

Soluble fermionic quantum critical point in 2D

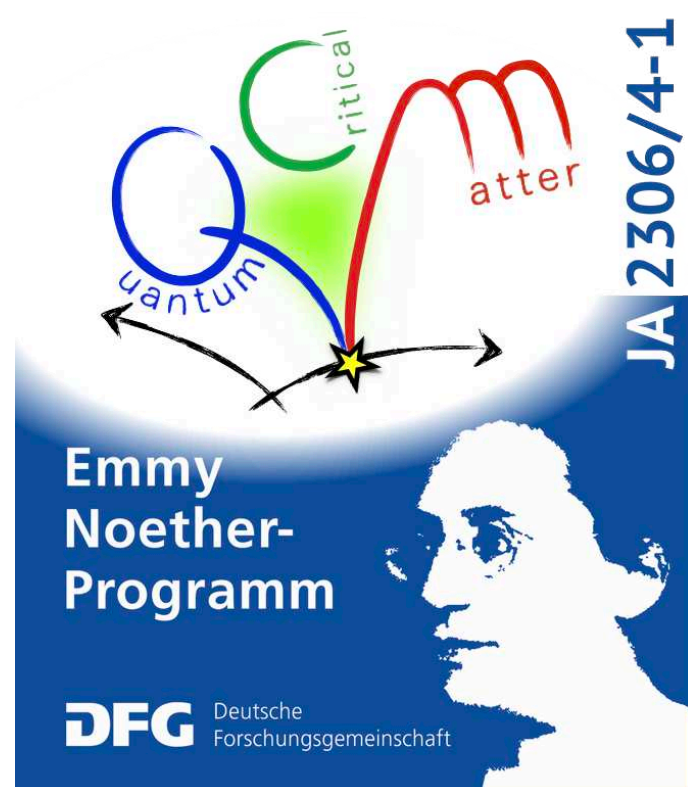
Lukas Janssen
(TU Dresden)



Shouryya Ray

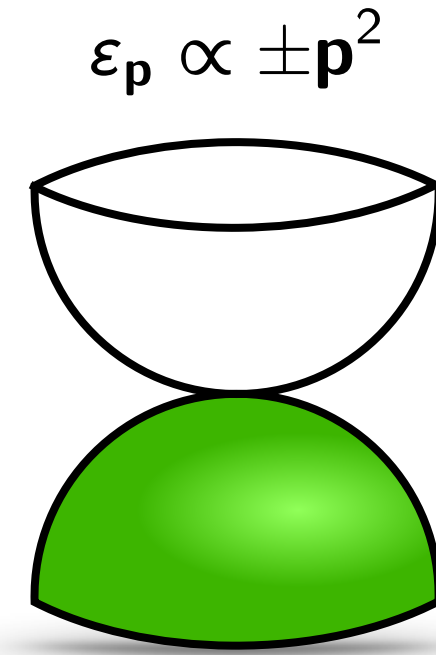


Matthias Vojtá



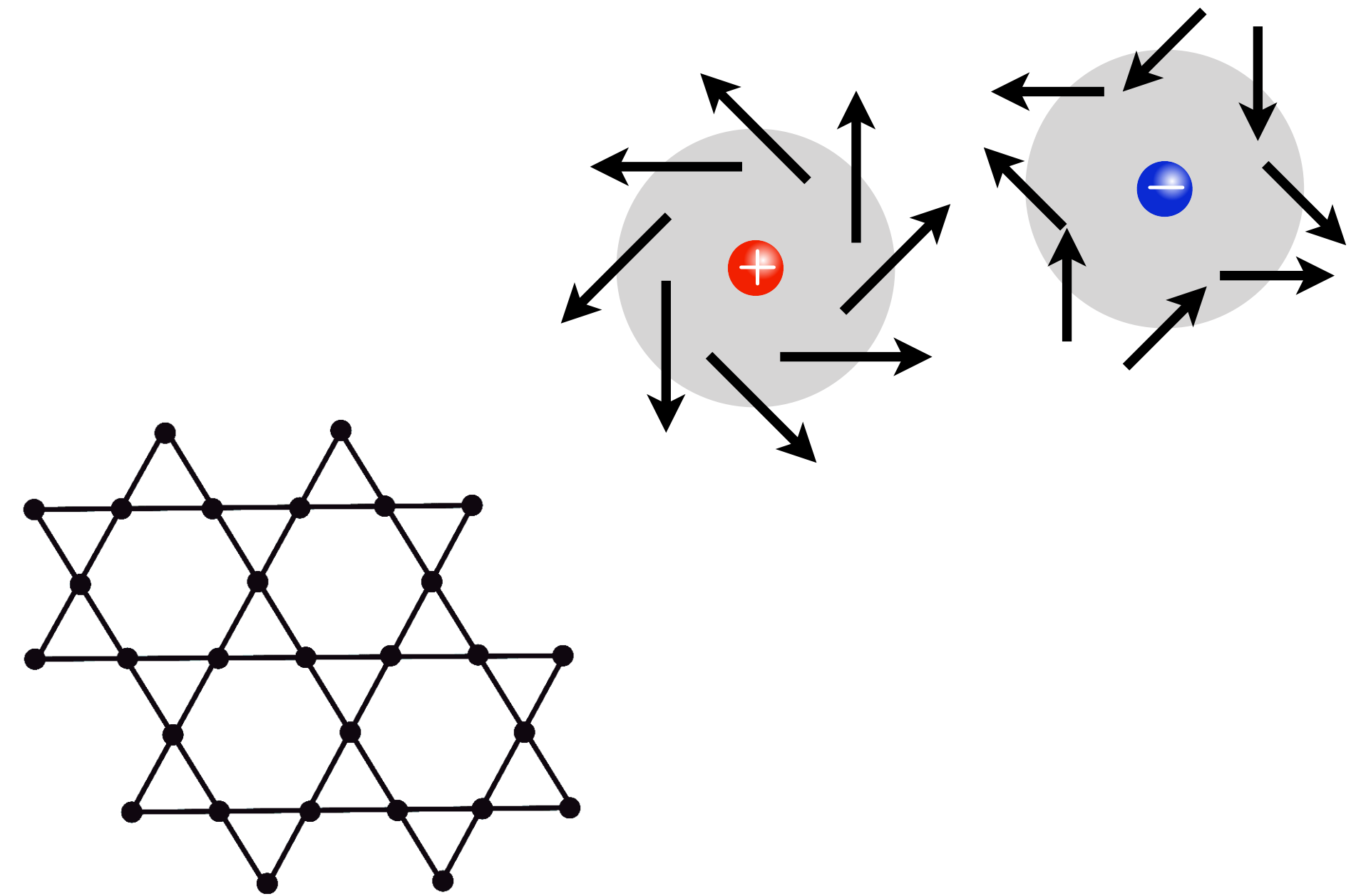
Outline

1) Introduction

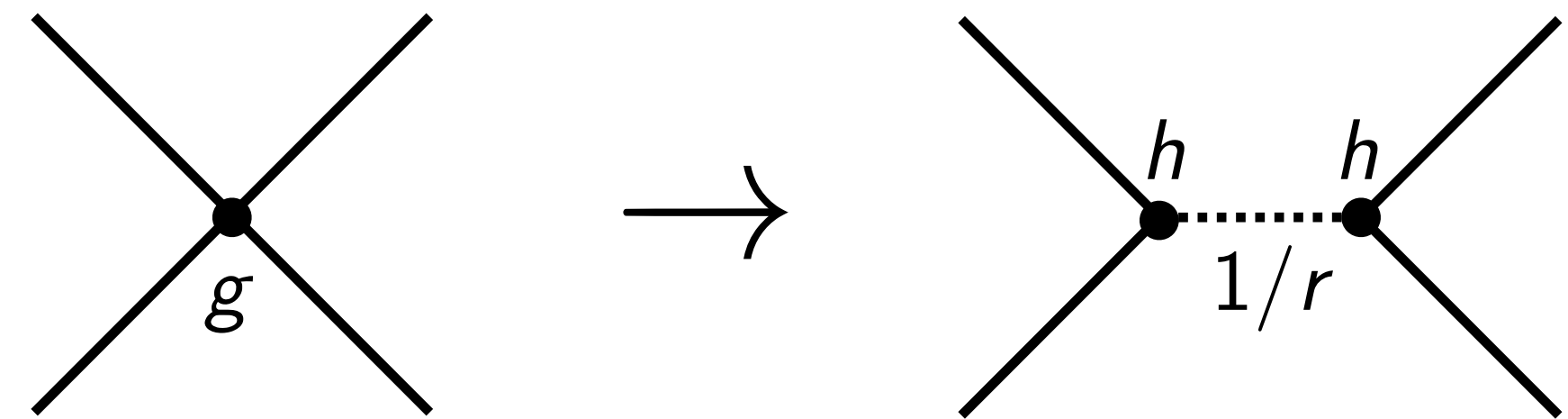


2) Review: BKT transition

3) “Luttinger” fermions on kagome lattice

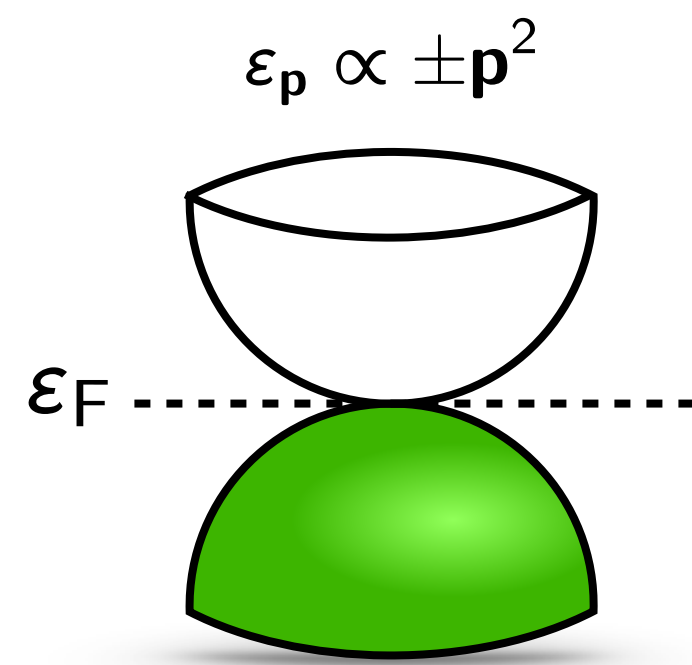


4) Critical behavior: “Luttinger-Yukawa” theory

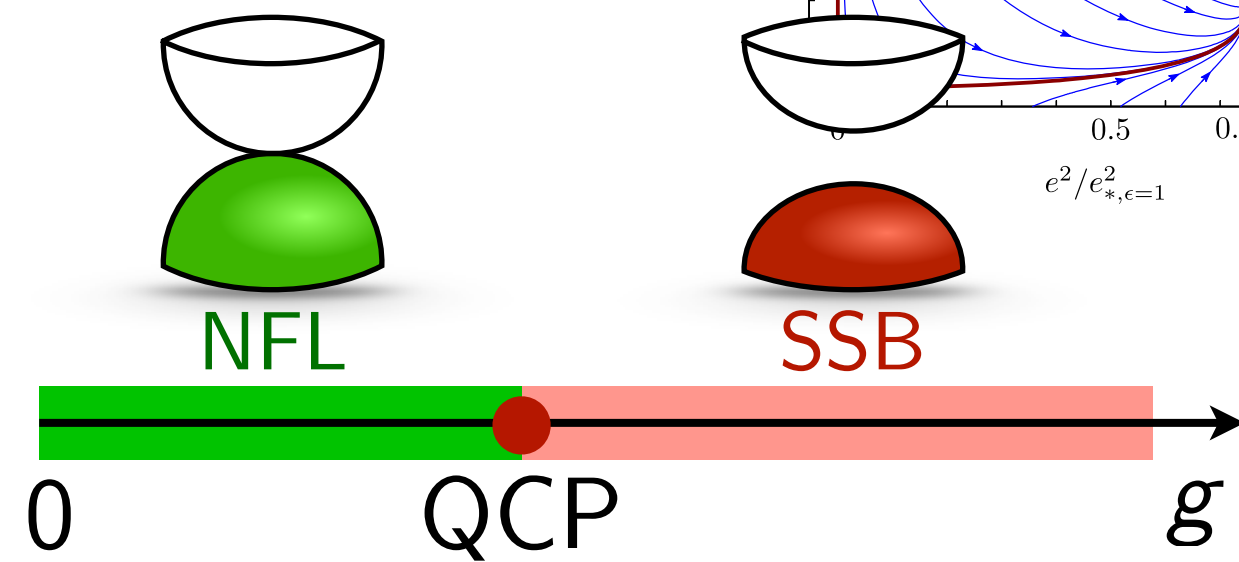
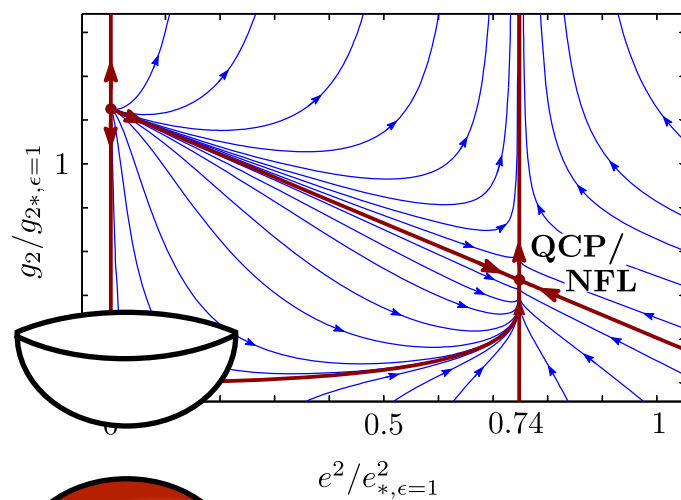


5) Conclusions

Introduction: Luttinger fermions



3D

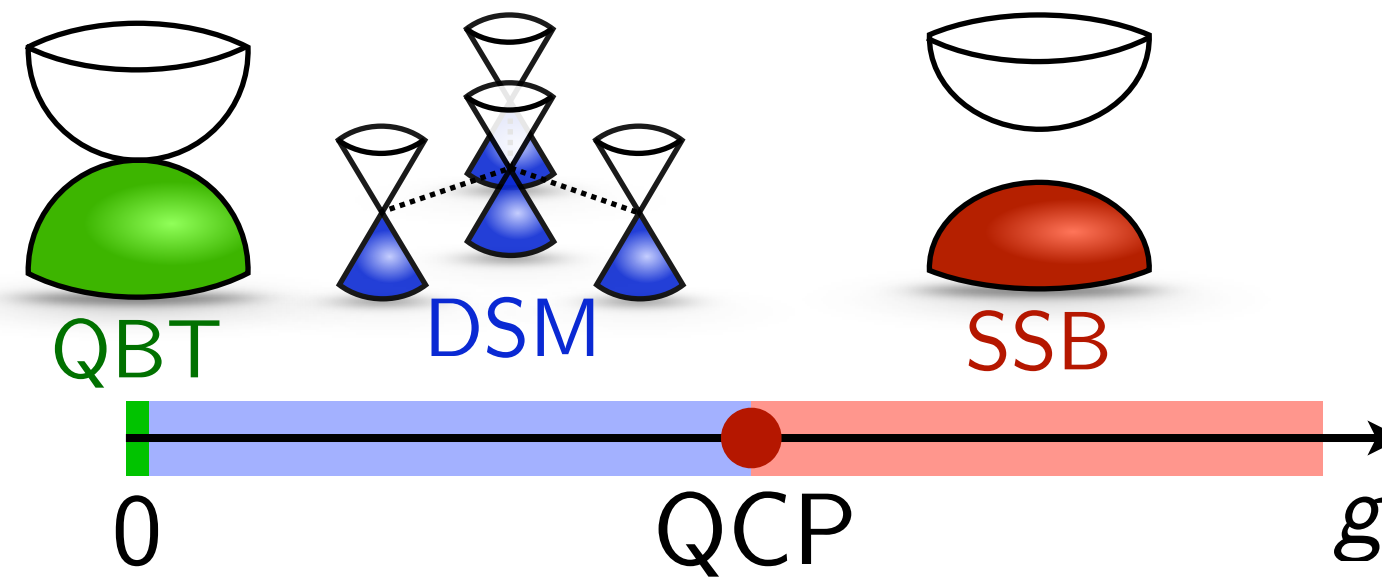


- ... non-Fermi liquids (large- M)
- ... quantum critical points
- ... fixed-point collisions

→ talk by I. Herbut

2D

(C_3 anisotropy)

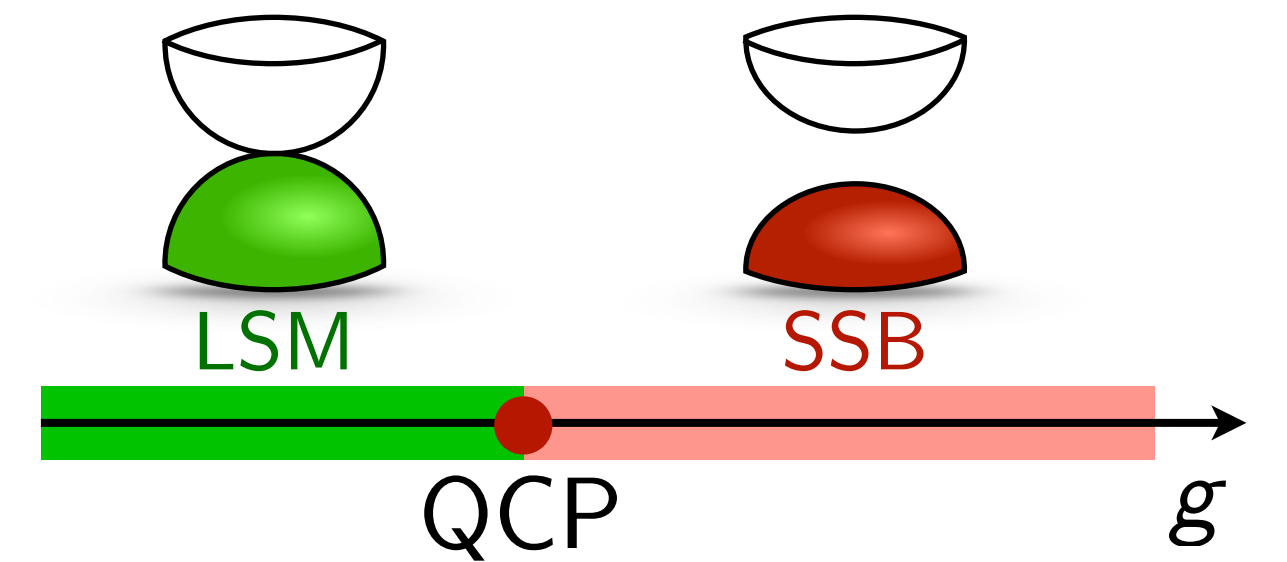


- ... dynamical generation of Dirac cones
- ... emergent Lorentz invariance
- ... Gross-Neveu criticality

→ talk by S. Ray

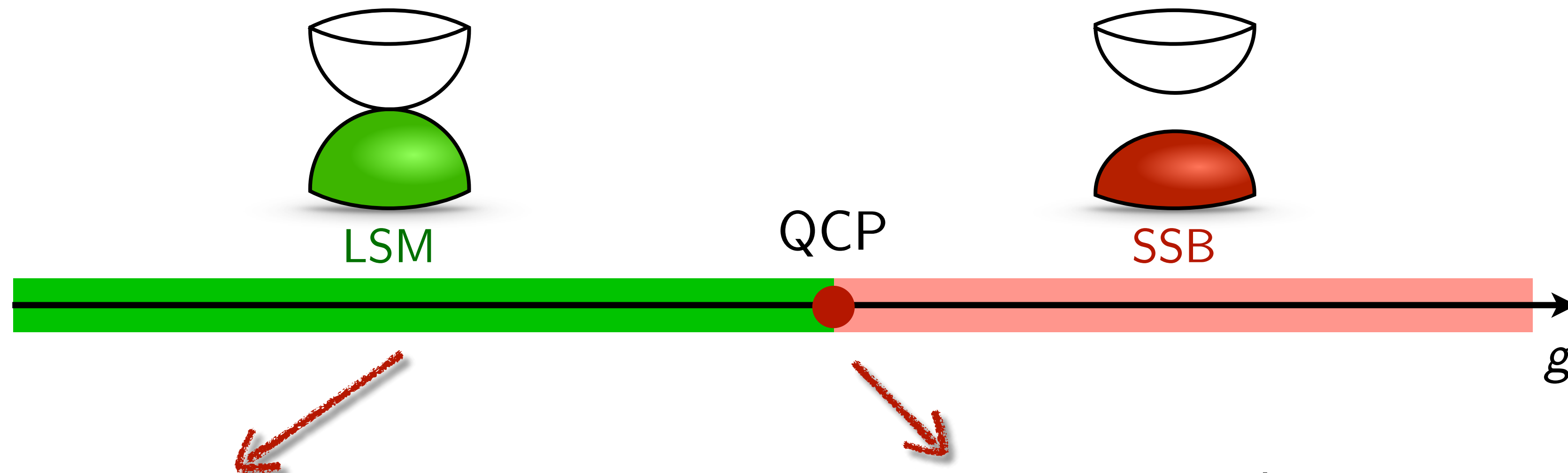
2D

(isotropic)



- ... solvable and nontrivial

Isotropic QBT in 2D



Disordered phase:

- Stable semimetal ... despite finite DOS

- Emergent scale invariance

$$\langle \phi(0, \mathbf{r}) \phi(0, 0) \rangle \propto \frac{1}{|\mathbf{r}|^{z+\eta_\phi}}$$

... with $z = 2$
... and $\eta_\phi = 2$

Quantum criticality:

- Essential singularities close to QCP

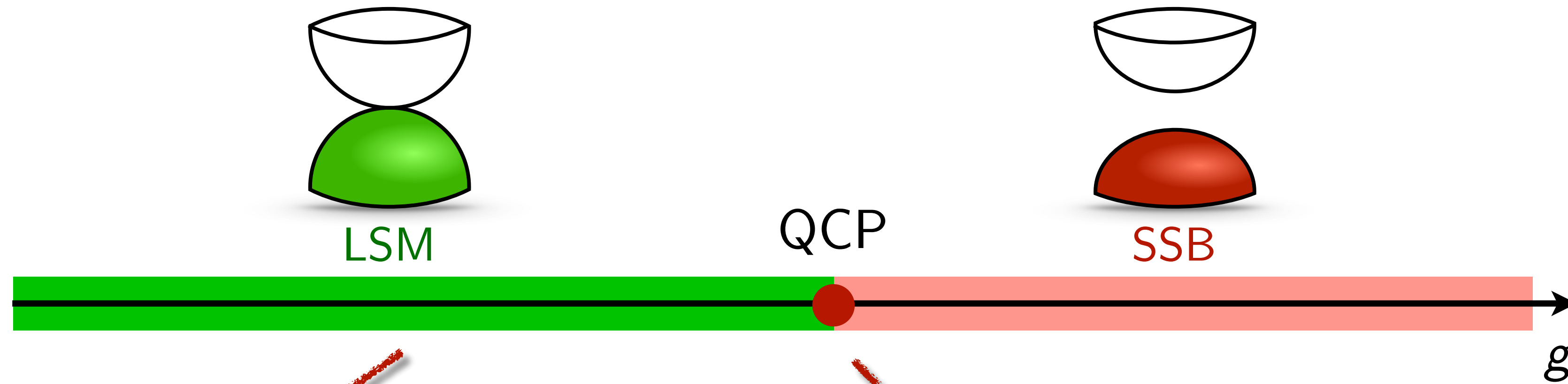
$$\xi \propto e^{4\pi/\delta g} \quad (\delta g \equiv g - g_c > 0)$$

- Power laws right at QCP

$$\langle \phi \rangle \propto h^{1/\delta} \quad (\delta g = 0)$$

... with $\delta = 1$ exactly

Isotropic QBT in 2D



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Phenomenology: Quantum version of BKT transition

Review: BKT transition

[Herbut, CUP '07]

Classical 2D XY model:

$$\mathcal{H}_{XY} = - \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$
$$\simeq \frac{1}{2} \int d^2\mathbf{r} (\nabla\theta(\mathbf{r}))^2$$

$$\text{with } \mathbf{S}_i \equiv \mathbf{S}(\mathbf{r}_i) \equiv \begin{pmatrix} \cos \theta(\mathbf{r}_i) \\ \sin \theta(\mathbf{r}_i) \end{pmatrix} : \mathbb{R}^2 \mapsto S^1$$

... in continuum limit

Review: BKT transition

[Herbut, CUP '07]

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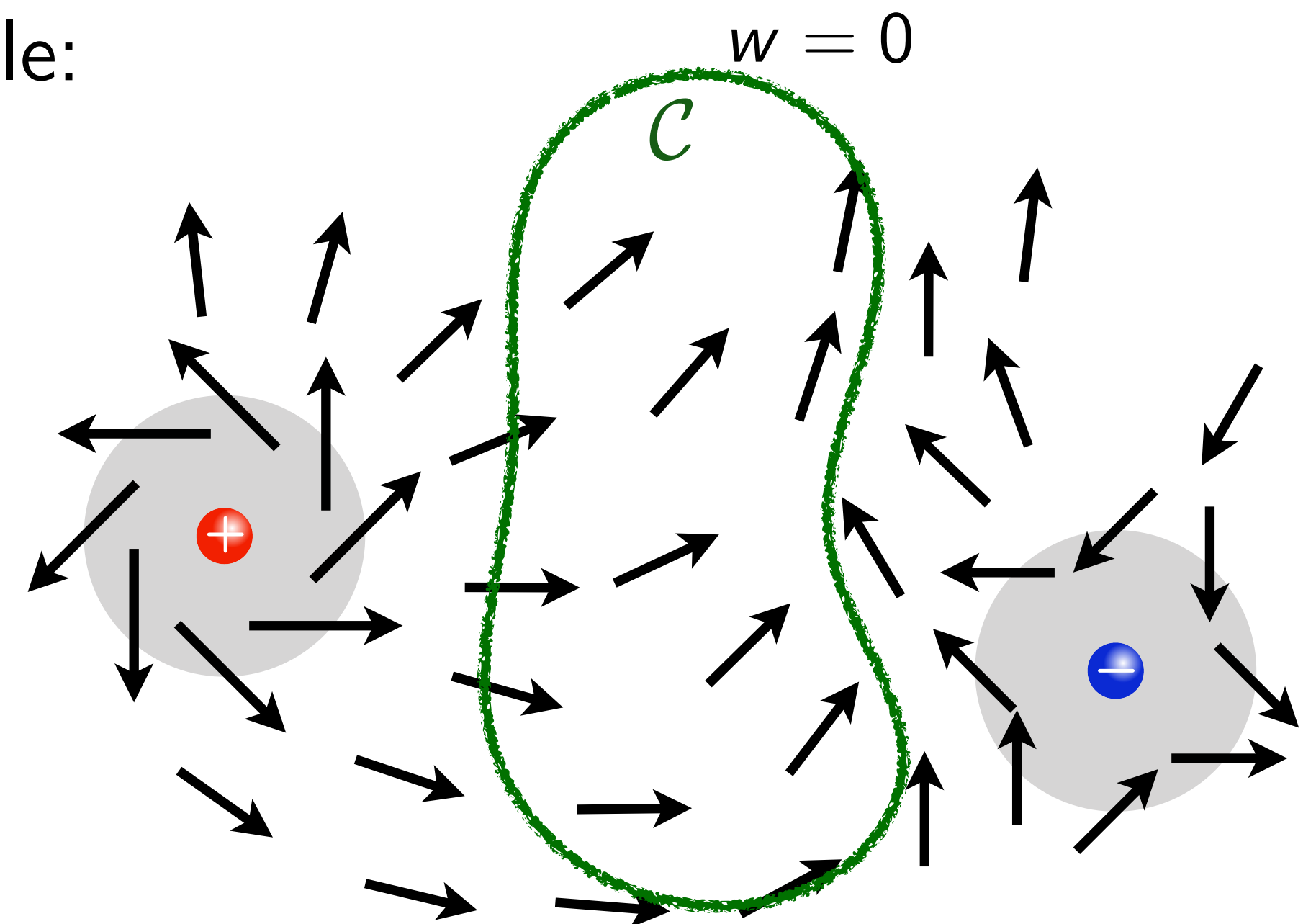
Closed contour $\mathcal{C} \in \mathbb{R}^2$:

$$w = \frac{1}{2\pi} \oint \mathbf{dr} \cdot \nabla\theta(\mathbf{r}) \in \mathbb{Z}$$
$$= \sum_{\text{vortices in } \mathcal{C}} q_i$$

... winding number

... q_i : vortex charges

Example:



Review: BKT transition

[Herbut, CUP '07]

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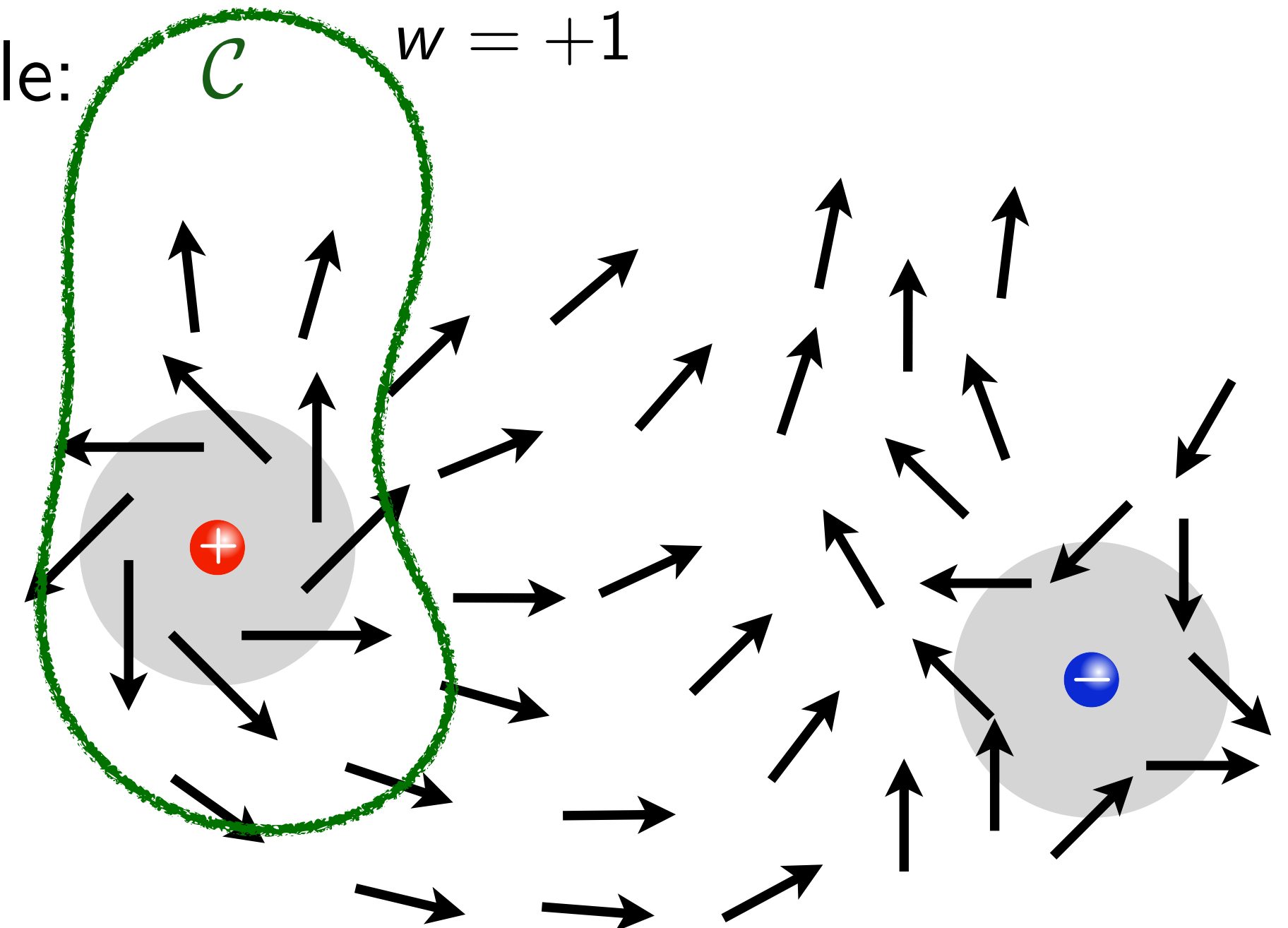
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Example: $w = +1$



Review: BKT transition

[Herbut, CUP '07]

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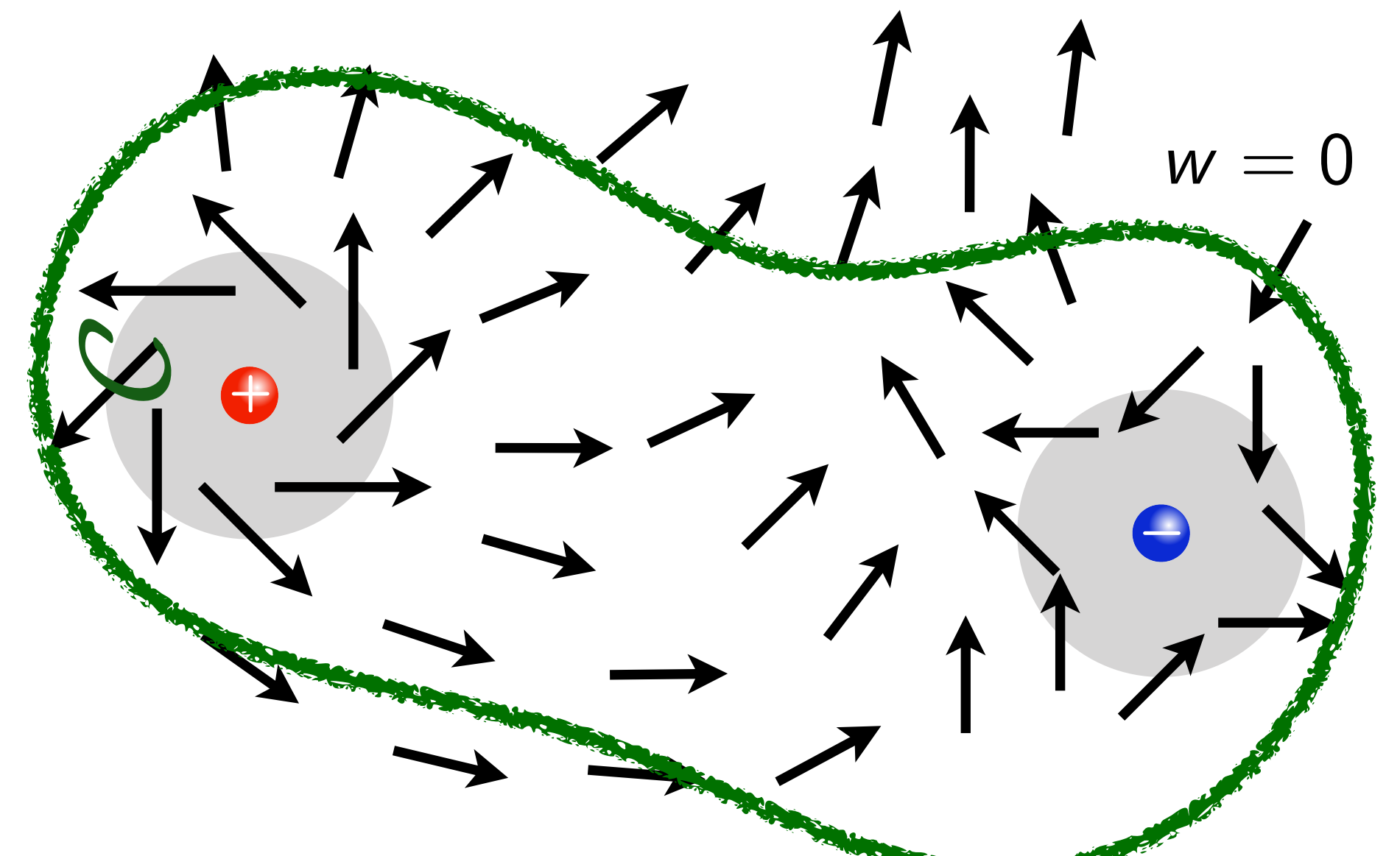
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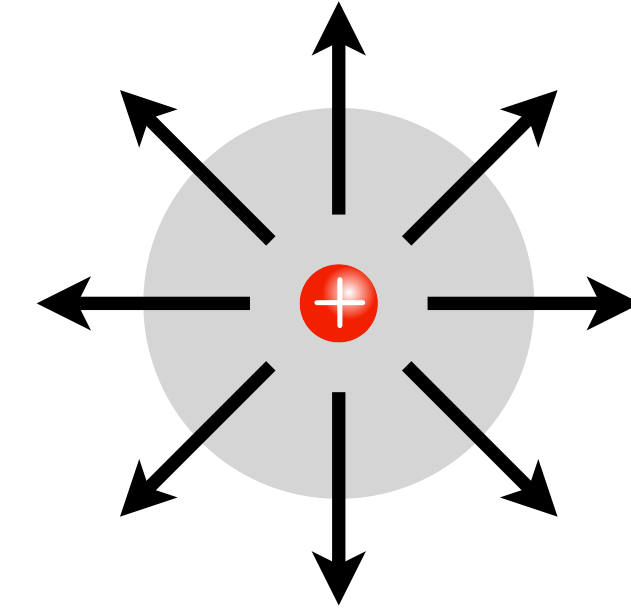
Example:



Vortex excitations

Isolated vortex:

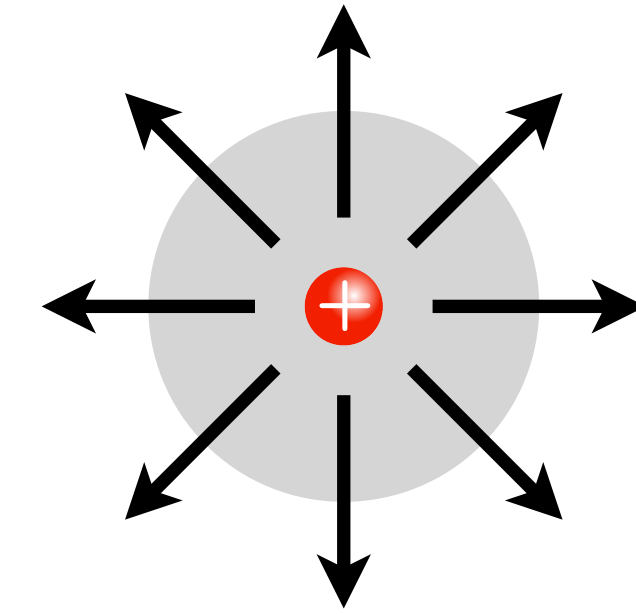
$$\theta(\mathbf{r}) = q\alpha \quad \text{where} \quad \mathbf{r} = r \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$$



Vortex excitations

Isolated vortex:

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Energy:

$$E_V = \frac{1}{2} \int d^2\mathbf{r} (\nabla\theta(\mathbf{r}))^2 = \pi q^2 \ln \frac{R}{r_0} \quad \xrightarrow{R \rightarrow \infty} \infty$$

system size

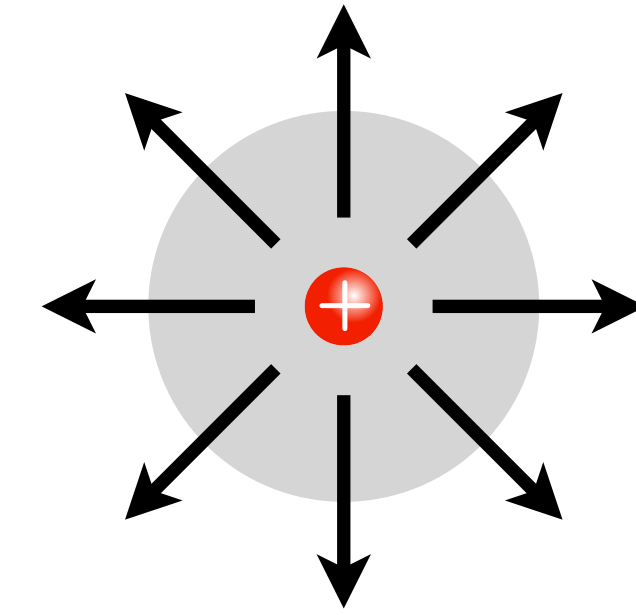
“vortex size” ... short-distance cutoff $r_0 \gtrsim a$

... vortices suppressed at low T

Vortex excitations

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system size

“vortex size” ... short-distance cutoff $r_0 \gtrsim a$

... vortices suppressed at low T

Entropy:

$$S_V = \ln \Omega \simeq \ln \left(\frac{R}{r_0} \right)^2 \xrightarrow{R \rightarrow \infty} \infty$$

... (same) logarithmic divergence
... vortices proliferate at high T

Vortex proliferation

Free energy (isolated vortex):

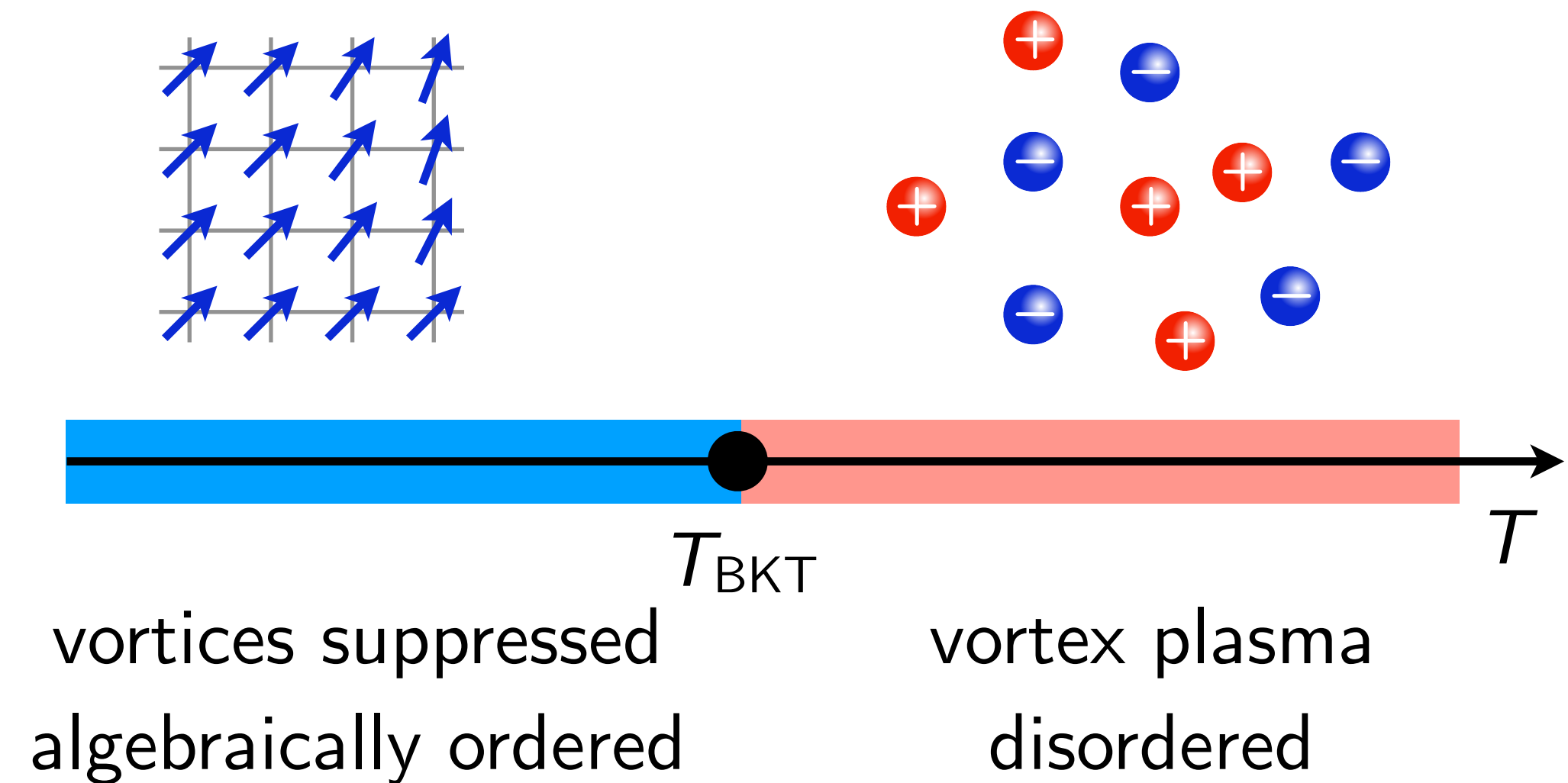
$$\begin{aligned}\Delta F &= E_V - TS_V \\ &= \pi q^2 \ln \frac{R}{r_0} - 2T \ln \frac{R}{r_0} \\ &\begin{cases} > 0 & \text{for } T < \frac{\pi}{2} q^2 \\ < 0 & \text{for } T > \frac{\pi}{2} q^2 \end{cases}\end{aligned}$$

Transition temperature:

$$T_{\text{BKT}} = \frac{\pi}{2}$$

... above which $q = \pm 1$ vortices proliferate

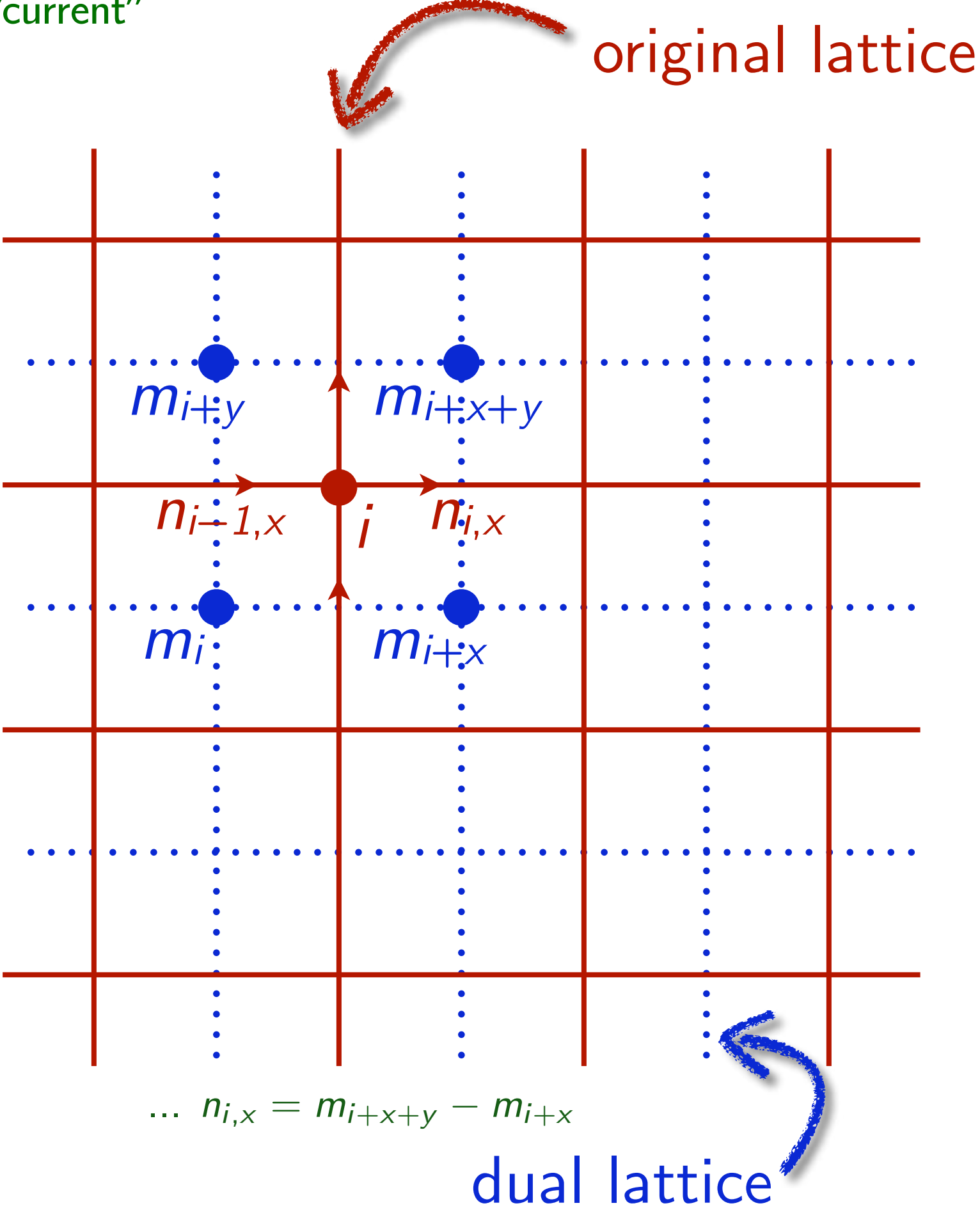
Phase diagram:



Duality transformation: Sine-Gordon model

XY model:

$$Z_{XY} = \prod_i \int_0^{2\pi} \frac{d\theta_i}{2\pi} e^{-\mathcal{H}_{XY}/T} = \prod_i \int_0^{2\pi} \frac{d\theta_i}{2\pi} e^{\frac{1}{T} \sum_{i,\hat{\mu}} \underbrace{\cos(\theta_i - \theta_{i+\hat{\mu}})}_{\sim n_{i,\mu} \text{ ... "current"}}$$



Duality transformation: Sine-Gordon model

XY model:

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Dual model:

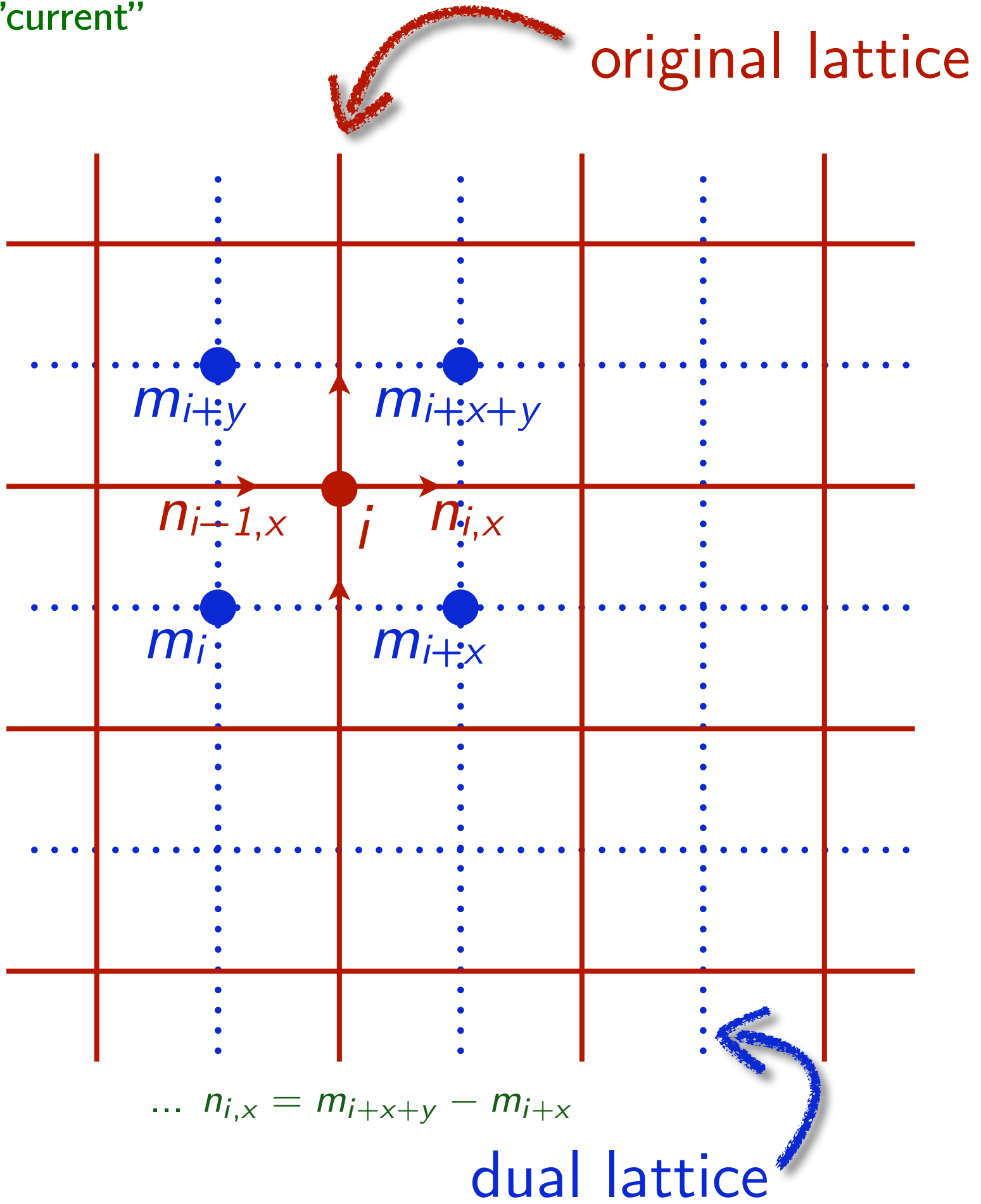
$$Z_{\text{dual}} = \sum_{\{m_i\}} e^{-\frac{T}{2} \sum_{i,\hat{\mu}} (m_{i+\hat{\mu}} - m_i)^2}$$

... using Villain approximation $\theta \approx 2\pi\mathbb{Z} + \delta\theta$
 ... and Hubbard-Stratonovich $n_{i,\mu} \sim \theta_i - \theta_{i+\hat{\mu}}$

$$\simeq \int \mathcal{D}\varphi e^{-\int d^2\mathbf{r} \mathcal{L}_{\text{SG}}(\varphi)}$$

with $\mathcal{L}_{\text{SG}} = \frac{T}{2} (\nabla\varphi(\mathbf{r}))^2 - 2y \cos(2\pi\varphi(\mathbf{r}))$

... "sine-Gordon model"
 ... assuming low "fugacity" $y \ll 1$



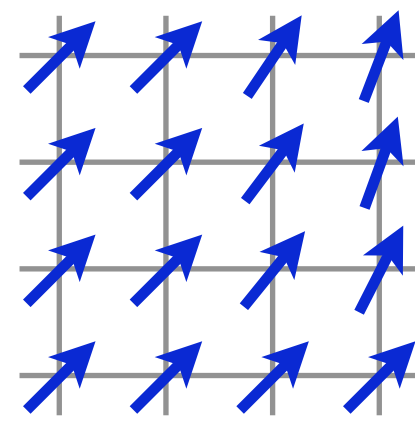
XY—Sine-Gordon duality

XY model

$$\mathcal{L}_{XY} = \frac{1}{2T} (\nabla\theta)^2$$

... with $\theta \equiv \theta + 2\pi$

spin angles



spin picture

... vortices gapped

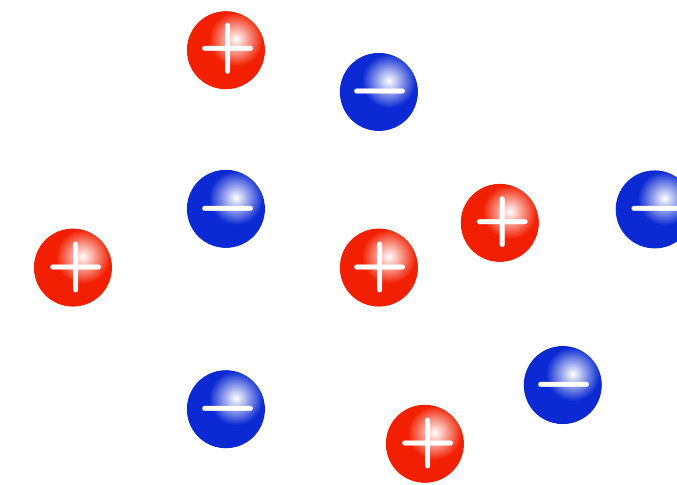
Sine-Gordon model

$$\mathcal{L}_{sG} = \frac{T}{2} (\nabla\varphi)^2 - 2y \cos(2\pi\varphi)$$

vortex density

vortex fugacity ... $e^{\mu/T}$

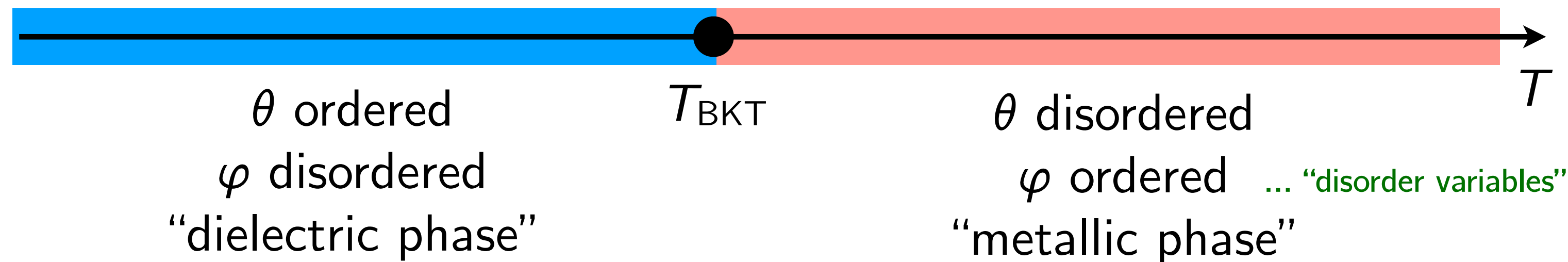
... without any constraint on φ



vortex picture

... "Coulomb plasma"

Phase diagram:



Renormalization group

[Herbut, CUP '07]

Flow equations:

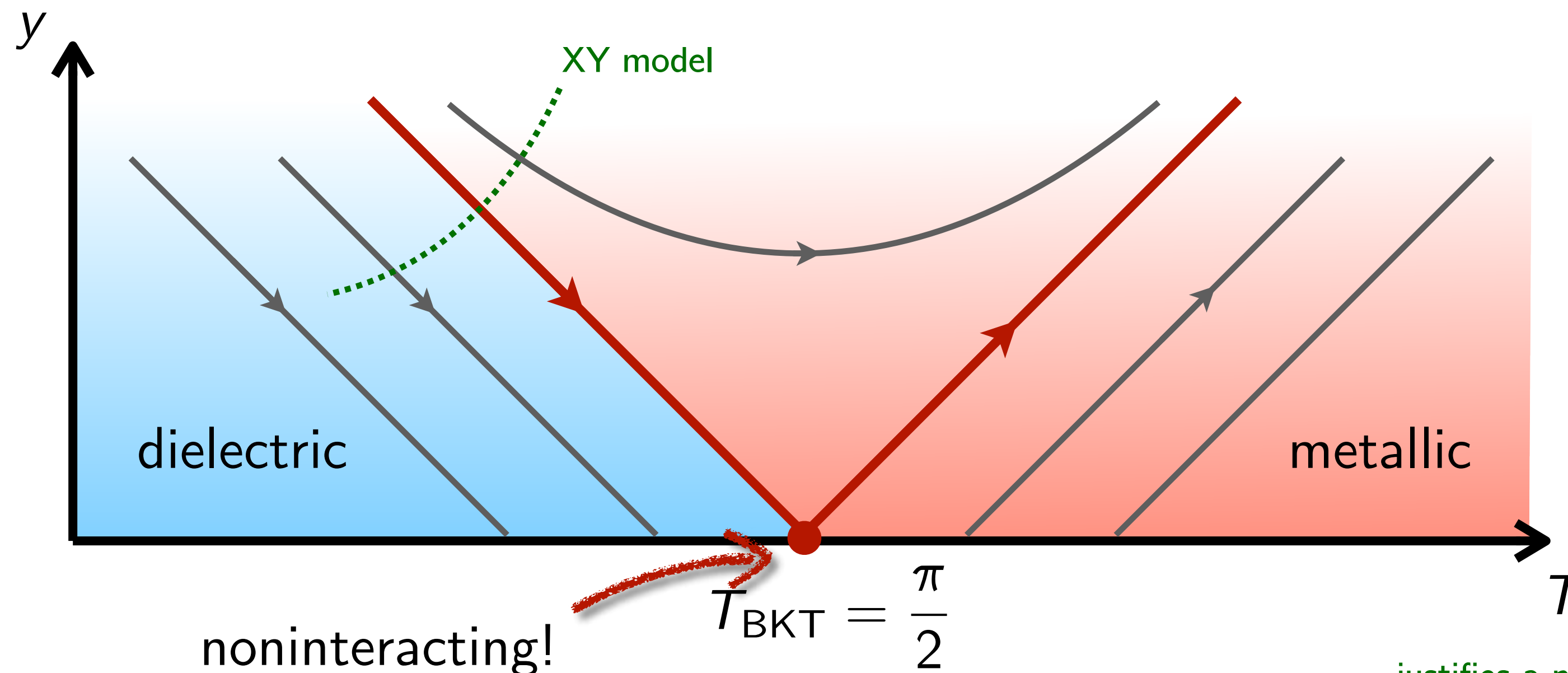
$$\frac{dy}{d \ln b} = \left(2 - \frac{\pi}{T}\right) y + \mathcal{O}(y^3)$$

$$\frac{dT}{d \ln b} = \frac{y^2}{2T} + \mathcal{O}(y^4)$$

... irrelevant for $T < \frac{\pi}{2}$
... relevant for $T > \frac{\pi}{2}$

... marginal for $y = 0$
... relevant for $y > 0$

Flow diagram:



... justifies a posteriori simple energy-entropy argument

Critical behavior and algebraic order

For $T < T_c$:

$$\langle e^{i\theta(\mathbf{r})} e^{-i\theta(0)} \rangle \propto \frac{1}{|\mathbf{r}|^{T_\infty/(2\pi)}} \quad \text{“algebraic order”}$$

$y \rightarrow 0$
 $T \rightarrow T_\infty < \frac{\pi}{2}$

... on line of fixed points

For $T = T_c$:

$$\langle e^{i\theta(\mathbf{r})} e^{-i\theta(0)} \rangle \propto \frac{1}{|\mathbf{r}|^{1/4}}$$

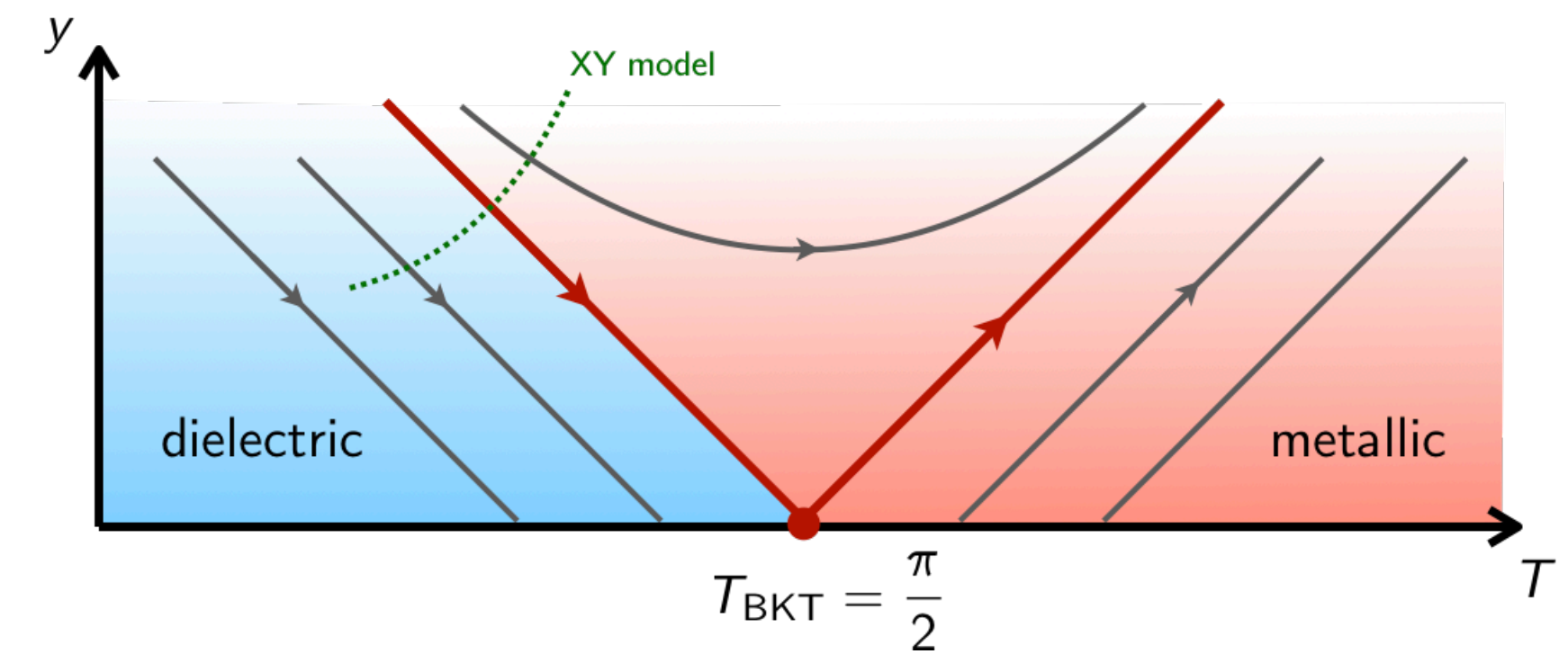
$y \rightarrow 0$
 $T \rightarrow \frac{\pi}{2}$

... i.e., $\eta = 1/4$

For $T > T_c$:

$$\langle e^{i\theta(\mathbf{r})} e^{-i\theta(0)} \rangle \propto e^{-|\mathbf{r}|/\xi} \quad \text{with correlation length } \xi \propto e^{C\sqrt{T_c/(T-T_c)}}$$

... essential singularity
... since T marginal at $y = 0$



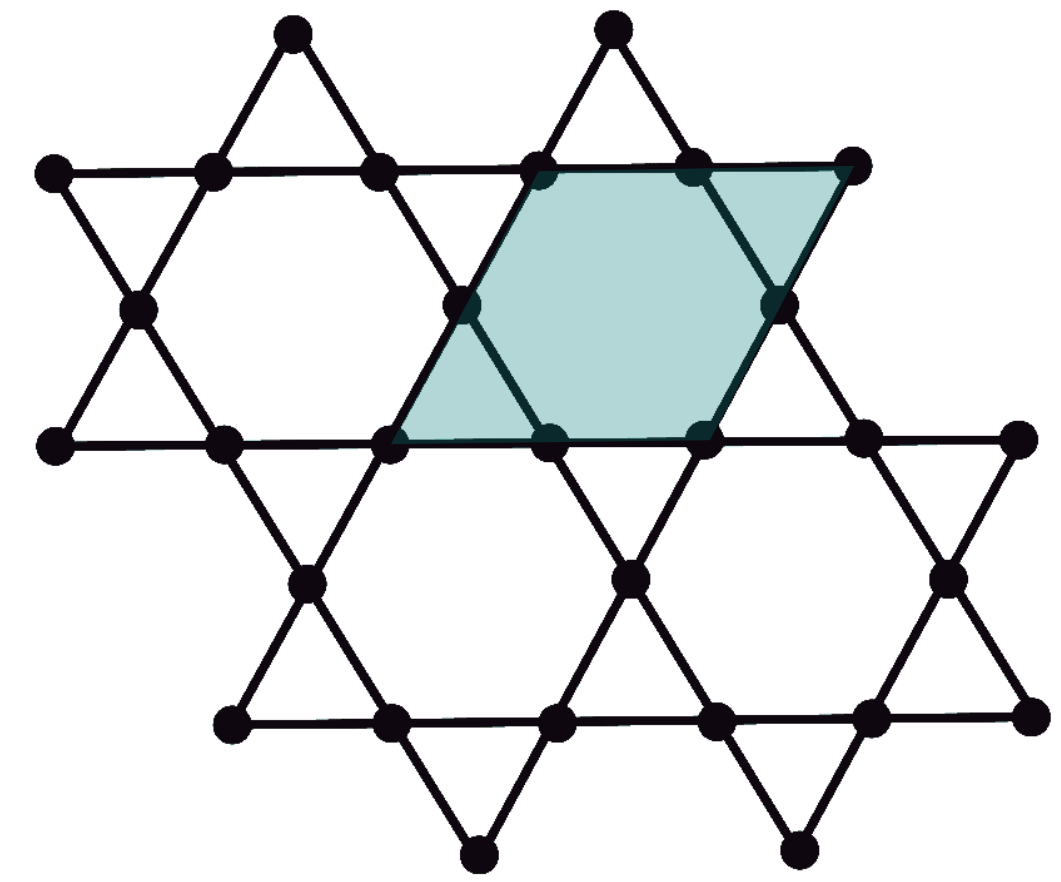
“Luttinger” fermions on kagome lattice

Hopping Hamiltonian (spinless fermions):

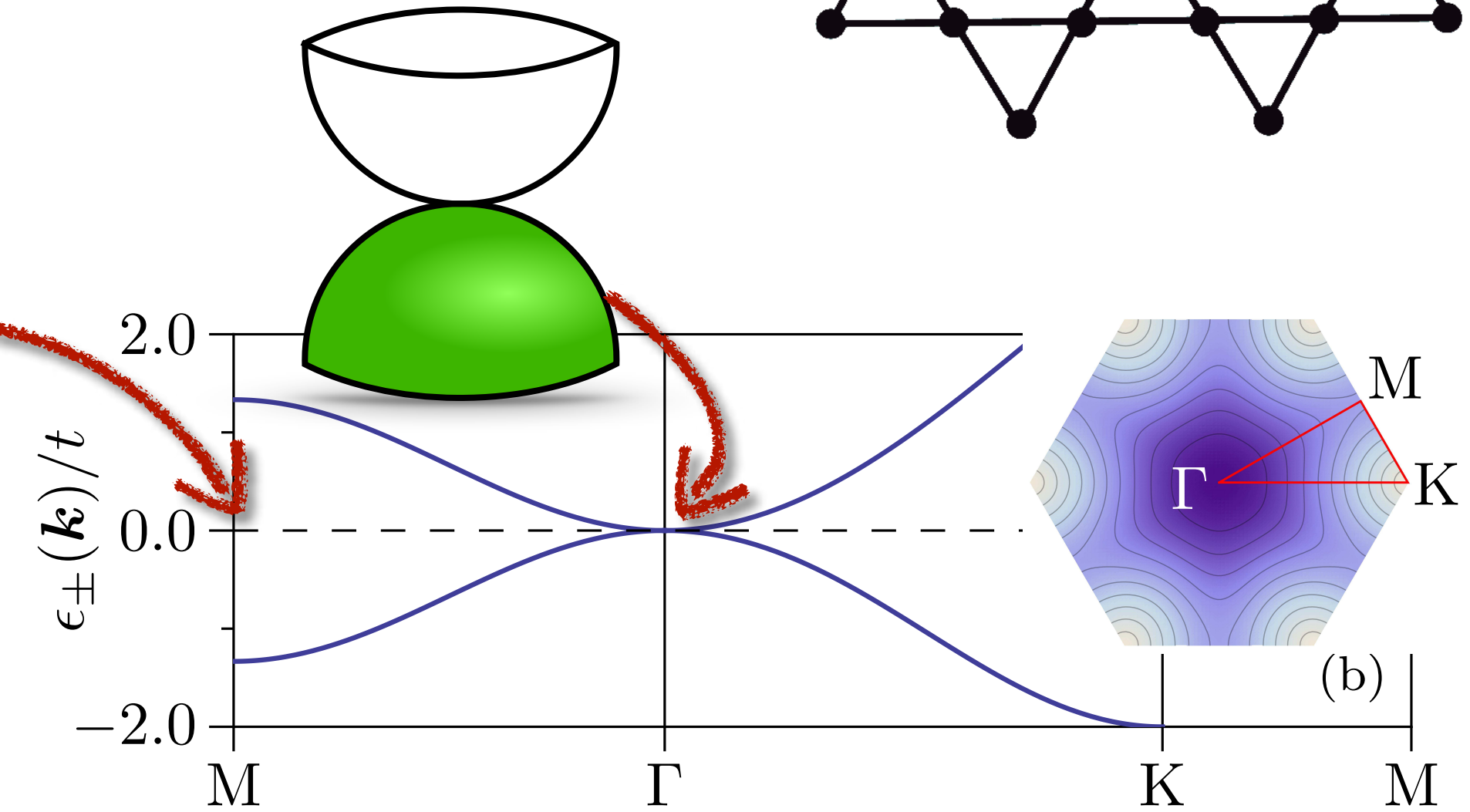
$$\mathcal{H}_0 = -t \sum_{\langle ij \rangle} c_i^\dagger c_j - t' \sum_{\langle\langle ij \rangle\rangle} c_i^\dagger c_j + \text{H.c.}$$

Spectrum:

$$\varepsilon_{\pm}(\mathbf{q}) = \frac{1}{2} [-(t + 3t') \pm (t - 3t')] \mathbf{q}^2 + \mathcal{O}(\mathbf{q}^4)$$



1/3 filling



... choose $t' = -t/3$ for simplicity

Interactions:

$$\mathcal{H}_{\text{int}} = V_1 \sum_{\langle ij \rangle} c_i^\dagger c_i c_j^\dagger c_j + V_2 \sum_{\langle\langle ij \rangle\rangle} c_i^\dagger c_i c_j^\dagger c_j$$

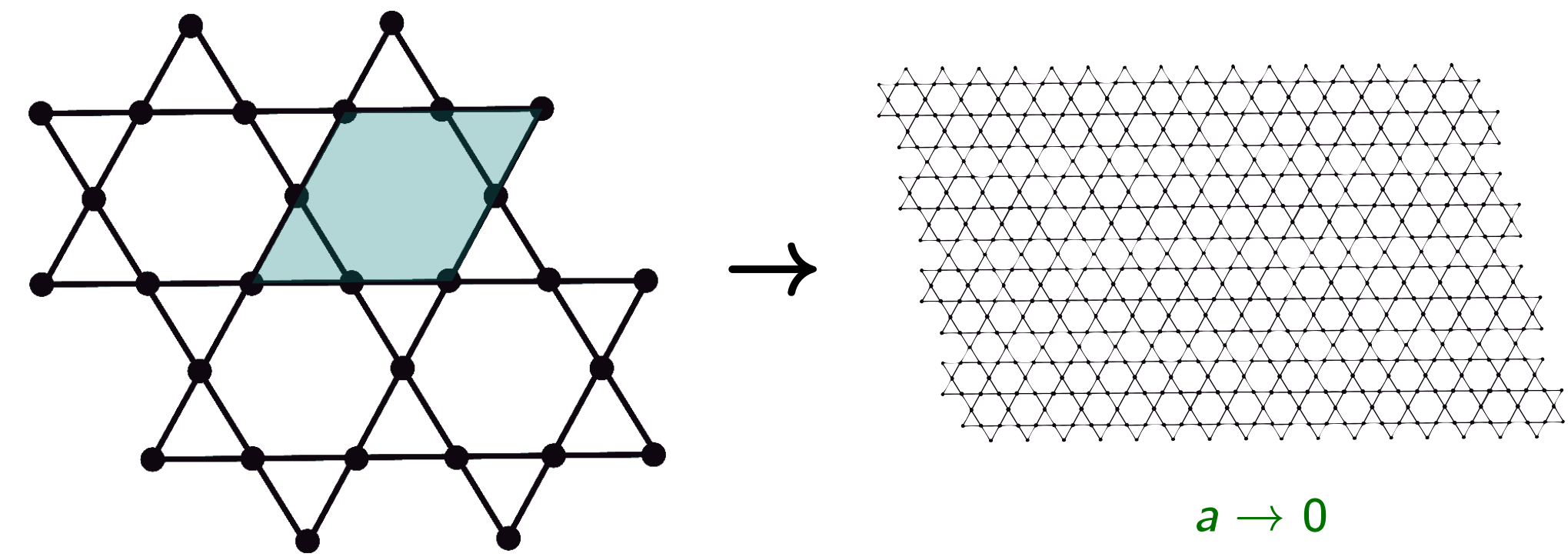
... $V_{1,2} > 0$ repulsive

... $V_{1,2} < 0$ attractive

Low-energy field theory

Effective Lagrangian:

... $g > 0$ repulsive
... $g < 0$ attractive



$$\mathcal{L}_\psi = \psi^\dagger \left[\partial_\tau + (\partial_x^2 - \partial_y^2)\sigma_1 + 2\partial_x\partial_y\sigma_3 \right] \psi - \frac{g}{2} (\psi^\dagger \sigma_2 \psi)^2 + \dots$$

$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ two-component

... higher-order terms, such as $g'(\psi^\dagger \sigma_\alpha \psi)\partial_i\partial_j(\psi^\dagger \sigma_\beta \psi)$
... irrelevant

Fierz completeness:

$$(\psi^\dagger \sigma_2 \psi)^2 = -2\psi_1^* \psi_1 \psi_2^* \psi_2 = -(\psi^\dagger \psi)^2 \propto (\psi \mathcal{O} \psi)^2$$

... single interaction channel

Comparison with microscopic model:

$$g \equiv \frac{2(V_1 + V_2)}{t}$$

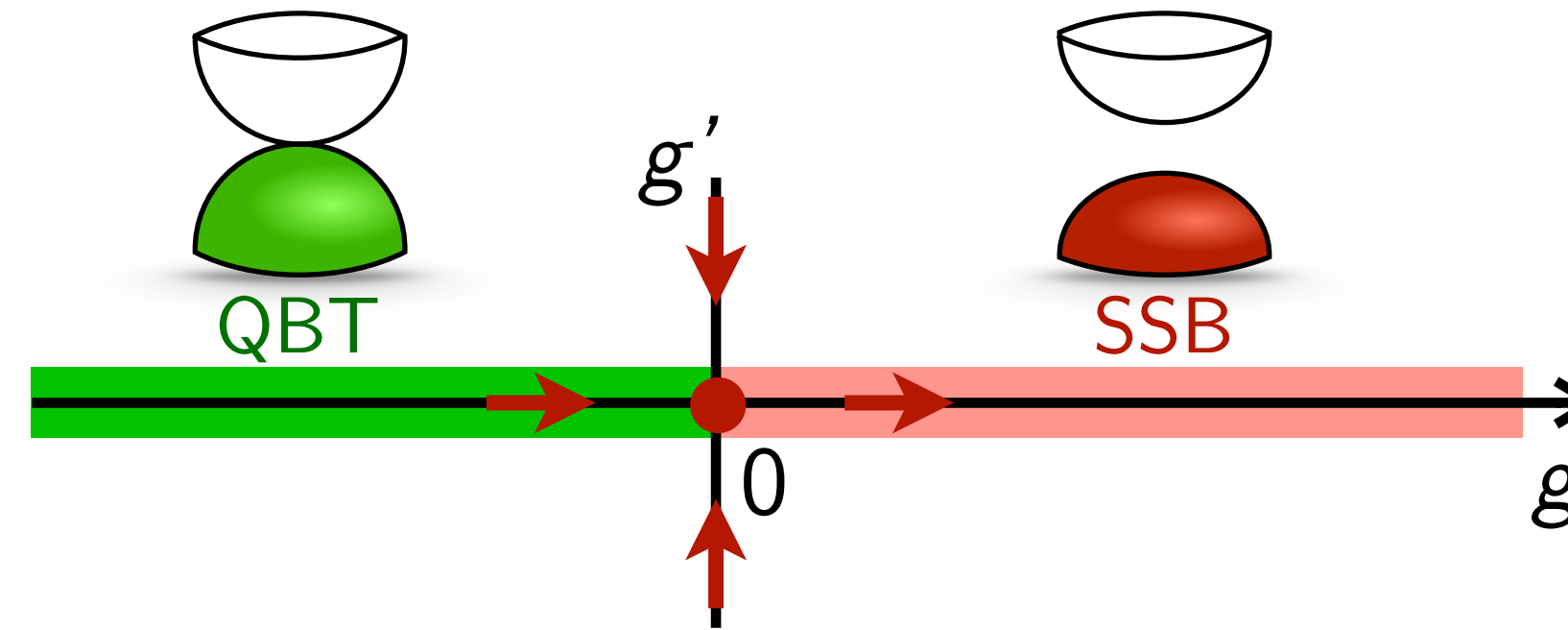
... $g \sim V_1 + V_2$ marginal
... $g' \sim V_1 - V_2$ irrelevant

Phase diagram

RG flow:

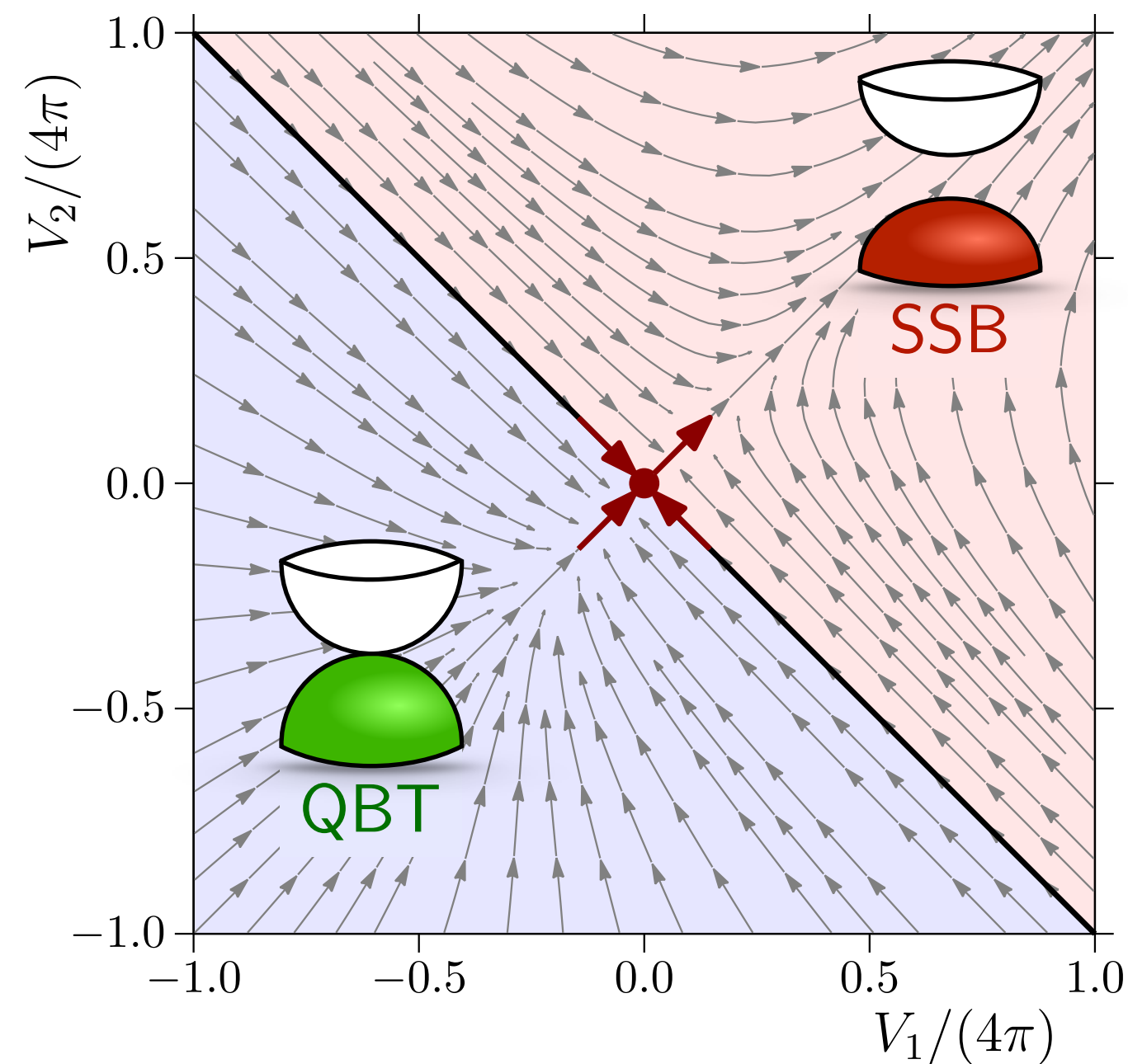
$$\frac{dg}{d \ln b} = \frac{g^2}{4\pi} + \dots$$

$$\frac{dg'}{d \ln b} = -2g' + \dots$$

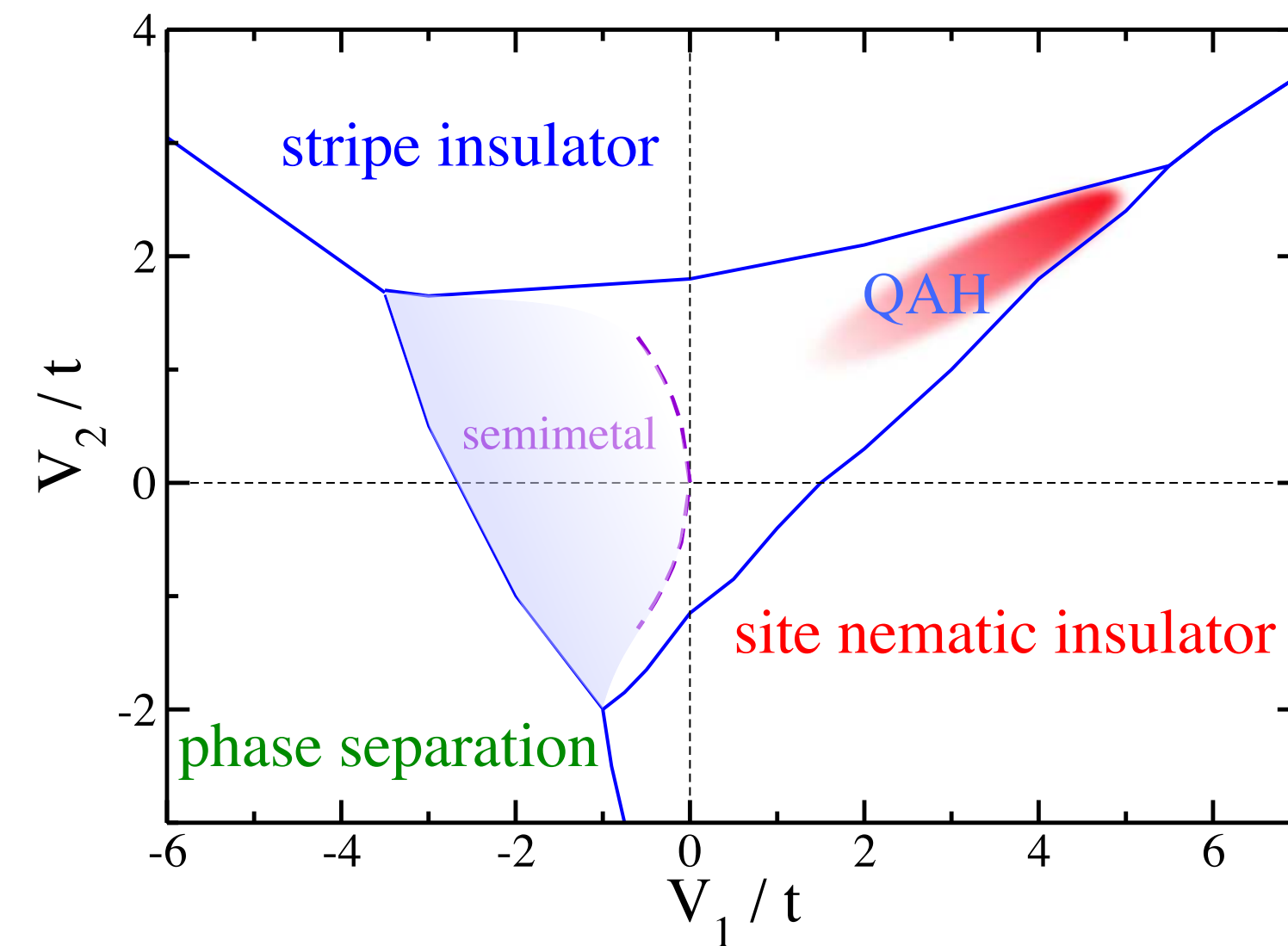


[Sun *et al.*, PRL '09]
[Zhu *et al.*, PRL '16]

Phase diagram:



DMRG (checkerboard):



[Sur *et al.*, PRB 18]

... $g \sim V_1 + V_2$
... $g' \sim V_1 - V_2$

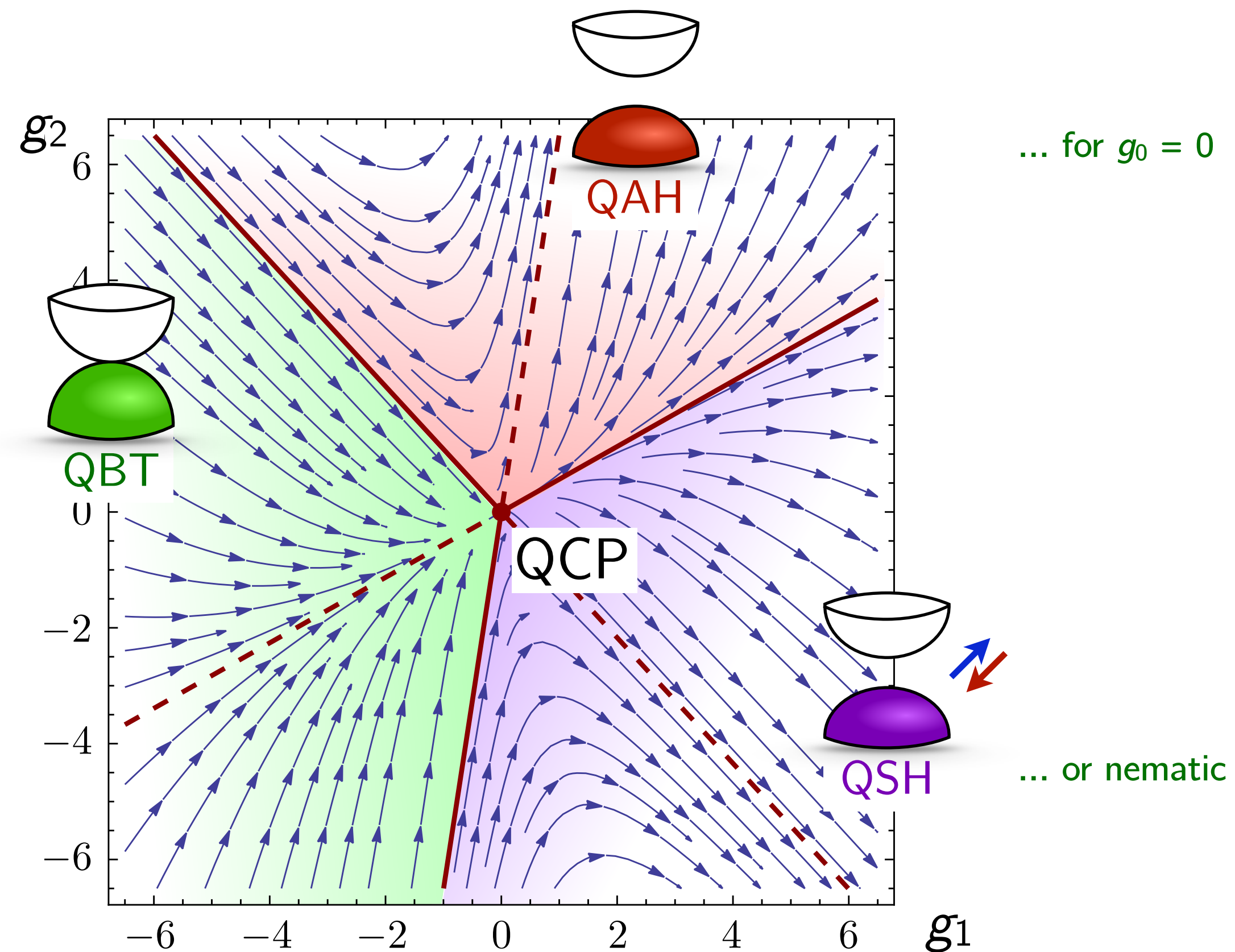
Spin-1/2 fermions ($N = 2$)

Interactions:

$$\mathcal{L}_{\text{int}} = g_0(\psi^\dagger\psi)^2 + g_1 [(\psi^\dagger\sigma_1\psi)^2 + (\psi\sigma_3\psi)^2] + g_2(\psi^\dagger\sigma_2\psi)^2$$

 four-component

Phase diagram:



Critical behavior: Partial bosonization

Technical trick:

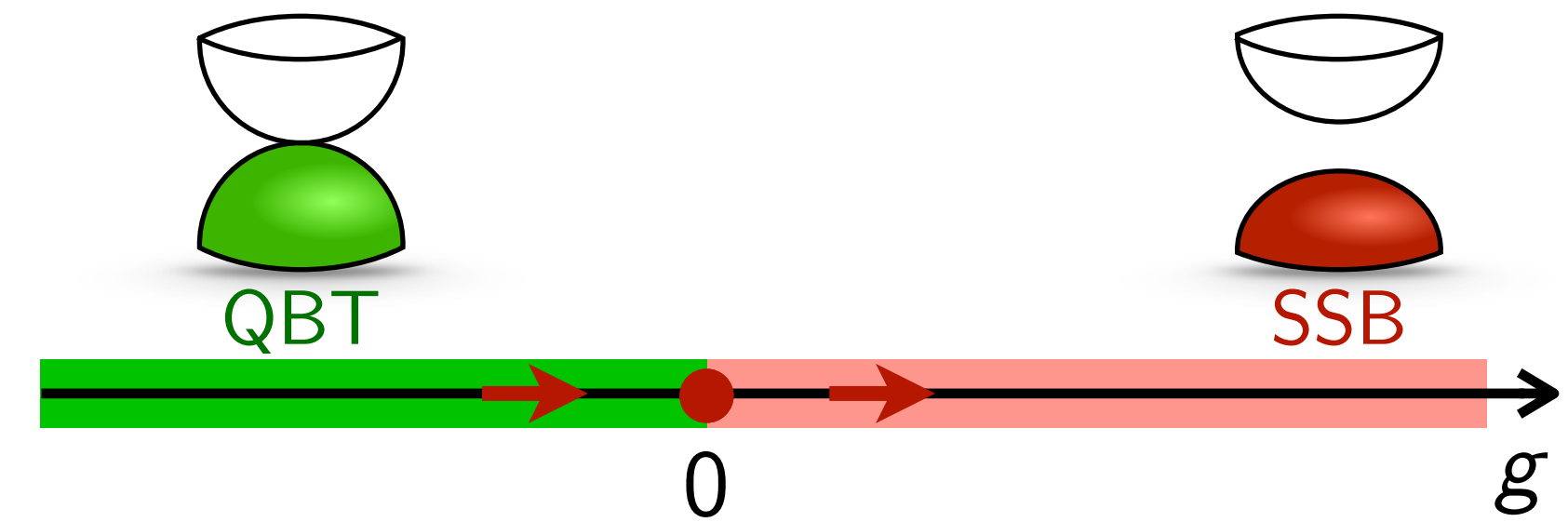
$$d = 2 \quad \mapsto \quad d = 2 + \epsilon$$

RG flow:

$$\frac{dg}{d \ln b} = -\epsilon g + \frac{g^2}{4\pi} + \dots$$

Fixed point: $g_\star = 4\pi\epsilon$

... for $0 < \epsilon \ll 1$



Critical behavior: Partial bosonization

Technical trick:

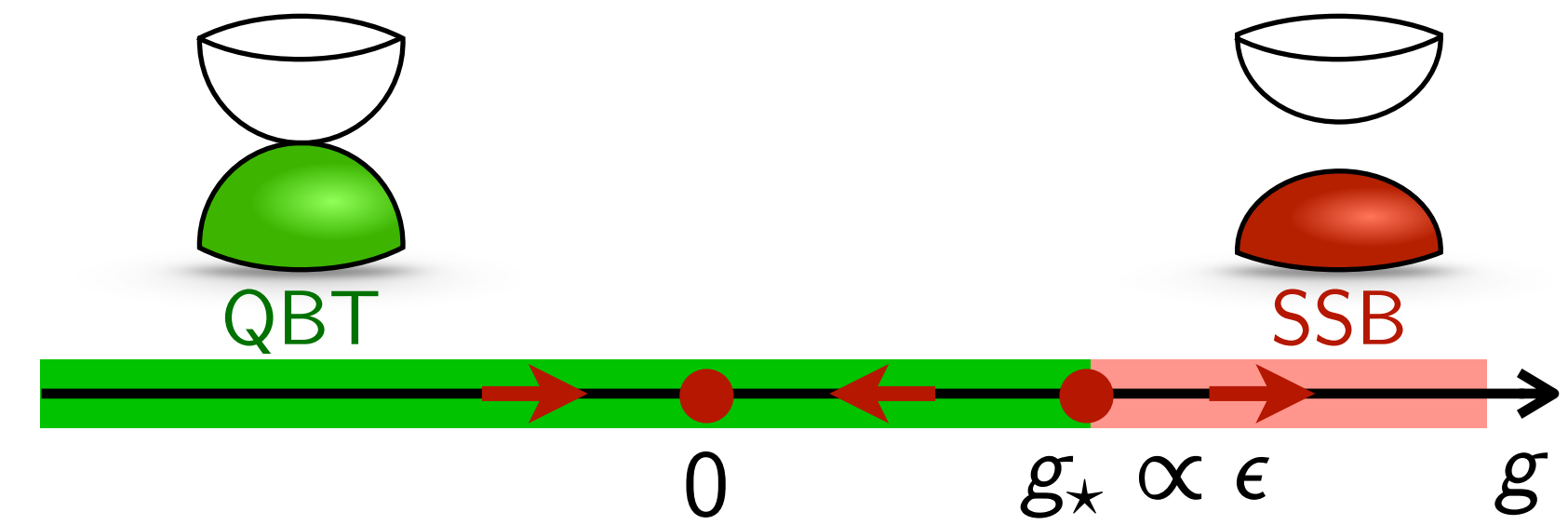
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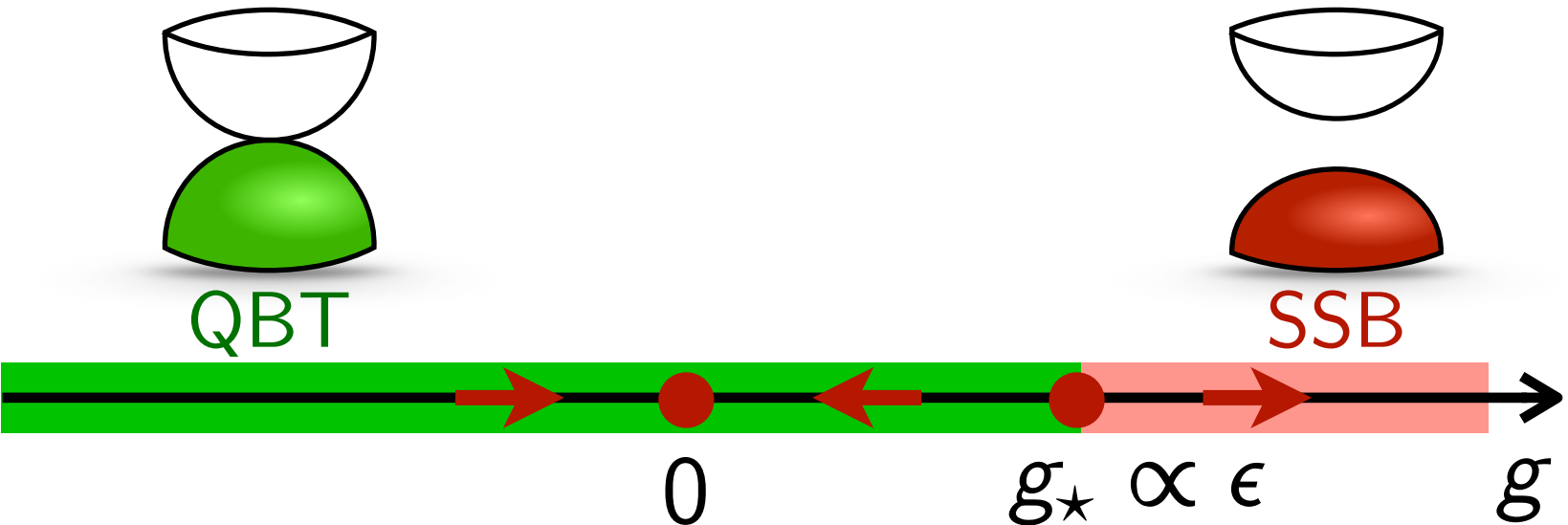
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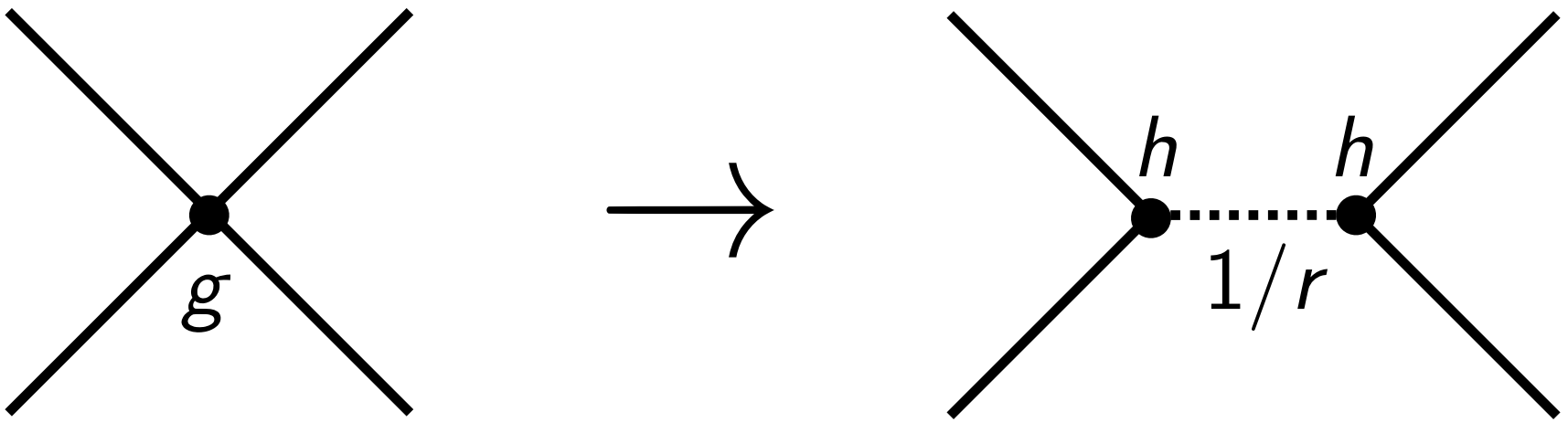
$$\frac{dg}{d \ln b} = -\epsilon g + \frac{g^2}{4\pi} + \dots$$



Fixed point: $g_* = 4\pi\epsilon$

... for $0 < \epsilon \ll 1$

Hubbard-Stratonovich transformation:



“Luttinger-Yukawa” theory

Lagrangian:

$$\mathcal{L}_\psi = \psi^\dagger \left[\partial_\tau + (\partial_x^2 - \partial_y^2)\sigma_1 + 2\partial_x\partial_y\sigma_3 \right] \psi + \frac{1}{2}\phi(r - c\partial_\tau^2 - \partial_x^2 - \partial_y^2)\phi - h\phi\psi^\dagger\sigma_2\psi$$

tuning parameter

parametrizes $z = 1$ (ϕ) vs. $z = 2$ (ψ)

“meson”

“Yukawa” coupling

... ϕ^4 coupling irrelevant for $\epsilon > 0$

Equivalence to \mathcal{L}_ψ :

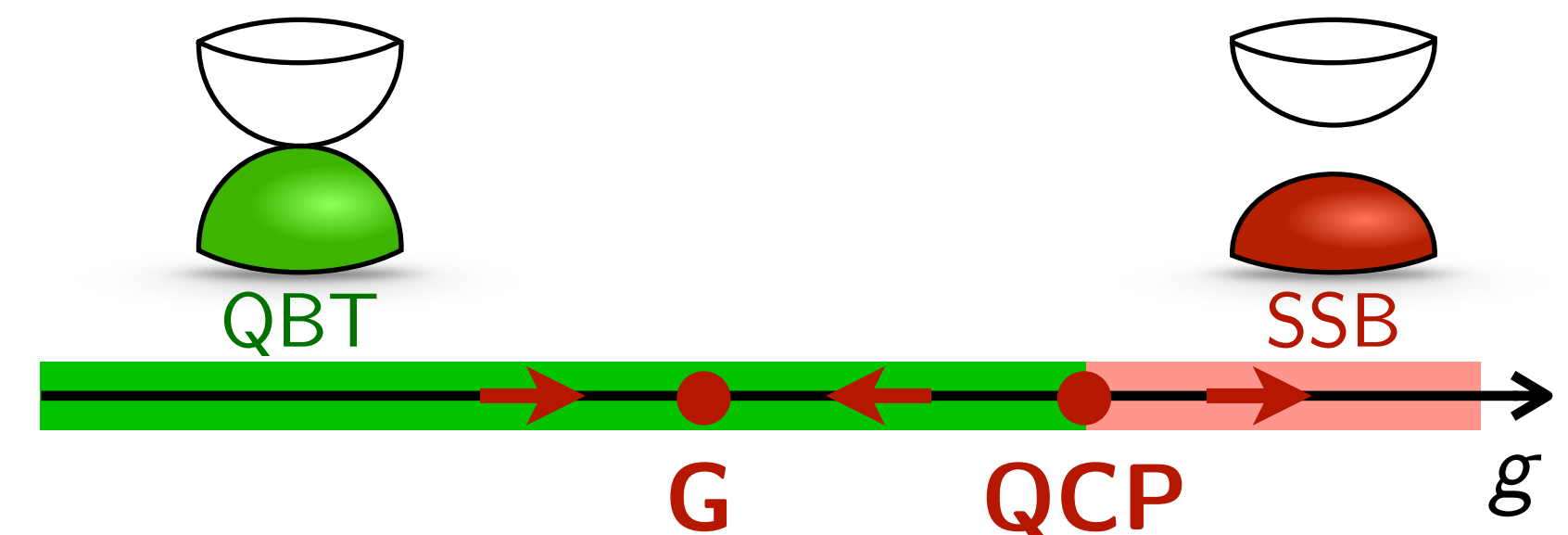
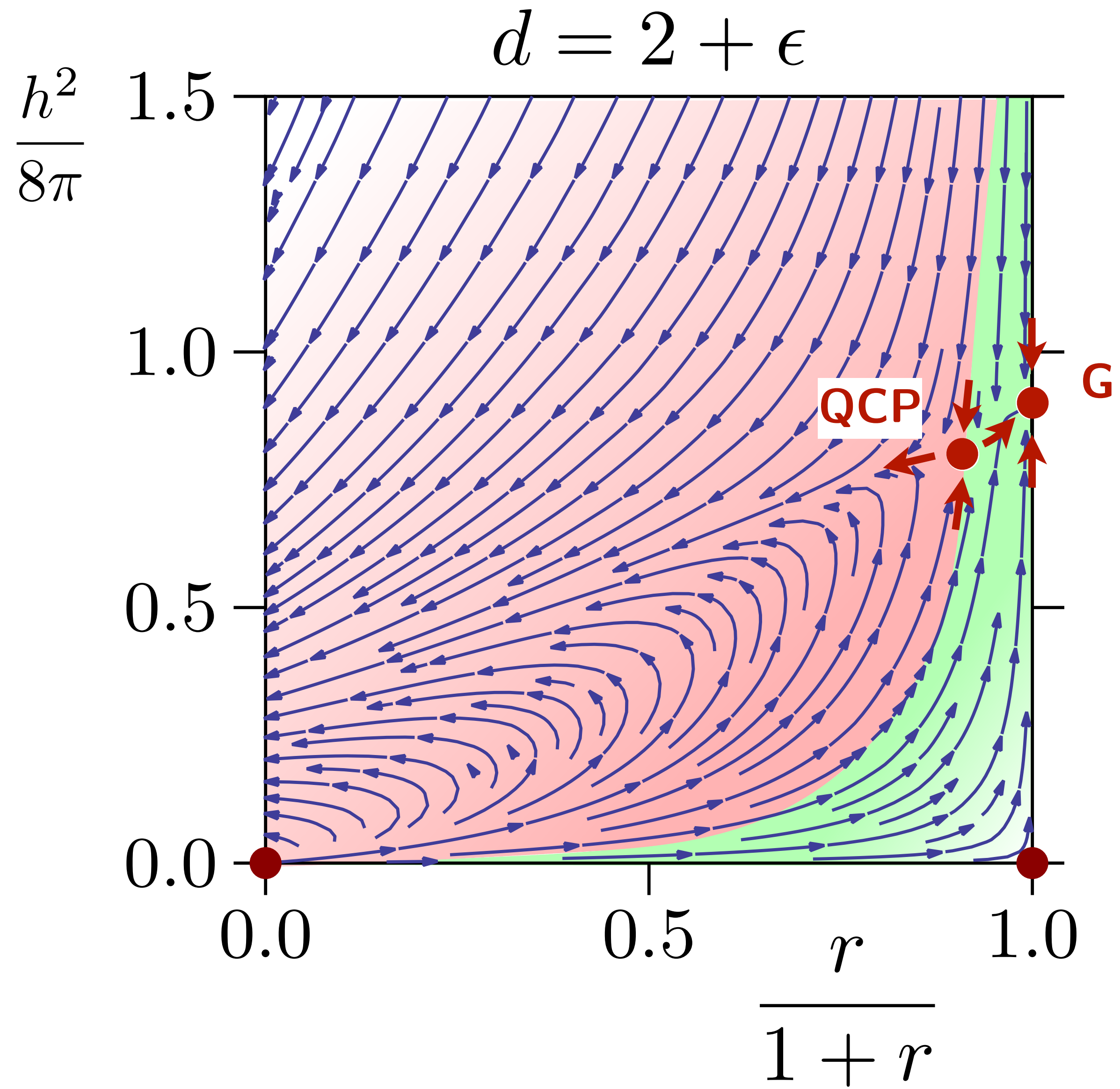
$$g \equiv \frac{h^2}{r}$$

... assume $g > 0$

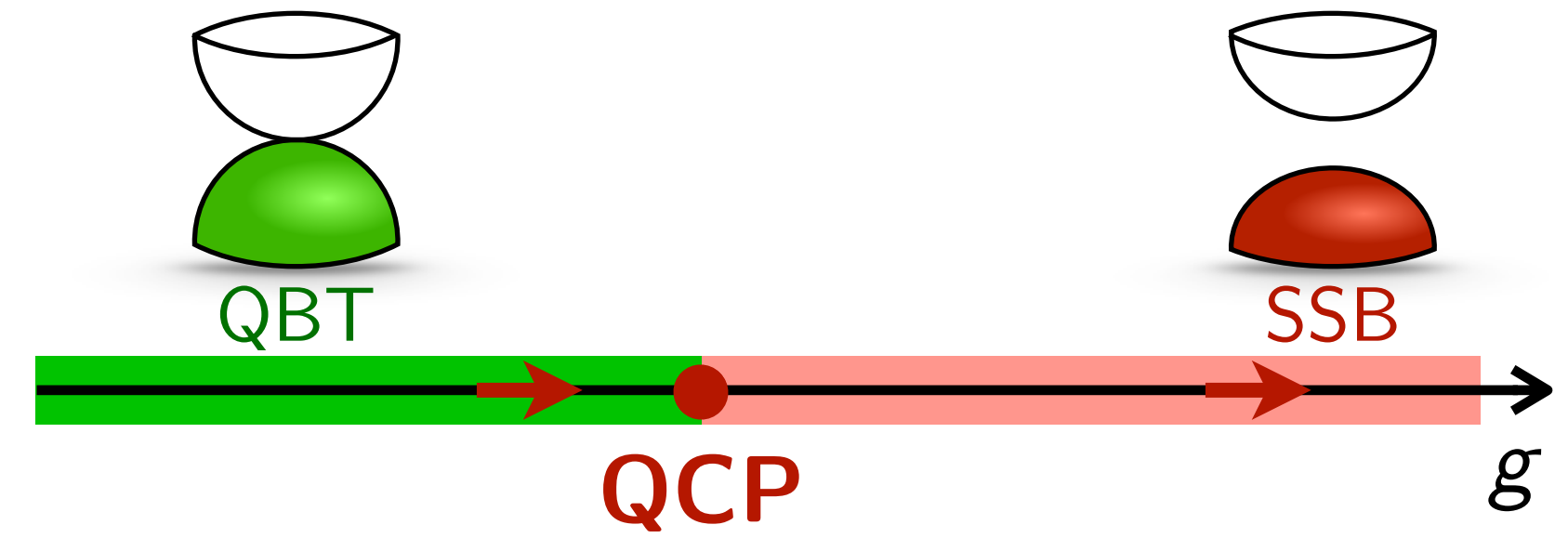
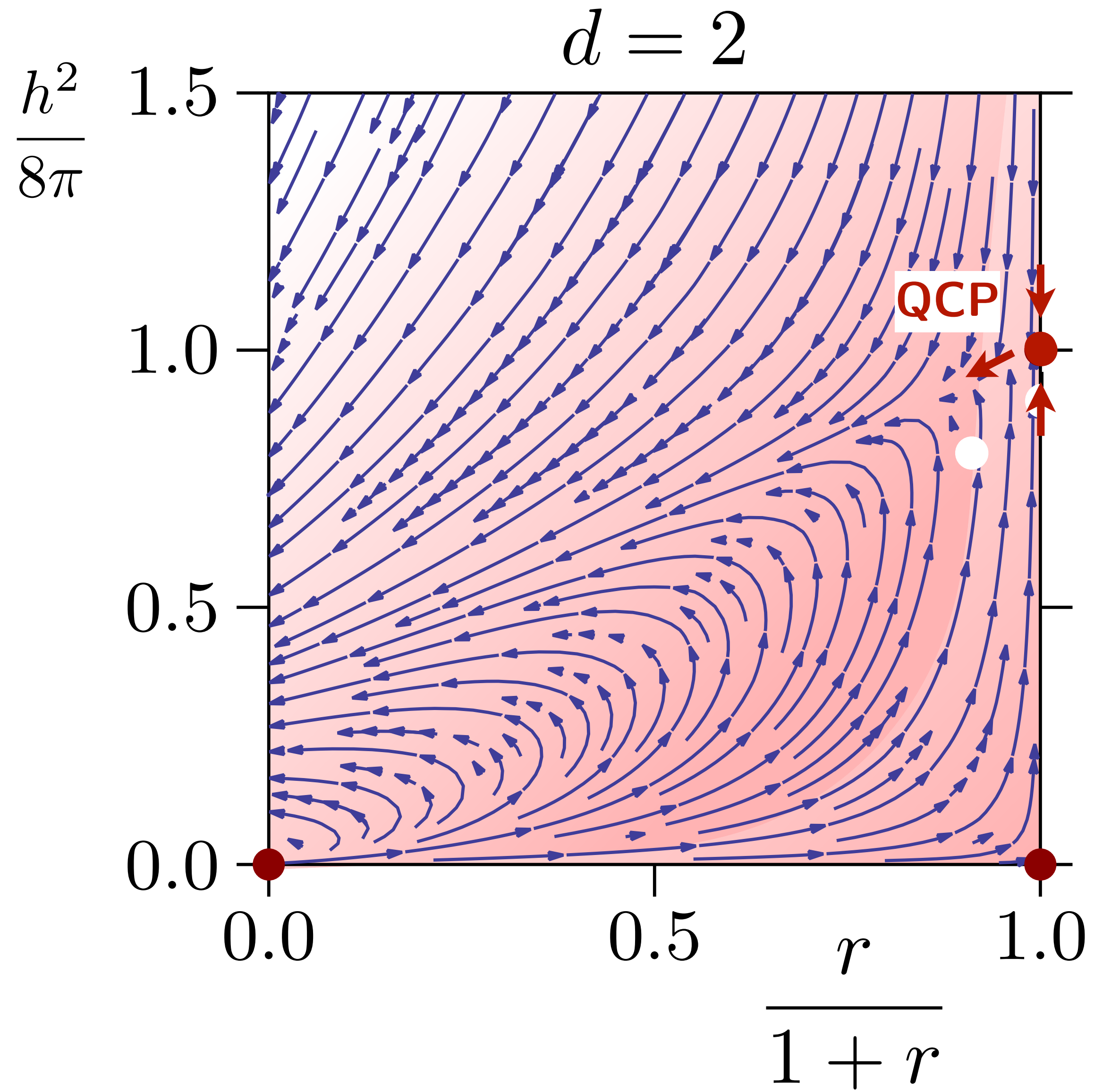
Order parameter:

$$\langle \phi \rangle = \frac{h}{r} \langle \psi^\dagger \sigma_2 \psi \rangle$$

RG flow



RG flow



Quantum critical fixed point ($\epsilon > 0$)

Couplings:

$$r_\star = \frac{2}{\epsilon}$$

$$c_\star = \frac{1}{4} - \frac{\epsilon}{8}$$

$$h_\star^2 = 8\pi(1 - \epsilon)$$

$$\rightarrow \text{equivalent to } g_\star = \frac{h_\star^2}{r} = 4\pi\epsilon$$

Critical exponents:

$$\eta_\phi = 2 - 2\epsilon$$

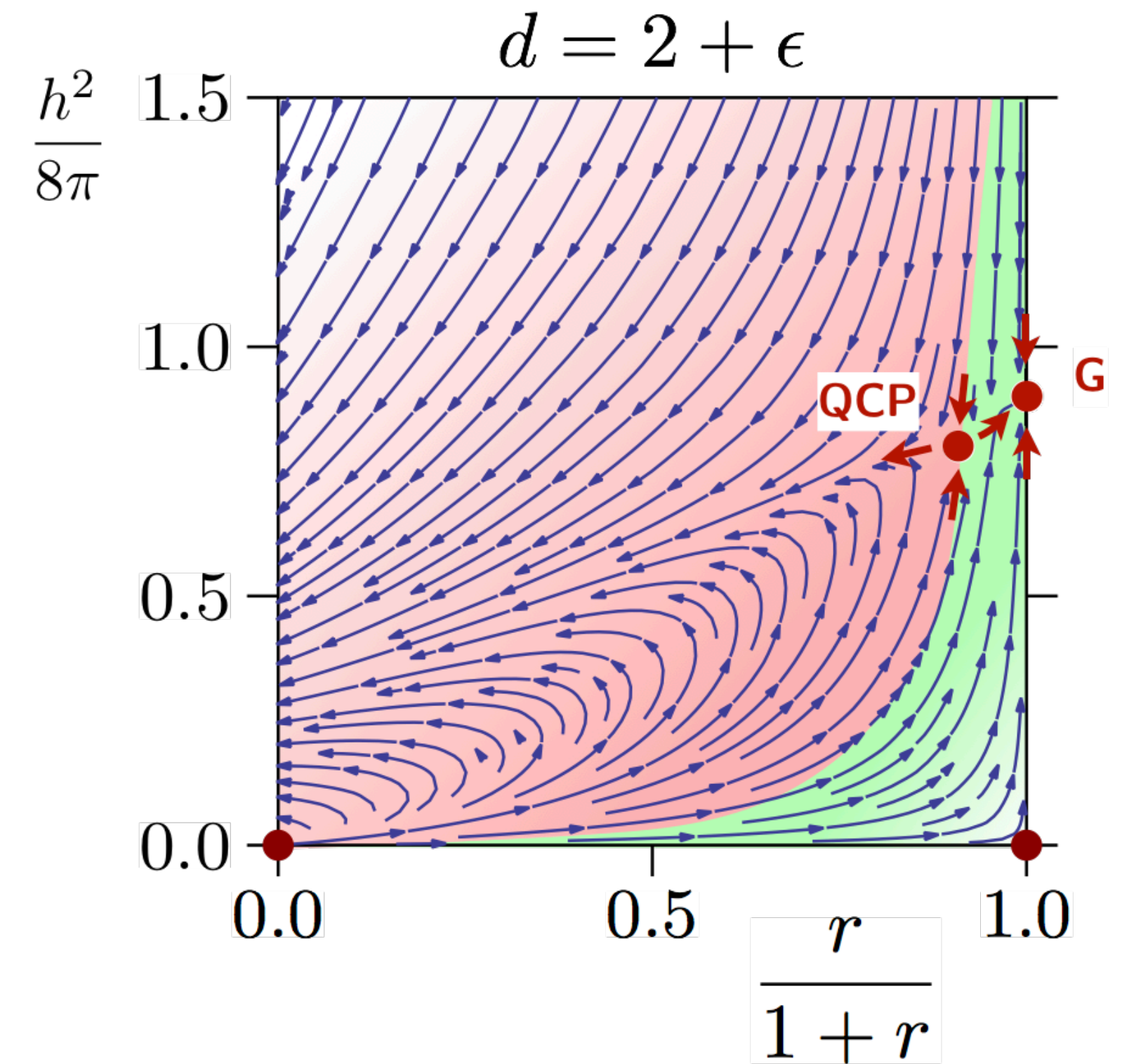
$$\xrightarrow{\epsilon \rightarrow 0} 2$$

$$\eta_\psi = 0$$

$$z = 2$$

$$\nu = 1/\epsilon$$

$$\xrightarrow{\epsilon \rightarrow 0} \infty$$

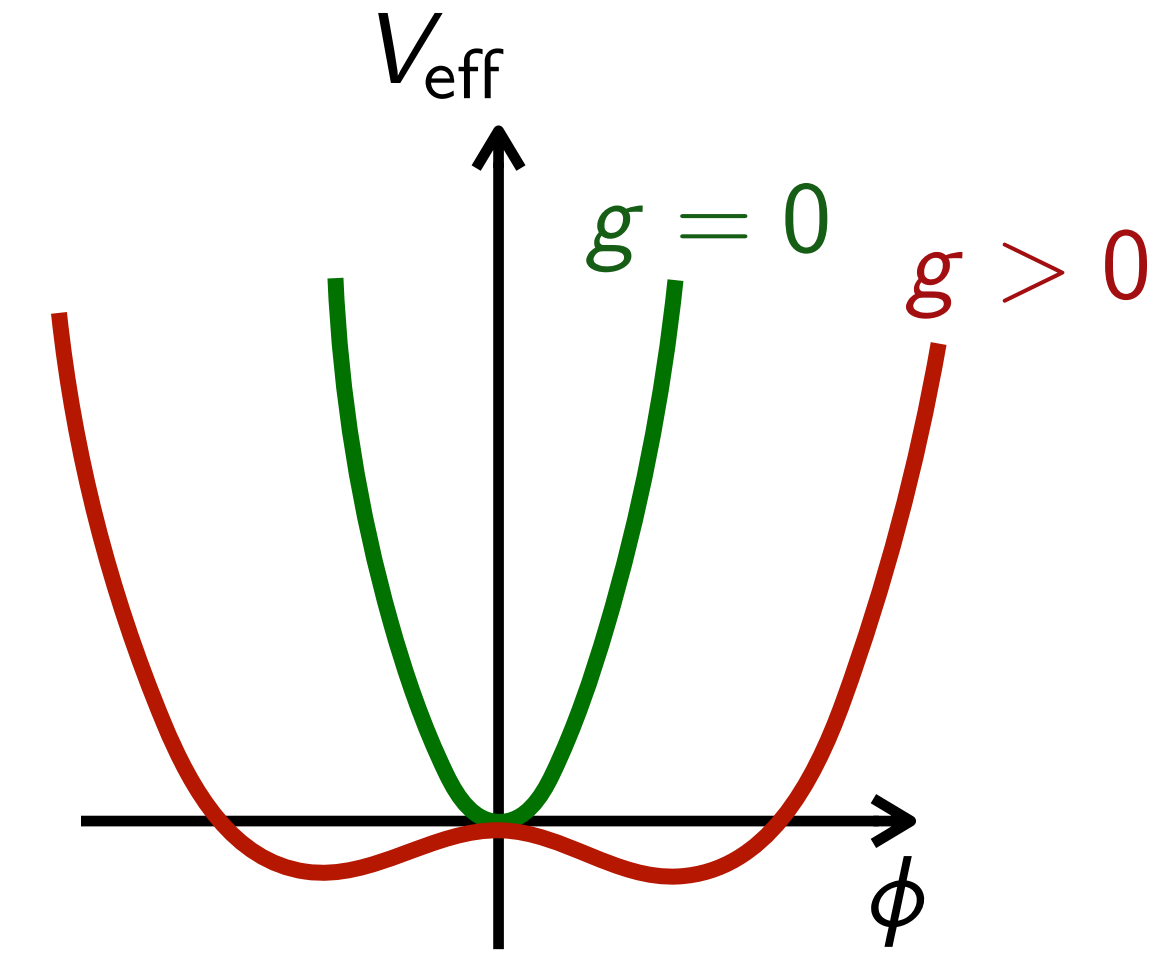


(Careful) 2D limit $\varepsilon \rightarrow 0$

Effective potential in $d = 2$:

$$V_{\text{eff}}(\phi) = \frac{1}{2g}\phi^2 - \frac{1}{16\pi}\phi^2 \left(1 - \ln \frac{\phi^2}{4}\right) + \frac{1}{(16\pi)^2} \frac{g}{2}\phi^2 \left(\ln \frac{\phi^2}{4}\right)^2$$

... from fully integrating out ψ



Order parameter:

$$\langle \phi \rangle \propto e^{-8\pi/g}$$

... essential singularity

... c.f. $\langle \phi \rangle \propto (\delta g)^{2/(d-2)}$ for $d > 2$

Critical “isotherm”:

$$h \propto \langle \phi \rangle^\delta \quad \text{with} \quad \delta = 1$$

... in agreement with hyperscaling

$$\delta = \frac{d+z+2-\eta_\phi}{d+z-2+\eta_\phi} = 1 \quad \text{for} \quad \eta_\phi = 2, \quad z = 2$$

Correlation length and hyperscaling

Correlator near criticality:

$$\langle \phi(0, \mathbf{p}) \phi(0, 0) \rangle^{-1} \sim \text{---} \mathbf{p} \text{---} \bigcirc \text{---} \mathbf{p} \text{---}$$
A Feynman diagram consisting of a central circle with two arrows on its circumference, one pointing clockwise and one pointing counter-clockwise. Two horizontal dotted lines extend from the left and right sides of the circle, each labeled with the letter 'p'.

Correlation length:

$$\xi \propto e^{4\pi/g} \quad (\text{"}\nu = \infty\text{"})$$

... for $g > 0$
... as expected from marginal g

Hyperscaling:

$$V_{\text{eff}}(\langle \phi \rangle) \propto \xi^{-(d+z)} \quad \text{with } z = 2 \text{ and } d = 2$$

... fulfilled despite marginally irrelevant ϕ^4 coupling

Semimetallic fixed point: Luttinger semimetal

Couplings:

$$r_{\star} = \infty$$

$$c_{\star} = \frac{1}{4} - \frac{\epsilon}{16}$$

$$h_{\star}^2 = 4\pi(2 - \epsilon)$$

→ equivalent to $g_{\star} = 0$

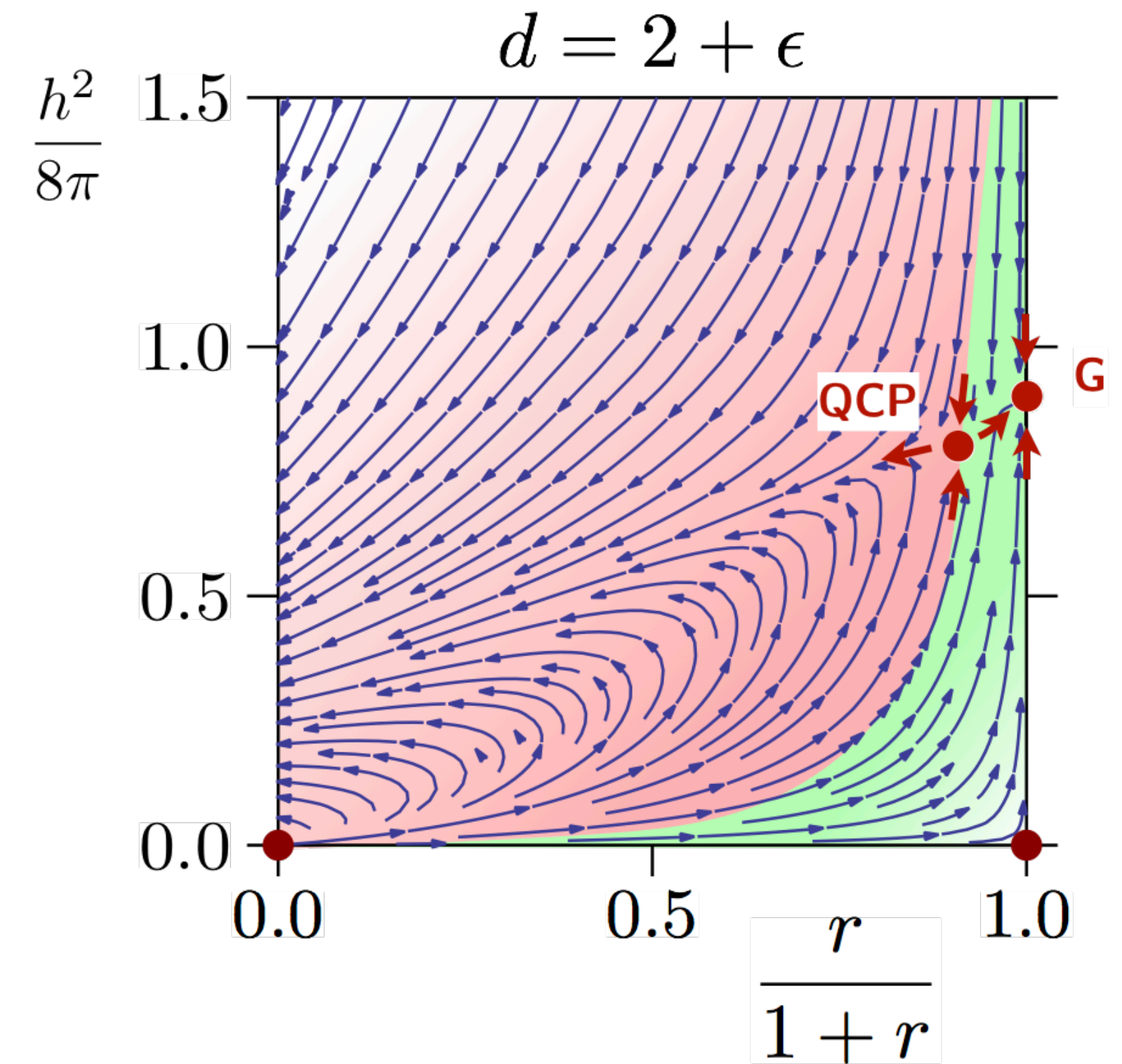
Critical exponents:

$$\eta_{\phi} = 2 - \epsilon \xrightarrow{\epsilon \rightarrow 0} 2$$

$$\eta_{\psi} = 0$$

$$z = 2$$

→ scale-invariant **phase**: “LSM”



... à la 3D QBT @ large N
[LJ & Herbut, PRB '17]

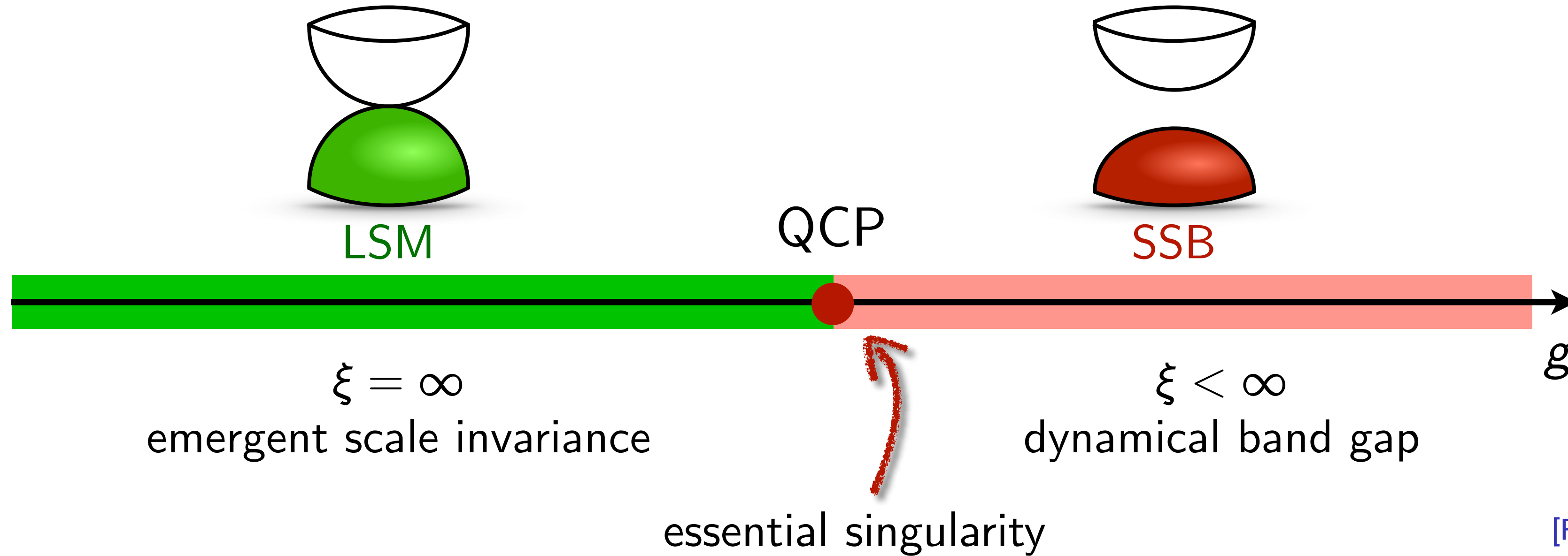
Algebraic “order”

Correlator in semimetallic phase:

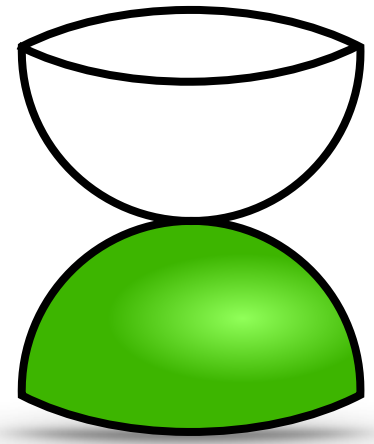
$$\langle \phi(0, \mathbf{r}) \phi(0, 0) \rangle \propto \frac{1}{|\mathbf{r}|^4} \quad \text{for all } g \leq 0$$

... i.e., $\eta_\phi = 2$

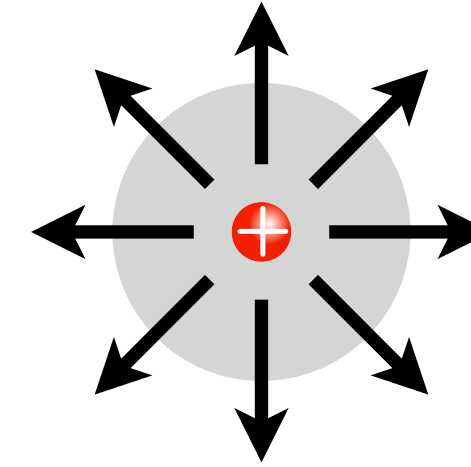
Phase diagram:



Phenomenological analogy: 2D QBT vs. BKT



2D QBT



BKT

Semimetallic phase ($g < 0$):

$$\langle \phi(0, \mathbf{r}) \phi(0, 0) \rangle \propto \frac{1}{|\mathbf{r}|^{z+\eta_\phi}}$$

... with $z = 2$ and $\eta_\phi = 2$

Ordered phase ($g > 0$):

$$\langle \phi(0, \mathbf{r}) \phi(0, 0) \rangle \propto e^{-|\mathbf{r}|/\xi}$$

Critical behavior:

$$\xi \propto e^{4\pi/g} \quad (g > 0)$$

$$h \propto \langle \phi \rangle^\delta \quad \text{with } \delta = 1 \quad (g = 0)$$

Dielectric phase ($t \equiv \frac{T-T_c}{T_c} < 0$):

$$\langle e^{i\theta(\mathbf{r})} e^{-i\theta(0)} \rangle \propto \frac{1}{|\mathbf{r}|^\eta}$$

... with $\eta = \frac{T_\infty}{2\pi} \leq \frac{1}{4}$

... emergent scale invariance

Metallic phase ($t > 0$):

$$\langle e^{i\theta(\mathbf{r})} e^{-i\theta(0)} \rangle \propto e^{-|\mathbf{r}|/\xi}$$

... gapped

Critical behavior:

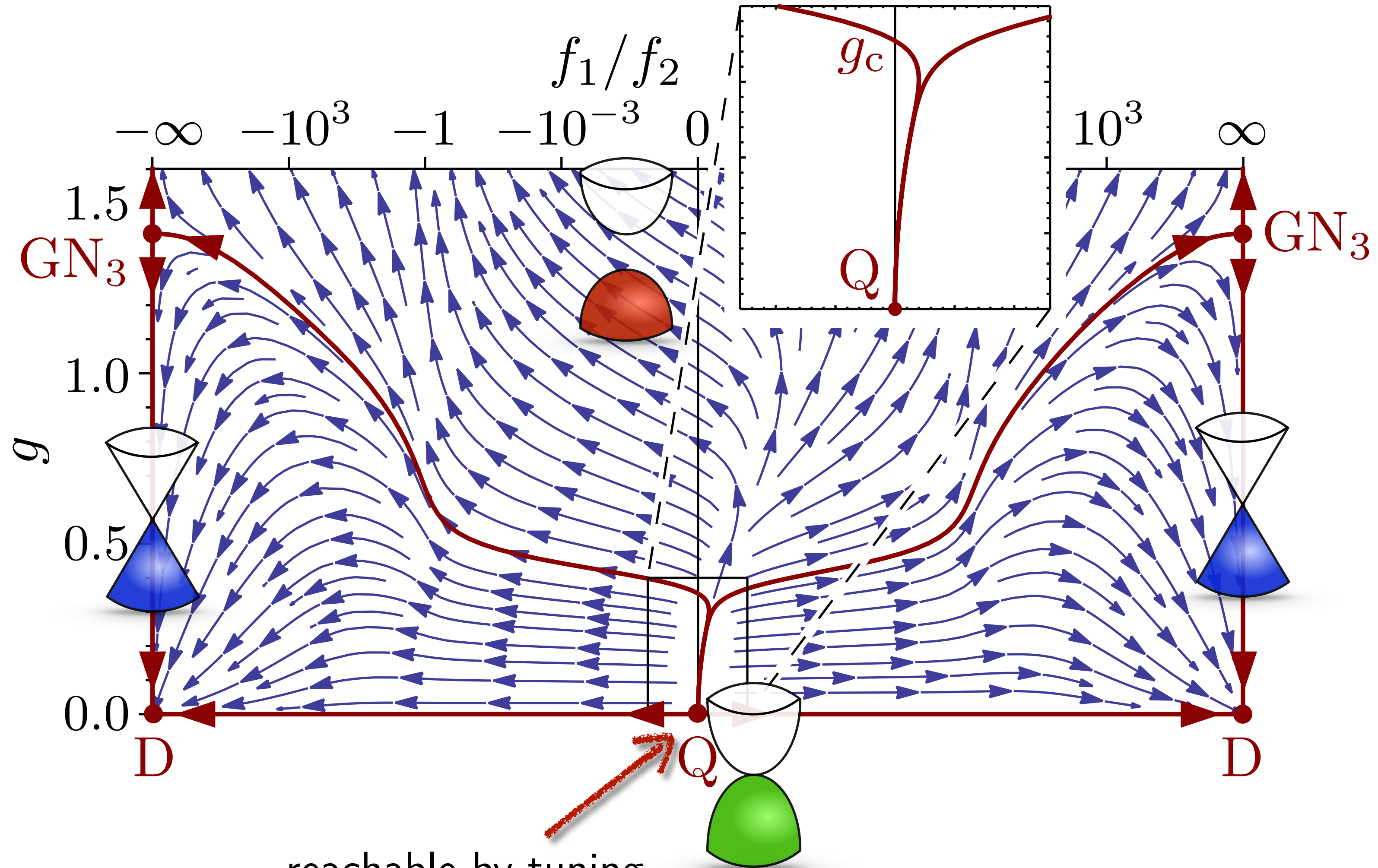
$$\xi \propto e^{C/\sqrt{t}} \quad (t > 0)$$

$$h \propto \langle e^{i\theta(\mathbf{r})} \rangle^\delta \quad \text{with } \delta = 15 \quad (t = 0)$$

... essential singularity

... power law

$z = 2$ QCP in bilayer honeycomb model



... accessible to QMC

[Ray, Vojta, LJ, PRB '18]

→ talk by S. Ray

Summary

