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One-dimensional physics
Problem set 1

Summer term 2017

1. Quantum confinement and one-dimensional states **3 Points**

Consider free three-dimensional electrons subject to a confining potential

$$V(\mathbf{r}) = V_y\,\Theta(|y| - y_0) + V_z\,\Theta(|z| - z_0) \tag{1}$$

with large positive potentials $V_y, V_z \to \infty$ and $y_0, z_0 > 0$.

a) **1 Point**

Give the expression of the real-space Hamiltonian describing this system in second quantization. Why is the product-state Ansatz

$$|k, m, n, \sigma\rangle = \int d^3r\, f_k(x)\, g_m(y)\, h_n(z)\, c_\sigma^\dagger(\mathbf{r})\,|0\rangle, \tag{2}$$

a clever choice if one searches eigenstates of the Hamiltonian (here, $c_\uparrow^\dagger(\mathbf{r})$ creates an electron of spin $\sigma =\uparrow,\downarrow$ at position $\mathbf{r}$, while $|0\rangle$ denotes the vacuum satisfying $c_\sigma(\mathbf{r})|0\rangle = 0$)?

b) **1 Point**

Using the Ansatz of Eq. (2), find the functions $g_m(y)$ and $h_n(z)$ such that $|k, m, n, \sigma\rangle$ is an eigenstate (with k, m, and n labelling different eigenstates). To which form can the Schrödinger equation be reduced for given functions $g_m(y)$ and $h_n(z)$?

c) **1 Point**

For $z_0 \ll y_0 \ll L$, where L is the length of the system along x, only the state with $h_n(z)$ corresponding to the lowest energy along z needs to be retained, which approximately corresponds to the situation realized in a two-dimensional electron gas. Under which further condition are the eigenstates effectively one-dimensional? Solve the Schrödinger equation now also along x (with k labelling the different eigenstates of the x-motion), plot the eigenenergies for the case that $z_0 \ll y_0$, and interpret this spectrum.

2. Conductance and spin-orbit coupling in a 1D chain **3 Points**

Consider a tight-binding model for a one-dimensional chain of spinful electrons with lattice spacing a. The electrons can move between the sites by a spin-conserving real hopping $t < 0$, and by a spin-flip hopping that allows the electrons to hop between sites while flippign their spin. This hopping has the purely imaginary amplitude $i\gamma$ for motion to the right. Each site furthermore has an on-site energy $-\mu$. Why is the name "spin-orbit coupling" appropriate for the imaginary hopping?

a) **1 Point**

Construct the full tight-binding Hamiltonian, and find the eigenenergies using momentum eigenstates. To simplify the expressions of the eigenenergies, you may find it helpful to decompose sinus and cosinus into exponentials. Plot the spectrum.

b) **1 Point**

Expand the Hamiltonian for small momenta to order $\mathcal{O}(k^2)$ and identify the effective mass. Add a

Zeeman energy E_{Zeeman} to the Hamiltonian that splits the energies of the up and down spins. Solve the approximated Hamiltonian, and plot its spectrum for $E_{\text{Zeeman}} = 0$ and finite E_{Zeeman}.

c) **1 Point**

An experimental way to distinguish whether the chemical potential μ is inside the (partial) gap opened by the Zeeman term is to measure the electric conductance through the wire. To determine the latter, assume that all electrons moving to the right are injected from a left electrode, in which all states up to a chemical potential $\mu_L = \mu + \delta\mu_L$ are filled, while all electrons moving to the left are injected from an electrode on the right with states filled up to $\mu_R = \mu + \delta\mu_R$. You can furthermore assume that $\delta\mu_{R,L} \ll \mu$, and that the injected electrons preserve their respective chemical potential $\mu_{R,L}$ as they move through the wire. Calculate the current in the wire as

$$I = \dot{Q} = \sum_n \rho_n \, v_n, \tag{3}$$

where n is the sum over all states, ρ_n is the charge density associated with state n, and v_n is the group velocity of state n (whose sign determines the direction of motion). To identify the conductance G, which is defined via $I = G\,V$ where V is the voltage drop across the wire, you can assume that chemical potential difference $\delta\mu_R - \delta\mu_l$ is precisely caused by the voltage drop.

Hint: you may assume the temperature to be zero, while the wire is of length L, and has periodic boundary conditions. Furthermore, you may find it helpful to take the limit $L \to \infty$ when explicitly calculating I (thus converting sums into integrals).