Exercises for "Quantum Phase Transitions" **Summer 24**

Dr. L. Janssen **Exercise 7 (22.07.24)**

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1. Shift exponent in the quantum ϕ^4 theory (5 points)

The ϕ^4 field theory with the action

$$S = \int d^d x \, d\tau \left\{ \frac{1}{2} \left[c^2 (\nabla \phi_\alpha)^2 + (\partial_\tau \phi_\alpha)^2 + r_0 \phi_\alpha^2 \right] + \frac{u_0}{4!} (\phi_\alpha^2)^2 \right\}, \tag{1}$$

($\alpha = 1, 2, \ldots, N$), has a quantum phase transition at $T = 0$, $r_0 = r_\mathrm{c}$. The shift exponent ψ is defined via the temperature-dependent phase boundary

$$T_\mathrm{c} \sim (r_\mathrm{c} - r_0)^\psi, \tag{2}$$

where T_c is the critical temperature. To calculate T_c, note that the phase transition occurs when the renormalized temperature-dependent mass $r(T)$ of the order parameter vanishes. The upper critical dimension for the quantum phase transition is $d_c^+ = 4 - z = 3$.

(a) Below the upper critical dimension d_c^+, use a simple scaling argument to relate ψ to other critical exponents.

(b) For $d > d_\mathrm{c}^+$, the naive scaling analysis above becomes invalid. However, a perturbative calculation of $r(T)$ becomes feasible. To this end, calculate the self-energy of the ϕ propagator in bare perturbation theory to first order in u_0. The temperature dependence of $r(T)$ at $r_0 = r_\mathrm{c}$ allows to obtain ψ in this case.

(c) Apply the procedure of (b) to a situation with $z = 2$ where the bare propagator is $G_\phi^{-1} = i\omega_n - c^2 \vec{k}^2 - r_0$ (instead of $G_\phi^{-1} = -\omega_n^2 - c^2 \vec{k}^2 - r_0$).

2. Quantum critical point in the dilute Bose gas (5 points)

Consider the quantum critical point in the dilute Bose gas with the action

$$S = \int d^d x d\tau \left(\Phi^* \partial_\tau \Phi + v |\partial_\tau \Phi|^2 + |\nabla \Phi|^2 - \mu |\Phi|^2 + \lambda |\Phi|^4 \right), \tag{3}$$

in $d = 2 - \epsilon$ dimensions.

(a) What is the scaling dimension of v?

(b) Show that RG flow of the quartic selfinteraction λ is given by

$$\frac{d\lambda}{d \ln b} = \epsilon \lambda - \lambda^2 \tag{4}$$

with suitably rescaled dimensionless λ.

(c) Determine the critical exponents ν, η, and z to the leading order in ϵ for $d < 2$.