UNIVERSITY OF Southampton

School of Physics and Astronomy

Group Theory

Day 1: Discrete Groups
G1: smaller groups
G2: larger groups

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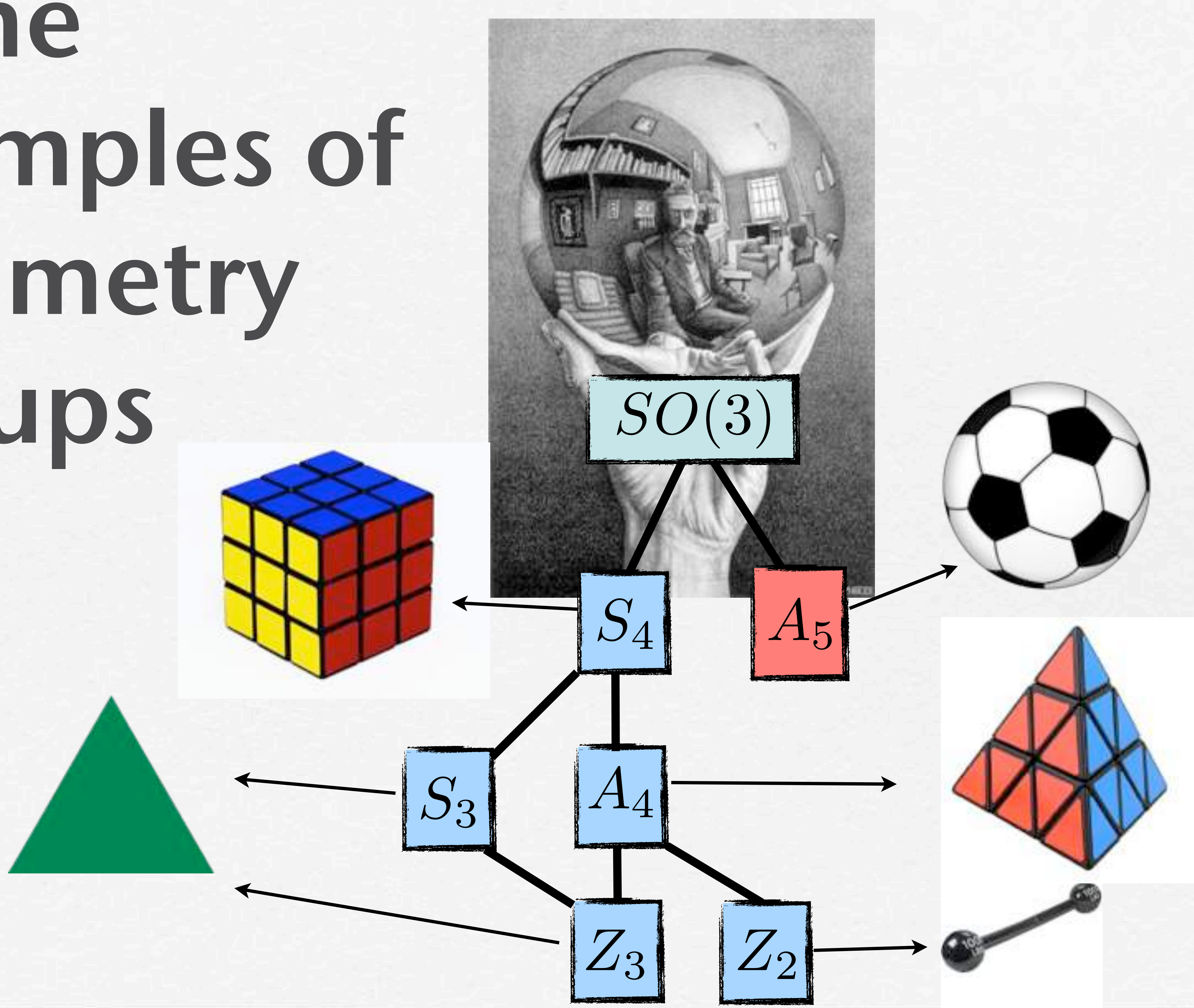
G1: smaller groups

- Symmetry in Nature
- Group axioms
- Z_2 a.k.a. C_2
- Z_3 a.k.a. C_3
- S_3 a.k.a. D_3 or Dih_3

Symmetry in Nature

- Let's play a game...
- I give you an object and then you must do something to it so that it looks the same
- The list of all things you can do to it is called a symmetry group or "group" for short
- The smallest group consists of doing nothing, that is called the "identity" and contains one element e, but that is boring...
- We will consider more interesting groups...

Some examples of symmetry groups

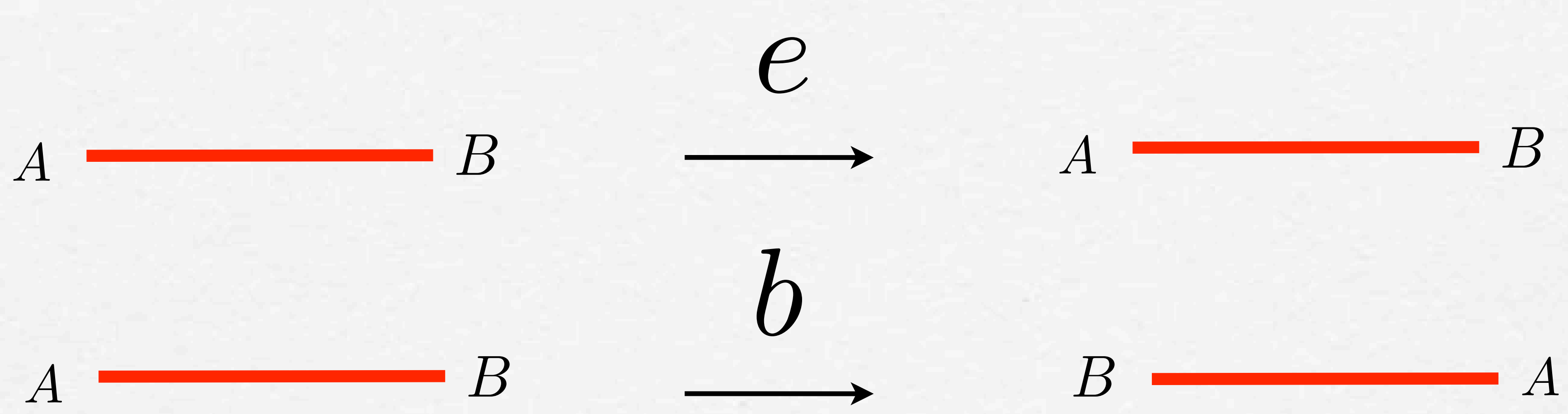


Group axioms

- A group is a set of elements a,b,... which can be combined together with ab inside the set
- (ab)c=a(bc)
- One element e satisfies ae=ea=a for all a
- For each element a there is an element a^{-1} which satisfies $aa^{-1}=a^{-1}a=e$
- e.g. square matrices form groups under matrix multiplication (see Appendix on matrices)

Z_2, the permutation group of 2 objects

- Play game with a line with two ends A,B



$$b^2 = e$$

$$b^{-1} = b$$

$$\begin{pmatrix} A \\ B \end{pmatrix} \xrightarrow{e} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

action of group — original state — new state

$$\begin{pmatrix} A \\ B \end{pmatrix} \xrightarrow{b} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} B \\ A \end{pmatrix}$$

- Matrix representation $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

- Matrix representation satisfies multiplication table

	e	b
e	e	b
b	b	e

- Two dimensional representation is reducible to diagonal form by a maximal mixing unitary matrix U

$$\mathbf{2} \rightarrow \mathbf{1} + \mathbf{1}'$$

$$b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow U^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

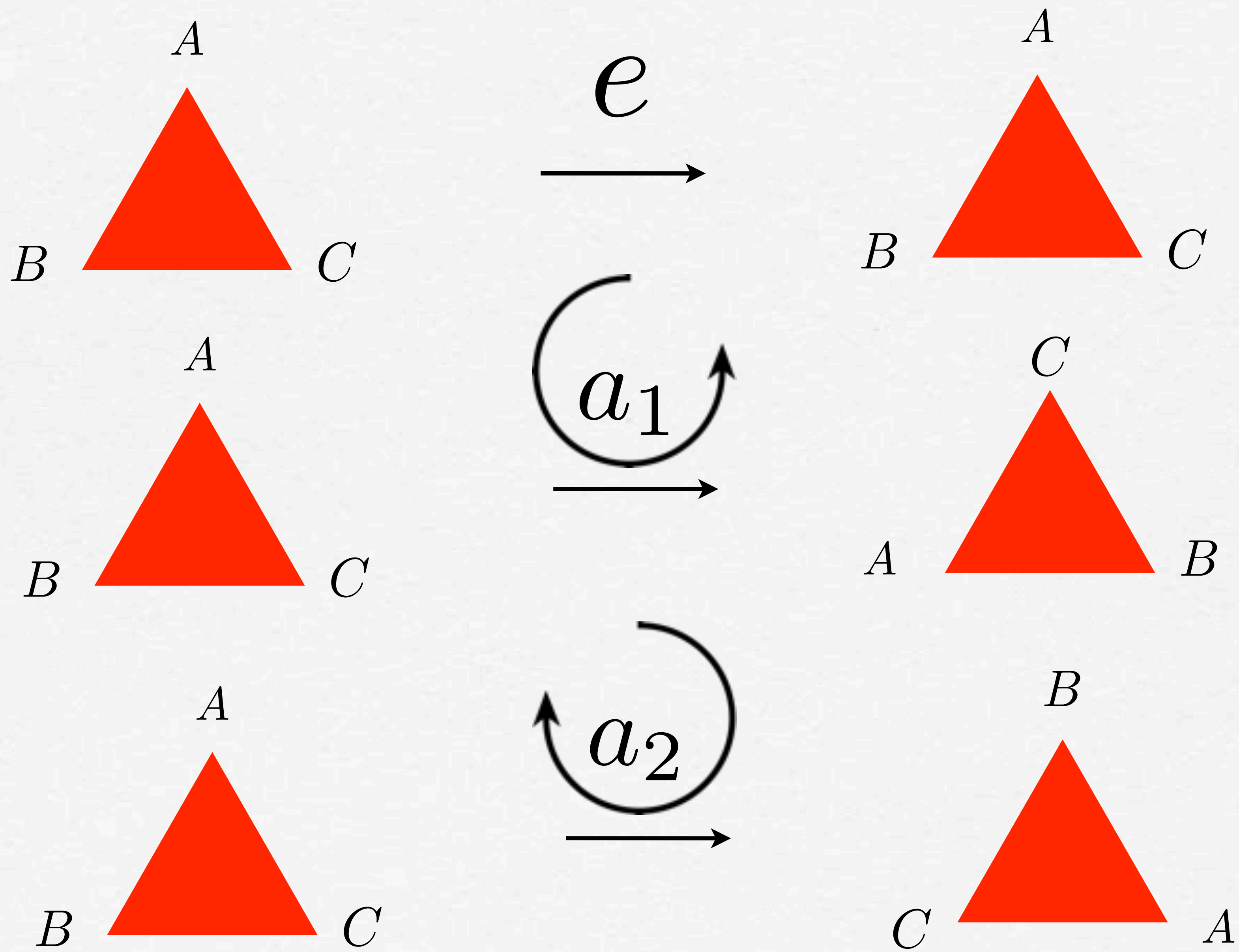
$$\mathbf{1}: \quad e = 1, \quad b = 1$$

$$\mathbf{1}': \quad e = 1, \quad b = -1$$

- Can write $\quad -1 = e^{i\pi} = \alpha \qquad \alpha^2 = 1$

- Can combine two irreducible reps $\quad \mathbf{1}' \times \mathbf{1}' = \mathbf{1}$

Z_3 is the symmetry group of 120° rotations of an equilateral triangle



$$e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$a_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$a_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

- Satisfies multiplication table →

	e	a_1	a_2
e	e	a_1	a_2
a_1	a_1	a_2	e
a_2	a_2	e	a_1

- Define "generator" $a=a_1$
- Then $\{e,a_1,a_2\}=\{e,a,a^2\}$
- Three dim rep is reducible to diagonal form $\mathbf{3} \to \mathbf{1} + \mathbf{1}' + \mathbf{1}''$

$$U^{-1}a_1U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \qquad U^{-1}a_2U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

	e	a_1	a_2
$\mathbf{1}$	1	1	1
$\mathbf{1}'$	1	ω	ω^2
$\mathbf{1}''$	1	ω^2	ω

"Character table"

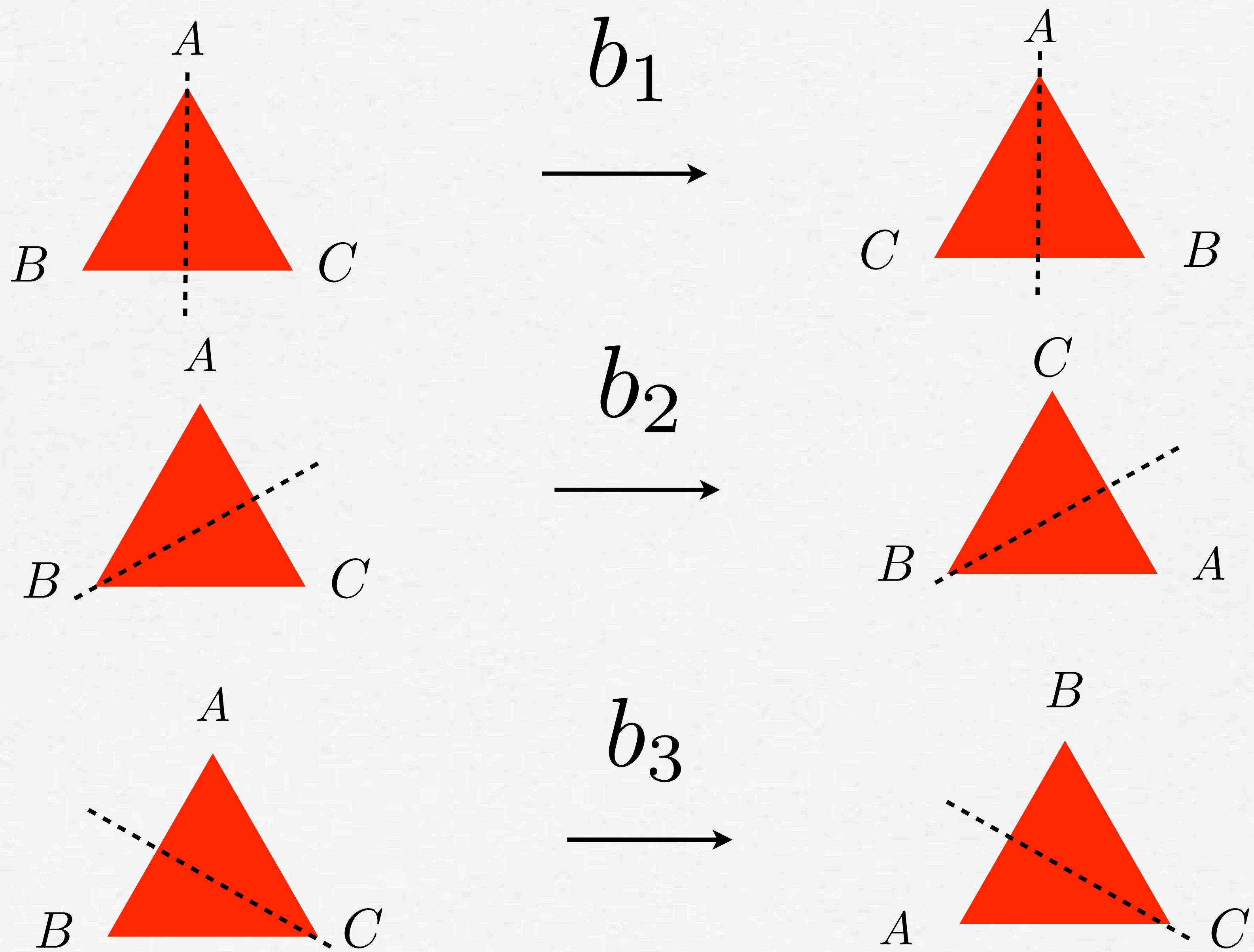
- We write $\omega = e^{i2\pi/3}$

$$\omega^3 = 1$$

- Can combine two irreducible reps

$$\mathbf{1}' \times \mathbf{1}' = \mathbf{1}''$$
$$\mathbf{1}' \times \mathbf{1}'' = \mathbf{1}$$

S_3 symmetry also includes the reflections:



$$b_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$b_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$b_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- S_3 is permutation group of 3 objects (A,B,C) → (A,B,C),(C,A,B),(B,C,A),(A,C,B),(C,B,A),(B,A,C)
- e a_1 a_2 b_1 b_2 b_3
- even even even odd odd odd
- even/odd refers to number of two-element swaps
- e: zero swaps, $\{a_1, a_2\}$:two swaps, $\{b_1, b_2, b_3\}$:one
- Z_3 rotation subgroup is $\{e, a_1, a_2\}$, the even perms
- Z_2 reflection subgroups: $\{e, b_1\}$, $\{e, b_2\}$, $\{e, b_3\}$
- Subgroups are subsets of $\{e, a_1, a_2, b_1, b_2, b_3\}$ which form a group by themselves

- S_3 can be defined by its multiplication table
- It is a non-Abelian group since its elements do not all commute e.g. $a_1b_1=b_2$, $b_1a_1=b_3$ so $a_1b_1 \neq b_1a_1$
- The order **of the group** is the number of elements = 6
- The order **of each element** is the power which gives e
- $a_i^3=e$ order 3, $b_i^2=e$ order 2

S_3	e	a_1	a_2	b_1	b_2	b_3
e	e	a_1	a_2	b_1	b_2	b_3
a_1	a_1	a_2	e	b_2	b_3	b_1
a_2	a_2	e	a_1	b_3	b_1	b_2
b_1	b_1	b_3	b_2	e	a_2	a_1
b_2	b_2	b_1	b_3	a_1	e	a_2
b_3	b_3	b_2	b_1	a_2	a_1	e

- Define "generators" $a=a_1$, $b=b_1$
- $\{e,a_1,a_2,b_1,b_2,b_3\}=\{e,a,a^2,b,ab,ba\}$

- S$_3$ multiplication table can be generated by a and b with the rules $a^3 = b^2 = e, \ (ab)^2 = e$
- Called "presentation" $< a, b | \ a^3 = b^2 = e, \ (ab)^2 = e >$
- The set of group elements g ∈ {e,a,a^2,b,ab,ba}
- fall into 3 "conjugacy classes" {e}, {a,a^2}, {b,ab,ba}
- corresponding to {geg^{-1}}, {gag^{-1}}, {gbg^{-1}} for all g
- Notation for classes: 1C^1(e), 2C^3(a), 3C^2(b)
- Each member of class has same order

#elements in class

- **Exercise: show that the rotations and reflections form separate conjugacy classes**

- Three dim rep is reducible to block diagonal form

$$a = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad U^{-1}aU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

Ex.

$$b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad U^{-1}bU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega^2 & 0 \end{pmatrix}$$

$$\mathbf{3} \to \mathbf{1} + \mathbf{2}$$

- irreducible complex doublet representation

$$\mathbf{2}: \qquad a = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \quad b = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}$$

irreducible representations of S_3

$\mathbf{1}:$ $a = 1, \quad b = 1$ unfaithful

$\mathbf{1}':$ $a = 1, \quad b = -1$ unfaithful

$$\mathbf{2}: \quad a = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}$$

- Rule 1: # irreps= #classes=3
- Rule 2: sum square irreps =group order $1^2+1^2+2^2=6$

- irreps are basis dependent but are characterised by their trace (N.B. $1+\omega+\omega^2=0$)

- In another basis the faithful doublet satisfies Tr(a)=-1 and Tr(b)=0 as in the original basis

$$\mathbf{2}: \quad a = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Shows that irrep **2** is real

- Character table of S_3:

	e	a	b
$\mathbf{1}$	1	1	1
$\mathbf{1}'$	1	1	-1
$\mathbf{2}$	2	-1	0

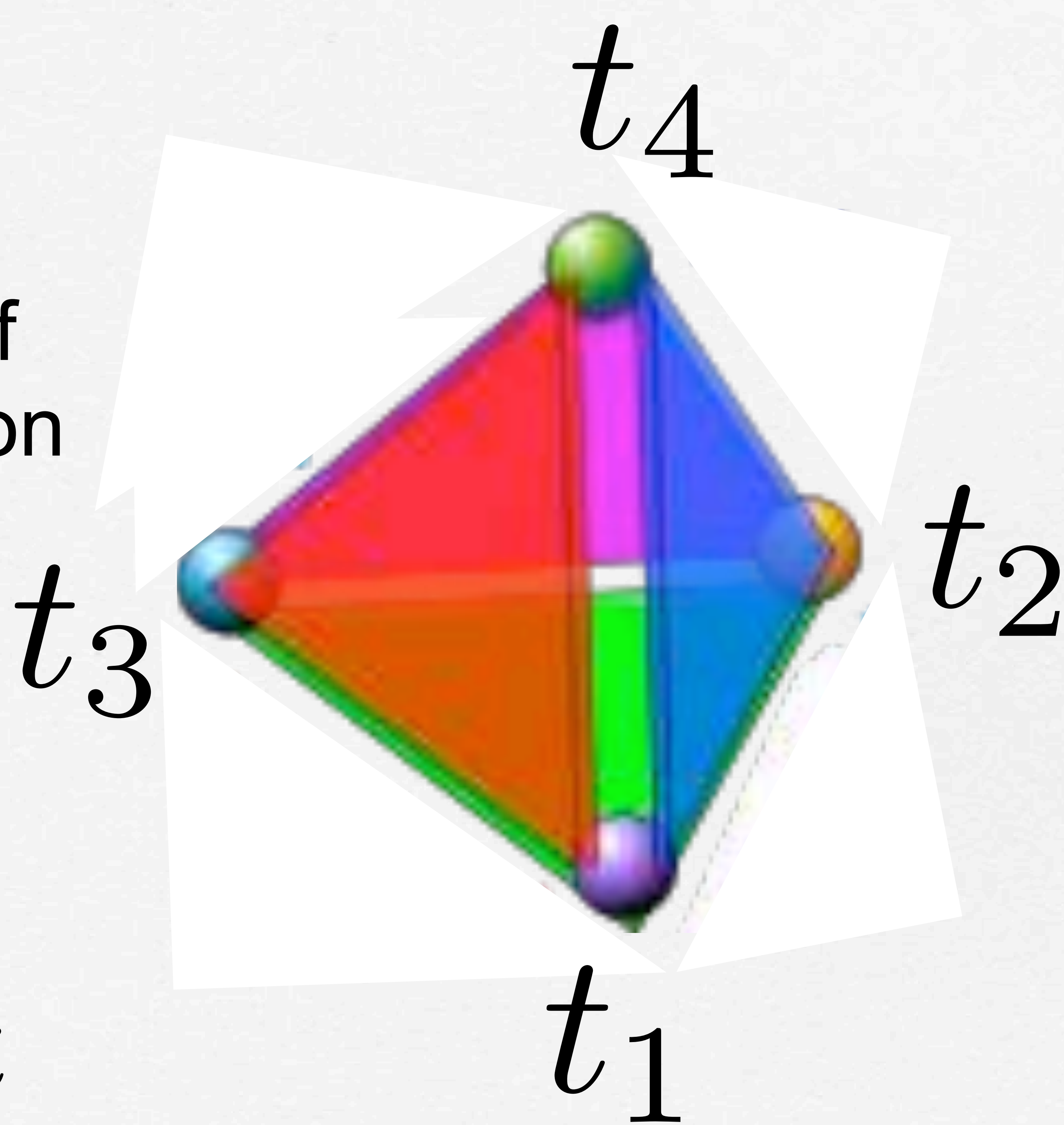
- Trace of elements as shown characterises that irrep
- Notation for characters=traces: $\chi_i^{[\mathbf{1}]}, \chi_i^{[\mathbf{1}']}, \chi_i^{[\mathbf{2}]}$
- E.g. irrep **2** has $\chi_e^{[\mathbf{2}]} = 2,\ \chi_a^{[\mathbf{2}]} = -1,\ \chi_b^{[\mathbf{2}]} = 0$
- One dimensional irreps have trivial traces
- All elements in same class have same trace
- $Tr(gag^{-1})=Tr(a)=-1$, $Tr(gbg^{-1})=Tr(b)=0$ for **2** irrep
- Recall $1C^1(e)=\{e\}$, $2C^3(a)=\{a,a^2\}$, $3C^2(b)=\{b,ab,ba\}$

G2: larger groups

- A_4 a.k.a. T
- Z_N a.k.a. C_N
- S_N
- S_4 a.k.a. O
- Subgroups
- D_N or Dih_N
- Symmetries in molecules and crystals

A_4

Symmetry of the tetrahedron



Vertices labelled by t_i

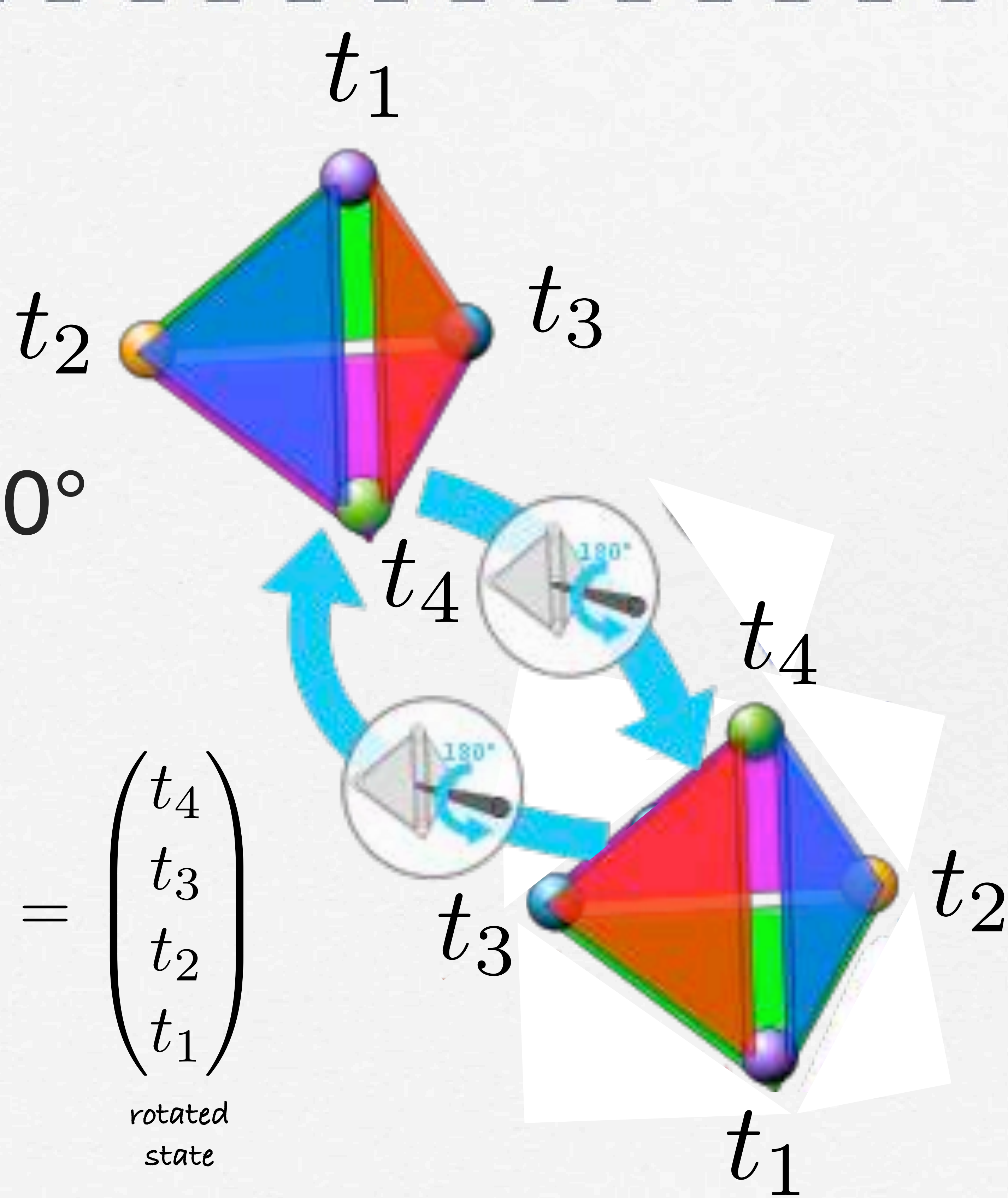
A_4

- rotation by 180°

$$S$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} t_4 \\ t_3 \\ t_2 \\ t_1 \end{pmatrix}$$

rotation matrix — original state — rotated state

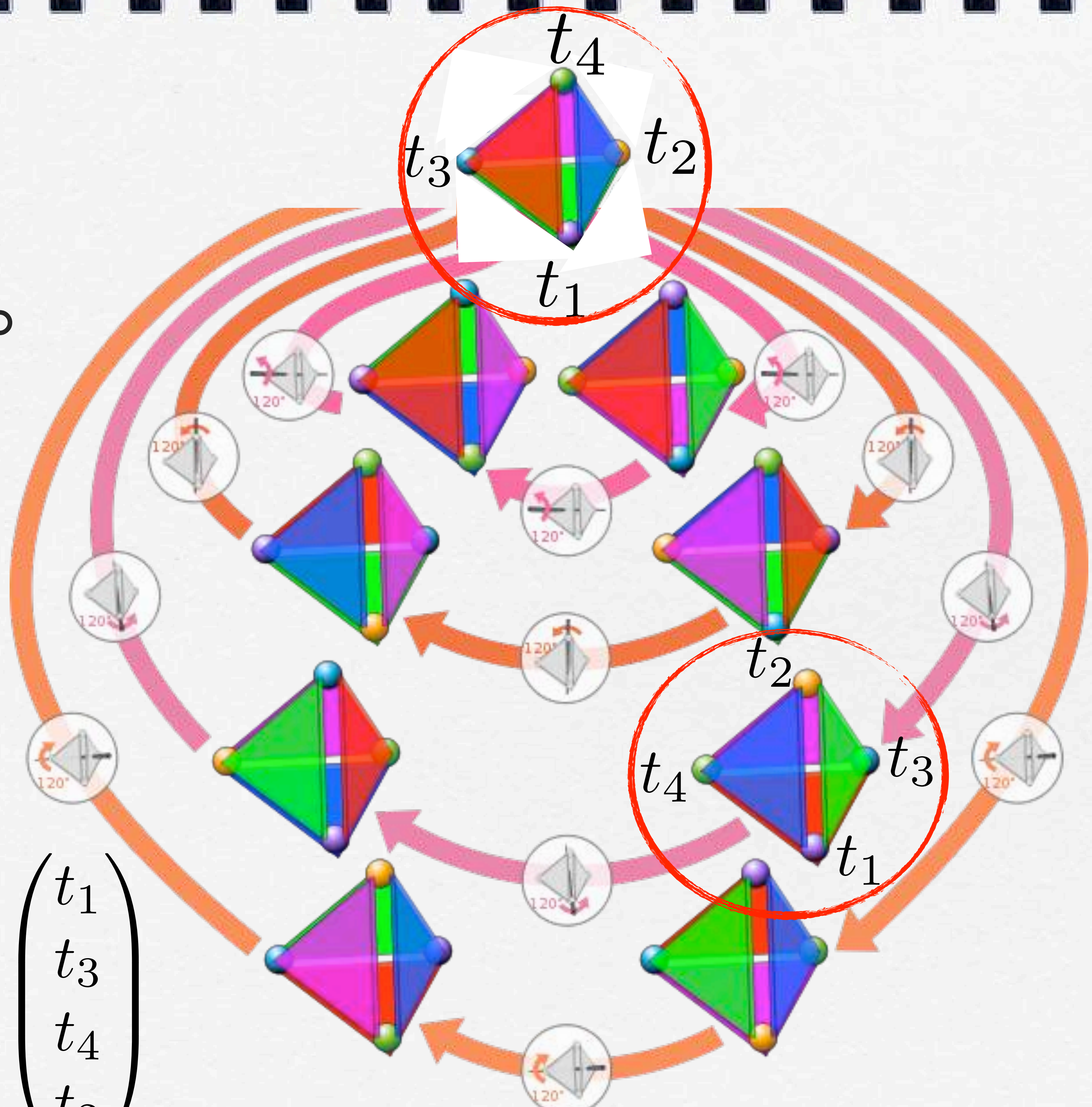


A_4

- rotation by 120° anti-clockwise (seen from a vertex)

$$T$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_3 \\ t_4 \\ t_2 \end{pmatrix}$$



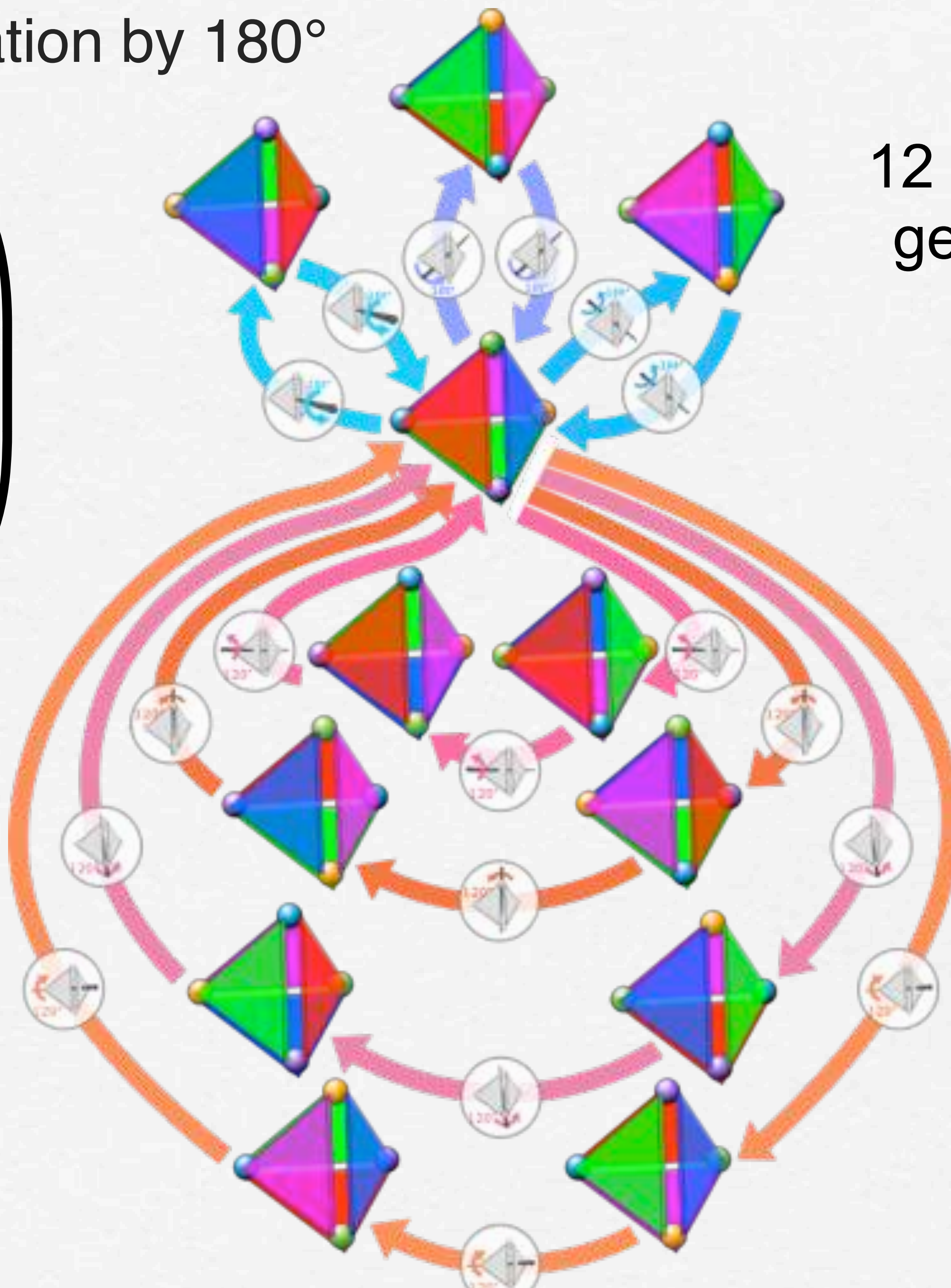
A_4

- 4 × rotation by 120° clockwise (seen from a vertex) T-type rotations
- 4 × rotation by 120° anti-clockwise (ditto) T-type rotations
- 3 × rotation by 180° S-type rotations

$$S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Diagonal

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



12 rotations ("group elements") generated by products of S,T ("generators")

$$S^2 = T^3 = I$$
$$(ST)^3 = I$$

Block diagonal (rotate about first vertex)

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Since S,T are block diagonal, the 4 dimensional matrix of vertex transformations is equivalent to a triplet plus singlet

$$4 \to 3 \oplus 1$$

A_4

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Writing $a_1 = e, \quad a_2 = S, \quad b_1 = T$ then multiplying S and T we generate 12 group elements

$$a_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; a_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; a_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; a_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$b_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \; b_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \; b_3 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \; b_4 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$c_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \; c_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \; c_3 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \; c_4 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

With eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \\ \pm\frac{1}{\sqrt{3}} \end{pmatrix}$$

- A_4 Presentation: $< S, T| \; S^2 = T^3 = e, \; (ST)^3 = e >$
- Group elements in four conjugacy classes:
- $1C^1(e)=\{e\}$, $3C^2(S)=\{S, TST^2, T^2ST\}=\{a_2, a_3, a_4\}$,

$4C^3(b_i)= \{T, TS, ST, STS\}$, $4C^3(c_i)=\{T^2, ST^2, T^2S, TST\}$

- Character table:

	e	S	T	T^2
$\mathbf{1}$	1	1	1	1
$\mathbf{1}'$	1	1	ω	ω^2
$\mathbf{1}''$	1	1	ω^2	ω
$\mathbf{3}$	3	-1	0	0

- Rule 1: # irreps= #classes=4
- Rule 2: sum square irreps =group order $1^2+1^2+1^2+3^2=12$
- Since $T^3=1$ it may be represented by any of the cube roots of unity: $\mathbf{1} = 1, \; \mathbf{1}' = \omega, \; \mathbf{1}'' = \omega^2$

A_4 Clebsch Gordan coefficients

Irreducible reps

$\mathbf{1},\ \mathbf{1'},\ \mathbf{1''},\ \mathbf{3}$

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$1 \otimes 1 = 1 \quad 1' \otimes 1'' = 1 \quad 1' \otimes 1' = 1'' \quad 1'' \otimes 1'' = 1'$$

$$(ab)_1 = a_1 b_1 + a_2 b_2 + a_3 b_3 \qquad 3 \otimes 3 = 1$$

$$(ab)_{1'} = a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3 \qquad \oplus\, 1'$$

$$(ab)_{1''} = a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3 \qquad \oplus\, 1''$$

$$(ab)_{3_1} = (a_2 b_3, a_3 b_1, a_1 b_2) \qquad \oplus\, 3_1$$

$$(ab)_{3_2} = (a_3 b_2, a_1 b_3, a_2 b_1) \qquad \oplus\, 3_2$$

where $\omega^3 = 1,\ a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$

Other groups

Z_N, rotation group of regular N-polygon

- Z_4 is square, Z_5 is pentagon, Z_6 hexagon, etc.
- Z_N generators a given by 2pi/N rotation
- Order = N group elements $\{e,a,a^2,...,a^{N-1}\}$
- We write $\rho = e^{i2\pi/N}, \quad \rho^N = 1$

	e	a	a^2	$\dots$
e	e	a	a^2	$\dots$
a	a	a^2	a^3	$\dots$
a^2	a^2	a^3	a^4	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

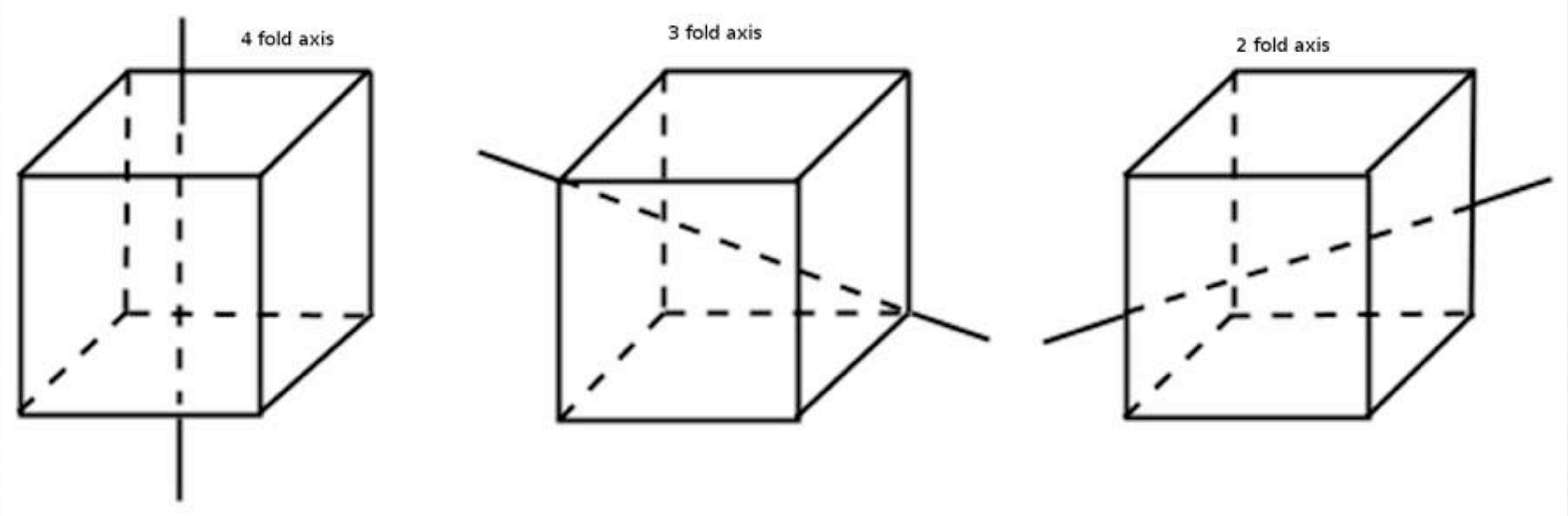
"Character table"

	e	a	a^2	$\dots$
$\mathbf{1}$	1	1	1	$\dots$
$\mathbf{1}'$	1	ρ	ρ^2	$\dots$
$\mathbf{1}''$	1	ρ^2	ρ	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

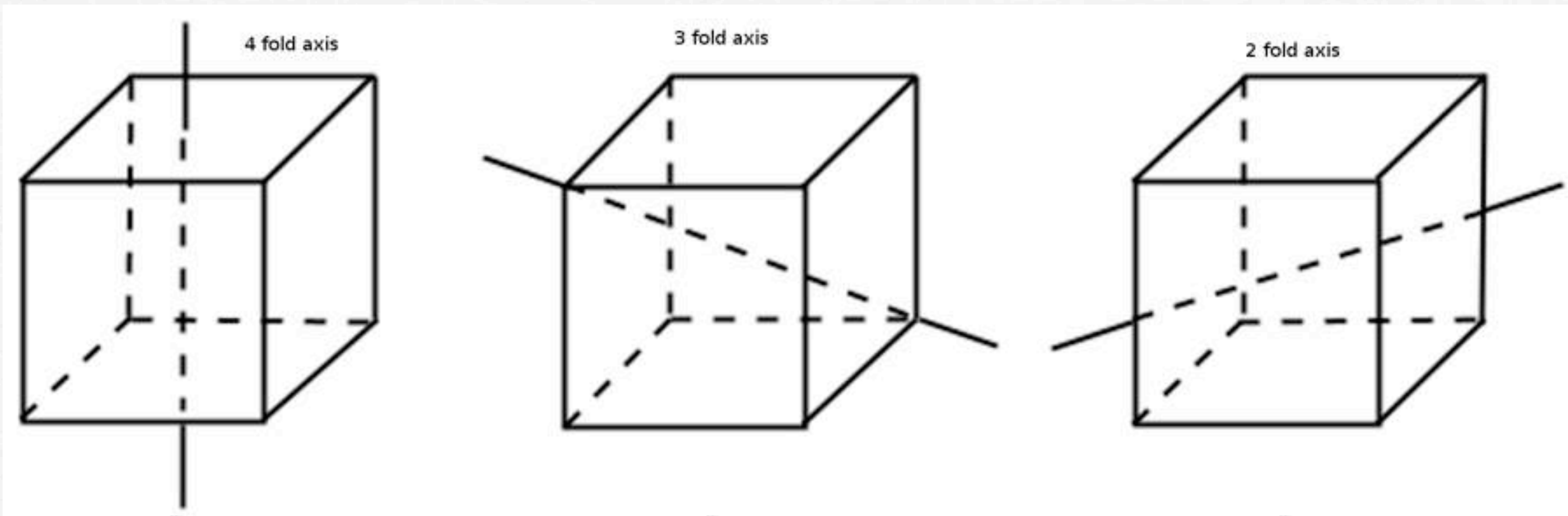
S_N, permutation group of N objects
A_N, its alternating subgroup

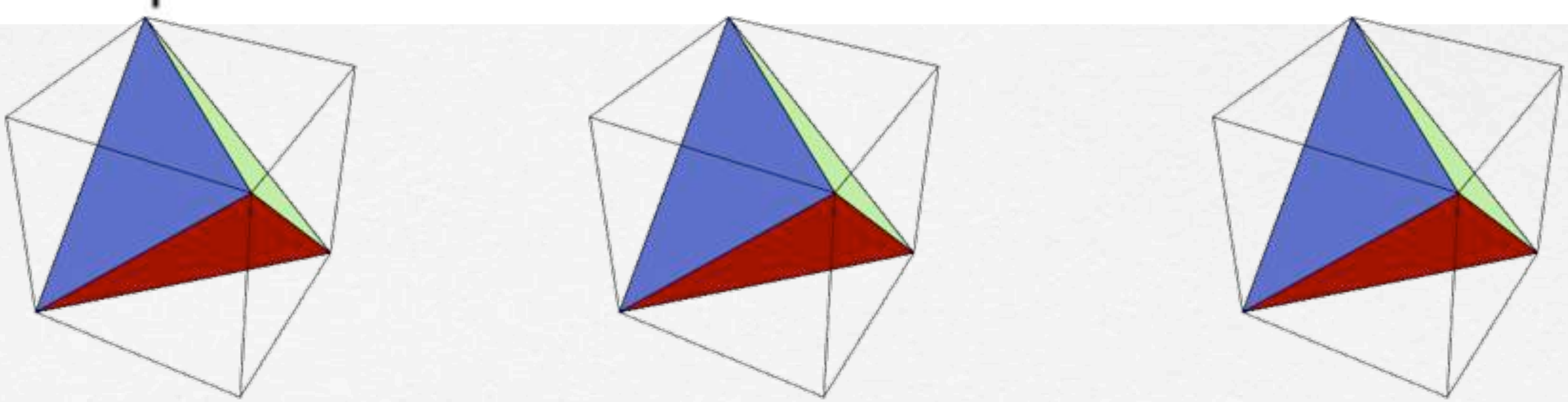
- (A,B,C,...) → (A,B,C,...), (A,C,B,...), (C,A,B,...),...
  even odd even
- N! group elements divided into even and odd
- even/odd refers to number of two-element swaps
- A_N subgroup consists of the N!/2 even perms
- A_N contains the alternating group elements of S_N
- E.g. $A_4 \subset S_4$ (also trivial example $A_3=Z_3 \subset S_3$)
- S_4 is the full symmetry group of the tetrahedron
- S_4 is also the rotation symmetry of a cube

- S_4 rotation symmetry of a cube



S_4 rotation symmetry of a cube





- 2 fold symmetry of the tetrahedron S
- 3 fold symmetry of the tetrahedron T
- Not a symmetry of the tetrahedron U

S_4

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Presentation

$$S^2 = T^3 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = (STU)^4 = 1$$

$$a_2 = S, \quad b_1 = T, \quad d_1 = U$$

Representation

$$a_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ a_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ a_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ a_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$b_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ b_2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \ b_3 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \ b_4 = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$c_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \ c_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \ c_3 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \ c_4 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$d_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ d_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \ d_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \ d_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ e_2 \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ e_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ e_4 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$f_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ f_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ f_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ f_4 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

A_4

Subgroups

- Subgroup H of group G are subsets of elements of G which form a group by themselves
- Order of H must be a divisor of the order of G
- E.g. if G is order 6 then H must be order 2 or 3
- E.g. S_3 is order 6 so H could be Z_2 or Z_3
- Normal subgroup N satisfies $gNg^{-1}=N$ for all $g \in G$
- Elements of N form complete conjugacy class + e
- N is sometimes called the Invariant subgroup

Example S_3:

- $1C^1(e)=\{e\}$, $2C^3(a)=\{a_1,a_2\}$, $3C^2(b)=\{b_1,b_2,b_3\}$
- Z_3 rotation subgroup is $\{e, a_1, a_2\}$, the even perms
- Z_3 is a normal subgroup satisfying $gNg^{-1}=N$
- This is because $\{e, a_1, a_2\} = e +$ complete a_i class
- Z_2 subgroups: $\{e, b_1\}$, $\{e, b_2\}$, $\{e, b_3\}$ not commute
- $\{b_1\}$ not complete class so Z_2 not normal
- S_3 is isomorphic to $Z_3 \rtimes Z_2 = \{e, a_1, a_2\} \rtimes \{e, b_1\}$
- Semi-direct product $\rtimes$ opens towards the normal subgroup Z_3 which does not commute with the Z_2

Example A_4:

- $1C^1(e)=\{e\}$, $3C^2(S)=\{S, TST^2, T^2ST\}=\{a_2, a_3, a_4\}$,

$4C^3(b_i)= \{T, TS, ST, STS\}$, $4C^3(c_i)=\{T^2, ST^2, T^2S, TST\}$

- A_4 is order 12 so H must be order 2,3,4,6
- $Z_2 \times Z_2$ normal subgroup: $\{e, a_2, a_3, a_4\}=e + a_i$ class
- Z_3 subgroup is $\{e, T, T^2\}$ not normal, $\{T, T^2\}$ not class
- A_4 is isomorphic to $Z_2 \times Z_2 \rtimes Z_3 = \{e, a_i\} \rtimes \{e, T, T^2\}$
- Semi-direct product $\rtimes$ opens towards the normal subgroup $Z_2 \times Z_2$ which does not commute with Z_3
- S_3 not subgroup of A_4 even perms (S_3 incl. odd)

Dihedral group D_n or Dih_n

$$D_n = \Delta(2n) = Z_n \rtimes Z_2$$

Symmetry group of regular n sided polygon including reflections

$$S_3 = D_3 = Z_3 \rtimes Z_2$$

Symmetry of equilateral triangle including reflections

$$D_4 = Z_4 \rtimes Z_2$$

Symmetry of square including reflections

Symmetries in molecules and crystals

Isometry groups	Abstract group
C_1	Z_1
C_2, C_i, C_s	Z_2
C_3	Z_3
C_4, S_4	Z_4
C_5	Z_5
C_6, S_6, C_{3h}	$Z_6 = Z_3 \times Z_2$
C_7	Z_7
C_8, S_8	Z_8
C_9	Z_9
C_{10}, S_{10}, C_{5h}	$Z_{10} = Z_5 \times Z_2$

Isometry groups	Abstract group
D_2, C_{2v}, C_{2h}	$Dih_2 = Z_2 \times Z_2$
D_3, C_{3v}	Dih_3
D_4, C_{4v}, D_{2d}	Dih_4
D_5, C_5v	Dih_5
D_6, C_{6v}, D_{3d}, D_{3h}	$Dih_6 = Dih_3 \times Z_2$
D_7, C_7v	Dih_7
D_8, C_8v, D_4d	Dih_8
D_9, C_9v	Dih_9
D_{10}, $C_{10}v$, D_5h, D_5d	$Dih_{10} = D_5 \times Z_2$

Isometry group	Abstract group
C_{4h}	$Z_4 \times Z_2$
C_{6h}	$Z_6 \times Z_2 = Z_3 \times Z_2^2 = Z_3 \times Dih_2$
C_{8h}	$Z_8 \times Z_2$
C_{10h}	$Z_{10} \times Z_2 = Z_5 \times Z_2^2 = Z_5 \times Dih_2$
D_{2h}	$Dih_2 \times Z_2$
D_{4h}	$Dih_4 \times Z_2$
D_{6h}	$Dih_6 \times Z_2 = Dih_3 \times Z_2^2$
D_{8h}	$Dih_8 \times Z_2$
T_h	$A_4 \times Z_2$
O_h	$S_4 \times Z_2$
I	A_5
I_h	$A_5 \times Z_2$
T_d, O	S_4
T	A_4

crystals

Graphic overview of the 32 crystallographic point groups

molecules

http://newton.ex.ac.uk/research/qsystems/people/goss/symmetry/Molecules.html