TECHNISCHE UNIVERSITÄT DRESDEN

School of Science International Summer School on Topological and Symmetry-Broken Phases

Phase Transitions and the Renormalization Group

An Introduction

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Outline

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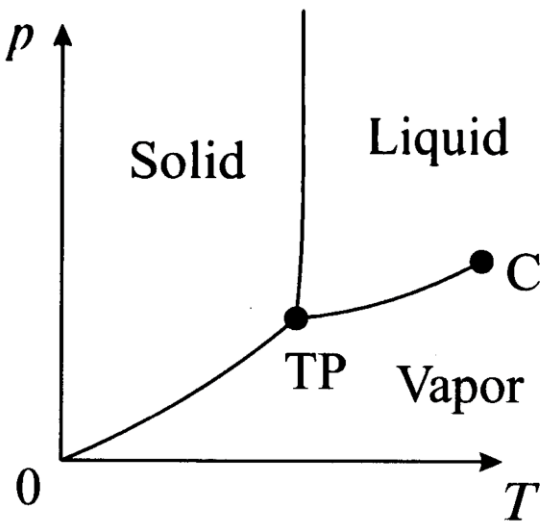
Tutorial: (1)…(5) illustrated on the example of 1-dim. Ising model

1. Introduction

It is a fact of everyday experience that matter in thermo-dynamic equilibrium exists in **different macroscopic phases**.

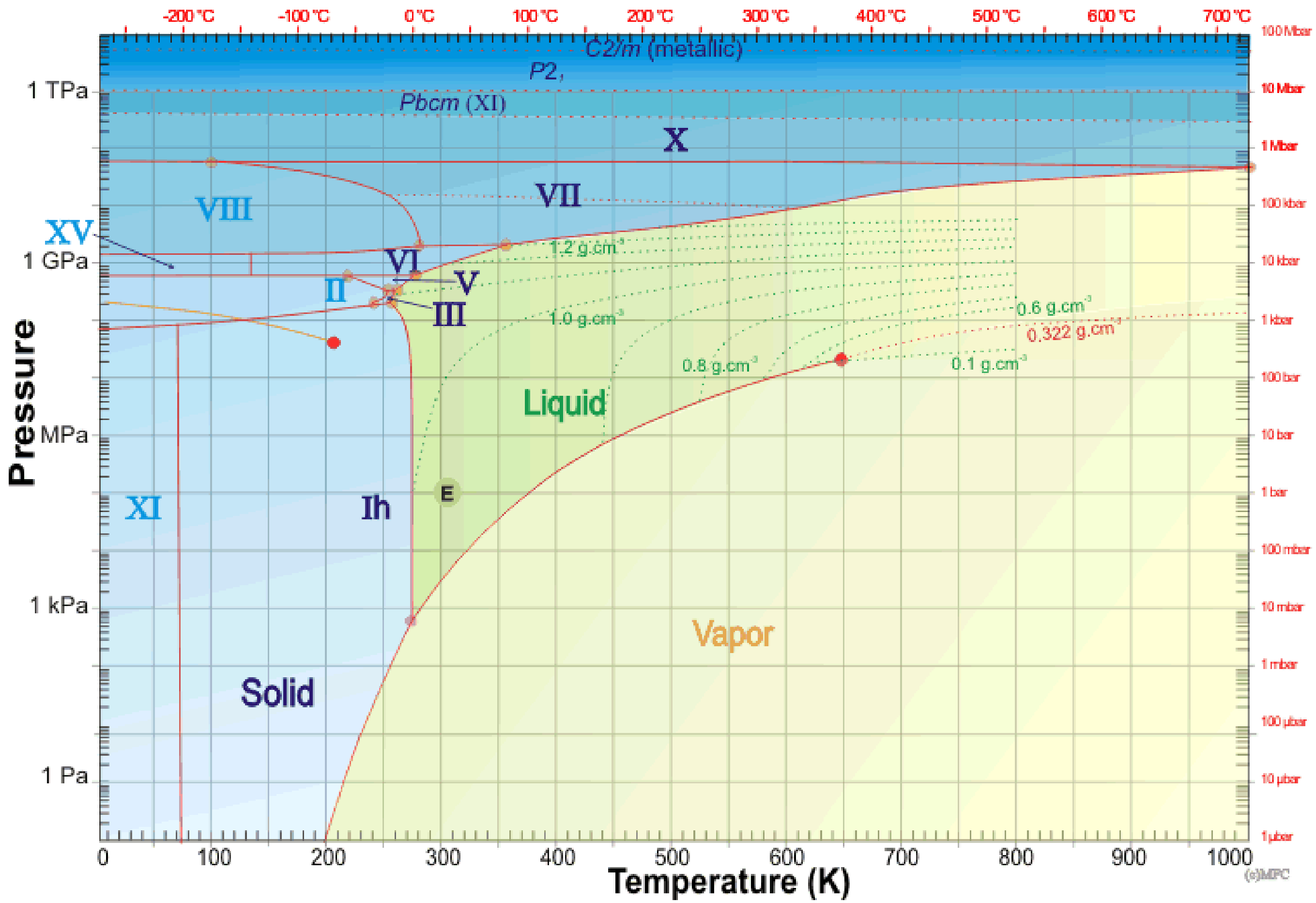
Example: Ice, liquid water, and water vapor are each a phase of water as a collection of macroscopic numbers of H_2O molecules



Schematic phase diagram of water

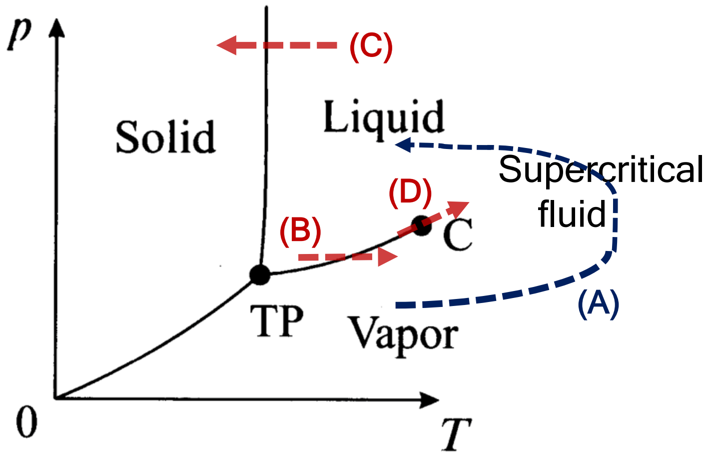
TP – triple point $(T_{\mathrm{tp}} = 273$ K, $p_{\mathrm{tp}} = 0.6$ kPa)

C - critical point $(T_{\mathrm{c}} = 647$ K, $p_{\mathrm{c}} = 22$ MPa)



More realistic pressure-temperature phase diagram of water

(from www.lsbu.ac.uk/water/phase.html)



The change of a phase can be

- **gradual** Path A: *continuous* crossover from liquid to gas (via a supercritical fluid)

or

- **abrupt** Path B: liquid/gas transition
  Path C: liquid/solid transition (symmetry breaking!)
  Path D: along the coexistence curve from a two-phase system into a single ("fluid") phase

In case that the change is abrupt, a **phase transition** **takes place** at well defined values of the parameters that determine the phase boundary

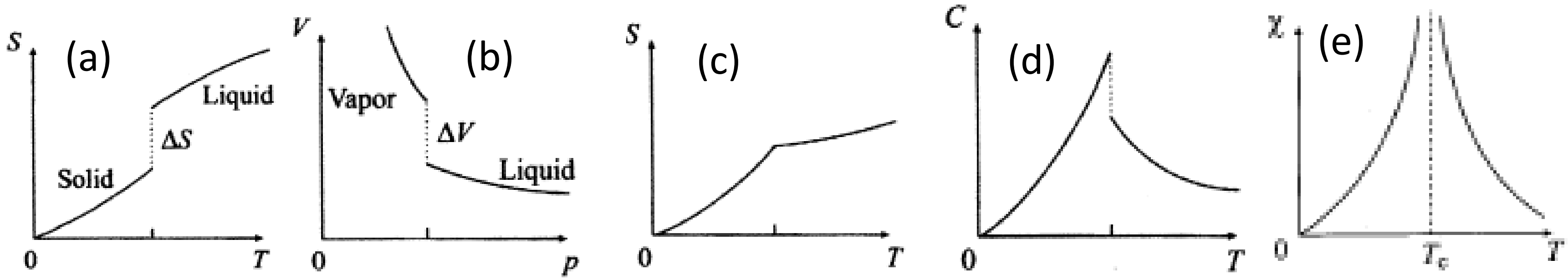
Phase

- state of matter in which all *macroscopically* physical properties of a material are uniform on a *macroscopic length scale*
- characterized by a *thermodynamic function*, typically the (Helmholtz) ***free energy*** $\mathcal{F}$ (or the Gibbs free energy $\mathcal{G} = \mathcal{F} + pV$)
- Equilibrium: the most stable state defined by lowest possible $\mathcal{G}(T, p)$

  ⇒ The description and analysis of phase transitions requires the use of thermodynamics and statistical physics

Phase Transition

- drastic (abrupt) change of *macroscopic system* properties as the system parameters (like temperature and pressure) are smoothly varied
- point in the parameter space where the thermodynamic potential becomes non-analytic



Physical origin of *thermal* (*"classical"*) phase transitions:

- driven by thermal fluctuations
- competition between internal energy $\boldsymbol{E}$ and entropy $\boldsymbol{S}$ which together determine the free energy $\boldsymbol{\mathcal{F} = E - TS}$

$\boldsymbol{E}$ favors order $\Leftrightarrow$ $\boldsymbol{S}$ privileges disorder

A **different** story: ***Quantum* Phase Transitions**

- phase transitions at absolute **zero temperature** triggered by varying some *non*-thermal control parameter (like magnetic field or pressure)
- QPT describes an abrupt change in the ground state of a many-body system due to its ***quantum* fluctuations**

⇝ S. Sachdev, Quantum phase transitions (Cambridge, 2011)

⇝ M. Vojta, Thermal and Quantum Phase transitions (Lectures at the Les Houches Doctoral Training School in Statistical Physics 2015, http://statphys15.inln.cnrs.fr)

Phase transition ⇝ point in parameter space where a thermodynamic potential non-analytic

Existence of Phase transitions ?

$$\text{Free energy } \mathcal{F} = -\frac{1}{kT} \ln \mathcal{Z}$$

$$\text{Partition function } \mathcal{Z} = \sum_{\text{states}} e^{-H/k_B T} \equiv \text{Tr}\, e^{-H/k_B T}$$

$\mathcal{Z}$ is a sum of exponentials of $(-\frac{H}{k_B T})$

⇒ in a ***finite*** system the partition function of any system is a finite sum of analytic functions of its parameters and is therefore ***always* analytic**

⇒ a **non-analyticity** can **only** arise in the ***thermodynamic limit***

A simple mathematical example:

Function $f_{1/2}(x,N) \equiv \sum_{k=1}^{N} \frac{e^{-xk}}{k^{1/2}}$

is **for finite $\boldsymbol{N}$** an infinitely differentiable function of x

$\Longrightarrow f_{1/2}(x,N)$ is **analytic**

However,

$$f_{1/2}(x) = \lim_{N\to\infty} f_{1/2}(x,N) = \underbrace{\sqrt{\frac{\pi}{x}}}_{\text{singular part}} + \underbrace{\text{rest}}_{\text{regular part}}$$

Physical relevance ? Energy of an ideal Bose gas $E = E\left(-\frac{\mu}{k_B T}\right)$

$$E''(x) \sim f_{1/2}(x)$$

Order Parameter

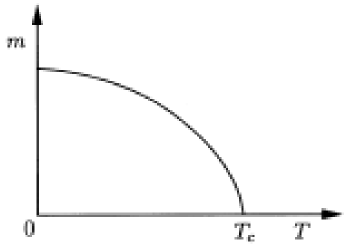
- observables whose values discriminate between the various phases
- measures how microscopic elements generating the macroscopic phase are ordered or in a similar state

Examples:

Phase Transition	*Order Parameter*
paramagnetic-ferromagnetic	spontaneous magnetization
liquid-gas transitions	difference of densities
liquid-solid	shear modulus
superfluid-normal liquid	superfluid density

Order parameter has not to be a scalar. There are phase transitions where the order parameter has the form of a complex number, a vector, … , a group element of a symmetry group.

- **order parameter** often associated with **breaking of a symmetry**
  (order parameter measures the "degree of asymmetry" in the *broken symmetry phase,* the "ordered" phase)



Behavior of the spontaneous magnetization $m = -\lim_{h\to 0} \frac{\partial g(T,h)}{\partial h}$ of a system exhibiting spontaneous ferromagnetism for $T < T_c$

Ordering of the particles has not be in real space, can also be in momentum space (example: superfluid transition of He-4 at 2.2K).

Correlation Length ξ

- distance over which the fluctuations of the microscopic degrees of freedom (e.g., in a magnetic system: local spins) are significantly correlated with each other

  [A spin at any site tends to align all adjacent spins in the same direction as itself to lower the energy – this tendency is opposed by that of entropy]

  $\Rightarrow$ the fluctuations in two parts of the material much further apart than ξ are effectively disconnected from each other

- ξ is usually of the order of a few interatomic spaces

  $\Rightarrow$ this is the reason why already small collection of atoms may give a good idea of the macroscopic behavior of the material (neglecting surface effects)

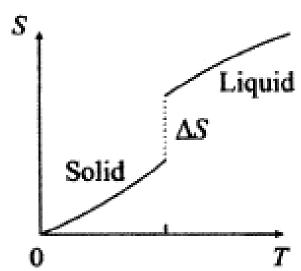
- actual value of ξ depends on the external conditions determining the state of the system (like temperature and pressure)
- near a critical point ξ has to grow (the system has to be prepared for a fully ordered state)

Classification of Phase Transitions

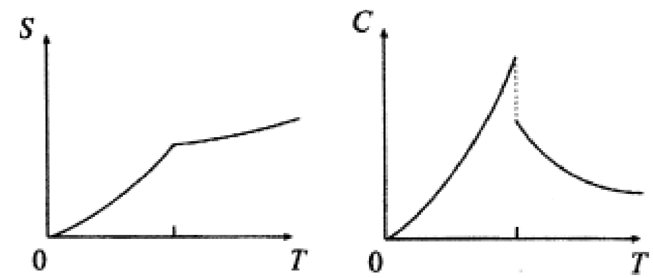
"Classical" classification (Paul Ehrenfest, 1933)

Phase transitions are named by the order of derivative of free energy that first shows a discontinuity

first order:



second order:



(Modern) classification (M.E. Fisher)

discontinuous **(*or first order*) transition ↔ *continuous* transition**

A material can show both discontinuous and continuous transitions depending on the conditions!

Discontinuous transitions

First order derivative of the free energy shows a discontinuity
(transitions involve a latent heat)

Examples: melting of a three-dimensional solid,
condensation of a gas into a liquid

Properties:

- two (or more) states on either side of the transition point coexistent exactly at the point of transition
  Slightly away, however, there is generically a unique phase whose properties are continuously connected to one of the co-exist phases at the transition point
  $\Rightarrow$ expect a discontinuous behavior in various thermodynamic quantities as we pass the coexistence line
- often hysteresis or memory effects are observed
  (since the continuation of a given state into the opposite phase may be metastable so that system may take a macroscopically long time to readjust)
- correlation length is (generally) finite

Continuous transition

Second or higher derivatives show a discontinuity or divergence
(*all* first order derivatives of free energy are continuous)

Examples: paramagnetic-ferromagnetic transition
liquid-gas transition at the critical point
superfluid transition

Properties:

- correlation length becomes effectively infinite
  ⇒ fluctuations are correlated over all distance scales
  ⇒ whole system is forced to be in a unique, critical phase
- fluctuations on all length scales (⇝ "critical opalescense")
  ⇒ system is scale invariant
- difference in energy density or magnetization (volume density) between the phases go (smoothly) to zero

Special type of continuous transitions: *"infinite-order"* phase transitions
Example: Kosterlitz–Thouless transition in 2-dim. XY model

2. Critical Phenomena

Continuous phase transitions $\rightsquigarrow$ **Critical Phenomena:**

Anomalous phenomena in the region around a critical point,
where two or more phases become indistinguishable

Essential feature of critical phenomena:
fluctuations at *all* length scales which occur simultaneously causing a non-analytic behavior of physical quantities

$\Longrightarrow$ **3 major factors that complicate a theoretical description:**

- non-analyticity of the thermodynamic potentials
- absence of small parameters (no normal perturbative methods are applicable)
- equal importance of all length scales

It was necessary a whole new way of thinking about such phenomena
$\Longrightarrow$ Renormalization Group

Degree of singularity or divergence of physical quantities near T_c is described by *critical exponents* (*critical indices*)

Experiments show that the relevant thermodynamic variables exhibit power-law dependences on the parameters specifying the distance away from the critical point.

$t = \frac{T-T_c}{T_c} \equiv t$ *reduced* temperature ("distance" from CP)

Definition of Critical exponents (here for magnetic materials):

Exponent	*Definition*	*Conditions*
α	Specific heat $c(t) \sim |t|^{-\alpha}$	$h = 0$
β	Spontaneous magnetization m(t) $\sim (-t)^{\beta}$	$T \leq T_c,\ h = 0$
γ	Magnetic susceptibility $\chi = \left(\frac{\partial m}{\partial h}\right)_T \sim |t|^{-\gamma}$	$h = 0$
δ	Critical Isotherm $m(h) \sim |h|^{1/\delta}\,\mathrm{sgn}(h)$	$t = 0$
ν	Correlation length $\xi \sim |t|^{-\gamma}$	$h = 0$
η	Correlation function $G(r) \sim r^{-d+2-\eta}$	$t = 0, h = 0$

(Two-point) correlation function

$$G(\vec{r}-\vec{r}') \equiv \langle \delta\hat{m}(\vec{r})\, \delta\hat{m}(\vec{r}')\rangle = \langle \hat{m}(\vec{r})\, \hat{m}(\vec{r}')\rangle - \langle \hat{m}(\vec{r})\rangle\langle \hat{m}(\vec{r}')\rangle$$

measures the correlation between fluctuations $\delta\hat{m}(\vec{r}) \equiv \hat{m}(\vec{r})$-$\langle\hat{m}(\vec{r})\rangle$ at point $\vec{r}$ and $\vec{r}$‘

here: $\hat{m}(\vec{r})$ operator of local magnetization density at point $\vec{r}$ (e.g., spin)

$\langle \dots \rangle$ − thermal average, $m = \langle \hat{m}(\vec{r})\rangle = \frac{1}{Z}\mathrm{Tr}\left(\hat{m}(\vec{r})e^{-H/k_B T}\right)$

Note:

For $T \neq T_c$: $G(r) \sim e^{-r/\xi}$ (for $r \gg \xi$) *exponential* decay

For $T = T_c$: $G(r) \sim r^{-d+2-\eta}$ (valid for $d > 2$) slow decay in a *power* manner (fluctuations at *all* length scales)

Why at $T = T_c$ the correlation function $G(r)$ cannot decrease exponentially with distance r ?

Relation between (uniform) susceptibility χ and $G(r)$:

$$\chi = -\left.\frac{\partial^2 g}{\partial h^2}\right|_{h=0} = \sim \int \mathrm{d}^d r\, G(r)$$

(from fluctuation-dissipation theorem)

Since χ diverges for $T \to T_c \Longrightarrow \mathrm{r.h.s.} = \mathrm{infinite}$ at $T = T_c$

$\Longrightarrow G(r) \sim r^{-\tau}$ at $T = T_c$ with $\tau \leq d$

and $\xi(T) \to \infty$

Although systems with large correlations lengths might to be very complex …. they also show some beautiful simplification.

One of these is the phenomenon of **universality**.

Many properties of a system close to T_c turn out to be largely **independent of *microscopic details* of the interaction**

Instead: Systems fall into one of a relatively small number of **different classes, each characterized only by global features**, such as the symmetries of underlying Hamiltonian

Critical exponents:
- pure numbers
- depend only on the **universality class**
  (materials consisting of very different microscopic constituents can have the same exponents)

Universality class is *only* determined by

(1) the dimensionality of the system

(2) the symmetry of its order parameter

(3) the range of interaction

$\Longrightarrow$ Theoretical challenge
- to explain why such non-trivial powers occur
- to predict their actual values
- microscopic understanding of universality

(final answer $\rightsquigarrow$ Renormalization Group)

The occurrence of power laws describing a system is a symptom of scaling behavior.

Not all exponents are independent!

There are simple relations between the exponents (scaling laws)

From thermodynamics (rigorous) relations:

$$\alpha + \beta + 2\gamma \geq 2$$

$$\beta(1 + \delta) \geq 2 - \alpha$$

Experiments and simulations for model systems show that the inequalities are rather **equations** and that there are additional relations

$\Longrightarrow$ **scaling hypothesis**

(final proof by Renormalization Group)

Scaling hypothesis (Benjamin Widom, 1965):

In the vicinity of a *continuous* phase transition the density of (Gibbs) free energy $g(t,h)$ can be written as the sum of a slowly varying regular part g_{reg} and a singular part g_{sing}

$$g(t,h) = g_{\text{reg}}(t,h) + g_{\text{sing}}(t,h)$$

with the **singular part** being a (generalized) **homogeneous function**

$$\boldsymbol{g_{\text{sing}}(t,h) = \lambda^{-n} g_{\text{sing}}\left(\lambda^{\Delta_t} t, \lambda^{\Delta_h} h\right)}$$

λ is an arbitrary (dimensionless) scale factor
exponents are charact. for a given universality class

Consequence: Since λ is arbitrary $\Longrightarrow$ may set $\lambda = |t|^{-1/\Delta_t}$

$$\Longrightarrow \boldsymbol{g_{sing}(t,h) = |t|^{n/\Delta_t} \psi_{\pm}\left(h/|t|^{\Delta_h/\Delta_t}\right)}$$

(singular part of **free energy** and of **any other thermodynamic quantity** have a ***homogeneous*** **form**, ψ is an arbitrary function)

Take derivatives $g_{\text{sing}}(t,h) = |t|^{n/\Delta_t}\psi_{\pm}\left(h/|t|^{\Delta_h/\Delta_t}\right)$

with respect to t, h

then set $h = 0$ and assume that close to T_c the derivatives of $g(t,h)$ are dominated by its singular part

⟹ Scaling Laws: $\boldsymbol{2-\alpha = 2\beta+\gamma}$ **Rushbrooke Identity**

$\boldsymbol{\gamma = \beta(\delta-1)}$ **Widom Identity**

From **assumption** that also ***correlation function*** is a ***homogeneous*** **function**:

$\boldsymbol{2-\alpha = d\nu}$ **Josephson Identity**

$\boldsymbol{\gamma = \nu(2-\eta)}$ **Fisher Identity**

(all scaling laws will be strictly proved by Renormalization Group)

For a microscopic approach to phase transitions and critical phenomena

… need of simple models that show a phase transition

Very useful: **Ising Model**

can be solved exactly in

$d = 1$ (⇝ tutorial)
$d = 2$ (without external field)

Heisenberg Model

Ising model (Ernst Ising 1925, Wilhelm Lenz 1920)

– **paradigm of a simple system exhibiting a well defined phase transition**

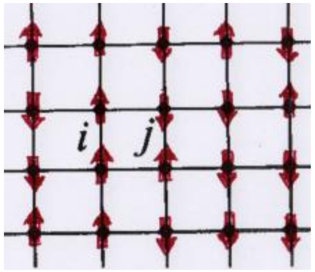
Model based on three assumptions:

(1) objects ("particles") are located on the sites of a d-dim. crystal lattice (one particle on each site)

(2) each particle "i" can be only in one of two possible states $s_i = \pm 1$ ("particle's spin")

(3) Hamiltonian (energy function) is given by

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_i s_i$$



$\langle i, j \rangle$ denotes the sum over all nearest neighbors i and j

J interaction parameter $\left(J \gtrless \begin{matrix}\text{ferromagnetic} \\ \text{antiferr.}\end{matrix} \text{ interaction}\right)$

h external magnetic field (expressed in units of energy)

Feature of the Ising model:

- global symmetry at $h = 0$ under the transformation $s_i \to -s_i$ at all sites

  $\Longrightarrow$ **$\mathbb{Z}_2$ symmetry** ("Ising symmetry")
  simplest discrete symmetry group: consisting only 2 elements {1,-1}

- $\mathbb{Z}_2$ symmetry also present in the **high temperature *paramagnetic* phase** with "magnetization" $\boldsymbol{m = \langle s_i \rangle = 0}$
  (any two configurations that have all spins reversed enter the partition function with equal weight)

- in **the low temperature ferromagnetic phase**: magnetization $\boldsymbol{m \neq 0}$

  $\Longrightarrow$ $\mathbb{Z}_2$ symmetry evidently broken.
  In the absence of an external magnetic field is nothing that explicitly breaks the symmetry in the Hamiltonian

Spontaneous symmetry breaking in the ordered phase !

To obtain correct results for the *ordered* phase within the formalism of *equilibrium* statistical mechanics:

- have to restrict the space of configuration over which the summation in the partition function

$$\mathcal{Z}_N(T,h) = \sum_{s_1=\pm 1} \sum_{s_2=\pm 1} \cdots \sum_{s_N=\pm 1} e^{-\mathcal{H}/k_B T}$$

  is performed.
- calculate first the magnetization in a finite external field and take then the limit of zero magnetic field **after** the thermodynamic limit has been taken

$$m = -\lim_{h\to 0} \left(\frac{\partial g(T,h)}{\partial h} \right)_T \quad \text{with} \quad g(T,h) := \lim_{N\to\infty} \frac{1}{N} \left(-k_\mathrm{B} T \ln \mathcal{Z}_N(T,h)\right)$$

Exact results for the Ising Model : $\boldsymbol{d = 2}$: $\boldsymbol{k_\mathrm{B} T_c = \frac{2J}{\ln(1+\sqrt{2})}}$

$\mathbf{d = 1}$: $\boldsymbol{k_\mathrm{B} T_c = 0}$ (⇝ tutorial)

Physical applications

- **uniaxial ferromagnets and antiferromagnets**
- **lattice Gas** (statistical model for the motion of atoms)
  occupation number $n_i = \frac{1+s_i}{2} = 0 \text{ or } 1$
- **lattice binary mixture** (cell i is occupied by atom “A” or “B”)
- **spin glasses** ($J \to J_{ij}$ with J_{ij} random distributed)

It is obvious to generalize the Ising model

- consider also interactions between second nearest neighbord, third nearest neighbors, …
  … model with weak interaction of infinite range $J_{ij} \to J/N$ for all i, j
- to higher discrete symmetries ⇝ “clock models”
- to continuous symmetries in the plane ⇝ XY model
  or in the space ⇝ Heisenberg model

Heisenberg model

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j - \vec{h} \sum_i \vec{s}_i$$

vector spin model , ${\vec{s}_i}^{\,2} = 1$

Hamiltonian is at $h = 0$ invariant under a simultaneous rotation of all spins by the same angle

$\Longrightarrow$ **$\mathbb{O}_3$ symmetry**

(rotations in three dimension ⇒ a continuous symmetry)

Symmetry of $\mathcal{H}$ Heisenberg different from Ising Hamiltonian
$\Longrightarrow$ different symmetry of order parameter $\Longrightarrow$ different critical indices

The Kadanoff construction (Leo Kadanoff, 1966)

- provides a heuristic explanation for the origin of scaling
- gives us an idea how to construct the renormalization group

Starting Point: correlation length diverges at the critical point, $\xi(T) \xrightarrow[T \to T_c]{} \infty$

$\Longrightarrow$ spins at different spatial positions are strongly correlated
$\Longrightarrow$ close to T_c fluctuations are present on all length scales
$\Longrightarrow$ scale invariance of the system

Consider an Ising model
(with spins $S_i = \pm 1$ at sites i on a d-dimensional hypercubic lattice)

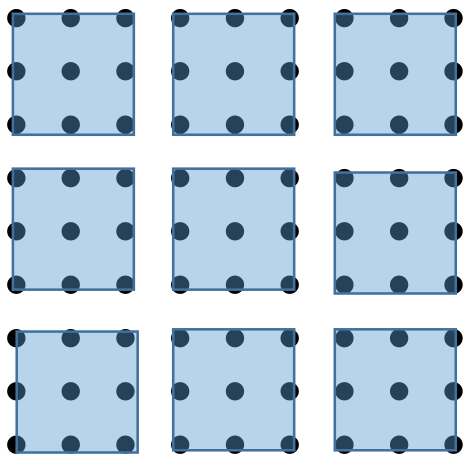
$$\tilde{\mathcal{H}} \equiv -\beta \mathcal{H} = +\tilde{J} \sum_{<i,j>} S_i S_j + \tilde{h} \sum_i S_i$$

with „reduced“ variables $\tilde{J} \equiv J/k_B T, \ \tilde{h} \equiv h/k_B T$

The Kadanoff construction consists of 3 stages:

Step 1:

Divide the original Ising lattice with lattice constant a into blocks with λ^d single spins (d = dimension of the lattice, $a < \lambda a \ll \xi$)



Grouping of site spins into blocks (here λ=3)

Step 2

Replace the λ^d spins inside each block by a single „block" spin S'_α (with $\alpha = 1, \dots, n$, where n = total number of blocks).

Assumption: Block spins S'_α behave exactly like original Ising spins S_i

(only S'_α ± 1 possible !)

$$S'_\alpha = \frac{1}{\lambda^d} \sum_{i \epsilon \alpha} S_i \approx \pm 1$$

(since $\lambda a \ll \xi$, we expect that most spins within a block will be of the same sign → majority rule is a very reasonable approximation)

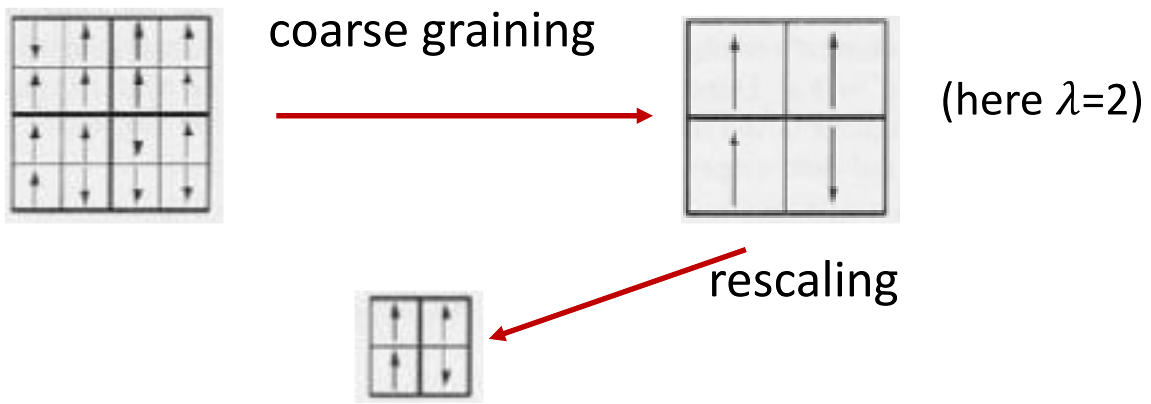
Assumption: Hamiltonian for the system of block spins with lattice constant λa has the same form as the original Hamiltonian, but with ***different*** coupling parameters $\tilde{J}'$, $\tilde{h}'$

$$\tilde{\mathcal{H}}' = +\tilde{J}' \sum_{<\alpha,\alpha'>} S'_\alpha S'_{\alpha'} + \tilde{h}' \sum_{\alpha} S'_\alpha$$

(assumption seems to be reasonable, since due to $\xi \to \infty$ the Ising system can be thought to consist of clusters of correlated individual spins as well as of clusters of correlated block spins)

Step 3:

„Return“ to the original site lattice by dividing all length by λ.



Goal of the procedure:

- make a ***„coarse graining“*** so as to reduce the number of degrees of freedom of the system
- even if we do not know the exact solution of the problem (for either the site lattice or the block lattice) a comparison of these two problems can provide us with valuable information

Hamiltonian has the ***same* structure in terms of both lattice sites *and* blocks**

$\Longrightarrow$ assume that the form of the partition function will be the same

$\Longrightarrow$ thermodynamic potentials for the two models are similar.

For the Gibbs free energy as function of $t \equiv (T - T_\mathrm{C})/T_\mathrm{C}$ and h holds:

$$\underbrace{g(t',h')}_{\text{Gibbs free energy}} = \underbrace{\lambda^d}_{\text{number of spins per block}} \cdot \underbrace{g(t,h)}_{\text{Gibbs free energ}}$$

per block **per site**

Now we need to relate h' to h and t' to t !

Parameters for the block lattice h', t' depend on h, t and on λ

Assumption of Kadanoff: $h' = \lambda^{\Delta_h} h \quad ; t' = \lambda^{\Delta_t} \cdot t$

(simplest possible relation consistent with the symmetry requirements)

$$\Longrightarrow \quad \boldsymbol{g\left(\lambda^{\Delta_t} t, \lambda^{\Delta_h} h\right) = \lambda^d g(t,h)}$$ **scaling hypothesis** **(with $n = d$)**

Kadanoff construction gives an intuitive explanation for the scaling hypothesis and the related properties

What remains to be done ??

- **to demonstrate explicitly that Kadanoff's assumptions are valid**
- **to obtain the values for all the critical exponents corresponding to a given model**

Mean-field theories (see lectures on statistical mechanics) like

Weiss' theory of ferromagnetism
(an effective external field replaces the interaction of all the other particles to an arbitrary particle)

Landau theory (an *effective* theory of the order parameter)

lose its internal consistency for spatial dimension $d < 4$ and lead to incorrect results for the critical exponents.

Need a better theory when fluctuations play vital roles!

Aim:

- **to understand, both qualitatively and quantitatively the critical phenomena**
- **to proof the scaling hypothesis for both free energy density and correlation function**
- **to find the critical exponents**

$\Longrightarrow$ Renormalization Group

Starting point:

- **system of N interacting particles (N large) described by a Hamiltonian**

$$\widetilde{\mathcal{H}} \equiv -\beta\mathcal{H}$$

Key idea of RG:

- **successive decimation of degrees of freedom**

- **RG is a *group of transformations* $\mathcal{R}$**

  (in strict sense a semi-group, since no inverse transformation)

  from a "*site* lattice" with lattice constant unity and Hamiltonian $\widetilde{\mathcal{H}}$
  to a "*block* lattice" with lattice constant λ and Hamiltonian $\widetilde{\mathcal{H}}'$
  without changing the form of the partition function

$$\mathcal{Z}_N[\widetilde{\mathcal{H}}\ ] = \mathcal{Z}_{N/\lambda^d}[\widetilde{\mathcal{H}}'\ ]$$

Formal steps of RG:

(1) Transformation of the Hamiltonian (decimation or "coarse graining")

$$\widetilde{\mathcal{H}} \longrightarrow \widetilde{\mathcal{H}}' = \mathcal{R}[\,\widetilde{\mathcal{H}}]$$

$\mathcal{R}$ is the RG (super-)operator, in generally, a complicated non-linear transformation of the coupling parameters

The RG transformation $\mathcal{R}$ reduces the total number of degrees of freedom by a factor λ^d, leaving $N' = N/\lambda^d$

$\Longrightarrow$ **decimation/"coarse graining"**

and does **not change the partition function** $\mathcal{Z}_{N'}[\widetilde{\mathcal{H}}\,'] = \mathcal{Z}_N[\widetilde{\mathcal{H}}\,]$

postulate !

examples: • partial trace over $N - N'$ degrees of freedom

$$e^{\widetilde{\mathcal{H}}'} = e^{\mathcal{R}[\,\widetilde{\mathcal{H}}]} = \mathrm{Tr}_{N-N'}\, e^{\widetilde{\mathcal{H}}}$$

• integrating out fluctuations of fields ψ which occur on length scales finer than $\boldsymbol{\lambda}$

(2) Rescaling of all lengths

To restore the spatial density of the degrees of freedom all lengths are rescaled by a factor $\boldsymbol{\lambda}$:

$$\vec{\boldsymbol{r}} \longrightarrow \vec{\boldsymbol{r}}' = \frac{\vec{\boldsymbol{r}}}{\boldsymbol{\lambda}} \quad \text{(for momenta: } \vec{\boldsymbol{p}} \longrightarrow \vec{\boldsymbol{p}}' = \boldsymbol{\lambda}\vec{\boldsymbol{p}} \text{)}$$

(3) Renormalization of the variables (spins, magnetization field)

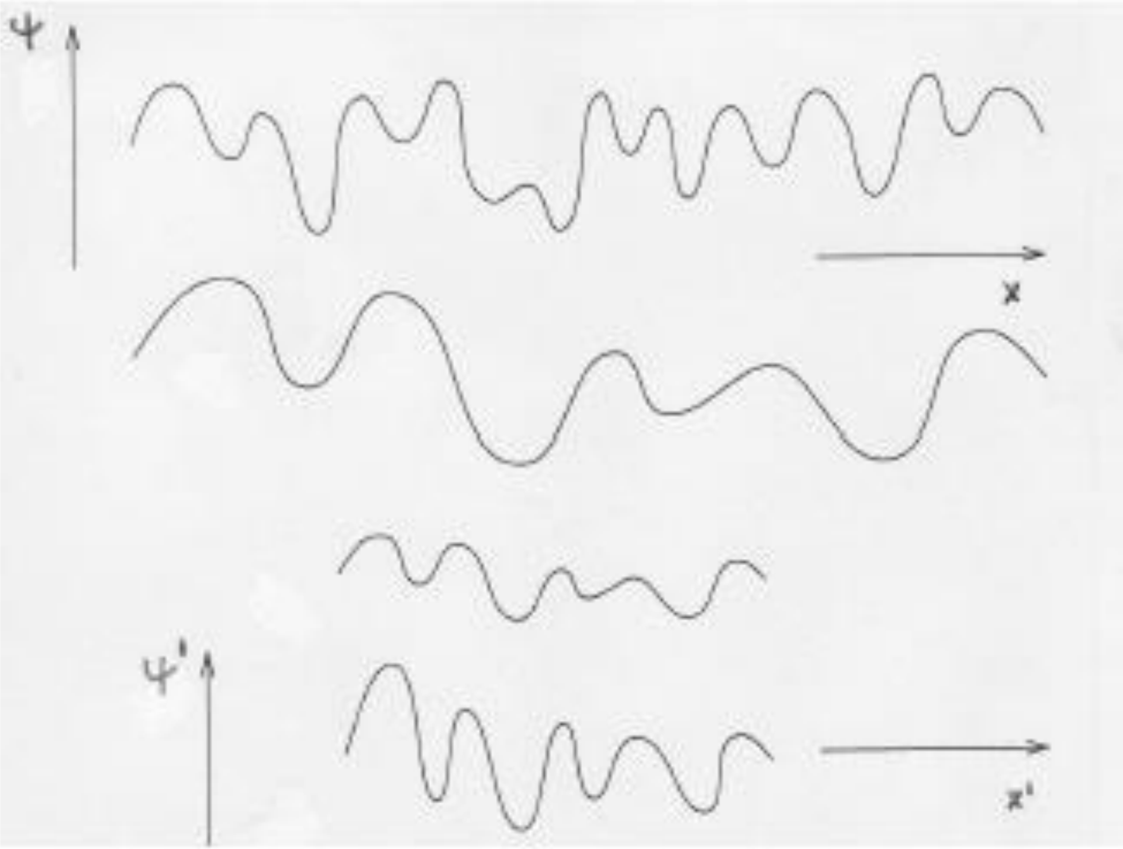
To restore the relative size of the fluctuations the variables will be renormalized (… **to restore the contrast of the original "picture")**

$$\boldsymbol{\psi}(\vec{\boldsymbol{r}}) \longrightarrow \boldsymbol{\psi}'(\vec{\boldsymbol{r}}') = \frac{\mathbf{1}}{\zeta}\boldsymbol{\psi}(\vec{\boldsymbol{r}}) \qquad (\zeta<1)$$

RG operator $\mathcal{R}$ depends on λ and ζ

(the real challenge: to find a transformation $\boldsymbol{\mathcal{R}}$)

Our first three steps:



(1) Coarse graining

(2) Rescaling

$$x' = \frac{1}{\lambda} x$$

(3) Renormalization

$$\psi' = \frac{\psi}{\zeta}$$

(4) Repeat the transformation

$$\boldsymbol{\widetilde{\mathcal{H}}''} = \boldsymbol{\mathcal{R}}[\,\boldsymbol{\widetilde{\mathcal{H}}'}] = \boldsymbol{\mathcal{R}}\left[\boldsymbol{\mathcal{R}}[\,\boldsymbol{\widetilde{\mathcal{H}}}]\right] = \cdots$$

(5) Since $\boldsymbol{\mathcal{H}}$ depends on the coupling parameters

(e.g., $\tilde{J} \equiv J/k_{\mathrm{B}}T, \tilde{h}$; more general a whole set of couplings)

the RG operator $\boldsymbol{\mathcal{R}}$ acts on the space of coupling parameters

A *fixed* point of RG transformation is a point in coupling parameter space (defining a fixed point Hamiltonian $\boldsymbol{\widetilde{\mathcal{H}}^*}$) where

$$\boldsymbol{\mathcal{R}}[\boldsymbol{\widetilde{\mathcal{H}}^*}] = \boldsymbol{\widetilde{\mathcal{H}}^*} \qquad (\widetilde{\mathcal{H}}^* \text{ is invariant under the transformation})$$

Why RG fixed points physical significant?

For the correlation length: $\xi = \xi[\,\widetilde{\mathcal{H}}] \longrightarrow \xi' = \xi[\,\widetilde{\mathcal{H}}'] = \frac{1}{\lambda}\xi[\,\widetilde{\mathcal{H}}]$

$\Longrightarrow$ **at the fixed point** $\xi[\underbrace{\mathcal{R}[\widetilde{\mathcal{H}}^*]}_{\widetilde{\mathcal{H}}^*}] = \frac{1}{\lambda}\xi[\widetilde{\mathcal{H}}^*] \;\; \Longrightarrow \boldsymbol{\xi = 0}$ **or** $\boldsymbol{\xi = \infty}$

Fixed point with $\xi = \infty$

describes a critical point ($T = T_\text{c}$)

$\Longrightarrow$ ***critical* fixed point**

Fixed point with $\xi = \mathbf{0}$

describes a system

in the high-temperature limit (completely *random*)

or

in the low-temperature limit (completely *ordered*)

$\Longrightarrow$ ***trivial* fixed point**

- (in general) a RG transformation has several fixed points
- **each fix point has its own basin of attraction**
  (all points within this basin ultimately reach the fixed point after an infinite number of transformations $\mathcal{R}$)

5. Renormalization Group

$T = T_c$

(A) Initial state:
Clusters of down spins (white points) and up spins (black points) of all sizes (due to the fact that at the critical point we have fluctuation at all length scales)

(B) Result after one block spin transformation (coarse-graining by "majority rule" for a 3x3 block + rescaling by a linear factor 3): picture looks very much like the first (clusters of all size), (A) and (B) are statistically the same

Figures from J. Cardy, *Scaling and Renormalization in Statistical Physics*

$T > T_{\text{c}}$

(C) Initial state for $T > T_{\text{c}}$: picture looks not so much different from (A) (system is only slightly above T_{c})

(D) Result after one block spin transformation of (C): picture looks more random (missing of large clusters)
$\Longrightarrow$ a few transformation already change the system state to a total random one

Figures from J. Cardy, *Scaling and Renormalization in Statistical Physics*

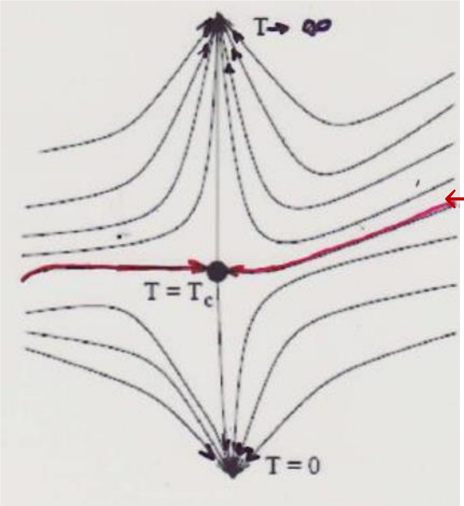
- **for all points in the basin of attraction of a critical fixed point: $\xi = \infty$**

$\Longrightarrow$ critical manifold (critical surface)

Proof: $\xi[\widetilde{\mathcal{H}}] = \lambda\xi[\widetilde{\mathcal{H}}'] = \lambda^2\xi[\widetilde{\mathcal{H}}''] = \cdots = \lambda^n\xi[\widetilde{\mathcal{H}}^{(n)}]$

Since $\lim_{n\to\infty} \widetilde{\mathcal{H}}^{(n)} = \widetilde{\mathcal{H}}^*$ and $\xi[\widetilde{\mathcal{H}}^*] = \infty$ for a critical fixed point,

r.h.s. becomes infinity $\Longrightarrow \xi[\widetilde{\mathcal{H}}] = \infty$



← **Critical surface**

(6) study of local RG flow close to a fixed Point

- linearization of $\boldsymbol{\mathcal{R}}$ in the vicinity of $\widetilde{\boldsymbol{\mathcal{H}}}^*$

$$\widetilde{\boldsymbol{\mathcal{H}}}' = \boldsymbol{\mathcal{R}}\big[\widetilde{\boldsymbol{\mathcal{H}}}\big] = \boldsymbol{\mathcal{R}}\big[\widetilde{\boldsymbol{\mathcal{H}}}^* + \Delta\widetilde{\boldsymbol{\mathcal{H}}}\big] = \widetilde{\boldsymbol{\mathcal{H}}}^* + \boldsymbol{\mathcal{L}}\Delta\widetilde{\boldsymbol{\mathcal{H}}} + \cdots$$

$\boldsymbol{\mathcal{L}}$ linear (super-)operator with eigenvectors (eigenoperators) Q_j and eigenvalues μ_j:

$$\boldsymbol{\mathcal{L}}Q_j = \mu_j Q_j$$

$\mathcal{R}$ and therefore $\mathcal{L}$ and μ_j depend on λ: $\mu_j = \mu_j(\lambda) = \lambda^{l_j}$
(it follows from the semi-group properties of RG)

- expansion of $\Delta\widetilde{\boldsymbol{\mathcal{H}}}$ by a set of eigenoperators Q_j:

$$\Delta\widetilde{\boldsymbol{\mathcal{H}}} = \sum_j h_j\, Q_j \Longrightarrow \widetilde{\boldsymbol{\mathcal{H}}} = \widetilde{\boldsymbol{\mathcal{H}}}^* + \sum_j h_j\, Q_j$$

coefficients h_j: characterize the properties of parameter space near the fixed point
⇝ called (linear) **scaling fields**
(by construction $h_j = 0$ at a fixed point)

- can express $\widetilde{\mathcal{H}}$ and therefore also the density of free energy

  as function of the scaling fields : $\tilde{\boldsymbol{g}}[\widetilde{\boldsymbol{\mathcal{H}}}] = \tilde{\boldsymbol{g}}(h_1, h_2, h_3, \dots)$

$$\boldsymbol{\mathcal{R}}[\widetilde{\boldsymbol{\mathcal{H}}}] = \boldsymbol{\mathcal{R}}[\widetilde{\boldsymbol{\mathcal{H}}}^* + \sum_j h_j\, Q_j] = \widetilde{\boldsymbol{\mathcal{H}}}^* + \sum_j h_j \boldsymbol{\mathcal{L}}\, Q_j = \widetilde{\boldsymbol{\mathcal{H}}}^* + \sum_j h_j\, \lambda^{l_j} Q_j$$

$$= \widetilde{\boldsymbol{\mathcal{H}}}' = \widetilde{\boldsymbol{\mathcal{H}}}^* + \sum_j h_j{}'\, Q_j$$

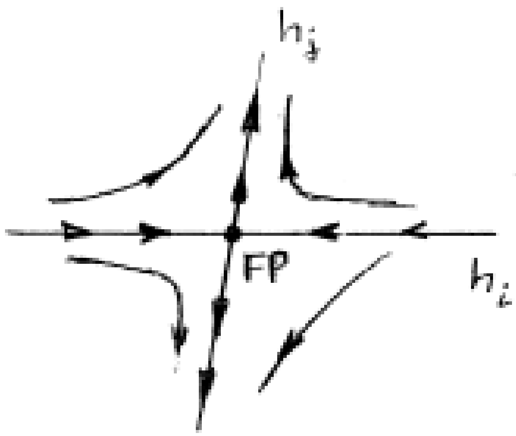
$\Longrightarrow h_j{}' = \lambda^{l_j}\, h_j$

$\Longrightarrow$ For $l_j > 0$: scaling field increases (for starting field $h_j^{(0)} \neq 0$) with iterations $\Longrightarrow$ RG flow is repelled from the FP $\Longrightarrow$ corresponding h_j called "**relevant"** scaling fields"

$l_j < 0$: scaling field decreases (for starting field $h_j^{(0)} \neq 0$) with iterations $\Longrightarrow$ h_j called "**irrelevant"** scaling fields

$l_j = 0$: $\Longrightarrow$ **"marginal"** scaling fields (has to retain corrections of quadratic order)

Note: The notion of relevance (irrelevance, …) is only relative to a particular FP



h_i **with** $l_i < 0 \Longrightarrow h_i$ **irrelevant**

h_j **with** $l_j > 0 \Longrightarrow h_j$ **relevant**

The irrelevant scaling fields correspond to directions of flow into the FP.

If we start near a **critical** FP C, but *not* at a critical manifold, then the flow away from C is determined by relevant scaling fields (with *relevant* eigenvalues) $\Longrightarrow$ **the exponents l_j associated with *relevant* couplings h_j of a critical FP are closely related to the critical exponents!**

Consequences from RG for free energy and correlation function

$$\tilde{\boldsymbol{g}}[\widetilde{\mathcal{H}}'] = \tilde{\boldsymbol{g}}\left[\mathcal{R}[\widetilde{\mathcal{H}}]\right] = \tilde{\boldsymbol{g}}\left[\widetilde{\mathcal{H}}^* + \sum_j h_j \lambda^{l_j} Q_j\right] = \tilde{\boldsymbol{g}}\left(\lambda^{l_1} h_1, \lambda^{l_2} h_2, \dots\right)$$

$$= \lim_{L'\to\infty} \frac{1}{(L')^d} \ln \mathcal{Z}_{N'}[\widetilde{\mathcal{H}}'] = \lim_{L\to\infty} \frac{1}{(L/\lambda)^d} \ln \mathcal{Z}_N[\widetilde{\mathcal{H}}] = \lambda^d \tilde{\boldsymbol{g}}(h_1, h_2, \dots)$$

For the singular part of $\tilde{\boldsymbol{g}}$:

$$\tilde{\boldsymbol{g}}_{\text{sing}}(h_1, h_2, \dots) = \lambda^{-d} \tilde{\boldsymbol{g}}_{\text{sing}}\left(\lambda^{l_1} h_1, \lambda^{l_2} h_2, \dots\right)$$

$\Rightarrow$ with $h_1 = t, h_2 = h$ $\Rightarrow$ proof of scaling hypothesis

$\Rightarrow$ with $\lambda^{l_1} h_1 = \lambda^{l_1}|t| = 1 \Longrightarrow \tilde{\boldsymbol{g}}_{\text{sing}}(t, h, h_3, \dots) = |t|^{d/l_1} \tilde{\boldsymbol{g}}_{\text{sing}}\left(h t^{-l_2/l_1}\right)$

$\Rightarrow$ critical exponents

$$\alpha = 2 - d/l_1, \gamma = \cdots$$

Using the renormalization rule $\psi(\vec{r}) = \zeta\, \psi'(\vec{r}\,')$ for the spin variables:

$\Rightarrow$ proof of scaling hypothesis for correlation function $G(r; t, h)$

Renormalization Group and universality ?

A universality class consist of all those critical models which flow into a particular critical fixed point

(To each universality class correspond a different critical FP)

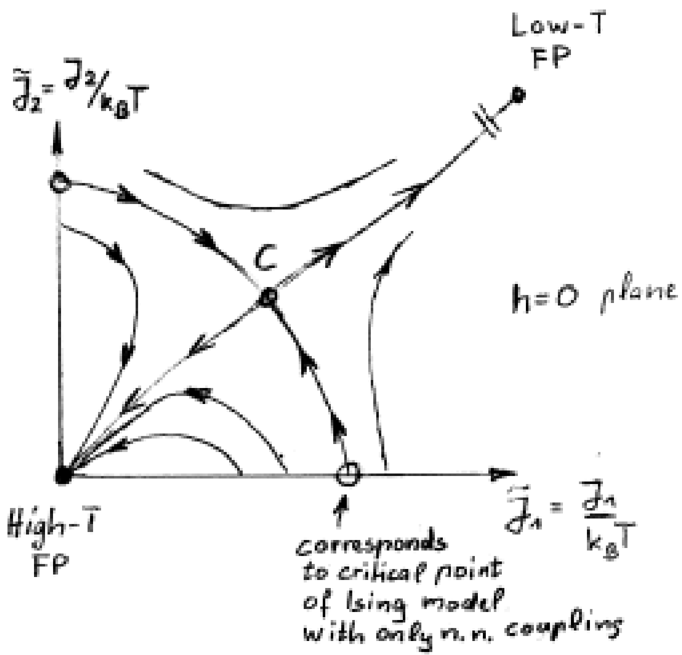
Example: Ferromagnetic system

- critical phenomena observed for $t \equiv \frac{T - T_C}{T_C} = 0$ and $h = 0$
  Slight deviations from $t = h = 0$ will drive the system away from the CP
  $\Longrightarrow$ $h_1 \equiv \boldsymbol{t}$ and $h_2 \equiv \boldsymbol{h}$ corresponds to ***relevant scaling fields***
- variables other than t and h, represented by the scaling fields $h_3, h_4, \ldots$, do *not* effect the essential features like critical exponents
  $\Longrightarrow$ $\boldsymbol{h_3, h_4, \ldots}$ correspond to ***irrelevant scaling fields***

Consider an Ising model with *nearest-neighbor* **and** *next-nearest-neighbor* interaction:

$$\tilde{\mathcal{H}} \equiv -\beta\mathcal{H} = +\tilde{J}_1 \sum_{<i,j>} S_i S_j + \tilde{J}_2 \sum_{i,j=n.n.n.} S_i S_j + \tilde{h} \sum_i S_i$$

with „reduced“ variables $\tilde{J}_{1,2} \equiv J_{1,2}/k_\mathrm{B}T,\ \tilde{h} \equiv h/k_\mathrm{B}T$



Schematic flow diagram for Ising model with n.n. and n.n.n. interaction: Since here $h = 0$ we have only one relevant scaling field + one irrelevant field (relative to C)

In practice many different ways of implementing the RG procedure:

Generally, 2 kinds: **real space RG** and **momentum-space RG**

(1) Real space RG

- **applied to discrete systems on a lattice in real space**
- **Migdal-Kadanoff real space RG**

  performing partial traces over the Hilbert spaces associated with certain block spins

  For 1-dimensional Ising model ⇝ tutorial

  a quite clever and illustrative approach for $d \geq 2$ is the "Bond-Moving Technique"

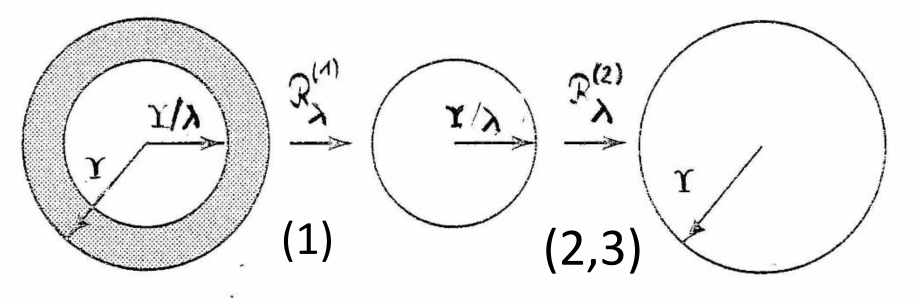
  (however with some uncontrolled approximations)

(2) Momentum-space RG

- **Applied to translationally invariant systems whose properties can described by an effective field theory**
- **Most popular implementation ("RG ala Wilson")**

  based on mode elimination in momentum space by introducing a cut-off separating long-wavelength fluctuations from short-wavelength fluctuations

  for a sharp cutoff ⟶ "Momentum shell RG"



Step 1: all fields with $\vec{k}$ in the shell between $\Upsilon/\lambda < k < \Upsilon$ are removed
Step 2: rescaling of all wave vectors $\vec{k} \longrightarrow \vec{k}' = \lambda\vec{k}$
Step 3: renormalization of the fields $\psi_{\vec{k}} \longrightarrow \psi'_{\vec{k}'} = \frac{1}{\zeta}\psi_{\vec{k}}$

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