An introduction to extreme value statistics
(part 2/2)

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Range of applications

Gallery of correlations

Illustrations of EVS in statistical physics

- DNA replication times in Xenopus Laevis
- Order parameter in percolation
- EVS in trees
- EVS of some Gaussian processes
- Integer partitions and the ideal Bose gas

Range of applications

Overview/statistical physics

J.-P. Bouchaud, M. Potters *Theory of Financial Risks* Cambridge University Press (2000)	Introduction
D. Sornette *Critical Phenomena in Natural Sciences* Springer (2006)	Introduction
M. Clusel, E. Bertin Int. J. Mod. Phys. B **22**, 3311 (2008)	Review
S. N. Majumdar, A. Pal arXiv:1406.6768v3 (2014)	Lecture notes
J.-Y. Fortin, M. Clusel J. Phys. A: Math. Theor. **48**, 183001 (2015)	Review

Range of applications

Modelling

R. W. Katz, M. B. Parlange, P. Naveau — Hydrology
Adv. Water Resour. **25**, 1287 (2002)

A. J. McNeil, R. Frey — Finance
J. Empir. Financ. **7**, 271 (2000)

T. Antal *et al.* — Astronomy
Eur. Phys. Lett. **88**, 59001 (2009)

J. Bechhoefer, B. Marshall — Cell biology
Phys. Rev. Lett. **98**, 098105 (2007)

S. Hallerberg, J. Bröcker, H. Kantz — Prediction/Forecasting
In *Nonlinear Time Series Analysis in the Geosciences,*
R. V. Donner, S. M. Barbosa (eds), Springer (2008)

Range of applications

Disordered systems

J.-P. Bouchaud, M. Mézard J. Phys. A: Math. Gen. **30**, 7997 (1997)	Random energy model
S. N. Majumdar, P. L. Krapivksy Physica A **318**, 161 (2003)	Hierarchical correlations
G. Biroli, J.-P. Bouchaud, M. Potters J. Stat. Mech. P07019 (2007)	Disordered systems
Y. V. Fyodorov Physica A **389**, 4229 (2010)	Logarithmic correlations
S. N. Majumdar, G. Schehr J. Stat. Mech. P01012 (2014)	Random matrices

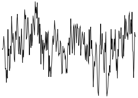
Range of applications

Dynamical systems

C. Nicolis, V. Balakrishnan, G. Nicolis Phys. Rev. Lett. 210602 (2006)	Extreme events in chaotic dynamics
M. Ghil *et al.* Nonlin. Processes Geophys. **18**, 295 (2011)	Review
J. M. Freitas Dynamical Systems **28**, 302 (2013)	EVS in chaotic dynamics
H. Aytaç, J. M. Freitas, S. Vaienti Trans. Amer. Math. Soc. **367**, 8229 (2015)	EVS in deterministic and random dynamical systems

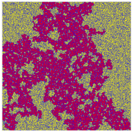
Gallery of correlations

Time series



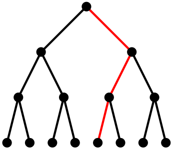
$$\langle x(t')x(t'+t)\rangle \sim t^{-\gamma}$$

Critical phenomena



$$g(r) \sim r^{-(d-2+\eta)}$$

Trees



$g(i,j) \propto$ length of shared path

Random matrices

$$\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$$P[\lambda_i] \propto \exp\left[-\beta\left(\frac{N}{2}\sum_{i=1}^{N}\lambda_i^2 - \frac{1}{2}\sum_{i\neq j}\ln|\lambda_i-\lambda_j|\right)\right]$$

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DNA replication in *Xenopus Laevis*

Bechhoefer, Marshall 2007

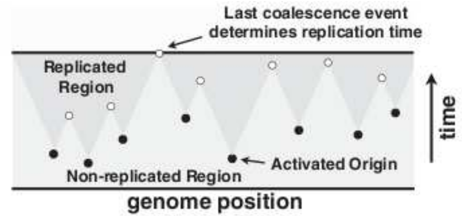

FIG. 1. Schematic of DNA replication model. Space-time diagram showing multiple origins (filled circles), each expanding symmetrically at constant velocity. Domains coalesce when they meet (open circles).

$\sim 3 \times 10^9$ bases
$\sim 10^5$ replication origins
~ 1kb/min fork velocity
~ 20 min replication time
$\sim 10^{-4}$ failure probability

DNA replication in *Xenopus Laevis*

Bechhoefer, Marshall 2007

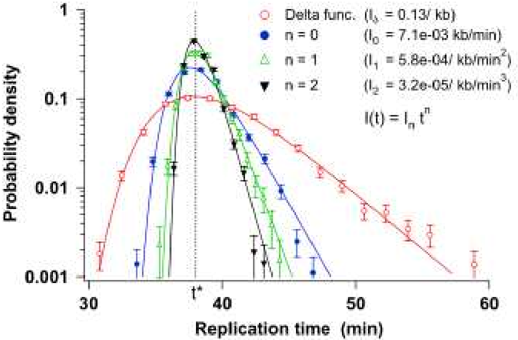

FIG. 2 (color online). Replication-time distribution function, fixing the mode to be $t^* = 38$ minutes. Markers are results from Monte Carlo simulations (3000 trials per simulation); solid lines are fits to the Gumbel distribution.

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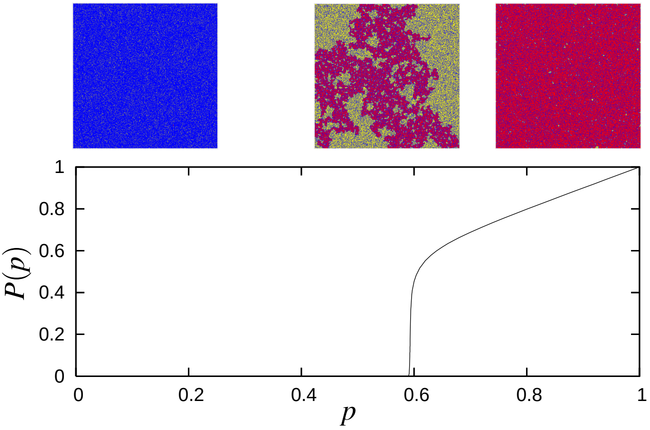
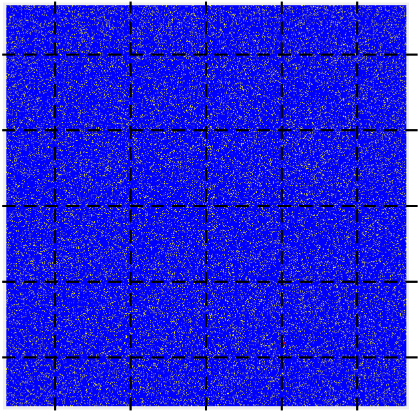
Percolation



Fraction occupied by the largest cluster is the order parameter.

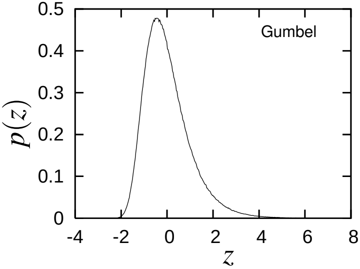
Subcritical percolation ($p \ll p_c$)

Duxbury, Leath 1987; Bazant 2000; Borgs et al. 2001; van der Hofstad, Redig 2006



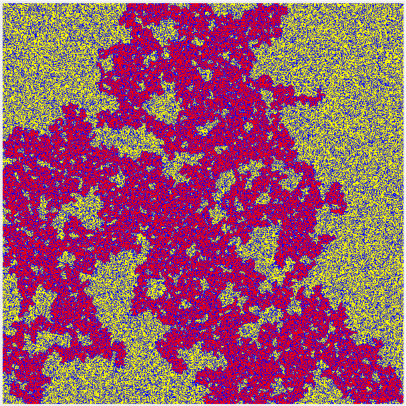
largest cluster $\sim \log L$

overall largest cluster is the largest cluster from uncorrelated blocks



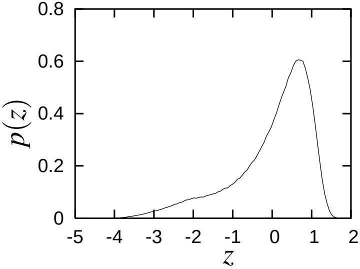
Critical percolation ($p = p_c$)

Hovi, Aharony 1996; Sen 2001; Borgs et al. 2001



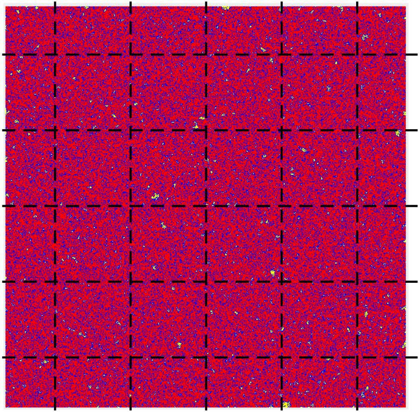
largest cluster $\sim L^D \quad (D = 91/48)$

correlations span the system!
non-trivial distribution emerges



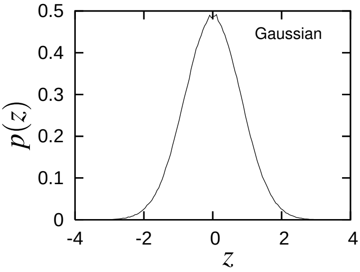
Supercritical percolation ($p \gg p_c$)

Borgs et al. 2001



largest cluster $\sim L^d$

largest cluster is extensive:
sum up contributions from
uncorrelated blocks



Percolation

P. M. Duxbury, P. L. Leath J. Phys. A: Math. Gen. **20**, L411 (1987)	Largest cluster in subcritical percolation
J.-P. Hovi, A. Aharony Phys. Rev. E **53**, 235 (1996)	Clusters in critical percolation
M. Z. Bazant Phys. Rev. E **62**, 1660 (2000)	Largest cluster in subcritical percolation
P. Sen J. Phys. A: Math. Gen. **34**, 8477 (2001)	Largest clusters in critical percolation
C. Borgs *et al.* Commun. Math. Phys. **224**, 153 (2001)	Finite-size scaling of largest cluster
R. van der Hofstad, F. Redig J. Stat. Phys. **122**, 671 (2006)	Largest clusters in non-critical percolation

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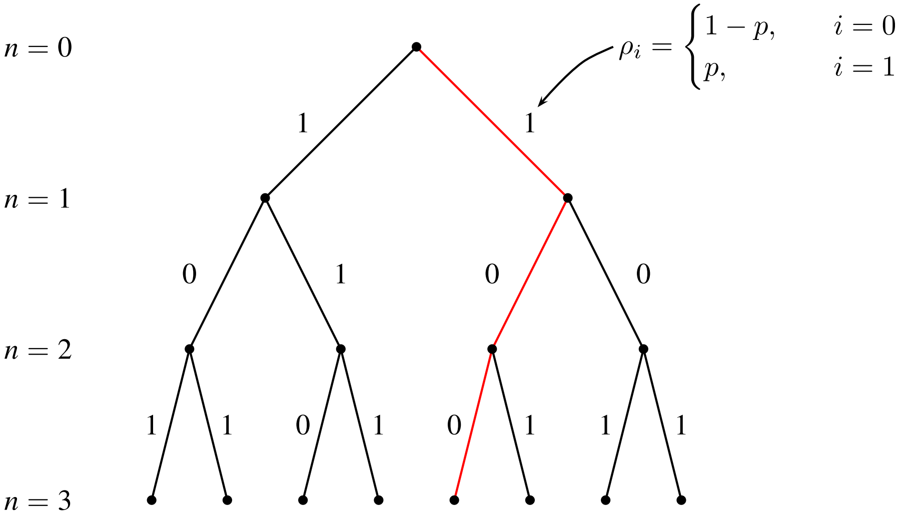
EVS in trees

Majumdar, Krapivsky 2000



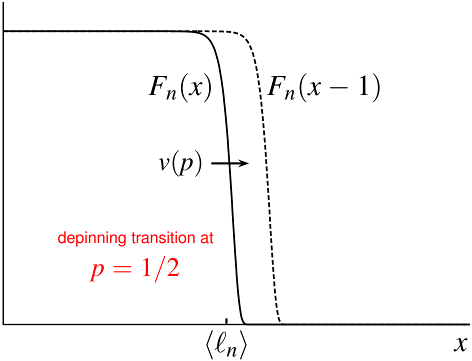
$\ell_3 =$ shortest path (at depth 3) $= 1 + 0 + 0 = 1$

EVS in trees

Analogy between 1-0 behaviour and wave fronts (c.f. $\partial_t \phi = \partial_{xx}\phi + \phi - \phi^2$)

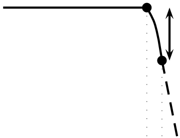
$$\mathbb{P}(\ell_{n+1} \geq x) = F_{n+1}(x) = [(1-p)F_n(x) + pF_n(x-1)]^2$$

initial condition: $F_0(x) = \begin{cases} 1, & x \leq 0 \\ 0, & x > 0 \end{cases}$



EVS in trees

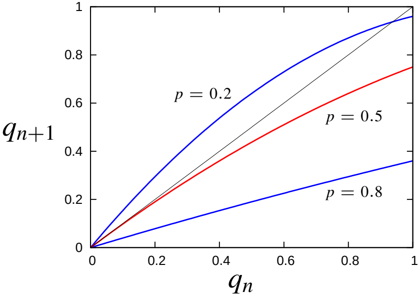
Tracking the tip of the front



$$q_n = F_n(0) - F_n(1) = 1 - F_n(1)$$

Substitute $q_n = 1 - F_n(1)$ into recursion relation for F_n:

$$q_{n+1} = 2(1-p)q_n - (1-p)^2 q_n^2$$



EVS in trees

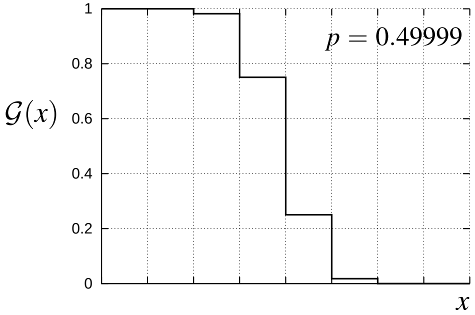
$p < 1/2$: wave front is 'pinned'

Wave front gets pinned at a finite $\langle \ell_n \rangle$ as $n \to \infty$.

$F_n(x)$ approaches a stationary distribution:

$$\mathcal{G}(x) = [(1-p)\mathcal{G}(x) + p\mathcal{G}(x-1)]^2$$

Iterate, starting with $\mathcal{G}(0) = 1$, to obtain $\mathcal{G}(1), \mathcal{G}(2), \ldots$



EVS in trees

$p \geq 1/2$: wave front is 'depinned'

Wave front travels as a fixed shape $\mathcal{F}^2$ located at $\langle \ell_n \rangle \approx v(p)n$:

$$
\begin{aligned}
F_{n+1}(x) &\to \mathcal{F}^2(x - v(n+1)) \\
F_n(x) &\to \mathcal{F}^2(x - vn) \\
F_n(x-1) &\to \mathcal{F}^2(x - v - 1)
\end{aligned}
$$

Substitute into recursion relation:

$$\mathcal{F}(z - v) = (1-p)\mathcal{F}^2(z) + p\mathcal{F}^2(z-1), \quad z = x - vn.$$

Behind the front, $1 - \mathcal{F}(z) \approx e^{\lambda z}$. Substitute and linearise:

$$v_\lambda(p) = -\frac{\log[2(1-p) + 2pe^{-\lambda}]}{\lambda}.$$

Front selects $v_{\lambda^\star}(p)$.

EVS in trees

Bramson 1978; Brunet, Derrida 1997; Ebert, van Saarloos 1998; Majumdar, Krapivsky 2000

(Majumdar & Krapivsky 2000)

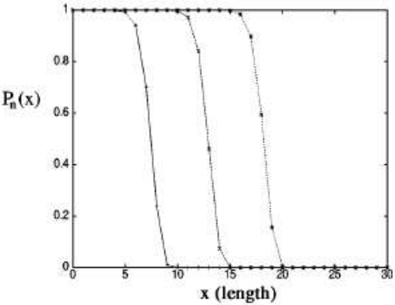


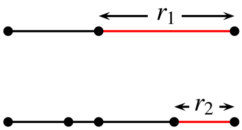
FIG. 1. The propagating front for the cumulative distribution $P_n(x)$ of the minimal length for the bimodal distribution with $p=0.8$.

$$\langle \ell_n \rangle = \begin{cases} c(p), & p < 1/2 \\ (\log 2)^{-1} \log \log n, & p = 1/2 \\ v(p) n, & p > 1/2. \end{cases}$$

EVS in trees

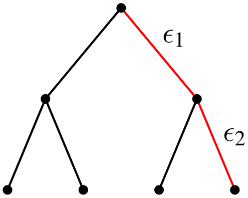
Hierarchichal structure is rather generic...

fragmentation process



$$-\log r_i = \epsilon_i$$

polymer/tree



$$\ell_k^{(n)} = \prod_{i=1}^n r_i \qquad E_k^{(n)} = \sum_{i=1}^n \epsilon_i$$

$$\mathbb{P}(\min_k E_k^{(n+1)} > x) = F_{n+1}(x) = \left[\int_0^\infty \mathrm{d}\epsilon\, F_n(x-\epsilon)\rho(\epsilon)\right]^2$$

- Write down recursion relation between F_n and F_{n+1}.
- Identify conditions for travelling wave solution.
- Linearise wave equation.
- Calculate wave speeds, etc.

EVS in trees

S. N. Majumdar, P. L. Krapivksy — Extremal paths on random Cayley trees
Phys. Rev. E **62**, 7735 (2000)

E. Ben-Naim, P. L. Krapivksy, S. N. Majumdar — Extremal paths on random Cayley trees
Phys. Rev. E **64**, 035101(R) (2001)

D. S. Dean, S. N. Majumdar — Polymers on random binary trees
Phys. Rev. E **64**, 046121 (2001)

S. N. Majumdar, P. L. Krapivksy — Polymers, binary search trees
Phys. Rev. E **65**, 036127 (2002)

D. S. Dean, S. N. Majumdar — Fragmentation, m-ary search
J. Phys. A: Math. Gen. **35**, L501 (2002)

S. N. Majumdar, P. L. Krapivksy — Overview article
Physica A **318**, 161 (2003)

Outline

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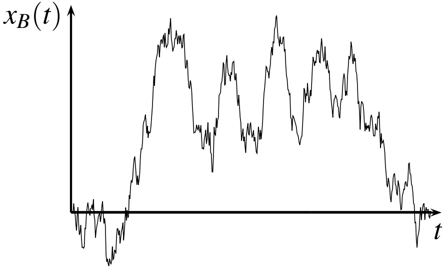
Maximum of a Brownian bridge



$$\langle X(s)X(t)\rangle = \min(s,t) - st$$

(Berman's condition fails)

$$\mathbb{P}(\max_t x_B(t) \le m) = \frac{W_m(x_t = 0, t \,|\, x_0 = 0, 0)}{W(x_t = 0, t \,|\, x_0 = 0, 0)},$$

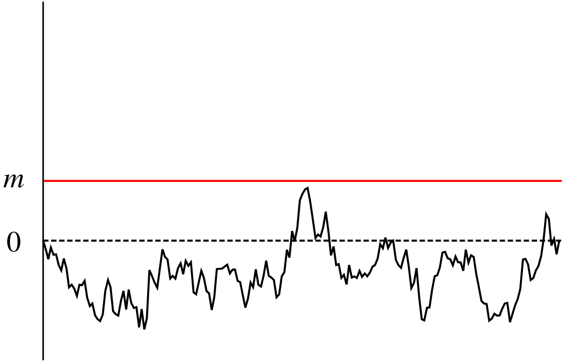
where transition probability $W_m(x_t, t \,|\, x_0 = 0, 0)$ satisfies

$$\left(\frac{\partial}{\partial t} - \frac{1}{2}\frac{\partial^2}{\partial x^2} + V_m(x)\right) W_m = 0$$

$$V_m(x) = \begin{cases} 0, & x < m, \\ \infty, & x \ge m. \end{cases}$$

Maximum of a Brownian bridge

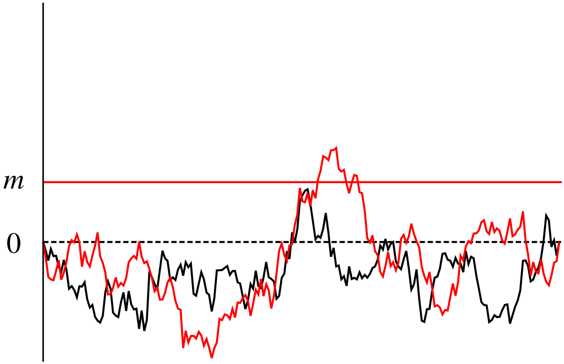
Reflection principle



$$W(x_t = 0, t \,|\, x_0 = 0, 0) = \frac{1}{\sqrt{2\pi t}} \exp(-x_t^2/2t)\Big|_{x_t=0}$$
$$= \frac{1}{\sqrt{2\pi t}}$$

Maximum of a Brownian bridge

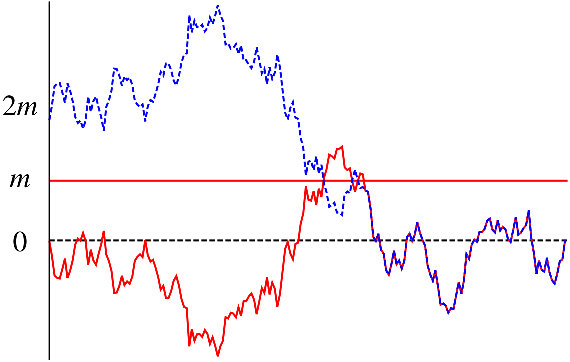
Reflection principle



$$W_m(0,t\,|\,0,0) = W(0,t\,|\,0,0) - \text{overcounting term}$$

Maximum of a Brownian bridge

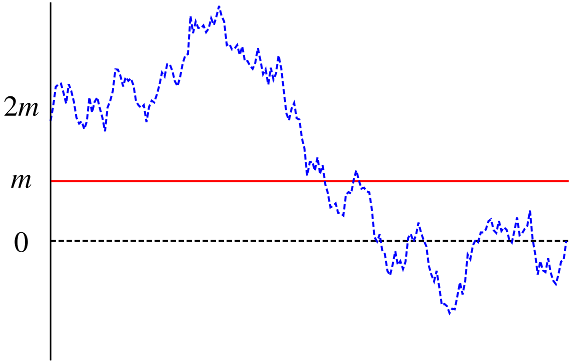
Reflection principle



$$W_m(0, t \,|\, 0, 0) = W(0, t \,|\, 0, 0) - \text{overcounting term}$$

Maximum of a Brownian bridge

Reflection principle



$$W_m(0,t\,|\,0,0) = W(0,t\,|\,0,0) - W(2m,t\,|\,0,0)$$
$$= \frac{1}{\sqrt{2\pi t}}\left(1 - e^{-2m^2/t}\right)$$

$$\mathbb{P}(\max_t x_B(t) \leq m) = 1 - e^{-2m^2/t}$$

Mean-subtracted Brownian bridge

Watson 1961

Define a new Gaussian process by substracting the bridge's mean:

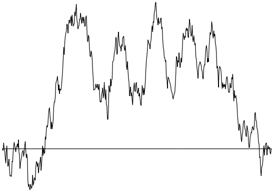
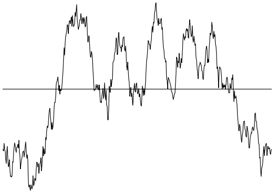
$$x_W(t) := x_B(t) - \int_0^t \mathrm{d}t'\, x_B(t').$$

'Watson' bridge has correlations

$$\langle X(s)X(t)\rangle = \min(s,t) - \frac{1}{2}(s+t) + \frac{1}{2}(s-t)^2 + \frac{1}{12}.$$

This depends only on time differences:

$$\langle X(s)X(s+t)\rangle = \frac{1}{12}\left[1 - 6|t|(1-|t|)\right].$$

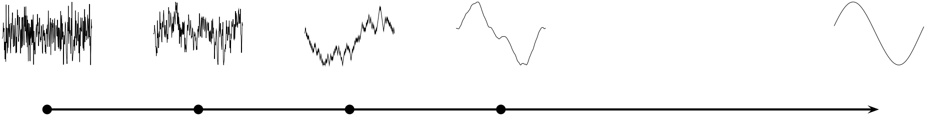
	$x_B(t)$	$x_W(t)$
realisation		
local variance	$t(1-t)$	$1/12$
Fourier decomposition $a_k, b_k \overset{d}{\sim} N(0,1)$	$\frac{\sqrt{2}}{\pi}\sum_{k=1}^{\infty}\frac{a_k}{k}\sin \pi k t$	$\frac{1}{\sqrt{2}\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left(a_k \sin 2\pi k t + b_k \cos 2\pi k t\right)$

Periodic Gaussian $1/f^{\alpha}$ noise

Signal construction

$$h(t) \propto \sum_{k=1}^{n} \frac{1}{k^{\alpha/2}} \left[a_k \cos\left(\frac{2\pi k}{T} t\right) + b_k \sin\left(\frac{2\pi k}{T} t\right) \right], \; a_k, b_k \overset{d}{\sim} N(0,1)$$

white noise	$1/f$ noise	Edwards Wilkinson	Mullins Herring	single mode
$\alpha = 0$	1	2	4	$\alpha \to \infty$

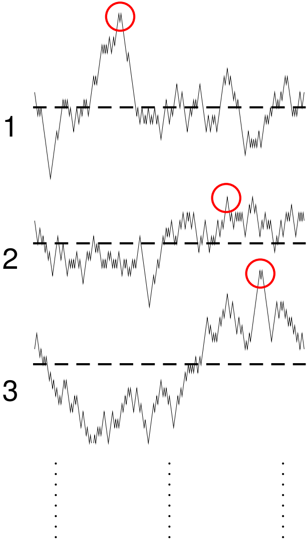


weaker ← correlations

stronger → correlations

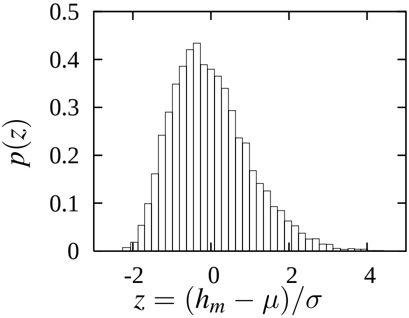
Periodic Gaussian $1/f^{\alpha}$ noise

Maximum relative height: Raychaudhuri *et al.* 2001



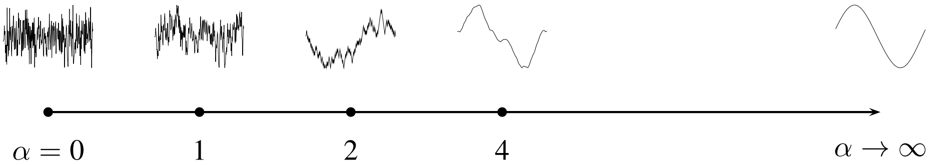
build histogram

$$h_m := \max_t h(t) - \overline{h(t)}$$



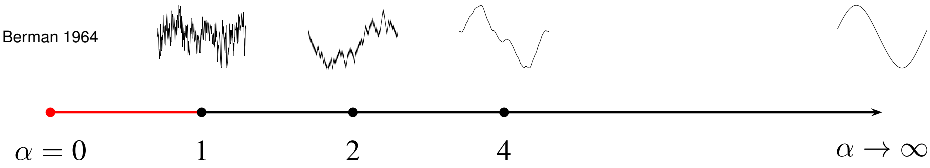
Periodic Gaussian $1/f^\alpha$ noise

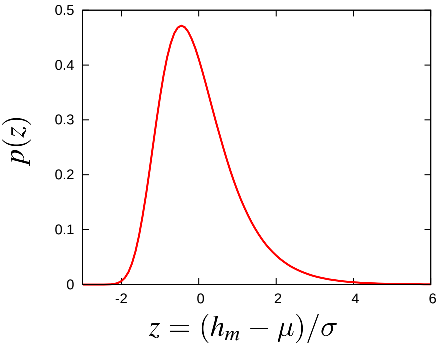
Maximum relative height

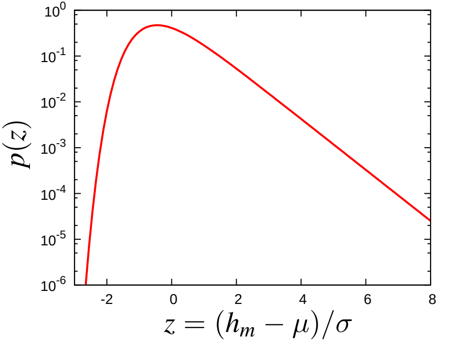


Periodic Gaussian $1/f^{\alpha}$ noise

Maximum relative height

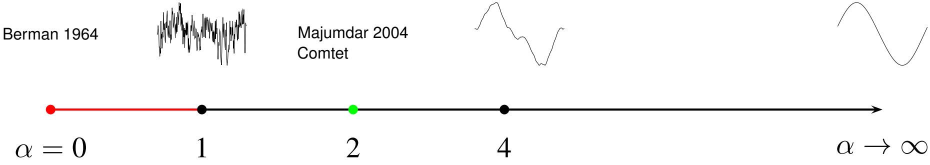


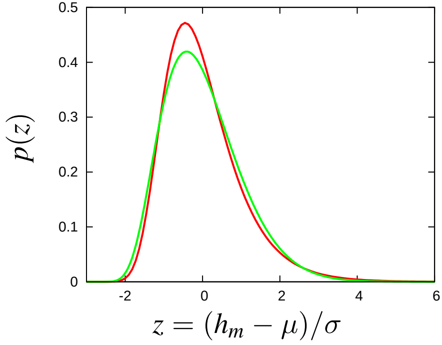


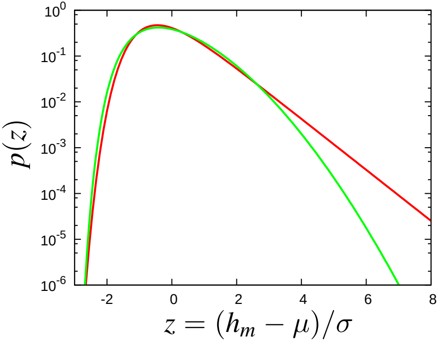


Periodic Gaussian $1/f^{\alpha}$ noise

Maximum relative height

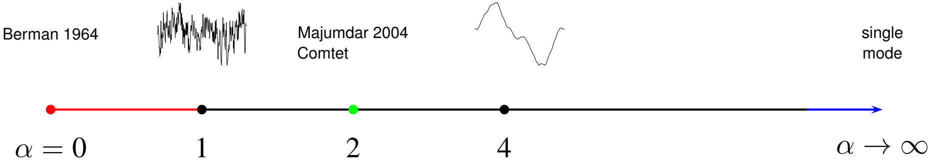


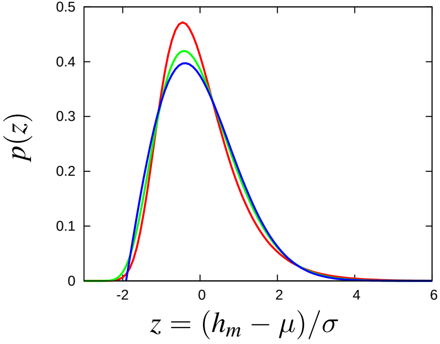


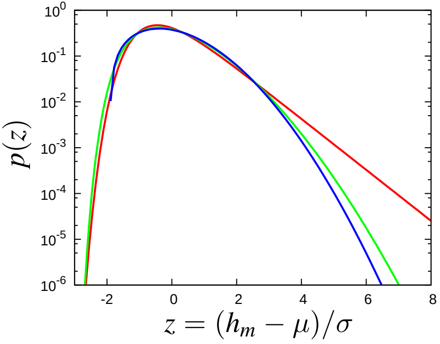


Periodic Gaussian $1/f^\alpha$ noise

Maximum relative height







Periodic Gaussian $1/f^{\alpha}$ noise

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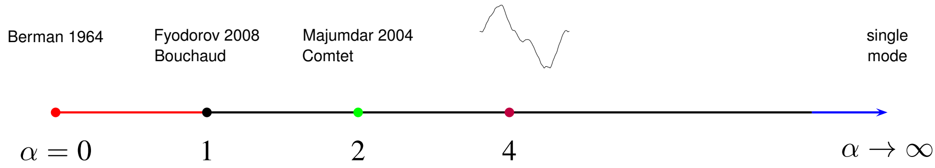
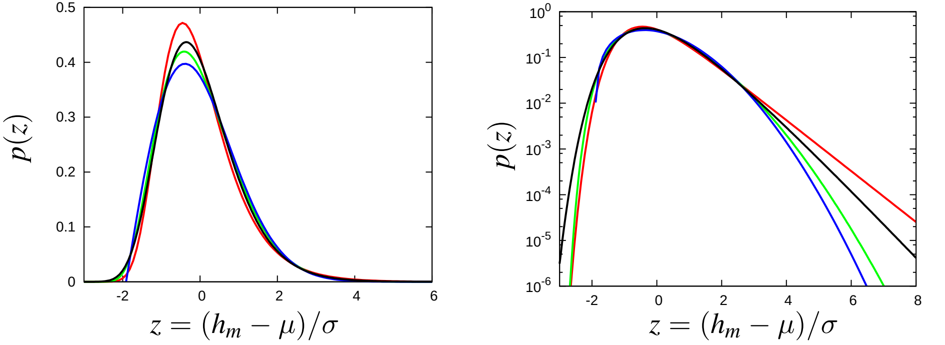
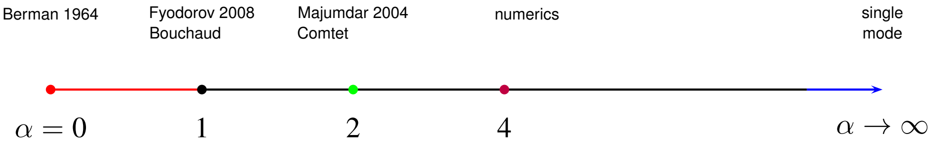
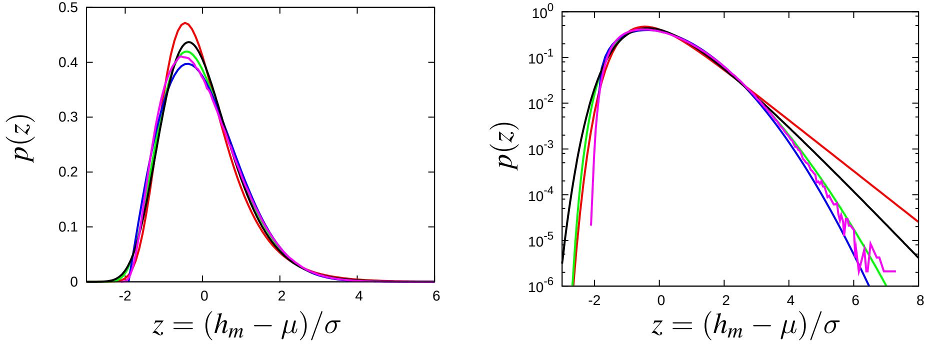




Periodic Gaussian $1/f^\alpha$ noise

Maximum relative height





Extremes of Gaussian processes

S. M. Berman — Berman's condition
Ann. Math. Stat. **33**, 502 (1964)

P. Biane, J. Pitman, M. Yor — Review
Bull. Amer. Math. Soc **38**, 435 (2001)

S. N. Majumdar, A. Comtet — MRH for Edwards-Wilkinson $(1/f^2)$
J. Stat. Phys. **119**, 777 (2005)

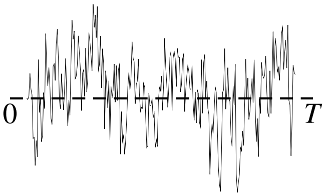
G. Györgyi *et al.* — MRH for $1/f^\alpha$ noise
Phys. Rev. E **75**, 021123 (2007)

Y. V. Fyodorov, J.-P. Bouchaud — MRH for $1/f$ noise
J. Phys. A: Math. Theor. **41**, 372001 (2008)

H. J. Hilhorst, P. Calka, G. Schehr — Random accelaration process $(1/f^4)$
J. Stat. Mech. P10010 (2008)

Roughness of $1/f$ noise

Antal *et al.* 2001



$$w^2 := \frac{1}{T}\int_0^T \mathrm{d}t\, [h(t) - \overline{h(t)}]^2 \propto \sum_{k=1}^{n} \frac{1}{k}(a_k^2 + b_k^2)$$

$$\propto \sum_{k=1}^{n} \frac{\varepsilon_k}{k},$$

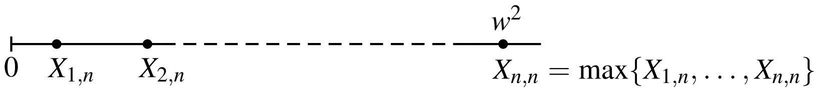
where ε_k are standard exponential variables.

Antal *et al.* observed that

$$w^2 \overset{d}{\sim} \text{Gumbel!}$$

Roughness of $1/f$ noise

Rényi 1953; Bertin 2005



Interpret the partial sums

$$X_{m,n} := \sum_{k=1}^{m} \frac{\varepsilon_k}{n-k+1}, \quad m \leq n$$

as marking the positions of a collection of ordered points $X_{m,n}$. In particular

$$w^2 = X_{n,n} = \max\{X_{1,n}, \ldots, X_{n,n}\}.$$

Roughness of $1/f$ noise

Rényi 1953; Bertin 2005

Starting from a collection of iid standard exponentials

$$f_{\varepsilon_1,\dots,\varepsilon_n}(y_1,\dots,y_n) = \prod_{k=1}^{n} \exp(-y_k) = \exp\left(-\sum_{k=1}^{n} y_k\right),$$

change variables with

$$x_m = \sum_{k=1}^{m} \frac{y_k}{n-k+1}, \quad \left|\frac{\partial y_i}{\partial x_j}\right| = n!$$

to obtain

$$f_{X_{1,n},\dots,X_{n,n}}(x_1,\dots,x_n) = n! \exp\left(-\sum_{k=1}^{n} x_k\right) \mathbf{1}_{\{x_1<\dots<x_n\}}$$

$$f_{X_1,\dots,X_n}(x_1,\dots,x_n) = \exp\left(-\sum_{k=1}^{n} x_k\right) = \prod_{k=1}^{n} \exp(-x_k).$$

Roughness of $1/f$ noise

Rényi 1953; Bertin 2005

Rényi's representation

$$\{X_{m,n},\, m = 1, \ldots, n\} \overset{d}{\sim} \left\{ \sum_{k=1}^{m} \frac{\varepsilon_k}{n-k+1},\, m = 1, \ldots, n \right\}$$

implies roughness of $1/f$ noise

$$w^2 = \max\{X_1, \ldots, X_n\} = X_{n,n} \overset{d}{\sim} \text{Gumbel},$$

where X_k, ε_k are standard exponentials.

Roughness of $1/f$ noise

A. Rényi — Order statistics

Acta Mathematica Scient. Hungar. **IV**, 191 (1953)

T. Antal *et al.* — Roughness of $1/f$ noise

Phys. Rev. Lett. **87**, 240601 (2001)

E. Bertin — Order statistics, global fluctuations

Phys. Rev. Lett. **95**, 170601 (2005)

E. Bertin, M. Clusel — Order statistics, global fluctuations

J. Phys. A: Math. Gen. **39**, 7607 (2006)

Outline

Range of applications

Gallery of correlations

Illustrations of EVS in statistical physics
- DNA replication times in Xenopus Laevis
- Order parameter in percolation
- EVS in trees
- EVS of some Gaussian processes
- Integer partitions and the ideal Bose gas

Integer partitions and the ideal Bose gas

$\Omega(E) =$ number of ways of partitioning an integer E into a sum of (indistinguishable) positive integers.

E.g. $\Omega(5) = 7$:

5
4 + 1
3 + 2
3 + 1 + 1
2 + 2 + 1
2 + 1 + 1 + 1
1 + 1 + 1 + 1 + 1

Integer partitions and the ideal Bose gas

$\Omega(E) =$ number of ways of partitioning an integer E into a sum of (indistinguishable) positive integers.

E.g. $\Omega(5) = 7$:

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4 + 1
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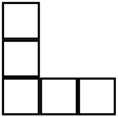
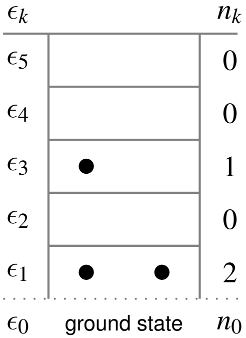
summands expansion



Young diagram



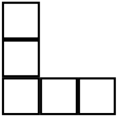
Bose gas

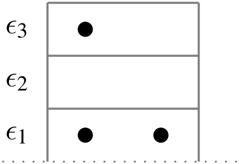
Integer partitions and the ideal Bose gas

$\Omega(E, N) =$ number of ways of partitioning E with N integers.

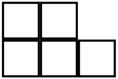
E.g. $\Omega(E = 5, N = 3) = 2$:

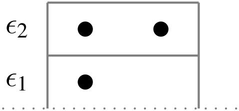
$3 + 1 + 1$





$2 + 2 + 1$





summands expansion | Young diagram | Bose gas

Integer partitions and the ideal Bose gas

Hardy, Ramanujan 1918; van Lier, Uhlenbeck 1937; Erdős, Lehner 1941; Auluck, Kothari 1946

- Number of partitions grows rapidly (e.g. $\Omega(E = 1000) \approx 2.4 \times 10^{31}$).

$$\Omega(E) \sim \frac{1}{4\sqrt{3}E} \exp\left(\pi\sqrt{\frac{2}{3}}E^{1/2}\right).$$

- Probability of partitioning E with N integers converges ($E, N \gg 1$) to Gumbel distribution!

$$\frac{\sum_{N'=1}^{N} \Omega(E, N')}{\Omega(E)} \sim \exp\left[-\exp\left(\frac{N - b_E}{a_E}\right)\right]$$

$$a_E = \frac{\sqrt{6}}{\pi} E^{1/2}$$

$$b_E = \frac{1}{\pi}\sqrt{\frac{3}{2}} E^{1/2} \log E.$$

Integer partitions and the ideal Bose gas

Auluck, Kothari 1946; Comtet, Leboeuf, Majumdar 2007

$$\Omega(E) = \sum_{\{n_k\}} \delta\left(E - \sum_{k=1}^{\infty} n_k \epsilon_k\right)$$

$$\mathcal{Z}(\beta) = \sum_{E} \Omega(E) e^{-\beta E} = \prod_{k=1}^{\infty} \frac{1}{1 - e^{-\beta \epsilon_k}}.$$

Saddle point approximation:

$$\Omega(E) \approx e^{S(\beta_0, E)},$$

where β_0 maximises

$$S(\beta, E) = \log \mathcal{Z}(\beta) + \beta E.$$

Using

$$E = \sum_{k=1}^{\infty} \frac{\epsilon_k}{e^{\beta_0 \epsilon_k} - 1} \approx \int_0^{\infty} \mathrm{d}\epsilon \, \frac{\epsilon}{e^{\beta_0 \epsilon} - 1} = \frac{\pi^2}{6\beta_0^2},$$

one recovers Hardy-Ramanujan:

$$\Omega(E) \approx \exp\left(\pi \sqrt{\frac{2}{3}} E^{1/2}\right).$$

Integer partitions and the ideal Bose gas

Auluck, Kothari 1946; Comtet, Leboeuf, Majumdar 2007

$$\sum_{N'=1}^{N} \Omega(E,N') = \tilde{\Omega}(E,N) = \sum_{\{n_k\}} \delta\left(E - \sum_{k=1}^{\infty} n_k \epsilon_k\right) \theta\left(N - \sum_{k=1}^{\infty} n_k\right)$$

$$\mathcal{Z}(\beta,z) = \sum_{E,N} \tilde{\Omega}(E,N) e^{-\beta E} z^N = \prod_{k=1}^{\infty} \frac{1}{1 - z e^{-\beta \epsilon_k}},$$

Saddle point approximation:

$$\tilde{\Omega}(E,N) \approx e^{S(\beta_0, z_0, E, N)},$$

where β_0, z_0 maximise

$$S(\beta,z,E,N) = \log \mathcal{Z}(\beta) + \beta E - N \log(z).$$

Saddle points fix

$$E = \sum_{k=1}^{\infty} \frac{\epsilon_k}{z_0^{-1} e^{\beta_0 \epsilon_k} - 1} \approx \int_0^{\infty} \mathrm{d}\epsilon \, \frac{\epsilon}{z_0^{-1} e^{\beta_0 \epsilon} - 1} = \frac{\mathrm{Li}_2(z_0)}{\beta_0^2}$$

$$N = \sum_{k=1}^{\infty} \frac{1}{z_0^{-1} e^{\beta_0 \epsilon_k} - 1} \approx \int_0^{\infty} \mathrm{d}\epsilon \, \frac{1}{z_0^{-1} e^{\beta_0 \epsilon} - 1} = -\frac{\log(1 - z_0)}{\beta_0}.$$

Integer partitions and the ideal Bose gas

Auluck, Kothari 1946; Comtet, Leboeuf, Majumdar 2007

Take large N limit ($z \uparrow 1$) to recover Erdős-Lehner:

$$\frac{\tilde{\Omega}(E,N)}{\Omega(E)} \sim \exp\left[-\exp\left(\frac{N-b_E}{a_E}\right)\right]$$
$$a_E = \frac{\sqrt{6}}{\pi}E^{1/2}$$
$$b_E = \frac{1}{\pi}\sqrt{\frac{3}{2}}E^{1/2}\log E.$$

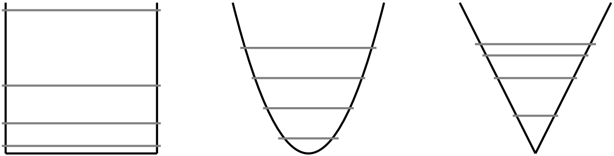
This calculations holds for equally spaced energy levels $\epsilon_1 = 1, \epsilon_2 = 2, \ldots$.

What about for $\rho(\epsilon) = \nu\epsilon^{\nu-1}$?

Integer partitions and the ideal Bose gas

Density of states $\rho(\epsilon) = \nu\epsilon^{\nu-1}$ for different $1d$ potentials

$a = \infty$ $\quad$ $a = 2$ $\quad$ $a = 1$ $\quad$ $V(x) \sim |x|^a$



$\rho(\epsilon) \sim \epsilon^{-1/2}$ $(\nu = 1/2)$

$\rho(\epsilon) \sim$ const. $(\nu = 1)$

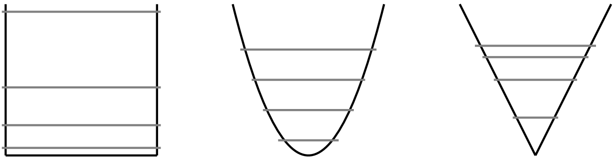
$\rho(\epsilon) \sim \epsilon^{1/2}$ $(\nu = 3/2)$

$\rho(\epsilon) \sim \epsilon^{(2-a)/2a}$

Integer partitions and the ideal Bose gas

Density of states $\rho(\epsilon) = \nu\epsilon^{\nu-1}$ for different $1d$ potentials

$a = \infty$ | $a = 2$ | $a = 1$ | $V(x) \sim |x|^a$



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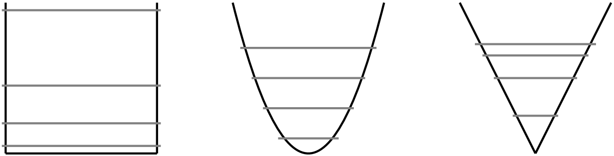
$\rho(\epsilon) \sim \epsilon^{(2-a)/2a}$

$0 < \nu < 1$	$\nu = 1$	$\nu > 1$
$\frac{\tilde{\Omega}(E,N)}{\Omega(E)} \longrightarrow$ Fréchet	Gumbel	Weibull

Integer partitions and the ideal Bose gas

Density of states $\rho(\epsilon) = \nu\epsilon^{\nu-1}$ for different $1d$ potentials

$a = \infty$ $\quad a = 2$ $\quad a = 1$ $\quad V(x) \sim |x|^a$



$\rho(\epsilon) \sim \epsilon^{-1/2}$ $(\nu = 1/2)$	$\rho(\epsilon) \sim$ const. $(\nu = 1)$	$\rho(\epsilon) \sim \epsilon^{1/2}$ $(\nu = 3/2)$	$\rho(\epsilon) \sim \epsilon^{(2-a)/2a}$

$0 < \nu < 1$	$\nu = 1$	$\nu > 1$
$\frac{\tilde{\Omega}(E,N)}{\Omega(E)} \to$ Fréchet	Gumbel	Weibull
e.g. $5 = 2^2 + 1^2$	$5 = 3 + 2$	$5 = 8^{2/3} + 1^{2/3}$
partition into powers > 1	partition into integers	partition into powers < 1

Integer partitions and the ideal Bose gas

G. H. Hardy, S. Ramanujan — Asymptotics of integer partitions
Proc. London Math. Soc. **17**, 75 (1918)

C. van Lier, G. E. Uhlenbeck — Ideal quantum gas, saddle point methods
Physica, **4**, 531 (1937)

P. Erdős, J. Lehner — Asymptotics of integer partitions, Gumbel
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F. C. Auluck, D. S. Kothari — Equally-spaced energy levels, Gumbel
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A. Comtet, P. Leboeuf, S. N. Majumdar — Power law-spaced energy levels, EVS
Phys. Rev. Lett. **98**, 070404 (2007)