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One-dimensional physics
Problem set 4

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1. Calculating bosonized correlation functions **7 Points**

a) **1 Point**

To establish a benchmark against which the bosonized correlation functions can be tested, we first calculate the following fermionic Green's functions

$$G_R(x-x',\tau-\tau') = \langle T_\tau\, R^\dagger(x',\tau')R(x,\tau)\rangle \quad \text{and} \quad G_L(x-x',\tau-\tau') = \langle T_\tau\, L^\dagger(x',\tau')L(x,\tau), \tag{1}$$

where τ denotes imaginary time, and T_τ is the imaginary time ordering operator. For fermionic operators $\mathcal{O}(\tau)$ and $\mathcal{O}'(\tau')$, its action can be summarized as

$$T_\tau\, \mathcal{O}(\tau)\mathcal{O}'(\tau') = \mathcal{O}(\tau)\mathcal{O}'(\tau')\,\theta(\tau-\tau') - \mathcal{O}'(\tau')\mathcal{O}(\tau)\,\theta(\tau'-\tau). \tag{2}$$

To find the expression of the Green's functions, start from the linearized Hamiltonians

$$H_r = \sum_q \hat{r} v_F q\, r_q^\dagger r_q \quad \text{with} \quad r = R, L \quad \text{and} \quad \hat{R} = +1,\ \hat{L} = -1. \tag{3}$$

You may then Fourier transform the creation and annihilation operators to momentum space, and use the Heisenberg picture in imaginary time,

$$\mathcal{O}(\tau) = e^{\tau H}\, \mathcal{O}\, e^{-\tau H}, \tag{4}$$

where H is the Hamiltonian. Finally, you can evaluate the Green's functions in Lehmann representation, which amounts to inserting a complete set of basis states in between the two fermionic operators. Restricting yourself to zero temperature, you can evaluate the expectation value with respect to the ground state, in which all states with negative energy are occupied. You may then take the continuum limit in momentum space to evaluate the momentum sums, but keep in mind that the linearized low-energy theory is only valid at sufficiently small momenta, which we have learned in the lecture can be taken into account by the introduction of an exponential cutoff factor $e^{-\alpha|q|}$ into the momentum sums. (Alternatively, you may use the coherent state path integral formalism, or the equation of motion technique).

b) **1 Point**

In the next steps, we study a similar expression using the bosonized Hamiltonian, and normal ordering. To this end, consider a spinless Luttinger liquid associated with the Hamiltonian

$$H = \int \frac{dx}{2\pi}\left(\frac{u}{K}(\partial_x\phi)^2 + uK(\partial_x\theta)^2\right), \tag{5}$$

and where the right and left moving modes are bosonized as

$$r^\dagger(x) = \frac{U_r^\dagger}{\sqrt{2\pi\alpha}} e^{i\Phi_r(x)} \quad \text{where} \quad \Phi_r(x) = \hat{r}\phi(x) - \theta(x) \quad \text{and} \quad [\phi(x),\theta(x')] = \frac{i\pi}{2}\mathrm{sgn}(x'-x). \tag{6}$$

In the lecture, we saw that normal ordering of the bosonized fields amounts to placing all bosonic creation operators "to the left", which can be implemented by a decomposition of Φ_r into parts $\Phi_r^\pm$ that contain only bosonic creation operators, or annihilation operators, respectively. These operators create bosonic excitations with respect to the non-interacting vacuum. Importantly, this vacuum is not the ground state for the interacting system. To solve this discrepancy in the next exercise, start by finding a transformation of the bosonized fields that maps the interacting Hamiltonian to a non-interacting one.

c) **2 Points**

Use this mapping, along with the finite-size expressions of the commutators

$$[\Phi_r(x), \Phi_{r'}(x')] = \delta_{r,r'} \ln\left(\frac{\alpha + i\hat{r}(x - x')}{\alpha - i\hat{r}(x - x')}\right), \tag{7}$$

$$[\Phi_r^+(x), \Phi_{r'}^-(x')] = \delta_{r,r'} \ln\left(1 - e^{-2\pi(\alpha + i\hat{r}(x-x'))/L}\right), \tag{8}$$

where L is the length of the wire, to show that normal ordering in the interacting system amounts to

$$r^\dagger(x')r(x) = \frac{-i}{2\pi}\frac{1}{\hat{r}(x - x') - i\alpha}\left(\frac{\alpha}{\sqrt{\alpha^2 + (x - x')^2}}\right)^{\frac{K+1/K-2}{2}} \tilde{\vdots}\, r^\dagger(x')r(x) \,\tilde{\vdots} \tag{9}$$

in the limit $L \to \infty$, and where $\tilde{\vdots}\ \tilde{\vdots}$ denotes normal ordering with respect to the ground state of the interacting system. Is the resulting expression consistent with the result obtained in the fermionic calculation?

d) **1 Point**

Finally, we turn to the evaluation of correlation functions using the field theoretical approach to bosonization. To this end, we recall that the Fourier transforms of the bosonized fields to momentum and Matsubara frequencies are given by

$$\phi(x,\tau) = \frac{1}{\sqrt{\beta L}}\sum_{q,\omega_n} e^{i(qx - \omega_n\tau)}\phi_{q,\omega_n} \quad \text{and} \quad \theta(x,\tau) = \frac{1}{\sqrt{\beta L}}\sum_{q,\omega_n} e^{i(qx-\omega_n\tau)}\theta_{q,\omega_n}, \tag{10}$$

in terms of which the action associated with the Luttinger liquid Hamiltonian of Eq. (5) reads

$$\mathcal{S} = \sum_{q,\omega_n}\left(\frac{iq\omega_n}{\pi}\phi_{-q,-\omega_n}\theta_{q,\omega_n} + \frac{u}{2\pi K}q^2\phi_{-q,-\omega_n}\phi_{q,\omega_n} + \frac{uK}{2\pi}\theta_{-q,-\omega_n}\theta_{q,\omega_n}\right). \tag{11}$$

Identify the matrix S that allows to bring the action to the form

$$\mathcal{S} = \frac{1}{2}\sum_{q,\omega_n}(\phi^*_{q,\omega_n}, \theta^*_{q,\omega_n})\, S_{q,\omega_n} \begin{pmatrix}\phi_{q,\omega_n}\\ \theta_{q,\omega_n}\end{pmatrix}. \tag{12}$$

In the next step, we will use the path integral formalism to evaluate the general correlation function

$$\mathcal{C} = \langle T_\tau \prod_j e^{i(A_j\phi(x_j,\tau_j) + B_j\theta(x_j,\tau_j))} = \langle T_\tau\, e^{i(A_1\phi(x_1,\tau_1)+B_1\theta(x_1,\tau_1))}\, e^{i(A_2\phi(x_2,\tau_2)+B_1\theta(x_2,\tau_2))}\,\ldots\rangle \tag{13}$$

involving a general number of bosonized *operators* at some positions x_j and imaginary times τ_j, and real prefactors A_j and B_j. Given the imaginary time evolution detailed in Eq. (4), show that the time evolution of an exponential of an operator $\mathcal{O}$ field can be written as

$$e^{iA\mathcal{O}(\tau)} = e^{\tau H}\, e^{iA\mathcal{O}}\, e^{-\tau H} \tag{14}$$

with $\mathcal{O} = \mathcal{O}(\tau = 0)$. Use this to sketch the most important steps in the construction of the path integral for $\mathcal{C}$, and state the path integral form of the correlation function (without evaluating it).

e) **1 Point**

After going to Fourier space, solve the path integral by a suitable shift of the integration variables. Show that this leads to

$$\mathcal{C} = \exp\left\{-\frac{1}{2}\frac{1}{\beta L}\sum_{k,l}\sum_{q,\omega_n} e^{i(q(x_k-x_l)-\omega_n(\tau_k-\tau_l))}(A_k, B_k)\, S^{-1}_{q,\omega_n}\begin{pmatrix}A_l\\ B_l\end{pmatrix}\right\}. \tag{15}$$

f) **1 Point**

As is discussed in detail in Appendix C of Giamarchi's book "Quantum Physics in One Dimension", the explicit evaluation of the Eq. (15) of yields $\mathcal{C} = 0$ unless $\sum_j A_j = 0 = \sum_j B_j$. For $\sum_j A_j = 0 = \sum_j B_j$ and at zero temperature, on the other hand, one finds

$$\mathcal{C} = \exp\left\{ -\frac{1}{2} \sum_{k<l} \big([-A_k A_l \, K - B_k B_l K^{-1}] \, F_1(x_k - x_l, \tau_k - \tau_l) + [A_k B_l + B_k A_l] \, F_2(x_k - x_l, \tau_k - \tau_l) \big) \right\} \tag{16}$$

with

$$F_1(x,\tau) = \ln\left(\frac{\sqrt{x^2 + (u|\tau| + \alpha)^2}}{\alpha} \right) \quad \text{and} \quad F_2(x,\tau) = -i \arg\left(u\tau + \alpha \, \mathrm{sgn}(\tau) + ix\right). \tag{17}$$

Check that this prescription reproduces the earlier results.[1]

[1] Note that there is a caveat concerning bosonic and fermionic imaginary time ordering that is taken into account correctly by the expressions of F_1 and F_2 as given in Eq. (17) - for more on that, see Appendix C of Giamarchi's book.