International Summer School: Symmetries and Phase Transitions from Crystals and Superconductors to the Higgs particle and the Cosmos

Steve King, 29th August to 2nd September 2016, Dresden, Germany

Group Theory Exercises: Properties of S_3 or D_3

1. By completing the table below, show that the rotations (a_i) and reflections (b_i) of S_3 (also known as D_3 or Dih_3) form separate conjugacy classes:

g	geg^{-1}	ga_1g^{-1}	ga_2g^{-1}	gb_1g^{-1}	gb_2g^{-1}	gb_3g^{-1}
e	e	a_1				
a_1	e					
a_2						
b_1						
b_2						
b_3						

2. Show that

$$U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega & \omega^2 \\ 1 & 1 & 1 \\ 1 & \omega^2 & \omega \end{pmatrix}, \qquad \omega = e^{i2\pi/3}$$

is a unitary matrix, $UU^\dagger = I$, but is not special, $\det U \neq 1$.

Hence show that the **3** representation of S_3 can be reduced to $\mathbf{1} \oplus \mathbf{2}$ by this matrix:

$$U^{-1}aU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \qquad U^{-1}bU = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega \\ 0 & \omega^2 & 0 \end{pmatrix},$$

where the generators of S_3 in the **3** representation are:

$$a = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$