_{j-1}\setminus Q_j) + \int_{Q_{j-1}\setminus Q_j} \big(\overline{W}(\tfrac{x}{\varepsilon}, D\overline{u}) + \overline{W}(\tfrac{x}{\varepsilon}, w^\varepsilon\nabla\psi_j^\intercal) + \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\big)\,\mathrm{d}x,
\end{aligned}\tag{5.29}$$

where in the last step again the Mild $\lambda_{\max}$-Convexity was used (this time with $F = D\overline{u}$, $G = Du^\varepsilon$, $t = 1-\psi_j$). We estimate the integral with the Growth Condition (5.2), and together with (5.28) we find

$$\begin{aligned}
&\overline{\mathcal{E}}_\varepsilon(\overline{u}+\phi^{\varepsilon,j}, Q_{j-1}\setminus Q_j) - \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q_{j_1}\setminus Q_1)\\
&\quad\lesssim \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q_{j-1}\setminus Q_j) + \int_{Q\setminus Q_N} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\big(|D\overline{u}(x)|^p+1\big)\,\mathrm{d}x\\
&\qquad+ \int_{Q_{j-1}\setminus Q_j} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})|w^\varepsilon(x)|^p|\nabla\psi_j(x)|^p\,\mathrm{d}x.
\end{aligned}$$

The first term fits perfectly as

$$\sum_{j=1}^N \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q_{j-1}\setminus Q_j) = \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q\setminus Q_N)$$

as required. The second term can be treated as in the step before using the arguments in Remark 5.3.7, which yields

$$\int_{Q\setminus Q_N} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\big(|D\overline{u}(x)|^p+1\big)\,\mathrm{d}x \lesssim \big(\|D\overline{u}\|^p_{L^\infty(Q)}+1\big)\mathcal{O}_N|Q|.$$

For the third term we can make use of the definition of ψ_j to get

$$\int_{Q_{j-1}\setminus Q_j} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})|w^\varepsilon(x)|^p|\nabla\psi_j(x)|^p\,\mathrm{d}x \le \tfrac{3N^2}{l}\int_{Q_{j-1}\setminus Q_j} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})|w^\varepsilon(x)|^p\,\mathrm{d}x.$$

The remainder of the proof is dedicated to show

$$\begin{aligned}
&\limsup_{\varepsilon\searrow 0}\tfrac{1}{N}\sum_{j=1}^N \tfrac{3N^2}{l}\int_{Q_{j-1}\setminus Q_j} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})|w^\varepsilon(x)|^p\,\mathrm{d}x\\
&\quad\lesssim \tfrac{1}{N}\limsup_{\varepsilon\searrow 0}\overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q\setminus Q_N) + \tfrac{3N^2}{l}\|u-\overline{u}\|^p_{L^p(Q)} + \big(1+\|D\overline{u}\|^p_{L^\infty(Q)}\big)\mathcal{O}_N|Q|,
\end{aligned}\tag{5.30}$$

because then (5.27) follows. To do so we would like to replace $\overline{\lambda}_{\max}$ by $\overline{\lambda}_{\min}$ and $|w^\varepsilon|$ by $|Dw^\varepsilon|$, since then with the definition of w^ε, the p-Growth Condition (5.2) and the fact that $\overline{\lambda}_{\min} \lesssim \overline{\lambda}_{\max}$ we would get

$$\int_{Q_{j-1}\setminus Q_j} \overline{\lambda}_{\min}(\tfrac{x}{\varepsilon})|Dw^\varepsilon(x)|^p \,\mathrm{d}x \lesssim \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q_{j-1}\setminus Q_j) + \int_{Q\setminus Q_N} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\big(|D\overline{u}(x)|^p + 1\big)\,\mathrm{d}x. \tag{5.31}$$

To reduce (5.30) to the situation of (5.31) with the Poincaré-Sobolev inequality we use basically the same arguments as for (5.19) in the proof of Lemma 5.4.2. We introduce a partition $\mathcal{Q}$ of $Q_{j-1}\setminus Q_j$ consisting of disjoint cubes with side length not greater than $\big(\frac{l}{3N^2}\big)^{\frac{1}{p}}$ and write

$$\begin{aligned}\int_{Q_{j-1}\setminus Q_j} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})|w^\varepsilon(x)|^p \,\mathrm{d}x &= \sum_{\overline{Q}\in\mathcal{Q}} \int_{\overline{Q}} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\big|w^\varepsilon(x) - \textstyle\fint_{\overline{Q}} w^\varepsilon\big|^p \,\mathrm{d}x \\ &\quad + \sum_{\overline{Q}\in\mathcal{Q}} \big|\textstyle\fint_{\overline{Q}} w^\varepsilon\big|^p \int_{\overline{Q}} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\,\mathrm{d}x.\end{aligned} \tag{5.32}$$

Since $u^\varepsilon \rightharpoonup u$ weakly in $L^1(Q)$ and thus $\fint_{\overline{Q}} w^\varepsilon \to \fint_{\overline{Q}}(u-\overline{u})$ we note with the arguments in Remark 5.3.7

$$\limsup_{\varepsilon\searrow 0} \tfrac{1}{N}\sum_{j=1}^N \sum_{\overline{Q}\in\mathcal{Q}} \big|\textstyle\fint_{\overline{Q}} w^\varepsilon\big|^p \int_{\overline{Q}} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\,\mathrm{d}x \lesssim \tfrac{1}{N}\sum_{j=1}^N \sum_{\overline{Q}\in\mathcal{Q}} \big|\textstyle\fint_{\overline{Q}}(u-\overline{u})\big|^p|\overline{Q}| \le \|u-\overline{u}\|^p_{L^p(Q)}. \tag{5.33}$$

For the first term in (5.32) we use Hölder's inequality to remove $\overline{\lambda}$ from the integrand, then recall that by assumption $\frac{\alpha-1}{\alpha}\frac{1}{p} \ge \frac{\beta+1}{\beta}\frac{1}{p} - \frac{1}{n}$ so we can use the Poincaré-Sobolev inequality to estimate w^ε by Dw^ε and take advantage of the side length of the cubes, and finally use Hölder's inequality again to smuggle $\overline{\lambda}_{\min}$ back into the integrand, i.e.

$$\begin{aligned}&\fint_{\overline{Q}} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\big|w^\varepsilon(x) - \textstyle\fint_{\overline{Q}} w^\varepsilon\big|^p \,\mathrm{d}x \\ &\quad\le \Big(\fint_{\overline{Q}} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})^\alpha \,\mathrm{d}x\Big)^{\frac{1}{\alpha}} \Big(\fint_{\overline{Q}} \big|w^\varepsilon(x) - \textstyle\fint_q w^\varepsilon\big|^{\frac{\alpha}{\alpha-1}p}\,\mathrm{d}x\Big)^{\frac{\alpha-1}{\alpha}} \\ &\quad\lesssim \tfrac{l}{3N^2}\Big(\fint_{\overline{Q}} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})^\alpha \,\mathrm{d}x\Big)^{\frac{1}{\alpha}} \Big(\fint_{\overline{Q}} |Dw^\varepsilon(x)|^{\frac{\beta}{\beta+1}p}\,\mathrm{d}x\Big)^{\frac{\beta+1}{\beta}} \\ &\quad\le \tfrac{l}{3N^2}\Big(\fint_{\overline{Q}} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})^\alpha \,\mathrm{d}x\Big)^{\frac{1}{\alpha}} \Big(\fint_{\overline{Q}} \overline{\lambda}_{\min}(\tfrac{x}{\varepsilon})^{-\beta}\,\mathrm{d}x\Big)^{\frac{1}{\beta}} \Big(\fint_{\overline{Q}} \overline{\lambda}_{\min}(\tfrac{x}{\varepsilon})|Dw^\varepsilon(x)|^p\,\mathrm{d}x\Big),\end{aligned}$$

which gives with the arguments in Remark 5.3.7 and (5.31)

$$\begin{aligned}
&\limsup_{\varepsilon\searrow 0} \tfrac{1}{N} \sum_{j=1}^{N} \sum_{\overline{Q}\in\mathcal{Q}} \int_{\overline{Q}} \overline{\lambda}_{\max}(\tfrac{x}{\varepsilon}) \big| w^\varepsilon(x) - \textstyle\fint_{\overline{Q}} w^\varepsilon \big|^p \,\mathrm{d}x \\
&\lesssim \tfrac{l}{3N^2} \tfrac{1}{N} \sum_{j=1}^{N} \sum_{\overline{Q}\in\mathcal{Q}} \limsup_{\varepsilon\searrow 0} \int_{\overline{Q}} \overline{\lambda}_{\min}(\tfrac{x}{\varepsilon}) |Dw^\varepsilon(x)|^p \,\mathrm{d}x \\
&= \tfrac{l}{3N^2} \limsup_{\varepsilon\searrow 0} \int_{Q\setminus QN} \overline{\lambda}_{\min}(\tfrac{x}{\varepsilon}) |Dw^\varepsilon(x)|^p \,\mathrm{d}x \\
&\lesssim \tfrac{l}{3N^2} \Big(\limsup_{\varepsilon\searrow 0} \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q\setminus Q_N) + \big(1 + \|D\overline{u}\|^p_{L^\infty(Q)}\big)\mathcal{O}_N |Q| \Big). \qquad \square
\end{aligned}$$

The crucial point where the assertion $\frac{\alpha-1}{\alpha}\frac{1}{p} \geq \frac{\beta-1}{\beta}\frac{1}{p} - \frac{1}{n}$ occurred above in the proof of Lemma 5.3.5 was when we used the Poincaré-Sobolev inequality to control $u^\varepsilon - u$ (Step 2.3). If u^ε and $\overline{u}$ are scalar valued functions, we can use a truncated version of $u^\varepsilon - \overline{u}$ instead, so the absolute is a priori bounded. But then we loose the fact that the left hand side of the assertion vanishes on the most part of the cube (Step 2.1), and we have to replace it by another argument. In the discrete setting of [NSS17] a mild convexity at infinity was demanded to handle this problem. Contrastingly in our continuous setting such additional assumption is not needed, we can use the fact that indicator functions has almost everywhere vanishing derivatives, and the respective argument (Step 2.1 in the proof of Lemma 5.3.6) can be done using the equi-integrability of $\overline{\lambda}_{\max}$ (Lemma 5.3.4).

Proof of Lemma 5.3.4.
Step 1: The first equality.
We set

$$\chi_N(y) := \begin{cases} 1, & \overline{\lambda}(y) > N, \\ 0, & \overline{\lambda}(y) \leq N. \end{cases}$$

Then $\chi_N(y)\overline{\lambda}(y)$ can be regarded as $\chi_N^{\tau_y}\overline{\lambda}^{\tau_y}$ and by Birkhoff's Pointwise Ergodic Theorem (Lemma 2.2.2) for every $N \in \mathbb{N}$ there is a set of full measure on which

$$\begin{aligned}
\lim_{\varepsilon\searrow 0} \int_{A_N^\varepsilon} \overline{\lambda}(\tfrac{x}{\varepsilon}) \,\mathrm{d}x &= \lim_{\varepsilon\searrow 0} \int_A \chi_N(\tfrac{x}{\varepsilon}) \overline{\lambda}(\tfrac{x}{\varepsilon}) \,\mathrm{d}x \\
&= |A| \mathbb{E}\Big[\fint_Y \chi_N(y) \overline{\lambda}(y) \,\mathrm{d}y \Big] \\
&= |A| \mathbb{E}\Big[\int_{Y_N} \overline{\lambda}(y) \,\mathrm{d}y \Big],
\end{aligned}$$

so on the intersection of these sets the desired equality holds.

Step 2: The second equality.
We first note that with Hölder's inequality

$$\mathbb{E}[|Y_N|] \leq \mathbb{E}\Big[|Y_N|^{1-\alpha}\Big(\int_{Y_N} \tfrac{\overline{\lambda}(y)}{N}\,\mathrm{d}y\Big)^{\alpha}\Big] \leq N^{-\alpha}\mathbb{E}\Big[\int_Y \overline{\lambda}(y)^{\alpha}\,\mathrm{d}y\Big] \leq CN^{-\alpha}, \tag{5.34}$$

since $\mathbb{E}[\int_Y \overline{\lambda}(y)^{\alpha}\,\mathrm{d}y] = \int_Y \mathbb{E}[\overline{\lambda}^{\alpha}]\,\mathrm{d}y \leq C$ by stationarity and the Moment Bound Condition (5.7). From this point the proof is standard. We define a sequence (g_n) of functions

$$g_n(y) := \min\{\overline{\lambda}(y), n\},$$

which monotonously converges pointwise to $\overline{\lambda}$. Then the Monotone Convergence Theorem (applied twice) first gives $\int_Y g_n \to \int_Y \overline{\lambda}$ and then, since this convergence is again monotone, $\mathbb{E}[\int_Y g_n] \to \mathbb{E}[\int_Y \overline{\lambda}]$. That means we can find $n_0 \in \mathbb{N}$ such that

$$\mathbb{E}\Big[\int_Y \overline{\lambda}(y) - g_{n_0}(y)\,\mathrm{d}y\Big] < \tfrac{\delta}{2}.$$

Now for $N \in \mathbb{N}$ large enough such that $n_0 C N^{-\alpha} \leq \frac{\delta}{2}$ we find with (5.34)

$$0 \leq \mathbb{E}\Big[\int_{Y_N} \overline{\lambda}(y)\,\mathrm{d}y\Big] \leq \mathbb{E}\Big[\int_Y \overline{\lambda}(y) - g_{n_0}(y)\,\mathrm{d}y\Big] + \mathbb{E}\Big[\int_{Y_N} g_{n_0}(y)\,\mathrm{d}y\Big] < \tfrac{\delta}{2} + n_0\mathbb{E}[|Y_N|] \leq \delta,$$

which gives the desired convergence. □

Proof of Lemma 5.3.6. As in the proof of Lemma 5.3.5 we will show the assertion of one concrete sequence $(\mathcal{O}_N)$ given by

$$\mathcal{O}_N := 1 - (\tfrac{N-1}{N})^m + \mathbb{E}\Big[\int_{\{y \in Y; \overline{\lambda}_{\max}(y) > N\}} \overline{\lambda}_{\max}(y)\,\mathrm{d}y\Big].$$

Obviously with Lemma 5.3.4 we immediately see $\lim_{N\to\infty} \mathcal{O}_N = 0$.

Step 1: Construction of ϕ_N^{ε}.
We define ϕ_N^{ε} the same way we did in the proof of Lemma 5.3.5, but instead of $u^{\varepsilon} - \overline{u}$ we use the truncated function

$$w^{\varepsilon}(x) := \begin{cases} -\frac{l}{3N^2}, & u^{\varepsilon}(x) - \overline{u}(x) < -\frac{l}{3N^2}, \\ u^{\varepsilon}(x) - \overline{u}(x), & |u^{\varepsilon}(x) - \overline{u}(x)| \leq \frac{l}{3N^2}, \\ \frac{l}{3N^2}, & u^{\varepsilon}(x) - \overline{u}(x) > \frac{l}{3N^2}. \end{cases}$$

Then, following the arguments in the proof Lemma 5.3.5, it remains to show

$$\begin{aligned} &\limsup_{\varepsilon \searrow 0} \tfrac{1}{N} \sum_{j=1}^{N} \Big(\overline{\mathcal{E}}_{\varepsilon}(\overline{u} + \phi^{\varepsilon,j}, Q_j) - \overline{\mathcal{E}}_{\varepsilon}(u^{\varepsilon}, Q_j)\Big) \\ &\quad \lesssim \big(1 + \|\nabla \overline{u}\|_{L^{\infty}(Q)}^{p}\big)\Big(\tfrac{N^3}{l}\|u - \overline{u}\|_{L^1(Q)} + \mathcal{O}_N|Q|\Big), \end{aligned} \tag{5.35}$$

$$\begin{aligned}\limsup_{\varepsilon\searrow 0}\tfrac{1}{N}\sum_{j=1}^{N}\Big(\overline{\mathcal{E}}_\varepsilon(\overline{u}+\phi^{\varepsilon,j},Q_{j-1}\setminus Q_j)-\overline{\mathcal{E}}_\varepsilon(u^\varepsilon,Q_{j-1}\setminus Q_j)\Big)\\ \lesssim \limsup_{\varepsilon\searrow 0}\overline{\mathcal{E}}_\varepsilon(u^\varepsilon,Q\setminus Q_N)+\big(\|\nabla\overline{u}\|^p_{L^\infty(Q)}+1\big)\mathcal{O}_N|Q|\end{aligned}\tag{5.36}$$

and

$$\limsup_{\varepsilon\searrow 0}\tfrac{1}{N}\sum_{j=1}^{N}\Big(\overline{\mathcal{E}}_\varepsilon(\overline{u}+\phi^{\varepsilon,j},Q\setminus Q_{j-1})-\overline{\mathcal{E}}_\varepsilon(u^\varepsilon,Q\setminus Q_{j-1})\Big)\lesssim\big(\|\nabla\overline{u}\|^p_{L^\infty(Q)}+1\big)\mathcal{O}_N|Q|.\tag{5.37}$$

Step 2.1: Proof of (5.35).
The key is for the indicator function

$$\chi^\varepsilon(x):=\begin{cases}1, & \text{if } w^\varepsilon(x)\neq u^\varepsilon(x)-\overline{u}(x),\\ 0, & \text{if } w^\varepsilon(x)=u^\varepsilon(x)-\overline{u}(x)\end{cases}$$

to observe for a.e. $x\in Q_j$

$$\begin{aligned}\chi^\varepsilon(x)=0 &\quad\Leftrightarrow\quad \nabla(\overline{u}+w^\varepsilon)(x)=\nabla u^\varepsilon(x),\\ \chi^\varepsilon(x)=1 &\quad\Leftrightarrow\quad \nabla(\overline{u}+w^\varepsilon)(x)=\nabla\overline{u}(x),\end{aligned}$$

and thus, as by definition $\phi^{\varepsilon,j}=w^\varepsilon$ on Q_j, with the Growth Condition (5.2) on each energy, and the fact that $\overline{\lambda}_{\min}\lesssim\overline{\lambda}_{\max}$

$$\begin{aligned}&\overline{\mathcal{E}}_\varepsilon(\overline{u}+\phi^{\varepsilon,j},Q_j)-\overline{\mathcal{E}}_\varepsilon(u^\varepsilon,Q_j)\\ &\quad=\int_{Q_j}\chi^\varepsilon(x)\,\overline{W}(\tfrac{x}{\varepsilon},\nabla\overline{u}(x))\,\mathrm{d}x-\int_{Q_j}\chi^\varepsilon(x)\,\overline{W}(\tfrac{x}{\varepsilon},\nabla u^\varepsilon(x))\,\mathrm{d}x\\ &\quad\le C\int_{Q_j}\chi^\varepsilon(x)\,\overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})(1+|\nabla\overline{u}(x)|^p)\,\mathrm{d}x-\int_{Q_j}\chi^\varepsilon(x)\,\overline{\lambda}_{\min}(\tfrac{x}{\varepsilon})(\tfrac{1}{C}|\nabla u^\varepsilon(x)|^p-C)\,\mathrm{d}x\\ &\quad\lesssim\big(1+\|\nabla\overline{u}\|^p_{L^\infty(Q)}\big)\int_Q\chi^\varepsilon(x)\overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\,\mathrm{d}x.\end{aligned}$$

Since $\chi^\varepsilon(x)=1$ means by construction $|u^\varepsilon(x)-\overline{u}(x)|\ge\frac{l}{3N^2}$, we have

$$\int_Q\chi^\varepsilon(x)\,\mathrm{d}x\le\int_Q\frac{|u^\varepsilon(x)-\overline{u}(x)|}{\frac{l}{3N^2}}\,\mathrm{d}x\lesssim\tfrac{N^2}{l}\|u^\varepsilon-\overline{u}\|_{L^1(Q)}$$

and hence with the equi-integrability of $\overline{\lambda}_{\max}$ (Lemma 5.3.4)

$$\begin{aligned}
&\limsup_{\varepsilon\searrow 0}\int_Q \chi^\varepsilon(x)\overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\,\mathrm{d}x\\
&\quad=\limsup_{\varepsilon\searrow 0}\int_{\{x\in Q;\overline{\lambda}_{\max}(\frac{x}{\varepsilon})\le N\}}\chi^\varepsilon(x)\overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\,\mathrm{d}x\\
&\qquad+\limsup_{\varepsilon\searrow 0}\int_{\{x\in Q;\overline{\lambda}_{\max}(\frac{x}{\varepsilon})>N\}}\chi^\varepsilon(x)\overline{\lambda}_{\max}(x,\tfrac{x}{\varepsilon})\,\mathrm{d}x\\
&\quad\lesssim\limsup_{\varepsilon\searrow 0}N\tfrac{N^2}{l}\|u^\varepsilon-\overline{u}\|_{L^1(Q)}+\mathbb{E}\Big[\int_{\{y\in Y;\overline{\lambda}_{\max}(y)>N\}}\overline{\lambda}_{\max}(y)\,\mathrm{d}y\Big]|Q|\\
&\quad\le\tfrac{N^3}{l}\|u-\overline{u}\|_{L^1(Q)}+\mathcal{O}_N|Q|,
\end{aligned}$$

so (5.35) follows.

Step 2.2: Proof of (5.37).
This step can be adopted from the proof of Lemma 5.3.5 without any changes. We use the Growth Condition (5.2) and the arguments in Remark 5.3.7 and see

$$\begin{aligned}
&\overline{\mathcal{E}}_\varepsilon(\overline{u}+\phi^{\varepsilon,j},Q\setminus Q_{j-1})-\overline{\mathcal{E}}_\varepsilon(u^\varepsilon,Q\setminus Q_{j-1})\\
&\quad\le C\int_{Q\setminus Q_{j-1}}\overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\big(|\nabla\overline{u}(x)|^p+1\big)\,\mathrm{d}x-\int_{Q\setminus Q_{j-1}}\overline{\lambda}_{\min}(\tfrac{x}{\varepsilon})\big(\tfrac{1}{C}|\nabla u^\varepsilon(x)|^p-C\big)\,\mathrm{d}x\\
&\quad\lesssim\int_{Q\setminus Q_N}\overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\big(|\nabla\overline{u}(x)|^p+1\big)\,\mathrm{d}x
\end{aligned}$$

and

$$\limsup_{\varepsilon\searrow 0}\int_{Q\setminus Q_N}\overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\big(|\nabla\overline{u}(x)|^p+1\big)\,\mathrm{d}x\lesssim\big(\|\nabla\overline{u}\|^p_{L^\infty(Q)}+1\big)\mathcal{O}_N|Q|. \tag{5.38}$$

Step 2.3: Proof of (5.36).
Once more we start with similar arguments as in the proof of Lemma 5.3.5 Step 2.3. From the definition of $\phi^{\varepsilon,j}$ on $Q_{j-1}\setminus Q_j$ and χ^ε we see a.e. on Q_{j-1} we see

$$\nabla(\overline{u}+\phi^{\varepsilon,j})=(1-\psi_j(1-\chi^\varepsilon))\nabla\overline{u}+w^\varepsilon\nabla\psi_j+\psi_j(1-\chi^\varepsilon)\nabla u^\varepsilon$$

and thus with the Mild $\lambda_{\max}$-Convexity (5.3), the Weak $\lambda_{\max}$-Δ_2-Property (5.4) and the p-Growth Condition (5.2) we have

$$\begin{aligned}
&\overline{\mathcal{E}}_\varepsilon(\overline{u}+\phi^{\varepsilon,j},Q_{j-1}\setminus Q_j)-\overline{\mathcal{E}}_\varepsilon(u^\varepsilon,Q_{j-1}\setminus Q_j)\\
&\quad\lesssim\int_{Q\setminus Q_N}\overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})\big(|\nabla\overline{u}|^p+|w^\varepsilon(x)|^p|\nabla\phi_j(x)|^p+1\big)\,\mathrm{d}x+\overline{\mathcal{E}}_\varepsilon(u^\varepsilon,Q_{j-1}\setminus Q_j).
\end{aligned}$$

The most part of the integral is again covered by (5.38). For the remaining part we note that by construction $\|\nabla\psi_j\|_{L^\infty(Q)}\le\frac{3N^2}{l}$ and $\|w^\varepsilon\|_{L^\infty(Q)}\le\frac{l}{3N^2}$, so we find with the arguments in Remark 5.3.7

$$\limsup_{\varepsilon\searrow 0}\int_{Q\setminus Q_N}\overline{\lambda}(\tfrac{x}{\varepsilon})|\nabla\psi_j(x)|^p|w^\varepsilon(x)|^p\,\mathrm{d}x\lesssim\mathcal{O}_N|Q|,$$

and thus (5.36) follows. □

5.4.4. Properties of the Homogenized Potential (Proposition 5.2.4)

Proof of Proposition 5.2.4 (a). The upper bound follows directly from the definition of W_{hom}, the Growth Condition (5.2) and the Moment Bound Condition (5.7). To be precise, for every $k\in\mathbb{N}$ we have

$$\inf_{\phi\in W_0^{1p}(kY)}\fint_{kY}W(x,y,F+D\phi(y))\,\mathrm{d}y\leq\fint_{kY}W(x,y,F)\,\mathrm{d}y\leq C(|F|^p+1)\fint_{kY}\lambda_{\max}(x,y)\,\mathrm{d}y$$

and thus with Hölder's inequality

$$\begin{aligned}W_{\mathrm{hom}}(x,F)&\leq C(|F|^p+1)\lim_{k\to\infty}\mathbb{E}\Big[\fint_{kY}\lambda_{\max}(x,y)^\alpha\,\mathrm{d}y\Big]^{\frac{1}{\alpha}}\\&\leq C^{\frac{\alpha+1}{\alpha}}(|F|^p+1),\end{aligned}$$

since $\mathbb{E}[\fint_{kY}\lambda_{\max}(x,y)^\alpha\,\mathrm{d}y]\leq C$ by the Moment Bound Condition (5.7).

Now for the lower bound we write

$$\mathcal{F}_\varepsilon:=\inf_{\phi\in W_0^{1,p}(\frac{1}{\varepsilon}Y)}\fint_{\frac{1}{\varepsilon}Y}W(x,y,F+D\phi(y))\,\mathrm{d}y$$

and fix $\omega\in\Omega_F(x)$, such that by the arguments in Remark 5.3.7 we have

$$W_{\mathrm{hom}}(x,F)=\lim_{\varepsilon\searrow 0}\mathcal{F}_\varepsilon.\tag{5.39}$$

If we additionally fix $\varepsilon>0$ and $\phi_\varepsilon\in W_0^{1,p}(\frac{1}{\varepsilon}Y)$ we have, because $\fint_{\frac{1}{\varepsilon}Y}D\phi_\varepsilon=0$,

$$|F|^p=\Big|\fint_{\frac{1}{\varepsilon}Y}F\,\mathrm{d}y\Big|^p\leq\Big(\fint_{\frac{1}{\varepsilon}Y}|F+D\phi_\varepsilon(y)|\,\mathrm{d}y\Big)^p$$

and we can use Hölder's inequality to smuggle in $\lambda_{\min}$, so we can apply the Growth Condition (5.2) and get

$$\begin{aligned}|F|^p&\leq\Big(\fint_{\frac{1}{\varepsilon}Y}\lambda_{\min}(x,y)^{-\frac{1}{p-1}}\,\mathrm{d}y\Big)^{p-1}\Big(\fint_{\frac{1}{\varepsilon}Y}\lambda_{\min}(x,y)|F+D\phi_\varepsilon(y)|^p\,\mathrm{d}y\Big)\\&\leq\Big(\fint_{\frac{1}{\varepsilon}Y}\lambda_{\min}(x,y)^{-\frac{1}{p-1}}\,\mathrm{d}y\Big)^{p-1}\\&\quad\cdot\Big(C\fint_{\frac{1}{\varepsilon}Y}W(x,y,F+D\phi_\varepsilon(y))\,\mathrm{d}y+C^2\fint_{\frac{1}{\varepsilon}Y}\lambda_{\min}(x,y)\,\mathrm{d}y\Big)\end{aligned}$$

Since $\phi_\varepsilon \in W_0^{1,p}(\frac{1}{\varepsilon}Y)$ was taken arbitrarily we can pass to the infimum and see

$$|F|^p \le \Big(\fint_{\frac{1}{\varepsilon}Y} \lambda_{\min}(x,y)^{-\frac{1}{p-1}} \,\mathrm{d}y \Big)^{p-1} \Big(C\mathcal{F}_\varepsilon + C^2 \fint_{\frac{1}{\varepsilon}Y} \lambda_{\min}(x,y) \,\mathrm{d}y \Big). \tag{5.40}$$

With the arguments in Remark 5.3.7 and Hölder's inequality we get

$$\limsup_{\varepsilon \searrow 0} \Big(\fint_{\frac{1}{\varepsilon}Y} \lambda_{\min}(x,y)^{-\frac{1}{p-1}} \,\mathrm{d}y \Big)^{p-1} \le \lim_{\varepsilon \searrow 0} \Big(\fint_{Y} \lambda_{\min}(x, \tfrac{y}{\varepsilon})^{-\beta} \,\mathrm{d}y \Big)^{\frac{1}{\beta}} \le C^{\frac{1}{\beta}}$$

and

$$\limsup_{\varepsilon \searrow 0} \fint_{\frac{1}{\varepsilon}Y} \lambda_{\min}(x,y) \,\mathrm{d}y \le \lim_{\varepsilon \searrow 0} \Big(\fint_{Y} \lambda_{\min}(x, \tfrac{y}{\varepsilon})^{\alpha} \,\mathrm{d}y \Big)^{\frac{1}{\alpha}} \le C^{\frac{1}{\alpha}},$$

so taking the limit $\varepsilon \searrow 0$ in (5.40), and remembering (5.39), we finally end up with the desired inequality

$$|F|^p \le C^{\frac{\beta+1}{\beta}} \Big(W_{\hom}(x,F) + C^{\frac{\alpha+1}{\alpha}} \Big). \qquad \square$$

Proof of Proposition 5.2.4 (b). We fix $k \in \mathbb{N}$. If we take a minimizing sequence (ϕ_δ) in $W_0^{1,p}(kY)$ such that

$$\lim_{\delta \searrow 0} \fint_{kY} W(x_2, y, F + D\phi_\delta(y)) \,\mathrm{d}y = \inf_{\phi \in W_0^{1,p}(kY)} \fint_{kY} W(x_2, y, F + D\phi(y)) \,\mathrm{d}y,$$

then we find by the Spatial Continuity of W (Condition (5.9))

$$\begin{aligned} &\inf_{\phi \in W_0^{1,p}(kY)} \fint_{kY} W(x_1, y, F + D\phi(y)) \,\mathrm{d}y \\ &\quad \le \fint_{kY} W(x_1, y, F + D\phi_\delta(y)) \,\mathrm{d}y \\ &\quad \le \fint_{kY} W(x_2, y, F + D\phi_\delta(y)) \,\mathrm{d}y \\ &\qquad + \rho'(|x_1 - x_2|) \fint_{kY} \big(1 + 2W(x_2, y, F + D\phi_\delta(y)) \big) \,\mathrm{d}y \end{aligned}$$

and thus, taking the limit $\delta \searrow 0$, the expectation value and then the limit $k \to \infty$

$$W_{\hom}(x_1, F) - W_{\hom}(x_2, F) \le \rho'(|x_1 - x_2|) \big(1 + 2W_{\hom}(x_2, F) \big).$$

Moreover, by interchanging the roles of x_1 and x_2, we also get

$$W_{\hom}(x_2, F) - W_{\hom}(x_1, F) \le \rho'(|x_1 - x_2|) \big(1 + 2W_{\hom}(x_1, F) \big).$$

Thus we have

$$\big| W_{\hom}(x_1, F) - W_{\hom}(x_2, F) \big| \le \rho'(|x_1 - x_2|) \big(1 + 2 \min\{ W_{\hom}(x_1, F), W_{\hom}(x_2, F) \} \big),$$

which gives the desired form of continuity (compare the discussion about the equivalence of Conditions (5.8) and (5.9)). $\square$

Proof of Proposition 5.2.4 (c). First for $\omega \in \Omega_F(x)$ Lemma 5.3.3 ensures that we can find a recovery sequence (u^ε) in $W^{1,p}(Y)$ such that

$$u^\varepsilon \to Fy \text{ in } L^{\frac{\beta}{\beta+1}p}(Y) \qquad \text{and} \qquad \lim_{\varepsilon \searrow 0} \int_Y W(x, \tfrac{y}{\varepsilon}, Du^\varepsilon(y)) \,\mathrm{d}y = W_{\mathrm{hom}}(x, F).$$

Then, since $F_j \to F$, by the Gluing Construction (Lemma 5.3.6) for every $\varepsilon > 0$ and all $j, N \in \mathbb{N}$ there is a function $\phi_N^{\varepsilon,j} \in W_0^{1,p}(Y)$ such that

$$\begin{aligned}\limsup_{N\to\infty} \limsup_{j\to\infty} \limsup_{\varepsilon \searrow 0} \fint_Y W(x, \tfrac{y}{\varepsilon}, F + D\phi_N^{\varepsilon,j}(y)) \,\mathrm{d}y &\le \lim_{\varepsilon\searrow 0} \fint_Y W(x, \tfrac{y}{\varepsilon}, Du^\varepsilon(y)) \,\mathrm{d}y \\ &= W_{\mathrm{hom}}(x, F).\end{aligned}$$

On the other hand for $\omega \in \bigcap_{j\in\mathbb{N}} \Omega_{F_j}(x)$ the arguments in Remark 5.3.7 give

$$\begin{aligned}W_{\mathrm{hom}}(x, F_j) &= \lim_{\varepsilon\searrow 0} \inf_{\phi \in W_0^{1,p}(\frac{1}{\varepsilon}Y)} \fint_{\frac{1}{\varepsilon}Y} W(x, y, F_j + D\phi(y)) \,\mathrm{d}y \\ &\le \limsup_{\varepsilon\searrow 0} \fint_Y W(x, \tfrac{y}{\varepsilon}, F_j + D\phi_N^{\varepsilon,j}(y)) \,\mathrm{d}y\end{aligned}$$

for every $j \in \mathbb{N}$. Together for $\omega \in \Omega_F(x) \cap \bigcap_{j\in\mathbb{N}} \Omega_{F_j}(x)$ we end up with

$$\limsup_{j\to\infty} W_{\mathrm{hom}}(x, F_j) = \limsup_{N\to\infty} \limsup_{j\to\infty} W_{\mathrm{hom}}(x, F_j) \le W_{\mathrm{hom}}(x, F),$$

which does not depend on ω anymore. Now interchanging the roles of F and F_j as well as taking the limes inferior as $j \to \infty$ instead of the lines superior yield also

$$W_{\mathrm{hom}}(x, F) \le \liminf_{j\to\infty} W_{\mathrm{hom}}(x, F_j),$$

so the assertion immediately follows. □

5.4.5. Recovery Sequence for Spatially Homogeneous Potentials and Sobolev Functions (Lemma 5.3.1)

Proof of Lemma 5.3.1.
Step 1: From affine functions to piecewise affine functions with rational derivatives.
Let u be a piecewise affine function on A, i.e. there is a finite partition $\{A_i\}$ of A consisting of bounded Lipschitz domains such that $Du = F_i \in \mathbb{Q}^{n\times m}$ on A_i. Hence on every A_i we are in the situation where we can apply Lemma 5.3.3, which gives a recovery sequence (u_i^ε) in $W^{1,p}(A_i)$ with $u_i^\varepsilon - u \in W_0^{1,p}(A_i)$ such that

$$u_i^\varepsilon \to u \text{ in } L^{\frac{\beta}{\beta+1}p}(A_i) \qquad \text{and} \qquad \lim_{\varepsilon\searrow 0} \overline{\mathcal{E}}_\varepsilon(u_i^\varepsilon, A_i) = \overline{\mathcal{E}}_{\mathrm{hom}}(F_i, A_i).$$

Using the convention $u_i^\varepsilon = u$ outside of A_i, we can put these recovery sequences together to

$$u^\varepsilon := u + \sum_i (u_i^\varepsilon - u) \in W^{1,p}(A),$$

which obviously satisfies $u^\varepsilon - u \in W_0^{1,p}(A)$ and $u^\varepsilon \to u$ in $L^{\frac{\beta}{\beta-1}p}(A)$, and

$$\lim_{\varepsilon \searrow 0} \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, A) = \sum_i \lim_{\varepsilon \searrow 0} \overline{\mathcal{E}}_\varepsilon(u_i^\varepsilon, A_i) = \sum_i \overline{\mathcal{E}}_{\text{hom}}(F_i, A_i) = \overline{\mathcal{E}}_{\text{hom}}(u, A).$$

Step 2: From piecewise affine functions to sobolev functions.
Let $u \in W^{1,p}(A)$. Since the piecewise affine functions on A are dense in $W^{1,p}(A)$ we can find for any $\delta > 0$ subsets $A_\delta \subseteq A^\delta \subseteq A$ and a function $u_\delta \in W^{1,p}(A)$ such that

- A_δ is open, A^δ is compact, and $|A \setminus A_\delta| \to 0$,
- u_δ is piecewise affine with rational derivative on A_δ and $u_\delta = u$ on $A \setminus A^\delta$,
- $u_\delta \to u$ in $W^{1,p}(A)$.

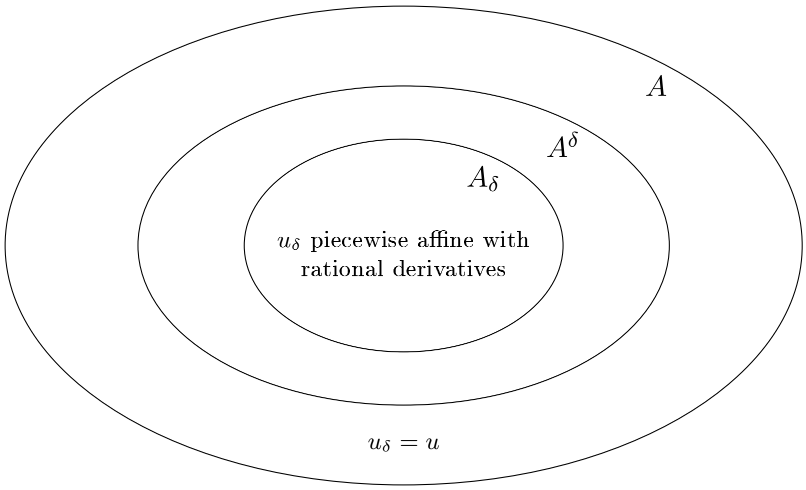


Figure 5.2.: Sketch of the sets $A_\delta \subseteq A^\delta \subseteq A$ and the function u^δ in the proof of Lemma 5.3.1 Step 2.

On the set A_δ we are in the situation of Step 1 so we can find a recovery sequence (v_δ^ε) in $W^{1,p}(A_\delta)$ with $v_\delta^\varepsilon - u_\delta \in W_0^{1,p}(A_\delta)$ such that

$$v_\delta^\varepsilon \to u_\delta \text{ in } L^{\frac{\beta}{\beta+1}p}(A_\delta) \qquad \text{and} \qquad \lim_{\varepsilon \searrow 0} \overline{\mathcal{E}}_\varepsilon(v_\delta^\varepsilon, A_\delta) = \overline{\mathcal{E}}_{\text{hom}}(u_\delta, A_\delta).$$

We set

$$u_\delta^\varepsilon(x) := \begin{cases} v_\delta^\varepsilon(x), & x \in A_\delta, \\ u_\delta(x), & x \in A \setminus A_\delta, \end{cases}$$

which gives functions $u_\delta^\varepsilon \in W^{1,p}(A)$ with $u_\delta^\varepsilon - u \in W_0^{1,p}(A)$ and

$$\lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\|u_\delta^\varepsilon - u\|_{L^{\frac{\beta}{\beta+1}p}(A)} = 0.$$

If we additionally could show

$$\lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\overline{\mathcal{E}}_\varepsilon(u_\delta^\varepsilon, A) = \overline{\mathcal{E}}_{\mathrm{hom}}(u, A), \tag{5.41}$$

then we could find a diagonal sequence $\delta(\varepsilon)$ such that $u^\varepsilon := u_{\delta(\varepsilon)}^\varepsilon$ fulfills all the desired assertions. We split the proof of (5.41) into the two sub-goals

$$\limsup_{\delta\searrow 0}\limsup_{\varepsilon\searrow 0}\left|\overline{\mathcal{E}}_\varepsilon(u_\delta^\varepsilon, A) - \overline{\mathcal{E}}_{\mathrm{hom}}(u_\delta, A)\right| = 0 \tag{5.42}$$

and

$$\limsup_{\delta\searrow 0}\left|\overline{\mathcal{E}}_{\mathrm{hom}}(u_\delta, A) - \overline{\mathcal{E}}_{\mathrm{hom}}(u, A)\right| = 0. \tag{5.43}$$

We first note that $\overline{\mathcal{E}}_\varepsilon(u_\delta^\varepsilon, A_\delta) \to \overline{\mathcal{E}}_{\mathrm{hom}}(u_\delta, A_\delta)$ by the definition of u_δ^ε, so for (5.42) it only remains to look at $A \setminus A_\delta$. But since $|A \setminus A_\delta| \to 0$ we see that $\overline{\mathcal{E}}_\varepsilon(u_\delta^\varepsilon, A \setminus A_\delta)$ and $\overline{\mathcal{E}}_{\mathrm{hom}}(u_\delta, A \setminus A_\delta)$ both vanish as $\varepsilon \searrow 0$ and $\delta \searrow 0$, because by the Growth Condition (5.2) and the arguments in Remark 5.3.7 we have

$$\begin{aligned}
&\limsup_{\delta\searrow 0}\limsup_{\varepsilon\searrow 0}\left|\overline{\mathcal{E}}_\varepsilon(u_\delta, A \setminus A_\delta)\right| \\
&\quad \le C \limsup_{\delta\searrow 0}\limsup_{\varepsilon\searrow 0}\int_{A\setminus A_\delta}\overline{\lambda}_{\max}(\tfrac{x}{\varepsilon})(|Du_\delta(x)|^p + 1)\,\mathrm{d}x \\
&\quad = 0,
\end{aligned}$$

and by the Uniform Ellipticity of $\overline{W}_{\mathrm{hom}}$ (Proposition 5.2.4 (a)) and the fact that $u_\delta \to u$ in $W^{1,p}(A)$ we have

$$\begin{aligned}
&\limsup_{\delta\searrow 0}\left|\overline{\mathcal{E}}_{\mathrm{hom}}(u_\delta, A \setminus A_\delta)\right| \\
&\quad \le C' \limsup_{\delta\searrow 0}\int_{A\setminus A_\delta}(|Du_\delta(x)|^p + 1)\,\mathrm{d}x \\
&\quad \le C' \limsup_{\delta\searrow 0}\Big(\int_{A\setminus A_\delta}(|Du(x)|^p + 1)\,\mathrm{d}x + \int_A |Du_\delta(x) - Du(x)|^p\,\mathrm{d}x\Big) \\
&\quad = 0.
\end{aligned}$$

To claim (5.43) we note $\overline{W}_{\mathrm{hom}}(Du_\delta(x)) \to \overline{W}_{\mathrm{hom}}(Du(x))$ for a.e. $x \in A$ as a consequence of $u_\delta \to u$ in $W^{1,p}(A)$ and the Continuity of $\overline{W}_{\mathrm{hom}}$ (Proposition 5.2.4 (c)). Now (5.43) follows from two applications of Fatou's Lemma, i.e.

$$\overline{\mathcal{E}}_{\mathrm{hom}}(u, A) = \int_A \lim_{\delta\searrow 0}\overline{W}_{\mathrm{hom}}(Du_\delta(x))\,\mathrm{d}x \le \liminf_{\delta\searrow 0}\overline{\mathcal{E}}_{\mathrm{hom}}(u_\delta, A),$$

and, since $C'(|Du_\delta(x)|^p+1)-\overline{W}_{\mathrm{hom}}(Du_\delta(x))$ is non-negative by the Uniform Ellipticity of $\overline{W}_{\mathrm{hom}}$ (Proposition 5.2.4 (a)),

$$\begin{aligned}\overline{\mathcal{E}}_{\mathrm{hom}}(u,A) &= -\int_A \lim_{\delta\searrow 0}\big(C'(|Du_\delta(x)|^p+1)-\overline{W}_{\mathrm{hom}}(Du_\delta(x))\big)\,\mathrm{d}x \\ &\quad+\int_A C'(|Du(x)|^p+1)\,\mathrm{d}x \\ &\geq -\liminf_{\delta\searrow 0}\int_A \big(C'(|Du_\delta(x)|^p+1)-\overline{W}_{\mathrm{hom}}(Du_\delta(x))\big)\,\mathrm{d}x \\ &\quad+\int_A C'(|Du(x)|^p+1)\,\mathrm{d}x \\ &= \limsup_{\delta\searrow 0}\overline{\mathcal{E}}_{\mathrm{hom}}(u_\delta,A). \end{aligned}$$

□

5.4.6. Lower Bound for Spatially Homogeneous Potentials (Lemma 5.3.2)

Proof.
Step 1: Definition of μ_ε and μ.
W.l.o.g. we can assume $(\overline{\mathcal{E}}_\varepsilon(u^\varepsilon,A))$ to be convergent, because if $\liminf_{\varepsilon\searrow 0}\overline{\mathcal{E}}_\varepsilon(u^\varepsilon,A)=\infty$ the assertion is trivial, and otherwise we can pass to a subsequence converging to $\liminf_{\varepsilon\searrow 0}\overline{\mathcal{E}}_\varepsilon(u^\varepsilon,A)$. Then by the Growth Condition (5.2) and the arguments in Remark 5.3.7 we can define uniformly bounded, non-negative Radon measures μ_ε by

$$\mu_\varepsilon(\mathrm{d}x) := \big(\overline{W}(\tfrac{x}{\varepsilon},Du^\varepsilon(x))+C\overline{\lambda}_{\min}(\tfrac{x}{\varepsilon})\big)\,\mathrm{d}x,$$

and the weak-$*$-compactness of Radon measures allows us to pass to a (not relabeled) subsequence and find another non-negative Radon measure μ such that $\mu_\varepsilon \overset{*}{\rightharpoonup} \mu$, which implies

$$\liminf_{\varepsilon\searrow 0}\mu_\varepsilon(U)\geq \mu(U) \qquad \text{for all open sets } U\subseteq A, \tag{5.44}$$

$$\lim_{\varepsilon\searrow 0}\mu_\varepsilon(K)=\mu(K) \qquad \text{for all compact sets } K\subseteq A. \tag{5.45}$$

Thus our definition of μ_ε and the arguments in Remark 5.3.7 (after applying Hölder's inequality) immediately yield

$$\mu(A)\leq \liminf_{\varepsilon\searrow 0}\mu_\varepsilon(A)\leq \lim_{\varepsilon\searrow 0}\overline{\mathcal{E}}_\varepsilon(u^\varepsilon,A)+C^{\frac{\alpha+1}{\alpha}}|A|,$$

so it will suffice to show

$$\mu(A)\geq \overline{\mathcal{E}}_{\mathrm{hom}}(u,A)+C^{\frac{\alpha+1}{\alpha}}|A|. \tag{5.46}$$

Step 2: Localization.
The Lebesgue Decomposition Theorem (see e.g. [EG92, 1.6 Theorem 3]) allows us to regard the non-negative Radon measure μ as the sum of a non-negative Radon measure $\mu_a \ll \mathrm{d}x$, the absolutely continuous part, and a non-negative Radon measure $\mu_s \perp \mathrm{d}x$, the singular part, with $D\mu_a = D\mu$ a.e. on A, where $D\mu_a$ and $D\mu$ denote the Lebesgue derivative

$$D\mu_a(x) = \lim_{l \searrow 0} \frac{\mu_a(Q_l(x))}{l^n} \quad \text{and} \quad D\mu(x) = \lim_{l \searrow 0} \frac{\mu(Q_l(x))}{l^n}, \quad \text{with } Q_l(x) := x + (-\tfrac{l}{2}, \tfrac{l}{2})^n.$$

Note that (5.45) implies $\lim_{\varepsilon \searrow 0} \mu_\varepsilon(B) = \mu(B)$ for all bounded Borel sets $B \subseteq A$ with $\mu(\partial B) = 0$. So if we fix a Lebesgue point $x \in A$ and let (l_j) be a decreasing sequence converging to zero such that $\mu(\partial Q_{l_j}(x)) = 0$, which is possible as $\mu(A) < \infty$, we find

$$D\mu(x) = \lim_{j \to \infty} \frac{\mu(Q_{l_j}(x))}{{l_j}^n} = \lim_{j \to \infty} \lim_{\varepsilon \searrow 0} \frac{\mu_\varepsilon(Q_{l_j}(x))}{{l_j}^n},$$

and hence our claim (5.46) reduces to

$$\lim_{j \to \infty} \lim_{\varepsilon \searrow 0} \frac{\mu_\varepsilon(Q_{l_j}(x))}{{l_j}^n} \geq \overline{W}_{\mathrm{hom}}(Du(x)) + C^{\frac{\alpha+1}{\alpha}}, \tag{5.47}$$

because $\mu(A) = \int_A D\mu_a(x)\,\mathrm{d}x + \mu_s(A) \geq \int_A D\mu(x)\mathrm{d}x$.

Step 3: Approximate L^p-differentiability.
First we note that by the definition of μ_ε and the arguments in Remark 5.3.7 (in connection with Hölder's inequality) we have

$$\lim_{j \to \infty} \lim_{\varepsilon \searrow 0} \frac{\mu_\varepsilon(Q_{l_j}(x))}{{l_j}^n} = \lim_{j \to \infty} \lim_{\varepsilon \searrow 0} \tfrac{1}{{l_j}^n} \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q_{l_j}(x)) + C^{\frac{\alpha+1}{\alpha}},$$

so claim (5.47) actually takes the form

$$\lim_{j \to \infty} \lim_{\varepsilon \searrow 0} \tfrac{1}{{l_j}^n} \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q_{l_j}(x)) \geq \overline{W}_{\mathrm{hom}}(Du(x)). \tag{5.48}$$

Since we assumed $\lim_{\varepsilon \searrow 0} \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, A) < \infty$, we are in the situation of the Compactness Result (Proposition 5.2.1), so we can follow $u \in W^{1,p}(A)$ and hence u is a.e. L^p-differentiable with L^p-derivative Du, i.e. for a.e. $x \in A$

$$\lim_{j \to \infty} \tfrac{1}{l_j} \Big(\fint_{Q_{l_j}(x)} \big| u(y) - u(x) - Du(x)(y-x) \big|^p \,\mathrm{d}y \Big)^{\frac{1}{p}} = 0.$$

Thus for almost every $x \in A$ we can find a sequence $(\overline{u}_j)_{j \in \mathbb{N}}$ of affine functions of the form $\overline{u}_j(y) := u(x) + F_j(y-x)$ with $F_j \in \mathbb{Q}^{n \times m}$, that satisfies $D\overline{u}_j \to Du(x)$ as well as

$$\lim_{j \to \infty} \tfrac{j^2}{{l_j}^{n+1}} \|u - \overline{u}_j\|^p_{L^p(Q_{l_j}(x))} = 0 \qquad \text{and} \qquad \lim_{j \to \infty} \tfrac{j^3}{{l_j}^{n+1}} \|u - \overline{u}_j\|_{L^1(Q_{l_j}(x))} = 0.$$

Thus, and because the Compactness Result (Proposition 5.2.1) also gives $u^\varepsilon \to u$ in $L^1(A)$, we are able to use the Gluing Construction (Lemma 5.3.5 or Lemma 5.3.6 resp.) to find functions $\phi_j^\varepsilon \in W_0^{1,p}(Q_{l_j}(x))$ such that

$$\limsup_{j\to\infty} \limsup_{\varepsilon\searrow 0} \tfrac{1}{l_j{}^n} \overline{\mathcal{E}}_\varepsilon(\overline{u}_j + \phi_j^\varepsilon, Q_{l_j}(x)) \le \lim_{j\to\infty} \lim_{\varepsilon\searrow 0} \tfrac{1}{l_j{}^n} \overline{\mathcal{E}}_\varepsilon(u^\varepsilon, Q_{l_j}(x)). \tag{5.49}$$

But now the Continuity of $\overline{W}_{\mathrm{hom}}$ (Proposition 5.2.4 (c)) and the arguments in Remark 5.3.7 give

$$\begin{aligned}
\overline{W}_{\mathrm{hom}}(Du(x)) &= \lim_{j\to\infty} \overline{W}_{\mathrm{hom}}(D\overline{u}_j) \\
&= \lim_{j\to\infty} \lim_{\varepsilon\searrow 0} \inf_{\phi\in W(\frac{1}{\varepsilon}Q_{l_j}(x))} \fint_{Q_{l_j}(x)} \overline{W}(y, D\overline{u}_j + \phi(y))\,\mathrm{d}y \\
&= \lim_{j\to\infty} \lim_{\varepsilon\searrow 0} \inf_{\phi\in W(Q_{l_j}(x))} \tfrac{1}{l_j{}^n} \overline{\mathcal{E}}_\varepsilon(\overline{u} + \phi, Q_{l_j}(x)) \\
&\le \limsup_{j\to\infty} \limsup_{\varepsilon\searrow 0} \tfrac{1}{l_j{}^n} \overline{\mathcal{E}}_\varepsilon(\overline{u}_j + \phi_j^\varepsilon, Q_{l_j}(x)),
\end{aligned}$$

which combined with (5.49) gives (5.48). □

5.4.7. Proof of Theorem 5.2.2

To lift the results in Lemmas 5.3.1 and 5.3.2 from spatially homogeneous potentials $\overline{W}$ to the general potential W we will part the set A into small cubes, i.e. for $\delta > 0$, $\delta \in \sqrt{n}\mathbb{Q}$ set

$$Z_\delta := \{z \in \tfrac{\delta}{\sqrt{n}}\mathbb{Z}^n; Q_\delta(z) \cap A \neq \emptyset\}, \qquad \text{with } Q_\delta(z) := z + \tfrac{\delta}{\sqrt{n}} Y.$$

Now on each cube $Q_\delta(z)$ we will replace W by the spatially homogeneous potential $\overline{W}_z := W(z, \cdot, \cdot)$ with the associated energies

$$\overline{\mathcal{E}}_{z,\varepsilon}(u, U) := \int_U \overline{W}_z(\tfrac{x}{\varepsilon}, Du(x))\,\mathrm{d}x = \int_U W(z, \tfrac{x}{\varepsilon}, Du(x))\,\mathrm{d}x$$

and

$$\overline{\mathcal{E}}_{z,\mathrm{hom}}(u, U) := \int_U \overline{W}_{z,\mathrm{hom}}(Du(x))\,\mathrm{d}x = \int_U W_{\mathrm{hom}}(z, Du(x))\,\mathrm{d}x.$$

The thereby occurring error can be controlled with the Spatial Continuity of W and W_{hom} (Condition (5.8) and Proposition 5.2.4 (b)) and vanishes with the diameter δ of the cubes.

Step 1: Recovery sequence.
By Lemma 5.3.1 we find for every $z \in Z_\delta$ a recovery sequence $(u_{\delta,z}^\varepsilon)$ in $W^{1,p}(Q_\delta(z) \cap A)$ with $u_{\delta,z}^\varepsilon - u \in W_0^{1,p}(Q_\delta(z) \cap A)$ such that

$$u_{\delta,z}^\varepsilon \to u \text{ in } L^{\frac{\beta}{\beta+1}p}(Q_\delta(z) \cap A) \quad \text{and} \quad \lim_{\varepsilon\searrow 0} \overline{\mathcal{E}}_{z,\varepsilon}(u_{\delta,z}^\varepsilon, Q_\delta(z) \cap A) = \overline{\mathcal{E}}_{z,\mathrm{hom}}(u, Q_\delta(z) \cap A)$$

for all $\omega \in \Omega_0$ since $z \in \mathbb{Q}^n$ by construction. Using the condition that $u^\varepsilon_{\delta,z} = u$ outside of $Q_\delta(z) \cap A$ we can put these sequences together and define u^ε_δ by

$$u^\varepsilon_\delta := u + \sum_{z \in Z_\delta} (u^\varepsilon_{\delta,z} - u) \in W^{1,p}(A),$$

which obviously satisfies $u^\varepsilon_\delta - u \in W^{1,p}_0(A)$ and $u^\varepsilon_\delta \to u$ in $L^{\frac{\beta}{\beta+1}p}(A)$. If we could show

$$\lim_{\delta \searrow 0} \lim_{\varepsilon \searrow 0} \mathcal{E}_\varepsilon(u^\varepsilon_\delta, A) = \mathcal{E}_{\mathrm{hom}}(u, A), \tag{5.50}$$

we could find a diagonal sequence $\delta(\varepsilon)$ such that $u^\varepsilon := u^\varepsilon_{\delta(\varepsilon)}$ is the desired recovery sequence. To show (5.50) it is sufficient to note, that for the energies we have by the Spatial Continuity of W and W_{hom} (Condition (5.9) and Proposition 5.2.4 (b))

$$\begin{aligned}
\mathcal{E}_\varepsilon(u^\varepsilon_\delta, A) &= \sum_{x \in Z_\delta} \mathcal{E}_\varepsilon(u^\varepsilon_{\delta,z}, Q_\delta(z) \cap A) \\
&\leq \sum_{x \in Z_\delta} \Big(\overline{\mathcal{E}}_{z,\varepsilon}(u^\varepsilon_{\delta,z}, Q_\delta(z) \cap A) + \int_{Q_\delta(z) \cap A} \rho'(|x-z|)\big(1 + \overline{W}_z(\tfrac{x}{\varepsilon}, Du^\varepsilon_{\delta,z}(x))\big)\,\mathrm{d}x \Big) \\
&\leq (1+\delta) \sum_{z \in Z_\delta} \overline{\mathcal{E}}_{z,\varepsilon}(u^\varepsilon_{\delta,z}, Q_\delta(z) \cap A) + \delta |A|
\end{aligned} \tag{5.51}$$

as well as

$$\begin{aligned}
&\mathcal{E}_{\mathrm{hom}}(u, A) \\
&\quad = \sum_{x \in Z_\delta} \mathcal{E}_{\mathrm{hom}}(u, Q_\delta(z) \cap A) \\
&\quad \leq \sum_{x \in Z_\delta} \Big(\overline{\mathcal{E}}_{z,\mathrm{hom}}(u, Q_\delta(z) \cap A) + \int_{Q_\delta(z) \cap A} \rho'(|x-z|)\big(1 + \overline{W}_{z,\mathrm{hom}}(\tfrac{x}{\varepsilon}, Du(x))\big)\,\mathrm{d}x \Big) \\
&\quad \leq (1+\delta) \sum_{z \in Z_\delta} \overline{\mathcal{E}}_{z,\mathrm{hom}}(u, Q_\delta(z) \cap A) + \delta |A|.
\end{aligned} \tag{5.52}$$

Thus, since $\overline{\mathcal{E}}_{z,\varepsilon}(u^\varepsilon_{\delta,z}, Q_\delta(z) \cap A) \to \overline{\mathcal{E}}_{z,\mathrm{hom}}(u, Q_\delta(z) \cap A)$, we end up with

$$\limsup_{\delta \searrow 0} \limsup_{\varepsilon \searrow 0} \big| \mathcal{E}_\varepsilon(u^\varepsilon_\delta, A) - \mathcal{E}_{\mathrm{hom}}(u, A) \big| = 0,$$

which is (5.50).

Step 2: Lower bound.
Using (5.52) and Lemma 5.3.2 we see

$$\begin{aligned}
\mathcal{E}_{\mathrm{hom}}(u, A) &\leq \limsup_{\delta \searrow 0} \sum_{x \in Z_\delta} \overline{\mathcal{E}}_{z,\mathrm{hom}}(u, Q_\delta(z) \cap A) \\
&\leq \limsup_{\delta \searrow 0} \liminf_{\varepsilon \searrow 0} \sum_{x \in Z_\delta} \overline{\mathcal{E}}_{z,\varepsilon}(u^\varepsilon, Q_\delta(z) \cap A)
\end{aligned}$$

for all $\omega \in \Omega_0$. Similar to (5.51) we find

$$\sum_{x \in Z_\delta} \overline{\mathcal{E}}_{z,\varepsilon}(u^\varepsilon, Q_\delta(z) \cap A) \leq (1+\delta)\mathcal{E}_\varepsilon(u^\varepsilon, A) + \delta|A|,$$

which immediately yields

$$\mathcal{E}_{\text{hom}}(u, A) \leq \liminf_{\varepsilon \searrow 0} \mathcal{E}_\varepsilon(u^\varepsilon, A),$$

and the theorem is proven. $\square$

6. Application to Rapidly Oscillating Riemannian Manifolds

As in Part II we want to apply our recent results to the case of bi-Lipschitz diffeomorphic manifolds. But now we drop the uniformity of the Lipschitz bounds and consider manifolds with rapidly oscillating random micro structure. We will discuss examples with degenerated geometries, like unbounded volume or collapse to singularities. Our results from the previous chapter will still yield Mosco-convergence, and in some cases even spectral convergence, to a deterministic manifold.

All proofs of this chapter are collected in Section 6.3.

The results of this chapter are new and unpublished.

6.1. Setting and Results

In Section 4.2 we considered examples of manifolds, which were all Euclidean submanifolds constructed from some reference manifold M_0 via adding normal perturbations with strength given by a periodic function f (see Section 4.2 for details). To give a flavor of how Theorem 5.2.2 reaches beyond the limitations of the uniformly bi-Lipschitz diffeomorphic manifolds and what situations may occur, we imagine that we manipulate the amplitude of the function f in a random way, independently for each period (see Section 6.2 for a concrete formulation). Then f and f' might no longer be uniformly bounded in $L^\infty(M_0)$, and since the diffeomorphisms h_ε directly depend on f, the so generated manifolds are not uniformly bi-Lipschitz diffeomorphic. However, if the coefficient fields $\mathbb{L}_\varepsilon$ can be controlled statistically in the sense of moment bounds on their eigenvalues, we are able to gain at least Mosco-convergence of the manifolds from the results in Chapter 5.

In the described situations the unboundedness of f and f' coincides with arbitrarily large volume of the perturbed manifold. In contrast one can also construct examples with uniformly bounded volume forms where the unboundedness of the diffeomorphisms is caused by singular points. One could for instance think of geometries similar to the bubble-like micro structures studied by Khrabustovskyi [Khr09]: In a flat surface small holes are cut, arranged on a grid of scale ε, and at these hole spheres with radius of order ε are attached. Note that if the radius of a hole tends to zero, the part where the sphere is attached collapses to a singular point. In [Khr09] spectral convergence is shown for the case that the radii of the holes were assumed to be deterministically bounded from below (with dependence of ε). We will gain spectral convergence in Section 6.2.3 for a quite similar setting, where the radii are chosen randomly so they can be arbitrarily close to zero.

In the spirit of Chapter 4 we want to adapt our Γ-convergence result Theorem 5.2.2 to the situation of sequences $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ of weighted Riemannian manifolds that are (not necessarily uniformly) bi-Lipschitz diffeomorphic to a weighted Riemannian reference manifold (M_0, g_0, μ_0). But since Theorem 5.2.2 is devoted to the flat Euclidean case, it would not be enough to pull the Dirichlet energy of M_ε back to an energy functional on M_0, but also to pull this energy functional back once more to $\mathbb{R}^n$ along local coordinate charts of M_0. However, for a bi-Lipschitz diffeomorphic chart (U, Ψ) of M_0 and a bi-Lipschitz diffeomorphism $h_\varepsilon \colon M_0 \to M_\varepsilon$, the composition $h_\varepsilon \circ \Psi^{-1} \colon \Psi(U) \to M_\varepsilon$ is a bi-Lipschitz diffeomorphic chart of M_ε, and we end up with the same energy functional on $\Psi(U) \subseteq \mathbb{R}^n$ if we pull the Dirichlet energy on M_ε back step by step along h_ε and Ψ^{-1} or directly along $h_\varepsilon \circ \Psi^{-1}$. Against this background it is much more natural to skip the intercalated reference manifold and regard the manifolds M_ε as locally (not necessarily uniformly) bi-Lipschitz diffeomorphic to subsets of $\mathbb{R}^n$. The only role of the reference manifold then is to ensure, that all these subsets in the end can be put together to one limiting manifold.

Definition 6.1.1 (Common Reference bi-Lipschitz Atlas). *Let $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ be a family of weighted Riemannian manifolds that are bi-Lipschitz diffeomorphic to a reference manifold (M_0, g_0, μ_0) via the diffeomorphisms $h_\varepsilon \colon M_0 \to M_\varepsilon$. Let further $\mathcal{A}$ be a pair-wise disjoint partition of M_0 and $\mathfrak{A}_0$ be a differentiable atlas for M_0 such that*

(i) for every $A \in \mathcal{A}$ there is a unique chart in $(U, \Psi) \in \mathfrak{A}_0$ with $A \subseteq U$, which is called the associated chart to *A,*

(ii) for $A_1, A_2 \in \mathcal{A}$ with associated charts $(U_1, \Psi_1), (U_2, \Psi_2) \in \mathfrak{A}_0$ we have

$$A_1 \neq A_2 \qquad \Rightarrow \qquad \Psi_1(U_1) \cap \Psi_2(U_2) = \emptyset.$$

Then $\mathfrak{A}_\varepsilon := \{(h_\varepsilon(U), \Psi \circ h_\varepsilon^{-1}); (U, \Psi) \in \mathfrak{A}_0\}$ is a differentiable atlas for M_ε and the family $(\mathfrak{A}_\varepsilon)$ is called a common reference atlas *for (M_ε) with the* corresponding set of reference cells *$\{\Psi(A) \subseteq \mathbb{R}^n; A \in \mathcal{A},\ (U, \Psi)$ associated chart to $A\}$.*

If $\mathcal{A}$ consists only of bounded Lipschitz domains and every chart included in $\mathfrak{A}_\varepsilon$ is bi-Lipschitz, the common reference atlas is called bi-Lipschitz. *It is called* countable, *if $\mathcal{A}$ is countable.*

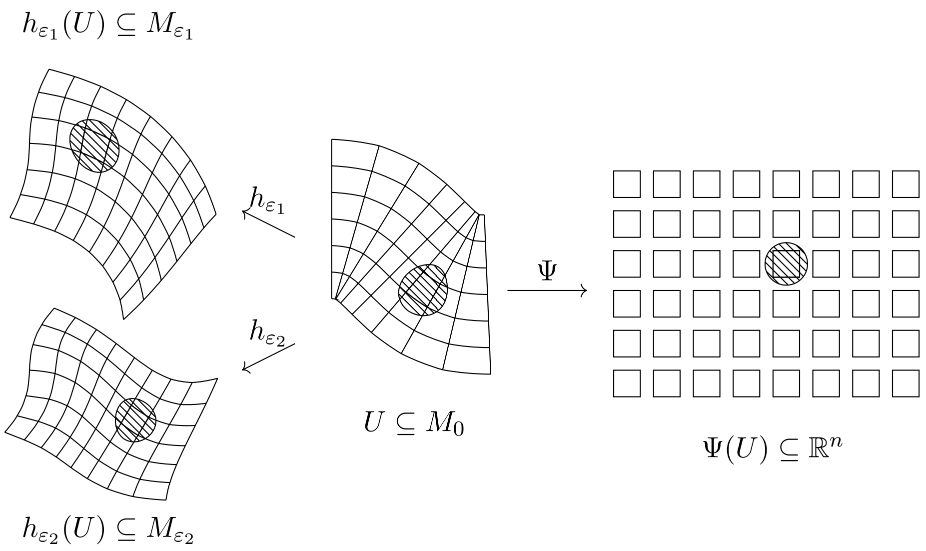


Figure 6.1.: Illustration of a common reference atlas, cf. Definition 6.1.1.

If a bi-Lipschitz diffeomorphic family (M_ε) provides a common bi-Lipschitz atlas $(\mathfrak{A}_\varepsilon)$ with corresponding set $\mathcal{A}$ of the reference cells, for every $A \in \mathcal{A}$ is a bounded Lipschitz domain in $\mathbb{R}^n$ and there is a unique chart $(U, \Psi) \in \mathfrak{A}_0$ with $A \subseteq \Psi(U)$. Moreover for every $\varepsilon > 0$ there is a unique chart $(U_\varepsilon, \Psi_\varepsilon) \in \mathfrak{A}_\varepsilon$ with $\Psi_\varepsilon = \Psi \circ h_\varepsilon^{-1}$, and for every such chart we find $(\Psi_\varepsilon \circ h_\varepsilon \circ \Psi^{-1})(A) = A$. Thus A can be interpreted as a (flat) reference manifold and for the (not necessarily uniformly) bi-Lipschitz diffeomorphic manifolds $h_\varepsilon(\Psi^{-1}(A)) \subseteq M_\varepsilon$ and the corresponding diffeomorphisms are given by Ψ_ε^{-1}.

In the examples in Section 6.2 the only reference cell will be the reference manifold M_0 itself and the associated chart is given by $(M_\varepsilon^\omega, (h_\varepsilon^\omega)^{-1})$.

While the existence of a common atlas yields a family of pulled back Dirichlet energies on bounded Lipschitz domains of $\mathbb{R}^n$, and these energies already have strictly convex quadratic potentials, we are still far from the setting of Theorem 5.2.2, as the potentials can depend on ε in every possible way. Thus we need to restrict to the special case of what we call locally rapidly oscillating manifolds. Therefor we recall the definition of the adjoint operator $d\Psi_\varepsilon^* \colon \mathbb{R}^n \to TM_\varepsilon$ of the differential $d\Psi_\varepsilon \colon TM_\varepsilon \to \mathbb{R}^n$:

$$g_\varepsilon(d\Psi_\varepsilon^*\xi, \eta)(\Psi_\varepsilon^{-1}(x)) = \xi \cdot d\Psi_\varepsilon \eta \qquad \text{for all } \xi \in \mathbb{R}^m,\ \eta \in T_{\Psi_\varepsilon^{-1}(x)} M_\varepsilon.$$

Definition 6.1.2 (Locally Rapidly Oscillating Random Manifolds). *Let $(M_\varepsilon, g_\varepsilon, \mu_\varepsilon)$ be a family of weighted Riemannian manifolds with a countable bi-Lipschitz common reference atlas $(\mathfrak{A}_\varepsilon)$ and corresponding set $\mathcal{A}$ of reference cells, and denote by σ_ε the density of μ_ε against the Riemannian volume measure associated to g_ε. The family (M_ε) is called* locally rapidly oscillating, *if for every reference cell $A \in \mathcal{A}$ there is a* *coefficient field* $\mathbb{L}\colon A \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ *such that for every* $\varepsilon > 0$ *and* $(U_\varepsilon, \Psi_\varepsilon) \in \mathfrak{A}_\varepsilon$ *with* $A \subseteq \Psi_\varepsilon(U_\varepsilon)$ *we have for all* $x \in A$, $\xi, \eta \in \mathbb{R}^n$

$$\rho_\varepsilon(x)\, \overline{g}_\varepsilon(d\Psi_\varepsilon^* \xi, d\Psi_\varepsilon^* \eta)(x) = \mathbb{L}(x, \tfrac{x}{\varepsilon})\xi \cdot \eta + o(\varepsilon), \qquad \rho_\varepsilon = \overline{\sigma}_\varepsilon \sqrt{\det \overline{g}_\varepsilon},$$

where $\overline{g}_\varepsilon = g_\varepsilon \circ \Psi_\varepsilon^{-1}$ *and* $\overline{\sigma}_\varepsilon = \sigma_\varepsilon \circ \Psi_\varepsilon^{-1}$*, and the error term* $o(\varepsilon)$ *is to be understood uniformly in* x *as* $\varepsilon \searrow 0$.

For a family $(M_\varepsilon^\omega, g_\varepsilon^\omega, \mu_\varepsilon^\omega)$ *which is also indexed by* $\omega \in \Omega$ *for same probability space* Ω*, the atlases* $\mathfrak{A}_\varepsilon^\omega$*, that means the charts* Ψ_ε^ω *but not the reference cells* $\mathcal{A}$*, also depend on* ω*. If the associated coefficient field* $\mathbb{L}^\omega$ *is stationary in the second argument (that is in* $\frac{x}{\varepsilon}$*), we call* (M_ε^ω) *locally rapidly oscillating* random *manifolds. (Note that the error term* $o(\varepsilon)$ *in the random case reads as "an error which for a.e.* ω *tends to zero uniformly in* x *as* $\varepsilon \searrow 0$*".)*

The definition of rapidly oscillating manifolds becomes much more transparent in the case where M_ε are submanifolds of the Euclidean space equipped with the induced metric and measure, because the condition then reads

$$\rho_\varepsilon(x)\, (d\Psi_\varepsilon(x) d\Psi_\varepsilon(x)^\intercal) = \mathbb{L}(x, \tfrac{x}{\varepsilon}) + o(\varepsilon), \qquad \rho_\varepsilon = \sqrt{\det(d\Psi_\varepsilon d\Psi_\varepsilon^\intercal)^{-1}}.$$

Now we are in position to formulate the assumptions on $\mathbb{L}$ in order to apply Theorem 5.2.2.

Proposition 6.1.3 (Mosco-Convergence of Rapidly Oscillating Random Manifolds)**.** *Let* $(M_\varepsilon^\omega, g_\varepsilon^\omega, \mu_\varepsilon^\omega)$ *be a family of* n*-dimensional rapidly oscillating random manifolds with reference manifold* (M_0, g_0, μ_0)*. Assume that for every reference cell* A *and the corresponding coefficient field* $\mathbb{L}^\omega$ *as in Definition 6.1.2 we have*

(i) the smallest reps. largest eigenvalue $\lambda_{\min}^\omega(x,y)$ *resp.* $\lambda_{\max}^\omega(x,y)$ *of* $\mathbb{L}^\omega(x,y)$ *satisfy*

$$\sup_{\substack{x\in A,\\ y\in\mathbb{R}^n}} \mathbb{E}[\lambda_{\max}(x,y)] < \infty \qquad \text{and} \qquad \sup_{\substack{x\in A,\\ y\in\mathbb{R}^n}} \mathbb{E}[\lambda_{\min}(x,y)^{-\frac{n}{2}}] < \infty,$$

(ii) there is a function $\rho\colon [0,\infty) \to (0,1)$ *with* $\lim_{\delta\searrow 0} \rho(\delta) = 0$ *such that for almost every* $\omega \in \Omega$ *and every* $x_1, x_2 \in A$, $y, \xi \in \mathbb{R}^n$ *we have*

$$\left|\mathbb{L}(x_1,y) - \mathbb{L}(x_2,y)\right| \le \rho(|x_1 - x_2|)\left(1 + |\mathbb{L}(x_1,y) + \mathbb{L}(x_2,y)|\right).$$

Then for every density ρ_0 *on* M_0 *there are a deterministic metric* $\hat{g}_0$ *and a measure* $\mathrm{d}\hat{\mu}_0 = \rho_0 \mathrm{d}\mu_0$ *on* M_0*, such that* M_ε^ω *Mosco-converge (w.r.t.* L^2*) for a.e.* ω *to* $(M_0, \hat{g}_0, \hat{\mu}_0)$ *as* $\varepsilon \searrow 0$.

Remark 6.1.4.

- *By going through the proof of Proposition 6.1.3 one can find an explicit formula for the metric $\hat{g}_0$, which we do not state here as in this abstract framework it is not very practical. But in the discussion of the case of Euclidean submanifolds below, we will address the explicit expression of $\hat{g}_0$. However, if the manifolds $(M_\varepsilon^\omega, g_\varepsilon^\omega, \mu_\varepsilon^\omega)$ are uniformly bi-Lipschitz diffeomorphic to a compact reference manifold, the volume forms ρ_ε weakly-$*$ converge in $L^\infty(M_0)$ to some ρ_0 and the corresponding metric $\hat{g}_0$ in Proposition 6.1.3 coincides with the one given in Lemma 4.1.3.*

- *While the formulation of assumption (i) in Proposition 6.1.3 is fitted to L^2-Mosco-convergence, the proof allows the following weaker variant of the statement: If one replaces the boundedness of the $\frac{n}{2}$th moments of the eigenvalue $\lambda_{\min}$ by*

$$\sup_{\substack{x\in A,\\ y\in\mathbb{R}^n}} \mathbb{E}[\lambda_{\min}(x,y)^{-\beta}] < \infty$$

  for some $\beta \geq 1$, one still can conclude Mosco-convergence of the manifolds, but with respect to L^p for $\frac{1}{p} = \frac{\beta+1}{2\beta} - \frac{1}{n}$ (cf. the compact embedding Proposition 5.2.1). However, we do not engage in this any further, since we will need L^2-convergence of the minimizers in order to deduce spectral convergence (cf. Proposition 6.1.5 below) and therefore L^2-Mosco-convergence is the natural notion of convergence for our studies.

One main difference to the case of uniformly bi-Lipschitz diffeomorphic manifolds is that the volume forms ρ_ε in general do not need to be uniformly bounded, so we cannot expect the measures μ_ε to converge, which is a necessary condition for spectral convergence. For instance in the situation discussed in the beginning of this chapter (cf. also Section 6.2) the family of volume forms might become unbounded in $L^\infty(M_0)$ for a.e. $\omega \in \Omega$, since the amplitude f can be arbitrarily large.

For that reason we are free to choose any density ρ_0 for the measure on the Mosco-limiting manifold. But if on a compact reference manifold the volume forms ρ_ε are bounded and do converge weakly-$*$ in L^∞, then their limit is the natural choice for ρ_0, since in this case, utilizing the compactness result Proposition 5.2.1, we gain spectral convergence, too.

Proposition 6.1.5 (Spectral Convergence of Rapidly Oscillating Manifolds). *Suppose that M_0 is compact. If in the setting of Proposition 6.1.3 $\rho_\varepsilon \overset{*}{\rightharpoonup} \rho_0$ weakly-$*$ in $L^\infty(M_0, g_0, \mu_0)$ for a.e. $\omega \in \Omega$, the family (M_ε^ω) spectral converges (w.r.t. L^2) to $(M_0, \hat{g}_0, \hat{\mu}_0)$ for a.e. $\omega \in \Omega$.*

As mentioned above, if the manifolds M_ε^ω are submanifolds of the Euclidean space the situation simplifies. We want to formulate our result for the even simpler case where the reference manifold M_0 is a subset of $\mathbb{R}^n$ equipped with the standard metric and the Lebesgue measure, and the oscillating structure of (M_ε) is natured such that M_0 itself is the only reference cell. In this situation we can turn back to the formulation used in Chapter 4 thinking of M_0 as the reference manifold and M_ε^ω being bi-Lipschitz diffeomorphic to M_0 via the diffeomorphism $h_\varepsilon^\omega \colon M_0 \to M_\varepsilon^\omega$.

Corollary 6.1.6. *Let $(M_0, \cdot, \mathrm{d}x)$ be an n-dimensional Riemannian manifold, which is a subset of $\mathbb{R}^n$ equipped with the induced metric and measure, and let $(M_\varepsilon^\omega, \cdot, \mathrm{d}x)$ be a family, indexed by $\omega \in \Omega$ and $0 < \varepsilon$, of n-dimensional submanifolds of the Euclidean space (with the induced metric and measure), being bi-Lipschitz diffeomorphic to the reference manifold M_0 via the diffeomorphisms $h_\varepsilon^\omega \colon M_0 \to M_\varepsilon$. Further assume that there is a stationary coefficient field $\mathbb{L} \colon \Omega \times M_0 \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that*

$$\rho_\varepsilon^\omega(x)\,(dh_\varepsilon^\omega(x)^\intercal dh_\varepsilon^\omega(x))^{-1} = \mathbb{L}^\omega(x, \tfrac{x}{\varepsilon}), \qquad \rho_\varepsilon^\omega = \sqrt{\det dh_\varepsilon^{\omega\intercal} dh_\varepsilon^\omega}.$$

If the following conditions are fulfilled

(i) the smallest resp. largest eigenvalue $\lambda_{\min}^\omega(x,y)$ resp. $\lambda_{\max}^\omega(x,y)$ of $\mathbb{L}^\omega(x,y)$ satisfy

$$\sup_{\substack{x\in M_0,\\ y\in\mathbb{R}^n}} \mathbb{E}[\lambda_{\max}(x,y)] < \infty \qquad \text{and} \qquad \sup_{\substack{x\in M_0,\\ y\in\mathbb{R}^n}} \mathbb{E}[\lambda_{\min}(x,y)^{-\frac{n}{2}}] < \infty,$$

(ii) there is a function $\rho \colon [0,\infty) \to (0,1)$ with $\lim_{\delta\searrow 0} \rho(\delta) = 0$ such that for almost every $\omega \in \Omega$ and every $x_1, x_2 \in A$, $y, \xi \in \mathbb{R}^n$ we have

$$\bigl|\mathbb{L}(x_1,y) - \mathbb{L}(x_2,y)\bigr| \le \rho(|x_1 - x_2|)\,\bigl(1 + |\mathbb{L}(x_1,y) + \mathbb{L}(x_2,y)|\bigr),$$

then for every density ρ_0 on M_0 there are a deterministic metric $\hat{g}_0$ and measure $\mathrm{d}\hat{\mu}_0 = \rho_0 \mathrm{d}x$ on M_0 such that for a.e. ω the manifolds M_ε^ω Mosco-converge to $(M_0, \hat{g}_0, \hat{\mu}_0)$ w.r.t. L^2 as $\varepsilon \searrow 0$. In particular, $\hat{g}_0$ is explicitly given by $\hat{g}_0(\xi,\eta) = \rho_0 \mathbb{L}_0^{-1}\xi \cdot \eta$ with

$$\mathbb{L}_0(x)\xi\cdot\eta = \lim_{k\to\infty} \mathbb{E}\Bigl[\inf_{\phi\in W_0^{1,p}([0,k)^n)} \fint_{[0,k)^n} \mathbb{L}(x,y)(\xi + \nabla\phi(y))\cdot(\eta + \nabla\phi(y))\,\mathrm{d}y\Bigr]. \quad (6.1)$$

Moreover, if the reference manifold M_0 is open and bounded with a (possibly empty) Lipschitz boundary and $\rho_\varepsilon \overset{*}{\rightharpoonup} \rho_0$ weakly in $L^\infty(M_0)$ for a.e. $\omega \in \Omega$, then for a.e. ω the manifolds M_ε^ω spectral converge to $(M_0, \hat{g}_0, \hat{\mu}_0)$ w.r.t. L^2 as $\varepsilon \searrow 0$.

Remark 6.1.7. *In the above case of the reference manifold M_0 being a flat subset of $\mathbb{R}^n$, with the standard metric and measure, a sufficient condition for spectral convergence* *can be formulated as follows: Let $(M_\varepsilon^\omega, g_\varepsilon^\omega, \mu_\varepsilon^\omega)$ be an n-dimensional submanifold of $\mathbb{R}^m$ being bi-Lipschitz diffeomorphic to M_0 via the diffeomorphisms $h_\varepsilon^\omega\colon M_0 \to M_\varepsilon^\omega$. Assume that there is a function $a\colon \Omega \times M_0 \to \mathbb{R}^{n\times n}$ such that*

$$dh_\varepsilon^\omega(x)^\intercal dh_\varepsilon^\omega(x) = a(\tau_{\frac{x}{\varepsilon}}\omega, x).$$

If we denote by $C_\varepsilon(\omega) > 0$ the Lipschitz constant of h_ε^ω, i.e.

$$\tfrac{1}{C_\varepsilon(\omega)}|\xi| \leq |dh_\varepsilon^\omega \xi| \leq C_\varepsilon(\omega)|\xi|$$

for all $\xi \in \mathbb{R}^n$, and if these Lipschitz constants have bounded moments in the sense of

$$\sup_{\varepsilon>0} \mathbb{E}[C_\varepsilon^{(n^2)}] < \infty, \tag{6.2}$$

then one can easily check that

$$C_\varepsilon(\omega)^{-n} \leq \rho_\varepsilon^\omega \leq C_\varepsilon(\omega)^n \qquad \text{and} \qquad C_\varepsilon(\omega)^{-n-2}|\xi| \leq |\mathbb{L}^\omega \xi| \leq C_\varepsilon(\omega)^{n+2}|\xi|,$$

and therewith for every eigenvalue λ of $\mathbb{L}^\omega$ one has

$$\lambda \leq C_\varepsilon(\omega)^{n+2} \leq C_\varepsilon(\omega)^{(n^2)} \qquad \text{and} \qquad \lambda^{-\frac{n}{2}} \leq C_\varepsilon(\omega)^{\frac{n^2}{2}+n} \leq C_\varepsilon(\omega)^{(n^2)},$$

so one finds the assumptions of Corollary 6.1.6 satisfied and is granted with Mosco-convergence. (Note that the exponent n^2 in (6.2) is far from being optimal.) If in addition the volume forms $(\rho_\varepsilon^\omega)$ are uniformly bounded in $\mathbb{L}^\infty(M_0)$ for a.e. $\omega \in \Omega$, we even deduce weak-$*$ convergence due to the stationarity, and therefore conclude even spectral convergence. Despite this condition looks natural with regard to the similarities to the uniformly bi-Lipschitz diffeomorphic case (cf. Definition 4.1.2), in practice it is often more convenient to examine the eigenvalues of $\mathbb{L}^\omega$ then the Lipschitz constant of h_ε^ω, since $\mathbb{L}^\omega$ needs to be calculated anyway to find the limiting manifold.

Remark 6.1.8 (Realizability of $(M_0, \hat{g}_0, \hat{\mu}_0)$). *If in the setting of Corollary 6.1.6 the density ρ_0 is chosen such that $\rho_0^{n-2} = \det \mathbb{L}_0$ and the corresponding metric $\hat{g}_0$ is smooth, then $\hat{\mu}_0$ is the Riemannian volume measure associated with the metric $\hat{g}_0$ and the limiting manifold $(M_0, \hat{g}_0, \hat{\mu}_0)$ can be isometrically embedded into $\mathbb{R}^m$ for some m large enough due to Nash's Embedding Theorem. Such an embedding $h_0\colon M_0 \to \mathbb{R}^m$ is characterized by the identity $dh_0^\intercal dh_0 = \rho_0 \mathbb{L}_{\mathrm{hom}}^{-1}$.*

6.2. Examples

As indicated in the beginning of Section 6.1, we want to modify the examples from Section 4.2 to demonstrate several aspects of the method provided by Corollary 6.1.6. The first two examples (Section 6.2.1) are laminate-like corrugated graphical surfaces over $\mathbb{R}^2$ and demonstrate the intersection of the method established above with the results for uniformly bi-Lipschitz diffeomorphic manifolds in Chapter 4. With the following two laminate-like corrugated spherical examples (Section 6.2.2) we leave the region of uniformly bi-Lipschitz diffeomorphic manifolds, since we allow the perturbation of the reference manifold to be arbitrarily large. Therewith we give up control over the volume and do not have weak-$*$ convergence of the volume form, hence our results provide only Mosco-convergence, but not spectral convergence. The last example (Section 6.2.3) steps out of the line, as it is a non-laminate-like example. We consider mushroom-shaped surfaces of revolution, repeatedly arranged as on a checker board, but whose stem width is chosen randomly (and independently) on each tile. Since the curvature (locally) becomes singular as the stem width tends to zero, the family of manifolds constructed this way is not uniform bi-Lipschitz diffeomorphic, but provides bounded volume forms and is therefore an example for spectral convergence of non-uniformly bi-Lipschitz diffeomorphic manifolds. In this sense, even though the limit is isotropic, this example is the most important.

6.2.1. Uniformly bi-Lipschitz Diffeomorphic Manifolds

The families of submanifolds considered in the following two examples are actually uniformly bi-Lipschitz diffeomorphic an could therefore be treated with the methods from Chapter 4. However, this would provide at first only spectral convergence along a subsequence, and in a second step one would observe that the limit is independent of the choice of the subsequence, which yields spectral convergence of the entire sequence. In contrast, Corollary 6.1.6 allows us to gain spectral convergence of the entire family immediately. Of course, both methods yield the same limiting manifold (up to isometry).

A graphical surface with concentric random corrugations

As already mentioned we want to manipulate the amplitude of the periodic corrugation independently in each period. To make this precise we consider a smooth, 1-periodic function $f\colon [0,\infty) \to \mathbb{R}$ of the form $f(y) := \psi(y - \lfloor y \rfloor)$ for some $\psi \in C_c^\infty(0,1)$, where $\lfloor y \rfloor$ denotes the integer part of y, i.e. $\lfloor y \rfloor \in \mathbb{Z}$ with $\lfloor y \rfloor \le y < \lfloor y \rfloor + 1$. Now we introduce the set of admissible corrugations

$$\Omega := \{\omega\colon \mathbb{R} \to \mathbb{R}; \omega(y) = a(\lfloor y_0 + y \rfloor) f(y_0 + y) \text{ for some } y_0 \in [0,1), a \in [0,2]^{\mathbb{Z}}\},$$

which is isomorphic to $\{(y_0, a); y_0 \in [0,1), a \in \mathbb{R}^{\mathbb{Z}}\}$. On this set we can define a probability measure by letting the offset $y_0 \in [0,1)$ be uniformly distributed and the amplitudes $a(k) \in [0,2]$ for $k \in \mathbb{Z}$ be independently uniformly distributed, and the push-forward of this measure defines a probability measure on Ω. Endowed with this measure and the Borel-σ-algebra, Ω forms a probability space, which is stationary and ergodic w.r.t. the group action $\tau_y\omega := \omega(y+\cdot)$ for $\omega \in \Omega$, $y \in \mathbb{R}$.

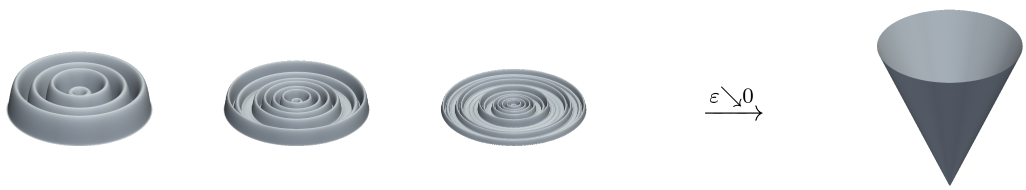
We now start with the reference manifold

$$M_0 := \{(r,\theta); r \in (\delta, R), \theta \in [0,1)\}$$

for some $R > \delta > 0$, and define the submanifolds $M_\varepsilon^\omega = h_\varepsilon^\omega(M_0)$ of $\mathbb{R}^3$ (with the induced metric and measure) via $h_\varepsilon^\omega \colon M_0 \to \mathbb{R}^3$,

$$h_\varepsilon^\omega(r,\theta) := \begin{pmatrix} r\sin 2\pi\theta \\ r\cos 2\pi\theta \\ \varepsilon\omega(\frac{r}{\varepsilon}) \end{pmatrix} \tag{6.3}$$

for $\omega \in \Omega$, $\varepsilon \in \{\frac{1}{k}; k \in \mathbb{N}\}$, which are displayed in Figure 6.2 with ψ as in (4.7) for some values of ε.



$\varepsilon = \frac{1}{4}$ $\varepsilon = \frac{1}{8}$ $\varepsilon = \frac{1}{16}$

Figure 6.2.: A family of graphical surfaces with concentric random corrugations. The three pictures on the left show a realization of M_ε^ω defined by (6.3) with ψ as in (4.7) and decreasing values of ε. The picture on the right shows the deterministic limiting surface N_0 defined via (6.4). As $\varepsilon \searrow 0$ the spectrum of the Laplace-Beltrami operator on M_ε converges to the spectrum of the Laplace-Beltrami operator on N_0.

Guided from Corollary 6.1.6 we calculate the density

$$\rho_\varepsilon^\omega = \sqrt{dh_\varepsilon^{\omega\intercal} dh_\varepsilon^\omega} = 2\pi r\sqrt{\omega'(\tfrac{r}{\varepsilon})^2+1}$$

and the coefficient field

$$\mathbb{L}_\varepsilon^\omega = \rho_\varepsilon^\omega (dh_\varepsilon^{\omega\intercal} dh_\varepsilon^\omega)^{-1} = \begin{pmatrix} \frac{2\pi r}{\sqrt{\omega'(\frac{r}{\varepsilon})^2+1}} & 0 \\ 0 & \frac{\sqrt{\omega'(\frac{r}{\varepsilon})^2+1}}{2\pi r} \end{pmatrix}.$$

Obviously, this coefficient field is of the desired form $\mathbb{L}_\varepsilon^\omega(r,\theta) = \mathbb{L}^\omega(r,\frac{r}{\varepsilon})$ with

$$\mathbb{L}^\omega(r,y) = \begin{pmatrix} \frac{2\pi r}{\sqrt{\omega'(y)^2+1}} & 0 \\ 0 & \frac{\sqrt{\omega'(y)^2+1}}{2\pi r} \end{pmatrix}$$

and, since ω' is uniformly bounded in $L^\infty(M_0)$ for all $\omega \in \Omega$, the eigenvalues of $\mathbb{L}$ are uniformly bounded and, in particular, have bounded moments as claimed in Corollary 6.1.6. Hence, there are a deterministic metric $\hat{g}_0$ and a measure $\hat{\mu}_0$ on M_0 such that the family (M_ε^ω) Mosco-converges to $(M_0, \hat{g}_0, \hat{\mu}_0)$ for a.e. $\omega \in \Omega$.

To identify the limiting metric and measure we first note that by Birkhoff's Ergodic Theorem (Lemma 2.2.2) we find for the average over any cube $Q \subseteq M_0$

$$\lim_{\varepsilon \searrow 0} \fint_Q \rho_\varepsilon^\omega(y)\,\mathrm{d}y = 2\pi r\,\tfrac{1}{2} \int_0^2 \int_0^1 \sqrt{a^2 f'(y)^2 + 1}\,\mathrm{d}y\,\mathrm{d}a =: \rho_0(r)$$

for a.e. $\omega \in \Omega$. Therefore, and since the densities ρ_ε^ω are uniformly bounded in $L^\infty(M_0)$, we conclude that $\rho_\varepsilon^\omega \overset{*}{\rightharpoonup} \rho_0$ weakly in $L^\infty(M_0)$ for a.e. $\omega \in \Omega$, which makes ρ_0 the natural choice for the density of the limiting measure $\hat{\mu}_0$, as it yields also spectral convergence of the manifolds.

Now in order to find the corresponding metric $\hat{g}_0$, we need to calculate the coefficient field $\mathbb{L}_0$ via (6.1), i.e.

$$\mathbb{L}_0 \xi \cdot \eta = \lim_{k \to \infty} \mathbb{E}\Big[\inf_{\phi \in W_0^{1,p}([0,k))} \tfrac{1}{k} \int_0^k \mathbb{L}(r, y)(\xi + \nabla\phi(y)) \cdot (\eta + \nabla\phi(y))\,\mathrm{d}y \Big].$$

This can be solved via the Euler-Lagrange-equation or by appealing to standard (stochastic) homogenization formulas for diagonal matrices associated to laminate-like structures, and we find

$$\mathbb{L}_0 = \begin{pmatrix} \frac{4\pi^2 r^2}{\rho_0} & 0 \\ 0 & \frac{\rho_0}{4\pi^2 r^2} \end{pmatrix}.$$

Therewith the limiting metric turns out to be

$$\hat{g}_0(\xi, \eta) = \rho_0 \mathbb{L}_0^{-1} \xi \cdot \eta = \begin{pmatrix} \frac{\rho_0^2}{4\pi^2 r^2} & 0 \\ 0 & 4\pi^2 r^2 \end{pmatrix} \xi \cdot \eta.$$

According to Remark 6.1.8, an isometric embedding $h_0 \colon M_0 \to \mathbb{R}^3$ of the limiting manifold $(M_0, \hat{g}_0, \hat{\mu}_0)$ can be found via $dh_0^\intercal dh_0 = \rho_0 \mathbb{L}_0^{-1}$, namely

$$h_0(r, \theta) = \begin{pmatrix} r \sin 2\pi\theta \\ r \cos 2\pi\theta \\ \int_0^r \sqrt{\frac{\rho_0(t)^2}{4\pi^2 t^2} - 1}\,\mathrm{d}t \end{pmatrix}. \tag{6.4}$$

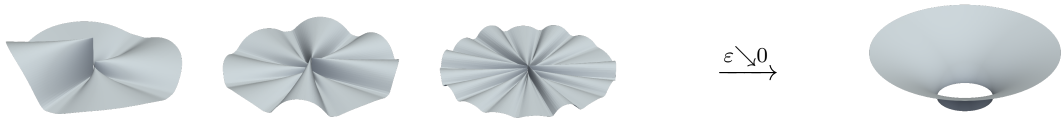
That means, the submanifold $N_0 := h_0(M_0)$ of $\mathbb{R}^3$ (with the induced metric and measure), pictured in Figure 6.2, is the spectral (and Mosco-) limit of the family (M_ε^ω) for a.e. $\omega \in \Omega$ as $\varepsilon \searrow 0$. Note that h_0 does not depend on the initial choice of the radius δ of the excluded circle around the origin in the manifolds M_ε^ω, and therefore we can pass to the reference manifold $M_0 = (0,1) \times [0,1)$, and the excluded origin of the manifolds M_ε^ω then coincides with the apex of the cone-shaped limiting manifold N_0.

A graphical surface with star-shaped random corrugations

In the same manner as above we can of course adapt the first example of Section 4.2.1. We start with the same probability space Ω and the same reference manifold M_0 as in the example above and define the submanifolds $M_\varepsilon^\omega = h_\varepsilon^\omega(M_0)$ of $\mathbb{R}^3$ via $h_\varepsilon^\omega\colon M_0 \to \mathbb{R}^3$,

$$h_\varepsilon^\omega(r,\theta) := \begin{pmatrix} r\sin 2\pi\theta \\ r\cos 2\pi\theta \\ \varepsilon\omega(\frac{\theta}{\varepsilon}) \end{pmatrix} \tag{6.5}$$

for $\omega \in \Omega$, $\varepsilon \in \{\frac{1}{k}; k \in \mathbb{N}\}$, which are pictured in Figure 6.3 with ψ as in (4.7) for some values of ε.



$\varepsilon = \frac{1}{4}$ $\quad\varepsilon = \frac{1}{8}$ $\quad\varepsilon = \frac{1}{16}$

Figure 6.3.: A family of graphical surfaces with star-shaped random corrugations. The three pictures on the left show a realization of M_ε^ω defined by (6.5) with ψ as in (4.7) and decreasing values of ε. The picture on the right shows the deterministic limiting surface N_0 defined via (6.6).

Doing the same calculations as above, we find the density

$$\rho_0(r) = \tfrac{1}{2}\int_0^2\int_0^1 \sqrt{a^2 f'(y)^2 + 4\pi^2 r^2}\,\mathrm{d}y\,\mathrm{d}a,$$

which is the a.s. weak limit of ρ_ε^ω in $L^2(M_0)$, and the metric

$$\hat{g}_0 = \rho_0\mathbb{L}_0^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \rho_0^2 \end{pmatrix}.$$

An isometric embedding $h_0\colon M_0 \to \mathbb{R}^3$ of the limiting manifold can be found via $dh_0^\intercal dh_0 = \rho_0\mathbb{L}_0^{-1}$, namely

$$h_0(r,\theta) = \begin{pmatrix} \frac{\rho_0(r)}{2\pi}\sin 2\pi\theta \\ \frac{\rho_0(r)}{2\pi}\cos 2\pi\theta \\ \int_0^r \sqrt{1 - \frac{\rho_0'(t)^2}{4\pi^2}}\,\mathrm{d}t \end{pmatrix}. \tag{6.6}$$

The submanifold $N_0 := h_0(M_0)$ of $\mathbb{R}^3$, shown in Figure 6.3, is the spectral (and Mosco-) limit of the family (M_ε^ω) for a.e. $\omega \in \Omega$ as $\varepsilon \searrow 0$. As in the example above we can pass to the reference manifold $M_0 = (0,1) \times [0,1)$, and the excluded origin of the manifolds M_ε^ω then coincides with a circle in the boundary of N_0.

6.2.2. Manifolds with Unbounded Volume

The previous two examples were uniformly bi-Lipschitz diffeomorphic families of manifolds and therefore also handable with the methods provided by Chapter 4. In the following we present two examples of manifolds with unbounded volume, so they cannot be uniformly bi-Lipschitz diffeomorphic.

A sphere with random radial perturbations oscillating with the latitude

Continuing the spirit of the examples above we start again with a smooth, 1-periodic function $f\colon [0,\infty) \to \mathbb{R}$ of the form $f(y) := \psi(y - \lfloor y \rfloor)$ for some $\psi \in C_c^\infty(0,1)$, but this time we get the set of admissible perturbations

$$\Omega := \{\omega\colon \mathbb{R} \to \mathbb{R}; \omega(y) = a(\lfloor y_0 + y \rfloor) f(y_0 + y) \text{ for some } y_0 \in [0,1), a \in [0,\infty)^{\mathbb{Z}}\}.$$

Note that in contrast to the examples above we allow arbitrary large amplitudes. We define a probability measure on Ω by letting the offset $y_0 \in [0,1)$ be uniformly distributed and the amplitudes $a(k) \in [0,\infty)$ for $k \in \mathbb{Z}$ be i.i.d. with finite expectation, i.e. $\mathbb{E}[a(k)] < \infty$. Together with the Borel-σ-algebra, and the group action $\tau_y \omega := \omega(y + \cdot)$ for $\omega \in \Omega$, $y \in \mathbb{R}$, the probability space Ω becomes stationary and ergodic.

Having the spherical examples in Section 4.2 in mind, we choose the reference manifold to be

$$M_0 = \{(\varphi, \theta); \varphi \in (\delta, 1-\delta), \theta \in [0,1)\}$$

for some $\delta > 0$, and define the family of submanifolds $M_\varepsilon^\omega := h_\varepsilon^\omega(M_0)$ of $\mathbb{R}^3$ via $h_\varepsilon^\omega\colon M_0 \to \mathbb{R}^3$,

$$h_\varepsilon^\omega(\varphi,\theta) = \left(1 + \varepsilon\omega(\tfrac{\varphi}{\varepsilon})\right) \begin{pmatrix} \sin \pi\varphi \sin 2\pi\theta \\ \sin \pi\varphi \cos 2\pi\theta \\ \cos \pi\varphi \end{pmatrix} \tag{6.7}$$

for $\omega \in \Omega$, $\varepsilon \in \{\frac{1}{k}; k \in \mathbb{N}\}$. In Figure 6.4 we illustrate M_ε^ω with ψ as in (4.7) and a χ^2-distribution with 3 degrees of freedom for some values of ε.

The same computations as in the previous example yield the density

$$\rho_\varepsilon^\omega(\varphi) = 2\pi(1 + \varepsilon\omega(\tfrac{\varphi}{\varepsilon})) \sin \pi\varphi \sqrt{\omega'(\tfrac{\varphi}{\varepsilon})^2 + \pi^2(1 + \varepsilon\omega(\tfrac{\varphi}{\varepsilon}))^2}$$

and the coefficient field

$$\mathbb{L}^\omega(\varphi, y) = \begin{pmatrix} \frac{2\pi \sin \pi\varphi}{\sqrt{\omega'(y)^2 + \pi^2}} & 0 \\ 0 & \frac{\sqrt{\omega'(y)^2+\pi^2}}{2\pi \sin \pi\varphi} \end{pmatrix},$$

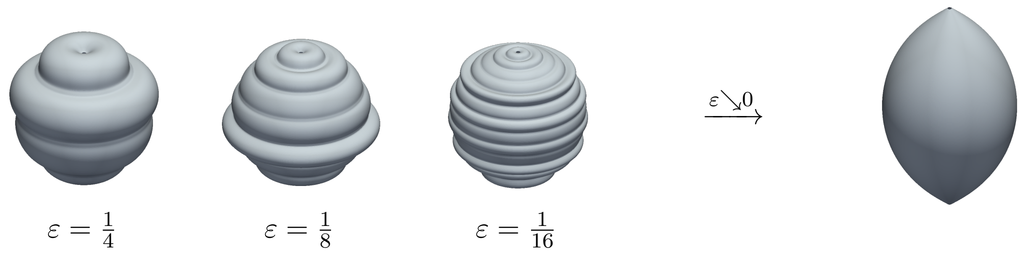


Figure 6.4.: A family of spheres with random radial perturbations oscillating with the latitude. The three pictures on the left show M_ε defined by (6.7) with ψ as in (4.7) and decreasing values of ε. The picture on the right shows the limiting surface N_0 defined via (6.8).

whose eigenvalues themselves are unbounded, but have bounded moments as claimed in Corollary 6.1.6, since on the one hand

$$\int_0^\infty \int_0^1 \frac{2\pi \sin \pi\varphi}{\sqrt{a^2 f'(y)^2 + \pi^2}} \,\mathrm{d}y \,\mathbb{P}(\mathrm{d}a) \leq 2 \sin \pi\varphi \leq 2,$$

where $\mathbb{P}$ denotes the probability measure on $[0,\infty)$ associated to the distribution of the amplitudes, and on the other hand, using the fact that $\sqrt{x^2+y^2} \leq |x| + |y|$,

$$\int_0^\infty \int_0^1 \frac{\sqrt{a^2 f'(y)^2 + \pi^2}}{2\pi \sin \pi\varphi} \,\mathrm{d}y \,\mathbb{P}(\mathrm{d}a) \leq \frac{1}{2\pi \sin \pi\delta} \Big(\int_0^\infty a \,\mathbb{P}(\mathrm{d}a) \int_0^1 |f'(y)| \,\mathrm{d}y + \pi \Big) < \infty.$$

Thus (6.1) yields the homogenized coefficient field

$$\mathbb{L}_0(\varphi) = \begin{pmatrix} \frac{2\pi \sin \pi\varphi}{\rho} & 0 \\ 0 & \frac{\rho}{2\pi \sin \pi\varphi} \end{pmatrix} \qquad \text{with } \rho = \int_0^\infty \int_0^1 \sqrt{a^2 f'(y)^2 + \pi^2} \,\mathrm{d}y \,\mathbb{P}(\mathrm{d}a).$$

Since the volume forms ρ_ε^ω are unbounded in $L^\infty(M_0)$, they cannot weakly-$*$ converge, so there is no natural choice for ρ_0. However, with respect to the periodic case (cf. Section 4.2) we decide for

$$\rho_0(\varphi) := 2\pi \sin \pi\varphi \int_0^\infty \int_0^1 \sqrt{a^2 f'(y)^2 + \pi^2} \,\mathrm{d}y \,\mathbb{P}(\mathrm{d}a),$$

which is the weak limit in $L^1(M_0)$, because then the limiting metric reads

$$\hat{g}_0 = \rho_0 \mathbb{L}_0^{-1} = \begin{pmatrix} \frac{\rho_0^2}{4\pi^2 \sin^2 \pi\varphi} & 0 \\ 0 & 4\pi^2 \sin^2 \pi\varphi \end{pmatrix},$$

which has the same form as in the periodic case. By Remark 6.1.8 we can find an isometric embedding $h_0 \colon M_0 \to \mathbb{R}^3$ via $dh_0^\intercal dh_0 = \rho_0 \mathbb{L}_0^{-1}$, namely

$$h_0(\varphi,\theta) = \begin{pmatrix} \sin \pi\varphi \sin 2\pi\theta \\ \sin \pi\varphi \cos 2\pi\theta \\ \int_0^\varphi \sqrt{\frac{\rho_0(t)^2}{4\pi^2 \sin^2 \pi t} - 4\pi^2 \cos^2 \pi t} \,\mathrm{d}t \end{pmatrix}. \tag{6.8}$$

The submanifold $N_0 := h_0(M_0)$ of $\mathbb{R}^3$, shown in Figure 6.4, is the Mosco-limit of the sequence (M_ε^ω) for a.e. $\omega \in \Omega$. As in the examples above we can pass to the reference manifold $M_0 = (0,1) \times [0,1)$ and find that the excluded poles of the manifolds M_ε coincide with the (excluded) poles of the manifold N_0.

A sphere with random radial perturbations oscillating with the longitude

With the same probability space Ω and reference manifold M_0 as in the previous example, we define the family $M_\varepsilon^\omega := h_\varepsilon^\omega(M_0)$ of submanifolds of $\mathbb{R}^3$ via $h_\varepsilon^\omega \colon M_0 \to \mathbb{R}^3$,

$$h_\varepsilon^\omega(\varphi, \theta) = \big(1 + \varepsilon\omega(\tfrac{\theta}{\varepsilon})\big) \begin{pmatrix} \sin \pi\varphi \sin 2\pi\theta \\ \sin \pi\varphi \cos 2\pi\theta \\ \cos \pi\varphi \end{pmatrix} \tag{6.9}$$

for $\omega \in \Omega$, $\varepsilon \in \{\frac{1}{k}; k \in \mathbb{N}\}$. These manifolds are illustrated in Figure 6.5 with ψ as in (4.7) and a χ^2-distribution with 3 degrees of freedom for some values of ε.

$\varepsilon = \frac{1}{4}$ $\qquad$ $\varepsilon = \frac{1}{8}$ $\qquad$ $\varepsilon = \frac{1}{16}$ $\qquad$ $\xrightarrow{\varepsilon \searrow 0}$

Figure 6.5.: A family of spheres with random radial perturbations oscillating with the longitude. The three pictures on the left show M_ε defined by (6.9) with ψ as in (4.7) and decreasing values of ε. The picture on the right shows the limiting surface N_0 defined via (6.10).

We find the density

$$\rho_\varepsilon^\omega(\varphi) = \pi\sqrt{\omega'(\tfrac{\theta}{\varepsilon})^2 + 4\pi^2 \sin^2 \pi\varphi}$$

and the coefficient field

$$\mathbb{L}^\omega(\varphi, y) = \begin{pmatrix} \frac{1}{\pi}\sqrt{\omega'(y)^2 + 4\pi^2 \sin^2 \pi\varphi} & 0 \\ 0 & \Big(\frac{1}{\pi}\sqrt{\omega'(y)^2 + 4\pi^2 \sin^2 \pi\varphi}\Big)^{-1} \end{pmatrix},$$

which yields the homogenized coefficient field

$$\mathbb{L}_0(\varphi) = \begin{pmatrix} \frac{\rho}{\pi} & 0 \\ 0 & \frac{\pi}{\rho} \end{pmatrix} \qquad \text{with } \rho = \int_0^\infty \int_0^1 \sqrt{a^2 f'(y)^2 + 4\pi^2 \sin^2 \pi\varphi}\, \mathrm{d}y\, \mathbb{P}(\mathrm{d}a),$$

where $\mathbb{P}$ denotes the probability measure on $[0,\infty)$ associated to the distribution of the amplitudes. Again, the volume forms ρ_ε^ω are unbounded in $L^\infty(M_0)$ and thus cannot weakly-$*$ converge, so there is no natural choice for ρ_0 and we choose

$$\rho_0(\varphi) := \pi \int_0^\infty \int_0^1 \sqrt{a^2 f'(y)^2 + 4\pi^2 \sin^2 \pi\varphi}\, \mathrm{d}y\, \mathbb{P}(\mathrm{d}a),$$

such that the limiting metric can be written in the same form as in the periodic case (cf. Section 4.2):

$$\hat{g}_0 = \rho_0 \mathbb{L}_0^{-1} = \begin{pmatrix} \pi^2 & 0 \\ 0 & \frac{\rho_0^2}{\pi^2} \end{pmatrix}.$$

An isometric embedding $h_0 \colon M_0 \to \mathbb{R}^3$ can be found via $dh_0^\intercal dh_0 = \rho_0 \mathbb{L}_0^{-1}$, namely

$$h_0(\varphi,\theta) = \begin{pmatrix} \sin \pi\varphi \sin 2\pi\theta \\ \sin \pi\varphi \cos 2\pi\theta \\ \int_0^\varphi \sqrt{\pi^2 - \frac{\rho_0'(t)^2}{4\pi^2}}\, \mathrm{d}t \end{pmatrix}. \tag{6.10}$$

The submanifold $N_0 := h_0(M_0)$ of $\mathbb{R}^3$, pictured in Figure 6.5, is the Mosco-limit of the family (M_ε^ω) for a.e. $\omega \in \Omega$. As in the examples above we can again pass to the reference manifold $M_0 = (0,1) \times [0,1)$ and find that the excluded poles of the manifolds M_ε coincide with the two circles forming the boundary of the manifold N_0.

6.2.3. Manifolds Locally Collapsing to Singular Points

In the examples above the degeneration of the geometry was achieved by blowing up the volume of the manifolds and therewith $|dh_\varepsilon^\omega \xi| \to \infty$. We finally want to present an example of manifolds with uniformly bounded volume, where the degeneration of $\mathbb{L}$ comes from collapsing to a singularity and therewith $|dh_\varepsilon^\omega \xi| \to 0$, still featuring spectral convergence.

To begin with we fix an even (in the sense of the symmetry $f(-t) = f(t)$) smooth function $f \colon \mathbb{R} \to [0,\infty)$, monotone decreasing on $[0,\infty)$, and with support $\operatorname{supp} f \subseteq [-\frac{1}{2}, \frac{1}{2}]$. A standard example of f is the mollifier

$$f(t) = \begin{cases} \exp(1 - \frac{1}{1-4t^2}), & t \in (-\frac{1}{2}, \frac{1}{2}), \\ 0, & \text{otherwise}. \end{cases} \tag{6.11}$$

Now for $\delta \in (0,1]$ we distort the graph of f in the way illustrated in Figure 6.6. This is achieved by re-parametrizing the domain of f with

$$\psi^\delta(t) = \begin{cases} t - (1-\sqrt{\delta})t \exp(2 - \frac{2}{1-(4|t|-1)^2}), & |t| \in (0, \frac{1}{2}), \\ t, & \text{otherwise} \end{cases}$$

for $\delta \in (0,1]$, which yields the smooth curve

$$\mathbb{R} \ni t \mapsto \begin{pmatrix} \psi^\delta(t) \\ f(t) \end{pmatrix} \in \mathbb{R}^2. \tag{6.12}$$

Due to the symmetry of f this curve can be utilized as the generatrix of a surface of revolution $M^\delta := h^\delta(\mathbb{R}^2)$, given by

$$h^\delta(x_1,x_2) = \begin{pmatrix} \psi^\delta(\sqrt{x_1^2+x_2^2})\frac{x_1}{\sqrt{x_1^2+x_2^2}} \\ \psi^\delta(\sqrt{x_1^2+x_2^2})\frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ f(\sqrt{x_1^2+x_2^2}) \end{pmatrix}, \tag{6.13}$$

as displayed in Figure 6.6. For convenience we denote by $h^\delta_\#\colon \mathbb{R}^2 \to \mathbb{R}^3$ the periodic continuation of h^δ given by

$$h^\delta_\#(x_1,x_2) = h^\delta(x_1-[x_1], x_2-[x_2]) + \begin{pmatrix} [x_1] \\ [x_2] \\ 0 \end{pmatrix},$$

where $[x]$ denotes the closest integer to x, i.e. $x - \frac{1}{2} \leq [x] < x + \frac{1}{2}$.

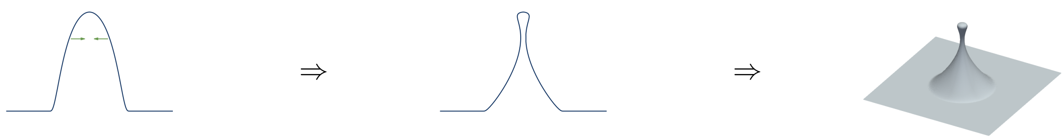


Figure 6.6.: Illustration of the process yielding the deformed surface of revolution. The picture to the left shows the function f defined in (6.11) and the intended deformation, the picture in the middle the deformed graph given by (6.12) for $\delta = \frac{1}{64}$, the picture to the right the corresponding surface of revolution M^δ defined via (6.13).

The manifolds we want to consider are obtained by repeating the surface M^δ as on a checkerboard, but with the parameter δ being randomly chosen on each tile. To that end we define the probability space

$$\Omega := \big\{\omega\colon \mathbb{R}^2 \to \mathbb{R}^3; \omega(y) = h^{a([x_0+y])}(x_0+y) \in \mathbb{R}^3 \text{ for some } x_0 \in [-\tfrac{1}{2},\tfrac{1}{2})^2, a \in (0,1)^{\mathbb{Z}\times\mathbb{Z}}\big\},$$

on which we define a probability measure by letting the offset $x_0 \in [-\frac{1}{2},\frac{1}{2})^2$ be uniformly distributed and the deformations $a(k) \in (0,1)$ for $k \in \mathbb{Z}^2$ be independently uniformly distributed, too. Together with the Borel-σ-algebra and the group action $\tau_y\omega := \omega(\cdot + y)$ for $y \in \mathbb{R}^2$, $\omega \in \Omega$, we can regard Ω as a stationary, ergodic probability space. We now simply set $M^\omega_\varepsilon := h^\omega_\varepsilon(M_0)$ for $\varepsilon > 0$, $\omega = (x_0,\delta) \in \Omega$, with

$$h^\omega_\varepsilon(x) := \varepsilon\omega(\tfrac{x}{\varepsilon}). \tag{6.14}$$

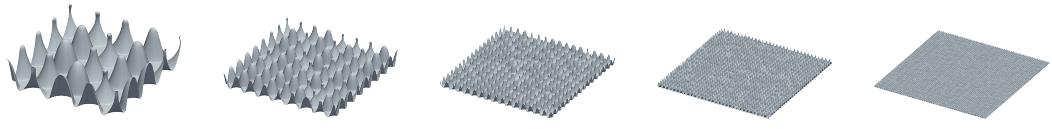


Figure 6.7.: A periodically repeated surface with random deformation parameter. The pictures show M_ε defined by (6.14) for decreasing values of ε. The spectrum of the Laplace-Beltrami operator on M_ε converges to the spectrum of the Laplace-Beltrami operator on a flat square.

See Figure 6.7 for an illustration of the defined manifolds.

Obviously the coefficient field

$$\sqrt{\det dh_\varepsilon^{\omega\intercal} dh_\varepsilon^\omega}\,(dh_\varepsilon^{\omega\intercal} dh_\varepsilon^\omega)^{-1} = \sqrt{\det d\omega(\tfrac{x}{\varepsilon})^\intercal d\omega(\tfrac{x}{\varepsilon})}\,(d\omega(\tfrac{x}{\varepsilon})^\intercal d\omega(\tfrac{x}{\varepsilon}))^{-1} = \mathbb{L}^\omega(\tfrac{x}{\varepsilon})$$

is of the desired form, and is spatially homogeneous so the continuity condition (ii) is trivially satisfied. To study the eigenvalues of $\mathbb{L}$ we note that $dh_\varepsilon^\omega(x) = d\omega(\frac{x}{\varepsilon}) = dh^\delta(y)$ for some $y \in [-\frac{1}{2}, \frac{1}{2}]^2$ and $0 < \delta < 1$, thus it is sufficient to understand the eigenvalues of

$$\mathbb{L}^\delta := \rho^\delta\,(dh^{\delta\intercal} dh^\delta)^{-1}, \qquad \text{with} \qquad \rho^\delta := \sqrt{\det dh^{\delta\intercal} dh^\delta}.$$

Due to the rotational symmetric structure of h^δ we can even further restrict to examining the eigenvalues of $\mathbb{L}^\delta$ on $[0, \frac{1}{2}] \times \{0\}$. Thus we calculate

$$dh^\delta(x_1, 0) = \begin{pmatrix} \psi'(x_1) & 0 \\ \psi'(x_1) & \frac{\psi(x_1)}{x_1} \\ f'(x_1) & 0 \end{pmatrix}$$

and

$$\rho^\delta(x_1, 0) = \tfrac{\psi(x_1)}{x_1}\sqrt{\psi'(x_1)^2 + f'(x_1)^2},$$

where we conveniently dropped the index δ of ψ. Note that for $x_1 = \frac{1}{4}$ we find $\frac{\psi(x_1)}{x_1} = \sqrt{\delta}$ and thus there is $\xi \in \mathbb{R}^2$ with $|dh^\delta(x_1, 0)\xi| \to 0$ as $\delta \to 0$, and we conclude that the manifolds M_ε^ω are not uniformly bi-Lipschitz diffeomorphic. (One can easily show with the Intermediate Value Theorem, that every parametrization yields degeneration of the diffeomorphisms.)

The eigenvalues of the coefficient field

$$\mathbb{L}^\delta(x_1, 0) = \tfrac{1}{\rho^\delta}\begin{pmatrix} \frac{\psi(x_1)^2}{x_1^2} & -\psi'(x_1)\frac{\psi(x_1)}{x_1} \\ -\psi'(x_1)\frac{\psi(x_1)}{x_1} & 2\psi'(x_1)^2 + f'(x_1)^2 \end{pmatrix}$$

can be calculated explicitly and are given by λ^δ and $(\lambda^\delta)^{-1}$ with

$$\lambda^\delta(x_1, 0) = \frac{\frac{\psi^2}{x_1^2} + 2\psi'^2 + f'^2 + \sqrt{4\psi'^2 \frac{\psi^2}{x_1^2} + \left(\frac{\psi^2}{x_1^2} - 2\psi'^2 - f'^2\right)^2}}{2\rho^\delta}.$$

Taking into account that by definition f' as well as ψ' are uniformly bounded and $\frac{\psi}{x_1}$ takes its only minimum at $x_1 = \frac{1}{4}$, which is $4\psi(\frac{1}{4}) = \sqrt{\delta}$, one can show that

$$\mathbb{E}[\lambda_{\max}(x_1, 0)] = \mathbb{E}[\lambda_{\min}(x_1, 0)^{-1}] \lesssim \int_0^1 \lambda^\delta(\tfrac{1}{4}, 0)\,\mathrm{d}\delta \lesssim \int_0^1 \sqrt{\delta}\,\mathrm{d}\delta,$$

where $\lesssim$ means $\leq$ up to a constant independent on x_1. Thus we are indeed in the situation where Corollary 6.1.6 applies and yields a coefficient field $\mathbb{L}_0$, which is spatially homogeneous, too. Moreover, we have $\det \mathbb{L}_0 = 1$ and for symmetry reasons we can conclude

$$\mathbb{L}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since ρ_ε^δ is uniformly bounded in $L^2(M_0)$ and by ergodicity for every cube $Q \subseteq \mathbb{R}^2$ we have

$$\fint_Q \rho_\varepsilon^\omega(x)\,\mathrm{d}x \to \int_0^1 \int_{M_0} \rho^\delta(x)\,\mathrm{d}x\,\mathrm{d}\delta =: \rho_0 < \infty \qquad \text{a.s.},$$

we get $\rho_\varepsilon^\omega \overset{*}{\rightharpoonup} \rho_0$ weakly in $L^\infty(M_0)$ a.s. Thus we can conclude that (M_ε^ω) for a.e. $\omega \in \Omega$ spectral converges to $(M_0, \hat{g}_0, \hat{\mu}_0)$ with $\hat{g}_0 = \rho_0 \mathbb{L}^{-1} = \rho_0 \mathbb{1}$ and $\mathrm{d}\hat{\mu}_0 = \rho_0 \mathrm{d}x$. This manifold can be isometrically embedded as a flat square with side length ρ_0.

6.3. Proofs

Proof of Proposition 6.1.3. For the sake of readability we will drop the index $\omega \in \Omega$ in the following where it is clear from the context.

Since the manifolds (M_ε) are rapidly oscillating, they admit a common reference atlas, so there is a countable tessellation $\mathcal{A}$ of the reference manifold M_0 consisting of reference cells, such that for every $A \in \mathcal{A}$ the following holds:

- There is a unique chart (U, Ψ) of M_0 such that $A \subseteq U$, and $\overline{A} := \Psi^{-1}(A)$ is a bounded Lipschitz domain in $\mathbb{R}^n$.

- For every $\varepsilon > 0$ there is a unique chart $(U_\varepsilon, \Psi_\varepsilon)$ of M_ε with $A_\varepsilon := \Psi_\varepsilon^{-1}(\overline{A}) \subseteq U_\varepsilon$.

- For every $\varepsilon > 0$ we have $A_\varepsilon = h_\varepsilon(A)$.

In the following we will use the notation $\overline{u}$ for the function u being pulled back to $\overline{A}$, by which we mean $\overline{u} := u \circ \Psi^{-1}$ for functions $u\colon A \to \mathbb{R}$, and $\overline{u} := u \circ \Psi_\varepsilon^{-1}$ for functions $u\colon A_\varepsilon \to \mathbb{R}$. This abuse of notation is justifiable as it will always be clear from the context which chart is used.

Step 1: Mosco-convergence on reference cells.
For $A \in \mathcal{A}$ we define the energy functional $\mathcal{E}_\varepsilon(\cdot, A_\varepsilon)\colon L^2(A_\varepsilon) \to \mathbb{R} \cup \{+\infty\}$ by

$$\mathcal{E}_\varepsilon(u, A_\varepsilon) := \begin{cases} \int_{A_\varepsilon} |\nabla_{g_\varepsilon} u|^2_{g_\varepsilon} \,\mathrm{d}\mu_\varepsilon, & \text{if } u \in H^1(A_\varepsilon), \\ +\infty, & \text{otherwise.} \end{cases}$$

Similar to the beginning of the proof of Lemma 4.1.1, we find for $u \in H^1(A_\varepsilon)$ due to the rapidly oscillating structure of M_ε

$$\int_{A_\varepsilon} |\nabla_{g_\varepsilon} u|^2_{g_\varepsilon} \,\mathrm{d}\mu_\varepsilon = \int_{\overline{A}} \overline{g}_\varepsilon(d\Psi_\varepsilon^* \nabla \overline{u}, d\Psi_\varepsilon^* \nabla \overline{u})\, \rho_\varepsilon \,\mathrm{d}x = \int_{\overline{A}} \mathbb{L}(x, \tfrac{x}{\varepsilon}) \nabla \overline{u}(x) \cdot \nabla \overline{u}(x) \,\mathrm{d}x + o(\varepsilon),$$

which implies $\mathcal{E}_\varepsilon(u, A_\varepsilon) = \overline{\mathcal{E}}_\varepsilon(\overline{u}, \overline{A}) + o(\varepsilon)$ with the pulled back energy functional $\overline{\mathcal{E}}_\varepsilon(\cdot, \overline{A})\colon L^2(\overline{A}) \to \mathbb{R} \cup \{+\infty\}$

$$\overline{\mathcal{E}}_\varepsilon(\overline{u}, \overline{A}) := \begin{cases} \int_{\overline{A}} \mathbb{L}(x, \tfrac{x}{\varepsilon}) \nabla \overline{u}(x) \cdot \nabla \overline{u}(x) \,\mathrm{d}x, & \text{if } \overline{u} \in H^1(\overline{A}), \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that the error term $o(\varepsilon)$ does not interfere with Mosco-convergence. Since $\mathbb{L}$ is a symmetric, positive definite coefficient field, the assumptions (i) and (ii) put us into the situation of Theorem 5.2.2 with $p = 2$, $\alpha = 1$ and $\beta = \frac{n}{2}$. Hence, regarding Remark 5.2.3, the functionals $\overline{\mathcal{E}}_\varepsilon(\cdot, \overline{A})$ Mosco-converge a.s. to some deterministic integral functional $\overline{\mathcal{E}}_0(\cdot, \overline{A})\colon L^2(\overline{A}) \to \mathbb{R} \cup \{+\infty\}$ by

$$\overline{\mathcal{E}}_0(\overline{u}, \overline{A}) := \begin{cases} \int_{\overline{A}} W_{\mathrm{hom}}(x, \nabla \overline{u}(x)) \,\mathrm{d}x, & \text{if } \overline{u} \in H^1(\overline{A}), \\ +\infty, & \text{otherwise,} \end{cases}$$

for W_{hom} given by (5.10). As it is the potential of the Gamma-limit of quadratic integral functions, $W_{\mathrm{hom}}(x, \cdot)$ is a symmetric, positive definite quadratic form (see e.g. [DM93, Theorem 22.1]), and thus for every $x \in \overline{A}$ we can find a bilinear form a_x such that

$$W_{\mathrm{hom}}(x, \nabla \overline{u}(x)) = \rho_0(x) a_x(d\Psi^* \nabla \overline{u}(x), d\Psi^* \nabla \overline{u}(x)).$$

If we set $\hat{g}_0(p) := a_{\Psi(p)}$ for $p \in A$, the same calculations as above yield

$$\int_{\overline{A}} W_{\mathrm{hom}}(x, \nabla \overline{u}(x)) \,\mathrm{d}x = \int_A |\nabla_{\hat{g}_0} u|^2_{\hat{g}_0} \,\mathrm{d}\hat{\mu}_0,$$

and thus $\overline{\mathcal{E}}_0$ is the pull-back of the Dirichlet energy $\mathcal{E}_0(\cdot, A)\colon L^2(A, \hat{g}_0, \hat{\mu}_0) \to \mathbb{R} \cup \{+\infty\}$,

$$\mathcal{E}_0(u, A) := \begin{cases} \int_A |\nabla_{\hat{g}_0} u|^2_{\hat{g}_0} \,\mathrm{d}\hat{\mu}_0, & \text{if } u \in H^1(A, \hat{g}_0, \hat{\mu}_0), \\ +\infty, & \text{otherwise.} \end{cases}$$

In fact, the application of Theorem 5.2.2 above even guarantees for every function $u \in H^1(A, \hat{g}_0, \hat{\mu}_0)$ the existence of a recovery sequence (u_ε), $u_\varepsilon \in H^1(A_\varepsilon)$, with $\overline{u}_\varepsilon - \overline{u} \in H_0^1(\overline{A})$ such that $\overline{\mathcal{E}}_\varepsilon(\overline{u}_\varepsilon, A_\varepsilon) \to \overline{\mathcal{E}}_0(\overline{u}, A)$ a.s.

Step 2: Construction of the limiting metric and Mosco-convergence of the manifolds. Since the reference cells are a disjoint covering of the reference manifold M_0, we can define a metric $\hat{g}_0$ on the entire reference manifold M_0 by letting $\hat{g}_0|_A$ be the metric for the reference cell $A \in \mathcal{A}$ constructed above. Moreover, from the countability of the covering we can conclude that there is a set $\Omega_0 \subseteq \Omega$ of full measure, such that $(\overline{\mathcal{E}}_\varepsilon^\omega(\cdot, \overline{A}_\varepsilon))$ Mosco-converges to $\overline{\mathcal{E}}_0(\cdot, \overline{A})$ for every $A \in \mathcal{A}$ and $\omega \in \Omega_0$.

It remains to show Mosco-convergence of the entire manifolds. To that end we note that, since $\mathcal{A}$ is a countable disjoint covering of M_0, for every $\varepsilon > 0$ the collection $\{A_\varepsilon = h_\varepsilon(A); A \in \mathcal{A}\}$ is a (countable) disjoint covering of M_ε and we find for the Dirichlet energies on M_ε and $(M_0, \hat{g}_0, \hat{\mu}_0)$

$$\mathcal{E}_\varepsilon(u, M_\varepsilon) = \sum_{A \in \mathcal{A}} \mathcal{E}_\varepsilon(u|_{A_\varepsilon}, A_\varepsilon) \qquad \text{and} \qquad \mathcal{E}_0(u, M_0) = \sum_{A \in \mathcal{A}} \mathcal{E}_0(u|_A, A).$$

We immediately see from the Mosco-convergence on the reference cells, that for every $\overline{u}_\varepsilon \rightharpoonup \overline{u}$ weakly in $L^2(M_0, g_0, \mu_0)$

$$\liminf_{\varepsilon \searrow 0} \overline{\mathcal{E}}_\varepsilon^\omega(\overline{u}_\varepsilon, M_0) \geq \overline{\mathcal{E}}_0(\overline{u}, M_0)$$

for all $\omega \in \Omega_0$. For the recovery sequence we fix $u \in H^1(M_0, \hat{g}_0, \hat{\mu}_0)$. Then for every $A \in \mathcal{A}$ and every $\omega \in \Omega_0$ there is a recovery sequence $(u_{A,\varepsilon})$, $u_{A,\varepsilon} \in H^1(A_\varepsilon^\omega)$, with $\overline{u}_{A,\varepsilon} - \overline{u}_0|_{\overline{A}} \in H_0^1(\overline{A})$ such that $\overline{\mathcal{E}}_\varepsilon^\omega(\overline{u}_{A,\varepsilon}, \overline{A}_\varepsilon^\omega) \to \overline{\mathcal{E}}_0(\overline{u}, \overline{A})$. Thus we can define functions

$$u_\varepsilon := \sum_{A \in \mathcal{A}} u_{A,\varepsilon} \mathbb{1}_{A_\varepsilon^\omega},$$

which are in $H^1(M_\varepsilon^\omega)$ due to the boundary conditions of $u_{A,\varepsilon}$. Now by summation follows

$$\overline{\mathcal{E}}_\varepsilon^\omega(\overline{u}_\varepsilon, M_0) \to \overline{\mathcal{E}}_0(\overline{u}, M_0)$$

for all $\omega \in \Omega_0$ and the proof is complete. □

Proof of Proposition 6.1.5. The following arguments hold for every $\omega \in \Omega$ for which the manifolds M_ε Mosco-converge to $(M_0, \hat{g}_0, \hat{\mu}_0)$, so we can tacitly drop the index ω.

If we denote by $\mathcal{E}_\varepsilon$ the Dirichlet energy on M_ε, i.e.

$$\mathcal{E}_\varepsilon(u) = \begin{cases} \int_{M_\varepsilon} g_\varepsilon(\nabla_\varepsilon u, \nabla_\varepsilon u), \mathrm{d}\mu_\varepsilon, & u \in H_0^1(M_\varepsilon), \\ +\infty, & \text{otherwise}, \end{cases}$$

and by $\overline{\mathcal{E}}_\varepsilon$ the pulled back energy on M_0, i.e. $\overline{\mathcal{E}}_\varepsilon(\overline{u}) = \mathcal{E}_\varepsilon(u)$ with $\overline{u} = u \circ h_\varepsilon$, we find that

$$\begin{aligned}\lambda_{\varepsilon,1} &= \inf\left\{\mathcal{E}_\varepsilon(u); u \in H_0^1(M_\varepsilon), \|u\|_{L^2(M_\varepsilon)} = 1\right\}\\ &= \inf\left\{\overline{\mathcal{E}}_\varepsilon(\overline{u}); \overline{u} \in H_0^1(M_0), \int_{M_0} |\overline{u}|^2 \rho_\varepsilon \,\mathrm{d}\mu_0 = 1\right\},\end{aligned}$$

and a similar representation holds for $\lambda_{0,1}$. In other words, if we define

$$\begin{aligned}H_\varepsilon &:= \left\{u \in H_0^1(M_0); \int_{M_0} |u|^2 \rho_\varepsilon \,\mathrm{d}\mu_0 = 1\right\} \qquad \text{and}\\ H_0 &:= \left\{u \in H_0^1(M_0); \int_{M_0} |u|^2 \rho_0 \,\mathrm{d}\mu_0 = 1\right\},\end{aligned}$$

we find that for every normalized eigenfunction $u_{\varepsilon,1} \in H_0^1(M_\varepsilon)$ of $-\Delta_{g_\varepsilon,\mu_\varepsilon}$ to the first eigenvalue $\lambda_{\varepsilon,1}$ the function $\overline{u}_{\varepsilon,1} = u_{\varepsilon,1} \circ h_\varepsilon$ is a minimizer of the functional $\overline{\mathcal{E}}_\varepsilon$ in H_ε. It is therefore natural to consider Mosco-convergence of $\overline{\mathcal{E}}_\varepsilon$ on H_ε in some sense, which we will make precise in step 1 below. Since the notion of Mosco-convergence on varying spaces is non-standard, we provide in step 2 the arguments for the convergence of minimizers in this context, which we will use in step 3, together with the compactness result Proposition 5.2.1, to deduce the convergence of the eigenpairs as claimed in the definition of spectral convergence.

Step 1: Mosco-convergence of $(\overline{\mathcal{E}}_\varepsilon)$ on the weighted spaces.
From Proposition 6.1.3 we know that $(\overline{\mathcal{E}}_\varepsilon)$ Mosco-converges on (M_0, g_0, μ_0) to $\overline{\mathcal{E}}_0$ w.r.t. $L^2(M_0, g_0, \mu_0)$. This implies immediately the following lower bound condition: For $u_\varepsilon \in H_\varepsilon$, $u_0 \in H_0$ with $u_\varepsilon \rightharpoonup u_0$ weakly in $L^2(M_0, g_0, \mu_0)$, we have

$$\liminf_{\varepsilon \searrow 0} \overline{\mathcal{E}}_\varepsilon(u_\varepsilon) \geq \overline{\mathcal{E}}_0(u_0).$$

For the recovery sequence we fix $u_0 \in H_0$ and a sequence (v_ε) in $H_0^1(M_0, g_0, \mu_0)$ with $v_\varepsilon \to u_0$ strongly in $L^2(M_0, g_0, \mu_0)$ and $\overline{\mathcal{E}}_\varepsilon(v_\varepsilon) \to \overline{\mathcal{E}}_0(u_0)$. If we set

$$u_\varepsilon := \frac{v_\varepsilon}{\int_{M_0} |v_\varepsilon|^2 \rho_\varepsilon \,\mathrm{d}\mu_0} \in H_\varepsilon,$$

we find

$$u_\varepsilon \to u_0 \quad \text{strongly in } L^2(M_0, g_0, \mu_0) \qquad \text{and} \qquad \overline{\mathcal{E}}_\varepsilon(u_\varepsilon) \to \overline{\mathcal{E}}_0(u_0),$$

since $\rho_\varepsilon \overset{*}{\rightharpoonup} \rho_0$ and therefore $\int_{M_0} |v_\varepsilon|^2 \rho_\varepsilon \,\mathrm{d}\mu_0 \to \int_{M_0} |u_0|^2 \rho_0 \,\mathrm{d}\mu_0 = 1$.

Step 2: Convergence of minimizers.
Let (u_ε) be a sequence of minimizers of $\overline{\mathcal{E}}_\varepsilon$ in H_ε with $u_\varepsilon \rightharpoonup u_0$ weakly in $L^2(M_0, g_0, \mu_0)$. We show that $\overline{\mathcal{E}}_\varepsilon(u_\varepsilon) \to \overline{\mathcal{E}}_0(u_0)$ and that $\overline{\mathcal{E}}_0(u_0) \leq \overline{\mathcal{E}}_0(v)$ for every $v \in H_0$. (Note that we do not claim $u_0 \in H_0$, since we do not assume strong convergence in $L^2(M_0)$.)

Indeed, the arguments are similar to the standard case. We first find with the Mosco-convergence in step 1, that

$$\liminf_{\varepsilon \searrow 0} \overline{\mathcal{E}}_\varepsilon(u_\varepsilon) \geq \overline{\mathcal{E}}_0(u_0).$$

On the other hand, there is a recovery sequence (v_ε) of u_0 with $v_\varepsilon \in H_\varepsilon$ and $\overline{\mathcal{E}}_\varepsilon(v_\varepsilon) \to \overline{\mathcal{E}}_0(u_0)$, and since u_ε is a minimizer, we can estimate

$$\limsup_{\varepsilon \searrow 0} \overline{\mathcal{E}}_\varepsilon(u_\varepsilon) \leq \lim_{\varepsilon \searrow 0} \overline{\mathcal{E}}_\varepsilon(v_\varepsilon) = \overline{\mathcal{E}}_0(u_0).$$

Together we can conclude $\overline{\mathcal{E}}_\varepsilon(u_\varepsilon) \to \overline{\mathcal{E}}_0(u_0)$. It remains to show, that $\overline{\mathcal{E}}_0(u_0)$ is smaller than $\overline{\mathcal{E}}_0$ on H_0. Therefore we assume the existence of $v_0 \in H_0$ with $\overline{\mathcal{E}}_0(v_0) < \overline{\mathcal{E}}_0(u_0)$. For v_0 we choose a recovery sequence (v_ε) with $v_\varepsilon \in H_\varepsilon$ as in step 1, and note that since u_ε is a minimizer

$$\overline{\mathcal{E}}_0(u_0) = \lim_{\varepsilon \searrow 0} \overline{\mathcal{E}}_\varepsilon(u_\varepsilon) \leq \lim_{\varepsilon \searrow 0} \overline{\mathcal{E}}_\varepsilon(v_\varepsilon) = \overline{\mathcal{E}}_0(v_0),$$

which contradicts the definition of v_0.

Step 3: Convergence of eigenpairs.
We first claim that the family of first eigenpairs $(\lambda_{\varepsilon,1}, u_{\varepsilon,1})$ as in Definition 1.3.4 satisfies $\lambda_{\varepsilon,1} \to \lambda_{0,1}$ and that there is a subsequence such that $u_{\varepsilon,1} \to v_1$ strongly in $L^2((M_\varepsilon, \mu_\varepsilon) \to (M_0, \hat{\mu}_0))$ for some $v_1 \in H_0^1(M_0, \hat{g}_0, \hat{\mu}_0)$ being an eigenfunction of $-\Delta_{\hat{g}_0, \hat{\mu}_0}$ to the eigenvalue $\lambda_{0,1}$. We therefore note that since $\int_{M_0} |\overline{u}_{\varepsilon,1}|^2 \rho_\varepsilon \, \mathrm{d}\mu_0 = 1$ and ρ_ε uniformly bounded in $L^\infty(M_0)$, the sequence $(\overline{u}_{\varepsilon,1})$ is bounded in $L^2(M_0, g_0, \mu_0)$ and thus provides a (not relabeled) subsequence with $\overline{u}_{\varepsilon,1} \rightharpoonup v_0$ weakly in $L^2(M_0, g_0, \mu_0)$. Since, as mentioned in the beginning of the proof, $\overline{u}_{\varepsilon,1}$ is a minimizer of $\overline{\mathcal{E}}_\varepsilon$ in H_ε, this implies with step 2

$$\lambda_{\varepsilon,1} = \overline{\mathcal{E}}_\varepsilon(\overline{u}_\varepsilon) \to \overline{\mathcal{E}}_\varepsilon(v_0) \leq \lambda_{0,1}.$$

Therewith, $(\overline{u}_{\varepsilon,1})$ is a sequence with bounded energy, and the compactness result Proposition 5.2.1 provides $\overline{u}_{\varepsilon,1} \to v_0$ strongly in $L^2(M_0, g_0, \mu_0)$. Due to the weak-$*$ convergence of ρ_ε, this implies $u_{\varepsilon,1} \to v_1$ strongly in $L^2((M_\varepsilon, \mu_\varepsilon) \to (M_0, \hat{\mu}_0))$. Moreover, we have $v_1 \in H_0$, and can conclude

$$\lambda_{\varepsilon,1} = \overline{\mathcal{E}}_\varepsilon(\overline{u}_\varepsilon) \to \overline{\mathcal{E}}_\varepsilon(v_1) = \lambda_{0,1},$$

which means that v_1 is an eigenfunction of $-\Delta_{\hat{g}_0, \hat{\mu}_0}$ to the eigenvalue $\lambda_{0,1}$. Note that the convergence $\lambda_{\varepsilon,1} \to \lambda_{0,1}$ is independent of the subsequence and therefore holds for the entire sequence.

We now note that this result can be inductively extended to eigenpairs $(\lambda_{\varepsilon,k}, u_{\varepsilon,k})$ with $k \geq 1$ with the Rayleigh-Ritz method: Assume that for $j = 1, \ldots, k$ we have already shown that there is a (not relabeled) subsequence with $\lambda_{\varepsilon,j} \to \lambda_{0,j}$ and $u_{\varepsilon,j} \to v_j$ strongly in $L^2((M_\varepsilon, \mu_\varepsilon) \to (M_0, \hat{\mu}_0))$. Then we define the spaces

$$H_{\varepsilon,k+1} := \Big\{ u \in H_\varepsilon; \int_{M_0} \overline{u} u_{\varepsilon,j} \rho_\varepsilon \, \mathrm{d}\mu_0 = 0 \text{ for } j = 1, \ldots, k \Big\} \qquad \text{and}$$
$$H_{0,k+1} := \Big\{ u \in H_0; \int_{M_0} \overline{u} v_j \rho_0 \, \mathrm{d}\mu_0 = 0 \text{ for } j = 1, \ldots, k \Big\},$$

and with the same arguments as above we can find a (not relabeled) subsequence with $\lambda_{\varepsilon,k+1} \to \lambda_{0,k+1}$ and $u_{\varepsilon,k+1} \to v_{k+1}$ strongly in $L^2((M_\varepsilon, \mu_\varepsilon) \to (M_0, \hat{\mu}_0))$. Here we used that for functions $u_\varepsilon, v_\varepsilon \in H_\varepsilon$ with $u_\varepsilon \to u_0$ and $v_\varepsilon \to v_0$ strongly in $L^2(M_0, g_0, \mu_0)$ from the orthogonality $u_\varepsilon \perp v_\varepsilon$ in H_ε follows the orthogonality $u_0 \perp v_0$ in H_0, due to the weak-$*$ convergence of ρ_ε.

To conclude the proof we let $s \geq 1$ be the multiplicity of $\lambda_{0,1}$. Then we can find a subsequence such that for $k = 1, \ldots, s$ we have $u_{\varepsilon,k} \to v_k$ strongly in $L^2((M_\varepsilon, \mu_\varepsilon) \to (M_0, \hat{\mu}_0))$ for some normalized eigenfunction v_k of $-\Delta_{\hat{g}_0, \hat{\mu}_0}$ to the eigenvalue $\lambda_{0,1}$. Note that by the argument above we can assume that $v_k \perp v_j$ in $L^2(M_0, \hat{g}_0, \hat{\mu}_0)$ for all $1 \leq j < k \leq s$. That means that $v_1, \ldots, v_s$ span the eigenspace associated with $\lambda_{0,1}$. So for every normalized eigenfunction $u_{\varepsilon,k}$ in this eigenspace (i.e. $k = 1, \ldots, s$), we find coefficients $\alpha_1, \ldots, \alpha_s$ such that $u_{\varepsilon,k} = \sum_{j=1}^s \alpha_j v_k$. But this implies that $u_{\varepsilon,k}$ is the strong L^2-limit of the linear combination $\sum_{j=1}^s \alpha_j u_{\varepsilon,j}$, and since this construction can be done with every subsequence (with different coefficients), the statement holds for the entire sequence. As above we can inductively step up to higher orders using the Rayleigh-Ritz method. □

Summary and Discussion

The intention of this thesis is to deepen the understanding of the asymptotic behavior of certain classes of Riemannian manifolds by appealing to techniques from the theory of homogenization. Especially we concentrate on Mosco- and spectral convergence w.r.t. L^2. Even though there are lots of other notions of convergence for manifolds, Mosco- and spectral convergence focus on the intrinsic geometry of the manifolds and might therefore be of particular interest in applications for instance in material science, as it models properties like heat flow on surfaces.

The main strategy of our studies is to reduce the problem of varying manifolds to a problem of varying geometries on one reference manifold. This is achieved via bi-Lipschitz diffeomorphisms between the manifolds. In practice these diffeomorphisms can be interpreted as deformations of a reference manifold such as perturbations of a surface from the equilibrium. Such situations occur for example in cellular and molecular biology, where diffusion processes on surfaces are studied, see e.g. [AG82; JII87; Sba+06; NRJ07].

We consider two different approaches. On the one hand we assume the Lipschitz constants of the diffeomorphisms to be uniformly bounded and established Mosco- and spectral convergence of the manifolds along a subsequence. These result is based on a H-compactness theorem for uniformly elliptic coefficient fields on a Riemannian manifold. Besides we established this compactness statement for the utilization in the studies of sequences of manifolds, it is of deep interest on its own. One consequence of our H-compactness theorem is Mosco- and spectral convergence (w.r.t. L^2) of the elliptic operators associated with the coefficient fields, which might have fruitful applications to the studies of partial differential equations and their evolution on manifolds. In the present form, our result implies only spectral convergence on manifolds with a strictly positive Dirichlet spectrum, but we demonstrate on the torus how it might be extended, and there might be room for more generalization.

Since the H-compactness results do not give any information about the limit (besides ellipticity), we demonstrate how special micro structures of the manifolds resp. the coefficient fields yield explicit homogenization formulas on the example of periodically perturbed surfaces. This is a good way to deepen the understanding of the limiting process and there is much more to learn from it, for instance in combination with numerical methods. However, the strength of the theorem is to be free of any assumption on the structure.

Our second approach is inherently different, because it is directly based on the special structure of the manifolds. We consider sequences of randomly oscillating bi-Lipschitz diffeomorphic manifolds, whose Lipschitz constants are not uniformly, but statistically bounded, and establish Mosco- and, under the condition that the volume forms are weakly-$*$ convergent in L^∞, spectral convergence w.r.t. L^2. The assumption of weak-$*$ convergence is used in the proof to ensure the space H_ε to be compactly embedded into $L^2(M_0)$, so that the eigenfunctions u_ε strongly converge in $L^2(M_0)$ and therefore $\int |u_\varepsilon|^2 \rho_\varepsilon \, d\mu_0 \to \int |u_0|^2 \rho_0 \, d\mu_0$. It might be possible to weaken the assumption of weak-$*$ convergence by directly thinking about the embedding of the space H_ε into $L^2((M_0, \rho_\varepsilon d\mu_0) \to (M_0, \rho_0 d\mu_0))$.

We also mention a variant of our result with weaker assumptions only providing Mosco-convergence w.r.t. L^p for some $1 < p < 2$, and with our method we are not able to conclude spectral convergence. The discussions in [Fle18; FHS19] for the discrete case give rise to the conjecture, that there are examples where the eigenfunctions concentrate in singular points and therefore spectral convergence is indeed not possible. However, it might be worth to study the structural differences behind these effects in terms of volume, curvature or distortion.

The background of this approach is given by a Γ-convergence statement for integral functionals on $\mathbb{R}^n$, whose integrals satisfy non-standard growth conditions. While we use this result on the pulled back Dirichlet energies of the manifolds, whose potentials are strictly convex quadratic forms, our Γ-convergence theorem covers much more general potentials, even of vector valued functions, which opens a wide range of applications. To continue the research in this field, a natural question for future work would be the generalization to variable growth conditions (see e.g. [Jik97; CM10]), i.e.

$$\lambda_{\min}(x)\left(\tfrac{1}{C}|F|^{p(x)} - C\right) \leq W(x, F) \leq \lambda_{\max}(x) C(|F|^{p(x)} + 1).$$

This would have applications in the study of electrorheological or thermorheological fluids (see e.g. [Růž00; RR01]).

The Γ-convergence theorem comes together with a corresponding compact embedding of L^p into the space of functions with bounded energy, where the exponent p depends on the conditions on W and the dimension. An interesting question would be if the conditions on the exponent p could be improved by a trick found in [BCD16; Bri+17], where the dimension is reduced via considering a radially symmetric parametrization of the considered functions. If this idea could be adopted to our setting, such that the application of the Gagliardo-Nirenberg-Sobolev inequality in the proof of the compactness statement would happen actually on an $n-1$-dimensional surface, it would reduce the considered exponent in the moment bounds in Proposition 6.1.3 to $\frac{n-1}{2}$.

The process of convergence in the stochastic framework is that for almost every realization the sequence of manifolds converges to the same deterministic limit. An interesting topic would be to reformulate the results for a spatially inhomogeneous probability measure, i.e. on every point of the reference manifold we consider a probability measure, and the sampling happens simultaneously with the convergence process. This would include for instance an-isotropic Poisson point processes, like point processes with constant parameter on the embedded torus (where, roughly speaking, in coordinates the inner points have a lower density than the outer ones).

The possibility to treat vector valued functions with the Γ-convergence result gives rise to study tangential valued functions on manifolds, like gradient fields. In contrast, since the consideration of tangential valued functions yields non-linear differential operators, the methods of our H-compactness approach cannot be directly adopted and need to be adjusted. A good starting point for such studies would be to consider the Bochner-Laplace operator instead of the Laplace-Beltrami operator.

In summary, both approaches are different, but have beneficial applications. The H-compactness theorem does not need any assumptions on the structure of the manifolds resp. the coefficient fields, but does not provide any specific information about the limit. It is therefore more of theoretical interest. The Γ-convergence theorem is restricted to oscillating manifold resp. potentials, but includes an explicit formula to find the limit. This should be interesting for the studies of geometric effects or the use in applied science.

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List of Figures

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Dresden, den 9. Oktober 2020 Helmer Hoppe