UNIVERSITY OF
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School of Physics
and Astronomy

Group Theory

Day 1: Solutions

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Exercise:

- By completing the table below, show that the rotations and reflections of S_3 form separate conjugacy classes.

S_3	e	a_1	a_2	b_1	b_2	b_3
e	e	a_1	a_2	b_1	b_2	b_3
a_1	a_1	a_2	e	b_2	b_3	b_1
a_2	a_2	e	a_1	b_3	b_1	b_2
b_1	b_1	b_3	b_2	e	a_2	a_1
b_2	b_2	b_1	b_3	a_1	e	a_2
b_3	b_3	b_2	b_1	a_2	a_1	e

g	geg^{-1}	ga_1g^{-1}	ga_2g^{-1}	gb_1g^{-1}	gb_2g^{-1}	gb_3g^{-1}
e	e	a_1	a_2	b_1	b_2	b_3
a_1	e	a_1	a_2	b_3	b_1	b_2
a_2	e	a_1	a_2	b_2	b_3	b_1
b_1	e	a_2	a_1	b_1	b_3	b_2
b_2	e	a_2	a_1	b_3	b_2	b_1
b_3	e	a_2	a_1	b_2	b_1	b_3

Exercise:

Show that $$U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega & \omega^2 \\ 1 & 1 & 1 \\ 1 & \omega^2 & \omega \end{pmatrix}$$ $$\omega = e^{i2\pi/3}$$ $$U^\dagger = (U^T)^*$$

is a unitary matrix $UU^\dagger = I$ but $\det U \neq 1$

Hence show

$$U^{-1} a U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad U^{-1} b U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega^2 & 0 \end{pmatrix}$$

where $$a = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Solution:

$$\omega = e^{i2\pi/3}$$

$$\omega^3 = 1$$

$$1 + \omega + \omega^2 = 0$$

$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega & \omega^2 \\ 1 & 1 & 1 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

$$U^\dagger = (U^T)^* = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega^* & 1 & \omega^{2*} \\ \omega^{2*} & 1 & \omega^* \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega^2 & 1 & \omega \\ \omega & 1 & \omega^2 \end{pmatrix}$$

$$UU^\dagger = \frac{1}{3} \begin{pmatrix} 1 & \omega & \omega^2 \\ 1 & 1 & 1 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ \omega^2 & 1 & \omega \\ \omega & 1 & \omega^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence $UU^\dagger = I$

Solution:

$$\det U = \left(\frac{1}{\sqrt{3}}\right)^3 \begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & 1 & 1 \\ 1 & \omega^2 & \omega \end{vmatrix}$$

$$\det U = \frac{1}{3\sqrt{3}} \begin{vmatrix} 1 & 1 \\ \omega^2 & \omega \end{vmatrix} - \frac{\omega}{3\sqrt{3}} \begin{vmatrix} 1 & 1 \\ 1 & \omega \end{vmatrix} + \frac{\omega^2}{3\sqrt{3}} \begin{vmatrix} 1 & 1 \\ 1 & \omega^2 \end{vmatrix}$$

$$\det U = \frac{1}{3\sqrt{3}}(\omega - \omega^2) - \frac{\omega}{3\sqrt{3}}(\omega - 1) + \frac{\omega^2}{3\sqrt{3}}(\omega^2 - 1)$$

$$\det U = \frac{1}{\sqrt{3}}(\omega - \omega^2)$$

Hence $\det U \neq 1$

Solution:

$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega & \omega^2 \\ 1 & 1 & 1 \\ 1 & \omega^2 & \omega \end{pmatrix} \qquad U^{-1} = U^\dagger = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega^2 & 1 & \omega \\ \omega & 1 & \omega^2 \end{pmatrix}$$

$$U^{-1} a U = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ \omega^2 & 1 & \omega \\ \omega & 1 & \omega^2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \omega & \omega^2 \\ 1 & 1 & 1 \\ 1 & \omega^2 & \omega \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

$$U^{-1} b U = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ \omega^2 & 1 & \omega \\ \omega & 1 & \omega^2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \omega & \omega^2 \\ 1 & 1 & 1 \\ 1 & \omega^2 & \omega \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega^2 & 0 \end{pmatrix}$$