Exercises for "Quantum Phase Transitions" Summer 24

Dr. L. Janssen **Exercise 2 (03.05.24)**

1. Tricritical point in an antiferromagnet (6 points)

An external magnetic field h applied to an antiferromagnet couples to the total magnetization m instead of the antiferromagnetic order parameter, the staggered magnetization n. Assume that the coupling between m and n is described phenomenologically by the Landau free energy density

$$f(n,m) = \frac{t}{2}n^2 + \frac{b}{4}n^4 + \frac{v}{2}m^2 + \frac{w}{2}n^2m^2 - hm, \tag{1}$$

where $t = (T - T_0)a$, and a, b, v, w are positive constants.

(a) Show that this model features a paramagnetic phase with magnetization m_0. Derive a temperature-independent relation $m = m(n^2)$ between the magnetization m and the staggered magnetization n in the antiferromagnetic phase, i.e., for $n^2 > 0$.

Hint: Equilibrium states are given by minima of $f(n,m)$.

(b) Consider the antiferromagnetic phase near the phase transition, i.e., for small values of n^2. Write $m = m_0 + \delta m$, expand $m(n^2)$ for small n^2, and derive a relation between δm and n^2.

(c) Show that the effective free energy density for the staggered magnetization $g(n) = f(n, m_0 + \delta m) - f(0, m_0)$ can be written as

$$g(n) = \frac{\bar{a}}{2}n^2 + \frac{\bar{b}}{4}n^4 + \frac{\bar{c}}{6}n^6 + \mathcal{O}(n^8), \tag{2}$$

with to-be-determined temperature- and field-dependent coefficients $\bar{a}$, $\bar{b}$, and $\bar{c}$.

(d) Show that the model (1) features a tricritical point at temperature T_t and field h_t, where

$$T_\mathrm{t} = T_0 - \frac{bv}{2aw}, \quad h_\mathrm{t}^2 = \frac{bv^3}{2w^2}. \tag{3}$$

Hint: Here we define a tricritical point as a point where first- and second-order transition lines meet. Depending on the sign of the coefficient $\bar{b}$ in $g(n)$, the transition between the paramagnetic and antiferromagnetic phases are first or second order, cf. Problem 1 on Exercise 1.

(e) Show that the second-order phase transition occurs for $h < h_\mathrm{t}$ at

$$T_\mathrm{c} = T_0 - \frac{wh^2}{av^2}, \tag{4}$$

and the first-order transition occurs for $h > h_\mathrm{t}$ at

$$T_\mathrm{c} = T_0 - \frac{3wh^2}{4av^2} - \frac{bv}{4aw} + \frac{b^2v^4}{16aw^3h^2}. \tag{5}$$

(f) Sketch the phase diagram in the (T, H) plane.

please turn over!

2. Static scaling hypothesis (4 points)

Consider the static scaling hypothesis for the free energy density

$$f_{\mathrm{s}}(t,h) = b^{-d} f_{\mathrm{s}}(b^{y_t} t, b^{y_h} h) \tag{6}$$

with scaling exponents y_t for the reduced temperature t and y_h for the external field h.

(a) Use the static caling hypothesis to derive the relation

$$\delta = \frac{d+2-\eta}{d-2+\eta} \tag{7}$$

between the critical-isotherm exponent δ and the anomalous dimension η.

Hint: Use the relation $y_t = 1/\nu$ and Fisher's law $\gamma = \nu(2-\eta)$ derived in class.

(b) In principle, critical exponents above and below a transition could differ from each other. Show for the example of the correlation-length exponents ν and ν' above and below the transition, respectively, that the static scaling hypothesis implies that they are equal, $\nu(T > T_{\mathrm{c}}) = \nu'(T < T_{\mathrm{c}})$.

Hint: For fixed $h \neq 0$, $f_{\mathrm{s}}(t,h)$ should be a smooth function of t, because the only singularity that we expect is at $t = h = 0$. Show that $f_{\mathrm{s}}(t,h)$ can be written in the form

$$f_{\mathrm{s}}(t,h) = h^{d/y_h} F_f^{\pm}\left(\frac{|t|}{h^{1/(\bar{\nu} y_h)}}\right), \tag{8}$$

and explain how the smoothness assumption mentioned above constrains the analytic form of the function $F_f^{\pm}(x)$.